EXCEPTIONAL GROUPS OF ORDER 243

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ABSTRACT. We describe all exceptional groups of order $243 = 3^5$, with explanations and proofs, adjusting a table that appears in a 2017 paper by Britnell, Saunders and Skyner. There are ten exceptional groups of order 243, each of minimal degree 18, with four distinguished quotients, each of order 81 and minimal degree 27. Using a sieve technique, we identify all preimages of each distinguished quotient. The minimal degrees of the preimages become either (a) 18, when the preimage is exceptional, (b) 27, when the preimage is almost exceptional, (c) 36, or (d) 54. Cases (a), (c) and (d) occur with an elementary abelian centre of order 9, but with contrasting intersection properties using subgroups of order 27, leading to minimal representations afforded by two subgroups. Case (b) occurs with a cyclic centre of order 3 and a transitive minimal representation. We prove that there are exactly two nonisomorphic exceptional groups of order 243 having more than one (in fact two) nonisomorphic distinguished quotients.

1. INTRODUCTION

Throughout this paper, all groups will be finite and the main focus will be on groups of order 243, their subgroups and quotients. The *minimal (faithful) degree* $\mu(G)$ of a group G is the least nonnegative integer such that G embeds in the symmetric group $\mathrm{Sym}(n)$ of permutations on a set of size n. If G is nontrivial then $\mu(G)$ is the minimal sum of indexes for any non-empty collection of subgroups $\mathscr{C} = \{H_1, \ldots, H_k\}$ with a trivial core intersection, in which case we say that $\mathscr{C}$ *affords a minimal (faithful) representation of* G. In this case, the subgroups $H_1, \ldots, H_k$ become the stabilisers of points in the respective orbits for the permutation action of G, and the orbits may be identified with the sets of cosets of $H_1, \ldots, H_k$ in G respectively. When $k = 1$, there is a single orbit and the representation is transitive. The following result, due to Karpilovsky [7], calculates minimal degrees of abelian groups, and will be used implicitly throughout:

Theorem 1.1. [7] *If $G = C_{p_1^{i_1}} \ldots C_{p_n^{i_n}}$ is an abelian group where $n, i_1, \ldots, i_n$ are positive integers and $p_1, \ldots, p_n$ are primes, then $\mu(G) = p_1^{i_1} + \ldots + p_n^{i_n}$.*

Johnson [6] proved a number of seminal results, including the following:

Theorem 1.2. [6, Theorem 3] *If p is an odd prime and G is a nontrivial p-group whose centre is minimally generated by d elements, then any minimal faithful representation of G is afforded by a collection of d subgroups. In particular, if the centre is cyclic then a minimal representation of G must be transitive.*

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Proposition 1.3. [6, Proposition 3] *If p is an odd prime and G is a p-group whose centre is cyclic or elementary abelian then*

$$p\mu\big(Z(G)\big) \;\le\; \mu(G) \;\le\; \frac{1}{p}|G:Z(G)|\mu\big(Z(G)\big)\,.$$

Wright [10] proved that taking minimal degrees is additive with respect to taking direct products of nilpotent groups (for which Theorem 1.1 becomes a special case):

Theorem 1.4. [10, Corollary 2] *If G and H are nilpotent, in particular if G and H are p-groups for some prime p, then $\mu(G\times H) = \mu(G)+\mu(H)$.*

Clearly if H is a subgroup of G then $\mu(H) \le \mu(G)$. However if N is a normal subgroup then $\mu(G/N)$ may be greater than $\mu(G)$. Neumann [9] observed that if $G = D_8^n$ is a direct product of n copies of the dihedral group D_8 then $\mu(G) = 4n$ whilst $\mu(G/N) = 2^{n+1}$ where N is chosen so that G/N becomes (isomorphic to) the n-fold central product of n copies of D_8. This shows that the minimal degree of the direct product of n groups may grow as a linear function of n, whilst the minimal degree of at least one of its quotients grows as an exponential function of n. Analogues of this result for odd primes p are exhibited also in [5] and, with respect to constructions related to wreath products, in [1].

Easdown and Praeger in [5] refer to a group G as *exceptional* if G has a normal subgroup N such that $\mu(G/N) > \mu(G)$, in which case N is called a *distinguished subgroup* and (any group isomorphic to) G/N is called a *distinguished quotient*. They prove that the smallest exceptional groups have order 32 and exhibit several classes of exceptional groups. Other examples and classes of exceptional groups have been studied, for example, by Lemieux [8], Britnell, Saunders and Skyner [2] and Chamberlain [4]. In [2], the authors study exceptional groups G of order p^5 where p is any odd prime. In particular they claim to have found all exceptional groups of order $3^5 = 243$, but do not provide proofs. We give a complete account here, making some corrections to their list. Recall from [1] that a group G is *almost exceptional* if it has a proper normal subgroup N such that $\mu(G) = \mu(G/N)$, and we call G/N an *almost distinguished quotient*.

Section 2 provides preliminary results, used extensively in the sieve process of the later sections. A summary of the main results appears in Section 3. There are ten exceptional groups of order 243, each of minimal degree 18, with four distinguished quotients, each of order 81 and minimal degree 27. Theorem 3.1 is adapted from [2, Table 1], whilst Theorems 3.3 and 3.5 provide new details or information, identifying, in a systematic way, all preimages of the possible distinguished quotients. Possible minimal degrees of preimages are 18, when the preimage is exceptional, 27, when the preimage is almost exceptional, 36, or 54. The cases when the degrees are 18, 36 or 54 occur with an elementary abelian centre of order 9, but with contrasting intersection properties using subgroups of order 27, leading to minimal representations afforded by two subgroups. These intersection properties rely on delicate interplay between general forms for cubes of typical elements and commutators, influenced by subtle alterations in the group relations (see Lemmas 2.6 and 2.7 below). By contrast, the almost exceptional preimages (Lemmas 5.1 and 5.3 below) occur when the centre is cyclic of order 3 and the minimal representation is transitive. We prove that there are exactly two nonisomorphic exceptional groups of order 243 having two nonisomorphic distinguished quotients, and in both cases these groups are almost exceptional (see Corollary 3.7 and Remark 7.2 below). These are the only groups of order 243 that are simultaneously exceptional and almost exceptional.

In Table 3, comprising the final section, we document all 67 groups of order 243, their minimal degrees and relationships to the 15 quotients of order 81. Calculations were made with the assistance of GAP and MAGMA computer algebra software, and the group identification numbers are common to both systems. From Table 3, one can see at a glance the positioning of the ten exceptional groups, highlighted in red, and ten almost exceptional groups, highlighted in blue, two of which have both properties simultaneously. Wherever these groups appear in the exposition as preimages of distinguised quotients, reference is made to this table, either directly or by means of remarks.

2. Preliminaries

Throughout let p be an odd prime. The following observations are well-known:

Lemma 2.1. *Let a,b be elements of a group K such that*

$$[a,b] \;=\; c$$

is central in K. Then, for any positive integer λ,

$$(ab)^{\lambda} \;=\; a^{\lambda}b^{\lambda}c^{-\binom{\lambda}{2}} \;, \tag{1}$$

where, as usual, $\binom{\lambda}{2} = \frac{\lambda(\lambda-1)}{2}$. In particular, if c has order p then

$$(ab)^p \;=\; a^p b^p \;. \tag{2}$$

Lemma 2.2. *Let a,b,c be elements of a group K such that the commutators $[a,b]$ and $[a,c]$ are central in K. Then,*

$$[a,bc] \;=\; [a,b][a,c] \qquad \text{and} \qquad [a^{\alpha},b^{\beta}] \;=\; [a,b]^{\alpha\beta} \;, \tag{3}$$

for all integers α and β.

Lemma 2.3. *If G is a non-abelian p-group of order p^3 then $\mu(G) = p^2$.*

The following result follows by double induction:

Lemma 2.4. *Let a, b, c be elements of a group K such that the commutators $[a,b]$ and $[b,c]$ are central in K. Suppose further that*

$$[a,c] \;=\; b^{\varepsilon} \;,$$

for some integer ε. Then, for all positive integers α and β,

$$[a^{\alpha},c^{\gamma}] \;=\; b^{\alpha\gamma\varepsilon}d \tag{4}$$

for some central element d in K.

We apply this lemma to prove the following useful technical result for controlling cubes of certain elements in a 3-group:

Lemma 2.5. *Let a, b, c be elements of a 3-group K such that the commutators $[a,b]$ and $[b,c]$ are central in K. Suppose further that a^3 is central in K, $b^3 = 1$ and*

$$[a,c] \;= b^{\varepsilon}$$

for some ε. Then, for all positive integers α β and γ,

$$(a^{\alpha}b^{\beta}c^{\gamma})^3 \;=\; (a^{\alpha}c^{\gamma})^3 \;=\; a^{3\alpha}c^{3\gamma}[a,b]^{\alpha^2\gamma\varepsilon}[b,c]^{\alpha\gamma^2\varepsilon} \;. \tag{5}$$

Proof. By (4), we have $[a^\alpha, c^\gamma] = b^{\alpha\gamma\varepsilon}d$, for some central element d. Using centrality of d and the commutators $[a,b]$ and $[b,c]$, and their powers, making free use of (3), and also using the fact that $(a^\alpha b^\beta)^3 = a^{3\alpha}$, by (2) when $p=3$, we have the following:

$$\begin{aligned}
(a^\alpha b^\beta c^\gamma)^3 &= a^\alpha\left(b^\beta c^\gamma\right)a^\alpha b^\beta\left(c^\gamma a^\alpha\right)b^\beta c^\gamma = a^\alpha\Big(c^\gamma b^\beta[b^\beta,c^\gamma]\Big)a^\alpha b^\beta\Big(a^\alpha c^\gamma[c^\gamma,a^\alpha]\Big)b^\beta c^\gamma \\
&= a^\alpha c^\gamma b^\beta a^\alpha b^\beta a^\alpha c^\gamma [a^\alpha,c^\gamma]^{-1} b^\beta c^\gamma [b^\beta,c^\gamma] \\
&= a^\alpha c^\gamma b^\beta a^\alpha b^\beta a^\alpha c^\gamma \Big(b^{-\alpha\gamma\varepsilon}d^{-1}\Big) b^\beta c^\gamma [b^\beta,c^\gamma] \\
&= \left(a^\alpha c^\gamma\right) b^\beta a^\alpha b^\beta a^\alpha \Big(c^\gamma b^\beta[b^\beta,c^\gamma]\Big) b^{-\alpha\gamma\varepsilon} c^\gamma d^{-1} \\
&= \Big(c^\gamma a^\alpha [a^\alpha,c^\gamma]\Big) b^\beta a^\alpha b^\beta a^\alpha \left(b^\beta c^\gamma\right) b^{-\alpha\gamma\varepsilon} c^\gamma d^{-1} \\
&= c^\gamma a^\alpha \Big(b^{\alpha\gamma\varepsilon}d\Big) b^\beta a^\alpha b^\beta a^\alpha b^\beta \Big(c^\gamma b^{-\alpha\gamma\varepsilon}\Big) c^\gamma d^{-1} \\
&= c^\gamma \Big(a^\alpha b^{\alpha\gamma\varepsilon}\Big) b^\beta a^\alpha b^\beta a^\alpha b^\beta \Big(b^{-\alpha\gamma\varepsilon} c^\gamma [c^\gamma, b^{-\alpha\gamma\varepsilon}]\Big) c^\gamma d d^{-1} \\
&= c^\gamma \Big(b^{\alpha\gamma\varepsilon} a^\alpha [a^\alpha, b^{\alpha\gamma\varepsilon}]\Big) b^\beta a^\alpha b^\beta a^\alpha b^\beta b^{-\alpha\gamma\varepsilon} c^{2\gamma} [b,c]^{\alpha\gamma^2\varepsilon} \\
&= c^\gamma b^{\alpha\gamma\varepsilon} a^\alpha [a,b]^{\alpha^2\gamma\varepsilon} b^\beta a^\alpha b^\beta a^\alpha b^\beta b^{-\alpha\gamma\varepsilon} c^{2\gamma} [b,c]^{\alpha\gamma^2\varepsilon} \\
&= c^\gamma b^{\alpha\gamma\varepsilon} \left(a^\alpha b^\beta\right)^3 b^{-\alpha\gamma\varepsilon} c^{2\gamma} [a,b]^{\alpha^2\gamma\varepsilon} [b,c]^{\alpha\gamma^2\varepsilon} \\
&= c^\gamma a^{3\alpha} \Big(b^{\alpha\gamma\varepsilon} b^{-\alpha\gamma\varepsilon}\Big) c^{2\gamma} [a,b]^{\alpha^2\gamma\varepsilon} [b,c]^{\alpha\gamma^2\varepsilon} = a^{3\alpha} c^{3\gamma} [a,b]^{\alpha^2\gamma\varepsilon} [b,c]^{\alpha\gamma^2\varepsilon} \,,
\end{aligned}$$

verifying (5), noting that the outcome is independent of β, completing the proof of the lemma. □

This leads to the following lemma, which is used frequently below in deducing information about minimal degrees of groups, exploring delicate interplay between cubes and commutators:

Lemma 2.6. *Let a,b,c be elements of a 3-group K such that $b^3 = 1$, a^3 and c^3 are central and*

$$[a,c] = b^\varepsilon\,, \qquad [a,b] = a^{3\sigma_1}c^{3\sigma_2}\,, \qquad [b,c] = a^{3\tau_1}c^{3\tau_2}$$

for some ε, σ_1, σ_2, τ_1 and τ_2. Then

$$(a^\alpha b^\beta c^\gamma)^3 = (a^\alpha c^\gamma)^3 = a^{3\alpha\left(1+\varepsilon\gamma(\sigma_1\alpha+\tau_1\gamma)\right)} c^{3\gamma\left(1+\varepsilon\alpha(\tau_2\gamma+\sigma_2\alpha)\right)} \tag{6}$$

and

$$[a^\alpha b^\beta c^\gamma, b] = a^{3(\alpha\sigma_1-\gamma\tau_1)} c^{3(\alpha\sigma_2-\gamma\tau_2)} \tag{7}$$

for any integers α, β and γ.

Proof. By (5), we have

$$\begin{aligned}
(a^\alpha b^\beta c^\gamma)^3 &= a^{3\alpha}c^{3\gamma}[a,b]^{\alpha^2\gamma\varepsilon}[b,c]^{\alpha\gamma^2\varepsilon} = a^{3\alpha}c^{3\gamma}a^{3\sigma_1\alpha^2\gamma\varepsilon}c^{3\sigma_2\alpha^2\gamma\varepsilon}a^{3\tau_1\alpha\gamma^2\varepsilon}c^{3\tau_2\alpha\gamma^2\varepsilon} \\
&= a^{3\alpha\left(1+\varepsilon\gamma(\sigma_1\alpha+\tau_1\gamma)\right)} c^{3\gamma\left(1+\varepsilon\alpha(\tau_2\gamma+\sigma_2\alpha)\right)}\,,
\end{aligned}$$

which verifies (6), and, by (3), we have

$$\begin{aligned}
[a^\alpha b^\beta c^\gamma, b] &= [a,b]^\alpha [c,b]^\gamma = [a,b]^\alpha [b,c]^{-\gamma} = a^{3\alpha\sigma_1}c^{3\alpha\sigma_2}a^{-3\gamma\tau_1}c^{-3\gamma\tau_2} \\
&= a^{3(\alpha\sigma_1-\gamma\tau_1)}c^{3(\alpha\sigma_2-\gamma\tau_2)}\,,
\end{aligned}$$

which verifies (7), completing the proof of the lemma. □

The following lemma is used, in Section 5, to analyse preimages of distinguished quotients that turn out to be neither exceptional nor almost exceptional. Cases (a) and (b) are related to minimal degree 54 (Lemmas 5.10 and 5.12), whilst cases (c) and (d) are related to minimal degree 36 (Lemmas 5.6 and 5.8).

Lemma 2.7. *Let G be the group given by the following presentation*

$$G \;=\; \langle x,y,z \mid x^9=y^3=z^9=1\,,\; z^3 \text{ central}\,,\; [x,y]=x^{3\sigma_1}z^{3\sigma_2}\,,\; [y,z]=x^{3\tau_1}z^{3\tau_2}\,,\; [x,z]=y^{-1}\rangle \tag{8}$$

for some $\sigma_1,\sigma_2,\tau_1,\tau_2\in\mathbb{Z}_3$. Then $|G|=243$ and

$$Z(G) \;=\; \langle x^3,z^3\rangle\,. \tag{9}$$

Suppose further that, in $\mathbb{Z}_3$,

$$\Big(\tau_1+\sigma_1\neq 1 \quad or \quad \tau_2+\sigma_2\neq 1\Big) \qquad and \qquad \Big(\tau_1-\sigma_1\neq 1 \quad or \quad \sigma_2-\tau_2\neq 1\Big)\,, \tag{10}$$

and let L be the subset of G consisting of all elements of order 1 or 3. Then L is a subgroup of G and

$$L \;=\; \langle x^3,y,z^3\rangle \;\cong\; C_3\times C_3\times C_3\,. \tag{11}$$

Let H be a subgroup of G of order 27. Then the following hold:

(a) *If $\sigma_1=-1$ and $\sigma_2=\tau_1=\tau_2=1$ then $Z(G)\subseteq H$.*
(b) *If $\sigma_1=0$, $\sigma_2=\tau_1=1$ and $\tau_2=-1$ then $Z(G)\subseteq H$.*
(c) *If $\sigma_1=\tau_1=\tau_2=0$ and $\sigma_2=\pm1$ then $z^3\in H$.*
(d) *If $\sigma_1=0$, $\sigma_2=\tau_2=1$ and $\tau_1=-1$ then $x^3z^3\in H$.*

Remark 2.8. The groups arising from cases (a), (b) and (d) are isomorphic to groups 243.9, 243.8 and 243.5 respectively in Table 3 below (in Section 8). The group arising from case (c) is isomorphic to group 243.15, when $\sigma_2=1$, and to group 243.14, when $\sigma_2=-1$, in Table 3.

Proof of Lemma 2.7. Let K be the group given by the following presentation:

$$K \;=\; \langle x,y,z,n \mid x^9=y^3=z^9=n^3=1\,,\; n \text{ central}\,,\; [x,y]=x^{3\sigma_1}n^{\sigma_2}\,,\; [y,z]=x^{3\tau_1}n^{\tau_2}\,,\; [x,z]=y^{-1}\rangle\,. \tag{12}$$

Then

$$K \;=\; \Big(\big(\langle x\rangle\times\langle n\rangle\big)\rtimes\langle y\rangle\Big)\rtimes\langle z\rangle \;\cong\; \big((C_9\times C_3)\rtimes C_3\big)\rtimes C_9\,,$$

which has order 3^6. It follows from the relations in (12) that x^3 and z^3 are also central in G and, moreover,

$$Z(K) \;=\; \langle n,x^3,z^3\rangle\,.$$

Put $N=\langle n^{-1}z^3\rangle$, which is a central subgroup of K of order 3, so that $K/N\cong G$, whence $|G|=243$, and (9) holds, noting also that the relations of (8) imply that x^3 is central.

Suppose further that (10) holds and let L be the subset of elements of G of order 1 or 3. Elements of G have the form

$$w \;=\; x^\alpha y^\beta z^\gamma \tag{13}$$

for some α,β,γ such that $0\le\alpha,\gamma\le 8$ and $0\le\beta\le 2$. By Lemma 2.6 and (6), with x, y and z in place of a, b and c respectively, and $\varepsilon=-1$,

$$w^3 \;=\; x^{3\alpha(1-\sigma_1\alpha\gamma-\tau_1\gamma^2)}z^{3\gamma(1-\tau_2\alpha\gamma-\sigma_2\alpha^2)}\,, \tag{14}$$

from which it is immediate that w has order 1, 3 or 9. Clearly, if α and γ are multiples of 3 then $x^3 = 1$. Suppose conversely that $w^3 = 1$. If α is a multiple of 3 then $1 = w^3 = z^{3\gamma}$, so that γ also must be a multiple of 3. Similarly, if γ is a multiple of 3 then α also must be a multiple of 3. Suppose that α and γ are both not multiples of 3, so that their squares evaluate to 1 in $\mathbb{Z}_3$. If $\alpha = \gamma$ in $\mathbb{Z}_3$ then, by (14),

$$1 \;=\; w^3 \;=\; x^{3\alpha(1-\sigma_1-\tau_1)}z^{3\alpha(1-\tau_2-\sigma_2)}$$

so that $\tau_1 + \sigma_1 = \tau_2 + \sigma_2 = 1$ in $\mathbb{Z}_3$, which contradicts the first part of (10). If $\alpha = -\gamma$ in $\mathbb{Z}_3$ then, by (14),

$$1 \;=\; w^3 \;=\; x^{3\alpha(1+\sigma_1-\tau_1)}z^{3\alpha(1+\tau_2-\sigma_2)}$$

so that $\tau_1 - \sigma_1 = \sigma_2 - \tau_2 = 1$ in $\mathbb{Z}_3$, which contradicts the second part of (10). This shows that both α and γ are multiples of 3. Thus we have proved that w has order 1 or 3 if and only if both α and γ are multiples of 3. Since x^3 and z^3 are central, it follows that L is a subgroup of G, which is generated by x^3, y and z^3, so that (11) holds.

Now let H be a subgroup of G of order 27. Note that each of the hypotheses of (a), (b), (c) and (d) guarantee that (10) holds, so that (11) holds in each case. If the exponent of H is 3 then $H \subseteq L$, so $H = L$, since both subgroups have the same size, and, in particular, $Z(G) \subseteq H$, and each of (a), (b), (c) and (d) holds automatically.

Suppose the exponent of H is not 3, so that H contains an element w of order 9, which we may take to be of the form (13), where at least one of α or γ is not a multiple of 3. If α is a multiple of 3 then γ is not a multiple of 3, so that, by (14),

$$z^3 \;\in\; \langle z^{3\gamma}\rangle \;=\; \langle w^3\rangle \;\subseteq\; H\,. \tag{15}$$

If γ is a multiple of 3 then α is not a multiple of 3, so that, by (14),

$$x^3 \;\in\; \langle x^{3\alpha}\rangle \;=\; \langle w^3\rangle \;\subseteq\; H\,. \tag{16}$$

It follows from general properties of groups of order 27 of exponent at most 9 that $H \cap L$ has order at least 9. If $H \cap L \subseteq Z(G)$ then $H \cap L = Z(G)$, by comparing sizes, so that $Z(G) \subseteq H$ and again each of (a), (b), (c) and (d) holds automatically. Hence we may suppose that $H \cap L$ has an element v that is not central, so, without loss of generality (replacing v by v^2 if necessary), we have

$$v \;=\; x^{3\delta}yz^{3\varepsilon}$$

for some δ, ε. But then

$$[w,v] \;=\; [x^\alpha y^\beta z^\gamma, x^{3\delta}yz^{3\varepsilon}] \;=\; [x^\alpha y^\beta z^\gamma, y] \;=\; x^{3(\alpha\sigma_1-\gamma\tau_1)}z^{3(\alpha\sigma_2-\gamma\tau_2)} \;\in\; H\,, \tag{17}$$

by (7), using x, y, z in place of a, b, c respectively.

Suppose first that α is a multiple of 3, so that γ is not a multiple of 3. Note that (c) holds automatically by (15). By (17),

$$x^{3\tau_1}z^{3\tau_2} \;\in\; \langle x^{-3\gamma\tau_1}z^{-3\gamma\tau_2}\rangle \;\subseteq\; H\,. \tag{18}$$

In case (a), (18) becomes $x^3z^3 \in H$, and in cases (b) and (d), (18) becomes $x^3z^{-3} \in H$, whence, combining each case with (15), we get $Z(G) \subseteq H$. Thus each of (a), (b), (c) and (d) holds.

Suppose next that γ is a multiple of 3, so that α is not a multiple of 3. By (17),

$$x^{3\alpha_1}z^{3\alpha_2} \;\in\; \langle x^{3\alpha\sigma_1}z^{3\alpha\sigma_2}\rangle \;\subseteq\; H\,. \tag{19}$$

In case (a), (19) becomes $x^{-3}z^3 \in H$, and in cases (b) and (d), (19) becomes $z^3 \in H$, whence, combined with (16), we get $Z(G) \subseteq H$. In case (c), (19) becomes $z^{\pm 3} \in H$. Thus, again, each of (a), (b), (c) and (d) holds.

Suppose now that neither α nor γ is a multiple of 3, so that $\alpha^2 = \gamma^2 = 1$. Consider first the case that $\alpha = \gamma$. From (14), we have

$$x^{3(1-\sigma_1-\tau_1)}z^{3(1-\tau_2-\sigma_2)} \in \langle x^{3\alpha(1-\sigma_1-\tau_1)}z^{3\alpha(1-\tau_2-\sigma_2)}\rangle \subseteq H \, , \tag{20}$$

and, from (17), we have

$$x^{3(\sigma_1-\tau_1)}z^{3(\sigma_2-\tau_2)} \in \langle x^{3\alpha(\sigma_1-\tau_1)}z^{3\alpha(\sigma_2-\tau_2)}\rangle \subseteq H \, . \tag{21}$$

In case (a), (20) and (21) yield $x^3z^{-3}, x^3 \in H$ respectively, whence $Z(G) \subseteq H$. In case (b), (20) and (21) yield $z^3, x^{-3}z^{-3} \in H$ respectively, whence $Z(G) \subseteq H$. In case (c), (21) yields $z^{\pm 3} \in H$. In case (d), (20) and (21) yield $z^3, x^3 \in H$ respectively, whence $Z(G) \subseteq H$, so, in particular, $x^3z^3 \in H$. Thus, again, each of (a), (b), (c) and (d) holds. Consider, secondly, the case that $\alpha = -\gamma$. From (14), we have

$$x^{3(1+\sigma_1-\tau_1)}z^{-3(1+\tau_2-\sigma_2)} \in \langle x^{3\alpha(1+\sigma_1-\tau_1)}z^{-3\alpha(1-\tau_2-\sigma_2)}\rangle \subseteq H \, , \tag{22}$$

and, from (17), we have

$$x^{3(\sigma_1+\tau_1)}z^{3(\sigma_2+\tau_2)} \in \langle x^{3\alpha(\sigma_1+\tau_1)}z^{3\alpha(\sigma_2+\tau_2)}\rangle \subseteq H \, . \tag{23}$$

In case (a), (22) and (23) yield $x^{-3}z^{-3}, z^{-3} \in H$ respectively, whence $Z(G) \subseteq H$. In case (b), (22) and (23) yield $z^3, x^3 \in H$ respectively, whence $Z(G) \subseteq H$. In case (c), (23) yields $z^{\pm 3} \in H$. In case (d), (22) yields $x^{-3}z^{-3} \in H$. Thus, again, each of (a), (b), (c) and (d) holds, completing the proof of the lemma. □

The following result is probably well-known, but we give a proof for completeness:

Lemma 2.9. *The socle, and hence also the centre, of a finite p-group G is not contained in any subgroup that appears in a subgroup collection that affords a minimal faithful representation of G.*

Proof. Suppose that $\mathscr{C} = \{H_1, \ldots, H_k\}$ is a subgroup collection that affords a minimal representation. Suppose that $\mathrm{socle}(G)$, is contained in one of the subgroups, which, by reordering if necessary, we may take to be H_1. If $k = 1$ then H_1 is not core-free, since $\mathrm{socle}(G)$ is nontrivial, contradicting that $\mathscr{C}$ is faithful. Hence $k > 1$. If $\{H_2, \ldots, H_k\}$ has a core-free intersection then $\mathscr{C}$ is not minimal, as we may delete H_1, which is a contradiction. Hence $N = \mathrm{core}(H_1 \cap \ldots \cap H_k)$ is a nontrivial normal subgroup of G. But N intersects $\mathrm{socle}(G)$ nontrivially, so that

$$\mathrm{core}(H_1 \cap H_2 \cap \ldots \cap H_k) \;=\; \mathrm{core}(H_1) \cap N \;\supseteq\; \mathrm{socle}(G) \cap N \;\neq\; \{1\} \, ,$$

contradicting that $\mathscr{C}$ affords a faithful representation. Hence $\mathrm{socle}(G)$ is not contained in any subgroup in $\mathscr{C}$. Note that the centre of a p-group always contains the socle. This completes the proof of the lemma. □

The following result follows from [2, Proposition 2.3] and also from results in [8]:

Lemma 2.10. *If G is an exceptional group of order p^5 with distinguished quotient G/N for some distinguished normal subgroup N, then N is a central subgroup of order p, G/N has order p^4 and $\mu(G/N) = p^3$. In particular, if G is a group of order p^5 and $\mu(G) \geq p^3$ then G is not exceptional.*

3. Distinguished quotients and their exceptional preimages

Throughout this article, we consider the following groups of order 81, which turn out to be the distinguished quotients of exceptional groups of order 243:

$$Q_1 \;=\; \langle\, a,b,c \mid a^9 = b^3 = c^3 = 1\,,\; a \text{ central}\,,\; [b,c] = a^3 \rangle \tag{24}$$

(which is group 81.14 in Table 3),

$$Q_2 \;=\; \langle\, a,b,c \mid a^9 = b^3 = c^3 = [a,b] = 1\,,\; [a,c] = b\,,\;\; [b,c] = a^{-3} \,\rangle \tag{25}$$

(which is group 81.9 in Table 3),

$$Q_3 \;=\; \langle\, a,b,c \mid a^9 = b^3 = [b,c] = 1\,,\; c^3 = a^3\,,\; [a,b] = a^3\,,\; [a,c] = b^{-1} \,\rangle \tag{26}$$

(which is group 81.10 in Table 3),

$$Q_4 \;=\; \langle\, a,b,c \mid a^9 = b^3 = [b,c] = 1\,,\; c^3 = a^{-3}\,,\; [a,b] = a^3\,,\; [a,c] = b^{-1} \,\rangle \tag{27}$$

(which is group 81.8 in Table 3). The group Q_1 is the group $Q(p)$ in [2] and the group labelled III(vii) on page 100 of Burnsides' list [3], when $p = 3$. The group Q_2 is the the group Q_{81} in [2] and the group labelled III(xv) on page 101 of [3], when $p = 3$. Both Q_1 and Q_2 have semidirect product decompositions corresponding to

$$\big(\langle a\rangle \times \langle b\rangle\big) \rtimes \langle c\rangle \;\cong\; \big(C_9 \times C_3\big) \rtimes C_3\,.$$

The groups Q_3 and Q_4 are the groups III(xii) and III(xiii) respectively on page 101 of [3], when $p = 3$. In fact, the group Q_3 is isomorphic to the group $Q_2(3)$ in [2]. Though this is not obvious, one can verify the isomorphism using the transformation $x = c$, $y = b^{-1}a^{-3}$, $z = a^{-1}$. Similarly, the group Q_4 is isomorphic to the group $Q_1(3)$ in [2], which one can verify using the transformation $x = c$, $y = b^{-1}c^3$, $z = a^{-1}$. We explain briefly why Q_3 and Q_4 have order $3^4 = 81$. To see this, put

$$G \;=\; \langle\, a,b,c \mid a^9 = b^3 = c^9 = [b,c] = 1\,,\; [a,b] = a^3\,,\;\; [a,c] = b^{-1} \,\rangle \tag{28}$$

(which is group 243.18 in Table 3). Then G has a semidirect product decomposition

$$G \;=\; \big(\langle a\rangle \rtimes \langle b\rangle\big) \rtimes \langle c\rangle \;\cong\; \big(C_9 \rtimes C_3\big) \rtimes C_9\,,$$

so that $|G| = 3^5$. It follows that $Z(G) = \langle a^3, c^3\rangle$, so that $N_1 = \langle a^3c^{-3}\rangle$ and $N_2 = \langle a^3c^3\rangle$ are central subgroups of G of order 3. Clearly, $Q_3 \cong G/N_1$ and $Q_4 \cong G/N_2$, so that $|Q_3| = |Q_4| = 3^4$.

It follows from the relations in (24) that

$$Z(Q_1) \;\cong\; C_9\,, \tag{29}$$

generated by the element a, and from the relations in (25), (26) and (27), that

$$Z(Q_2) \;\cong\; Z(Q_3) \;\cong\; Z(Q_4) \;\cong\; C_3\,, \tag{30}$$

generated in each of these cases by the element corresponding to a^3. By Theorem 1.3, a minimal faithful representation of each of Q_1, Q_2, Q_3 and Q_4 is transitive. For each of Q_1 and Q_2, a minimal representation is afforded by the core-free subgroup corresponding to $\langle c\rangle$. For each of Q_3 and Q_4, a minimal representation is afforded by the core-free subgroup corresponding to $\langle b\rangle$. It follows quickly from Proposition 1.3, and as noted in [2], that

$$\mu(Q_1) \;=\; \mu(Q_2) \;=\; \mu(Q_3) \;=\; \mu(Q_4) \;=\; 3^3 \;=\; 27\,. \tag{31}$$

The following result from [2] characterises exceptional preimages of Q_1 of order 243. These groups appear in [2, Table 1] when $p = 3$, though listed in a different order. The presentations given here differ slightly from [2], though easily checked to be equivalent. The form of presentations chosen here is consistent with the shape and style of presentations given in Theorems 3.3 and 3.5, for which we provide proofs in later sections.

Theorem 3.1. [2, Table 1] *The following groups have order* 243 *and have* Q_1 *defined by* (24) *as a distinguished quotient:*

(i) $G_1 = \langle x,y,z \mid x^9 = y^9 = z^3 = 1\,,\ x \text{ central}\,,\ [y,z] = x^3y^3\rangle$,
(ii) $G_2 = \langle x,y,z \mid x^9 = y^9 = z^3 = [x,y] = 1\,,\ [x,z] = y^3\,,\ [y,z] = x^3\rangle$,
(iii) $G_3 = \langle x,y,z \mid x^9 = y^3 = z^9 = [x,y] = 1\,,\ [x,z] = z^3\,,\ [y,z] = x^3z^3\rangle$,
(iv) $G_4 = \langle x,y,z,n \mid x^9 = y^3 = z^3 = n^3 = 1\,,\ x \text{ central}\,,\ n \text{ central}\,,\ [y,z] = x^3n\rangle$,
(v) $G_5 = \langle x,y,z,n \mid x^9 = y^3 = z^3 = n^3 = [x,y] = 1\,,\ n \text{ central}\,,\ [x,z] = n\,,\ [y,z] = x^3n\rangle$.

Suppose that G is an exceptional group of order p^5 *with distinguished quotient* Q_1*. Then there is a distinguished normal subgroup N, generated by a central element of G, such that* $G/N \cong Q_1$*, and G is isomorphic to* G_1*,* G_2*,* G_3*,* G_4 *or* G_5*. Moreover,* $\mu(G) = 18$.

Remark 3.2. The groups G_1, G_2, G_3, G_4 and G_5 are groups 243.36, 243.43, 243.41, 243.35 and 243.39 respectively in Table 3.

The following result is proved in Section 6 and classifies exceptional preimages of Q_2 of order 243, up to isomorphism. These are the same groups as those listed in [2, Table1], but given without proof. Again, the presentations are slightly different to those given in [2], but easily seen to be equivalent.

Theorem 3.3. *The following two groups have order* $3^5 = 243$ *and have* Q_2 *defined by* (25) *as a distinguished quotient:*

(i) $G_6 = \langle x,y,z \mid x^9 = y^3 = z^9 = [x,y] = 1\,, [x,z] = y\,,\ [y,z] = x^{-3}z^3\rangle$,
(ii) $G_7 = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^{-3}\,,\ [x,z] = y\,,\ [y,z] = x^{-3}\rangle$.

Suppose that G is an exceptional group of order p^5 *with distinguished quotient* Q_2*. Then there is a distinguished normal subgroup N, generated by a central element of G, such that* $G/N \cong Q$*, and G is isomorphic to* G_6 *or* G_7*. Moreover,* $\mu(G) = 18$.

Remark 3.4. The groups G_6 and G_7 are groups 243.17, 243.3 respectively in Table 3.

The following result is proved in Section 7 and classifies exceptional preimages of Q_3 and Q_4 of order 243, up to isomorphism.

Theorem 3.5. *The following groups have order* $3^5 = 243$*:*

(i) $\widetilde{G}_6 = \langle x,y,z \mid x^9 = y^3 = z^9 = [y,z] = 1\,,\ [x,y] = x^{-3}z^3\,,\ [x,z] = y^{-1}\rangle$,
(ii) $G_8 = \langle x,y,z \mid x^9 = y^3 = z^9 = [y,z] = 1\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1}\rangle$,
(iii) $G_9 = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^{-3}z^3\rangle$,
(iv) $G_{10} = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^{-3}z^{-3}\rangle$.

The groups G_8 *and* G_9 *are exceptional with distinguished quotient* Q_3*, and the groups* $\widetilde{G}_6$*,* G_8 *and* G_{10} *are exceptional with distinguished quotient* Q_4*. Let G be an exceptional group of order* 243*. If G has distinguished quotient* Q_3 *then G is isomorphic to* G_8 *or* G_9*. If G has distinguished quotient* Q_4 *then G is isomorphic to* $\widetilde{G}_6$*,* G_8 *or* G_{10}*. In all cases,* $\mu(G) = 18$.

Remark 3.6. The groups $\widetilde{G}_6$, G_8, G_9 and G_{10} are groups 243.17, 243.18, 243.7 and 243.4 respectively in Table 3. The group $\widetilde{G}_6$ is isomorphic to the group G_6 described in Theorem 3.3, under the isomorphism induced by the mapping $x \mapsto z$, $y \mapsto y$, $z \mapsto x$. The presentation chosen here for $\widetilde{G}_6$ highlights similarities with the other presentations in the statement of Theorem 3.5, and also to facilitate the flow of the proof in Section 7, which uses a sieve technique.

By inspection, from Theorems 3.1, 3.3 and 3.5, we deduce the following:

Corollary 3.7. *There are exactly two nonisomorphic exceptional groups of order* 243*, namely G_6 and G_8, with the property that they have two nonisomorphic distinguished quotients, namely Q_2 and Q_4 for G_6, and Q_3 and Q_4 for G_8.*

4. Exceptional groups

In this section, we document a sequence of propositions, providing proofs that G_6, G_7, G_8, G_9 and G_{10} are exceptional. These results are then applied in the proofs in Sections 6 and 7 below.

Proposition 4.1. *Let G_6 be the group defined by the following presentation:*

$$G_6 = \langle x,y,z \mid x^9 = y^3 = z^9 = [x,y] = 1\,, [x,z] = y\,,\ [y,z] = x^{-3}z^3 \rangle\ . \tag{32}$$

Then $|G_6| = 243$, $\mu(G_6) = 18$,

$$Z(G_6) = \langle x^3, z^3 \rangle \ \cong\ C_3 \times C_3\ , \tag{33}$$

and G_6 is exceptional, with distinguished quotient Q_2.

Proof. It follows from the relations of (32) that x^3 and z^3 are central in G_6. By Lemma 2.7, interchanging the roles of x and z, with corresponding adjustments to the commutator relations, we have that $|G_6| = 243$ and (33) holds. In particular, $\mu\big(Z(G_6)\big) = 6$, so that, by Proposition 1.3,

$$\mu(G_6)\ \geq\ 3\mu\big(Z(G_6)\big)\ =\ 18\ .$$

Consider the following subgroups of order 27:

$$H\ =\ \langle x,y \rangle \qquad \text{and} \qquad K\ =\ \langle y,z,x^{-3}z^3 \rangle\ .$$

Then

$$\operatorname{core}(H \cap K)\ =\ \operatorname{core}(H) \cap \operatorname{core}(K)\ =\ \langle x^3 \rangle \cap \langle x^{-3}z^3 \rangle\ =\ \{1\}\ ,$$

so that $\{H,K\}$ affords a faithful representation of G_6 of degree $9+9 = 18$. Thus, also, $\mu(G_6) \leq 18$, so that $\mu(G_6) = 18$. Let $N = \langle z^3 \rangle$, so that $|G_6/N| = 3^4 = 81$. By adding the relation $z^3 = 1$, the presentation (32) quickly reduces to the equivalent presentation (25) of Q_2, so that $G_6/N \cong Q_2$. By (31), we have $\mu(G_6) = 18 < 27 = \mu(Q_2) = \mu(G_6/N)$, so that G_6 is exceptional with distinguished quotient Q_2, completing the proof. ☐

Proposition 4.2. *Let G_7 be the group defined by the following presentation:*

$$G_7\ =\ \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^{-3}\,,\ [x,z] = y\,,\ [y,z] = x^{-3} \rangle\ . \tag{34}$$

Then $|G_7| = 243$, $\mu(G_7) = 18$,

$$Z(G_7) = \langle x^3, z^3 \rangle \ \cong\ C_3 \times C_3\ , \tag{35}$$

and G_7 is exceptional, with distinguished quotient Q_2.

Proof. It follows from the relations of (34) that x^3 and z^3 are central in G_7. By Lemma 2.7, interchanging the roles of x and z, with corresponding adjustments to the commutator relations, we have that $|G_7| = 243$ and (35) holds. As in the proof of the previous proposition, $\mu(G_7) \geq 18$. Put

$$H \;=\; \langle y, x^3z^{-3}, xz\rangle \;=\; \big(\langle y\rangle \times \langle x^3z^{-3}\rangle\big) \rtimes \langle xz\rangle \;\cong\; (C_3 \times C_3) \rtimes C_3 \;.$$

Clearly $|H| = 27$ and $\operatorname{core}(H) = \langle x^3z^{-3}\rangle$. Now put

$$K \;=\; \langle y, x^3z^3, xz^{-1}\rangle \;.$$

Observe that, by Lemma 2.2 and the relations of (34),

$$(xz^{-1})^3 \;=\; 1 \qquad \text{and} \qquad [y, xz^{-1}] \;=\; [y,x][y,z]^{-1} \;=\; z^3x^3$$

so that

$$K \;=\; \big(\langle y\rangle \times \langle x^3z^3\rangle\big) \rtimes \langle xz^{-1}\rangle \;\cong\; (C_3 \times C_3) \rtimes C_3 \;.$$

Clearly $|K| = 27$ and $\operatorname{core}(K) = \langle x^3z^3\rangle$, so that

$$\operatorname{core}(H \cap K) \;=\; \langle x^3z^{-3}\rangle \cap \langle x^3z^3\rangle \;=\; \{1\} \;.$$

Hence $\{H,K\}$ affords a faithful representation of G_7 of degree $9+9 = 18$. Thus, also, $\mu(G_7) \leq 18$, so that $\mu(G_7) = 18$. Let $N = \langle z^3\rangle$, so that $|G_7/N| = 3^4 = 81$. By adding the relation $z^3 = 1$, the presentation (34) quickly reduces to the equivalent presentation (25) of Q_2, so that $G_7/N \cong Q_2$. By (31) we have $\mu(G_7) = 18 < 27 = \mu(Q_2) = \mu(G_7/N)$, so that G_7 is exceptional with distinguished quotient Q_2, completing the proof. □

Proposition 4.3. *Let G_8 be the group defined by the following presentation:*

$$G_8 \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = [y,z] = 1\,,\; [x,y] = x^3\,,\; [x,z] = y^{-1}\rangle \;. \tag{36}$$

Then $|G_8| = 243$, $\mu(G_8) = 18$,

$$Z(G_8) = \langle x^3, z^3\rangle \;\cong\; C_3 \times C_3 \;, \tag{37}$$

and G_8 is exceptional, with distinguished quotients Q_3 and Q_4.

Proof. Observe that G_8 is isomorphic to the group G given by presentation (28), discussed earlier, identifying x,y,z with a,b,c respectively. From that discussion, $|G_8| = 243$, (37) holds and

$$G_8/N_1 \;\cong\; Q_3 \qquad \text{and} \qquad G_8/N_2 \;\cong\; Q_4 \;, \tag{38}$$

where $N_1 = \langle x^3z^{-3}\rangle$ and $N_2 = \langle x^3z^3\rangle$ are central subgroups of G_8. As before, $\mu(G_8) \geq 18$. Consider the following subgroups of G_8 of order 27:

$$H = \langle x,y\rangle \qquad \text{and} \qquad K = \langle y,z\rangle \;.$$

Then $\operatorname{core}(H \cap K) = \langle x^3\rangle \cap \langle z^3\rangle = \{1\}$, so that $\{H,K\}$ affords a faithful representation of G_8 of degree $|G_8 : H| + |G_8 : K| = 18$. Hence $\mu(G_8) \leq 18$ so that $\mu(G_8) = 18$. But $\mu(Q_3) = \mu(Q_4) = 27$, by (31), so that, by (38), G_8 is exceptional with respective distinguished quotients Q_3 and Q_4, completing the proof. □

Proposition 4.4. *Let G_9 be the group defined by the following presentation:*

$$G_9 \;=\; \langle x,y,z \mid x^9=y^3=z^9=1\,,\ [x,y]=x^3\,,\ [x,z]=y^{-1}\,,\ [y,z]=x^{-3}z^3\rangle\;. \tag{39}$$

Then $|G_9|=243$, $\mu(G_9)=18$,

$$Z(G_9) \;=\; \langle x^3,z^3\rangle \;\cong\; C_3\times C_3\;, \tag{40}$$

and G_9 is exceptional, with distinguished quotient Q_3.

Proof. It follows from the relations of (39) that x^3 and z^3 are central in G_9. By Lemma 2.7, we have that $|G_9|=243$ and (40) holds. As before, $\mu(G_9)\geq 18$. Consider the following subgroups of G_9:

$$H=\langle x,y\rangle \qquad \text{and} \qquad K=\langle y,xz^2\rangle\;.$$

Clearly,

$$H \;=\; \langle x\rangle\rtimes\langle y\rangle \;\cong\; C_9\rtimes C_3$$

is a subgroup of G_9 of order 27, and $\mathrm{core}(H)=\langle x^3\rangle$. Observe that, in G_9,

$$xz^2 \;=\; zxy^{-1}z \;=\; zxzy^{-1}x^3z^{-3} \;=\; z^2xyx^3z^{-3}\;,$$

and

$$z^2x \;=\; zxzy \;=\; xzyzy \;=\; xz^2y^{-1}x^{-3}z^3\;,$$

so that, by (6) and (7) of Lemma 2.6, with x, y, z in place of a, b and c respectively,

$$(xz^2)^3 \;=\; z^3 \qquad \text{and} \qquad [xz^2,y] \;=\; z^3 \;=\; (xz^2)^3\;,$$

from which it follows that

$$K \;=\; \langle xz^2\rangle\rtimes\langle y\rangle \;\cong\; C_9\rtimes C_3$$

is also a subgroup of G_9 of order 27, and $\mathrm{core}(K)=\langle z^3\rangle$. Hence

$$\mathrm{core}(H\cap K) \;=\; \langle x^3\rangle\cap\langle z^3\rangle \;=\; \{1\}\;,$$

so that $\{H,K\}$ affords a faithful representation of G_9 of degree $|G_9:H|+|G_9:K|=18$. Hence $\mu(G_9)\leq 18$, so that $\mu(G_9)=18$. Put $C=\langle x^{-3}z^3\rangle$, which is a central subgroup of G_9 of order 3. Using the mapping $x\mapsto a$, $y\mapsto b$, $z\mapsto c$, and comparing relations, it follows quickly that

$$G_9/C \;\cong\; Q_3\;.$$

But $\mu(Q_3)=27$, by (31), so that G_9 is exceptional with distinguished subgroup C and distinguished quotient Q_3, completing the proof. □

Remark 4.5. We may examine algebraic properties to demonstrate that the group G_9 in Lemma 4.4 is a new group that does not appear in the classification of exceptional groups of order 243 in [2], or may have been inadvertently excluded. A straightforward analysis shows that G_9 of Lemma 4.4 has 13 subgroups of order 3 and commutator subgroup isomorphic to $C_3\times C_3\times C_3$. However, in [2, Table 1], the groups named G_3 and G_4 have 67 and 40 subgroups of order 3, respectively, the groups named G_5, G_6, G_7, E_3, E_4 and E_5 have commutator subgroup isomorphic to $C_3\times C_3$, and the groups named E_1 and E_2 have commutator subgroup isomorphic to C_3. This shows that G_9 of Lemma 4.4 cannot be isomorphic to any of the groups listed in that table.

Proposition 4.6. *Let G_{10} be the group defined by the following presentation:*

$$G_{10} \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\; [x,y] = x^3\,,\; [x,z] = y^{-1}\,,\; [y,z] = x^{-3}z^{-3}\rangle\;. \tag{41}$$

Then $|G_{10}| = 243$, $\mu(G_{10}) = 18$,

$$Z(G_{10}) \;=\; \langle x^3, z^3\rangle \;\cong\; C_3 \times C_3\;, \tag{42}$$

and G_{10} is exceptional, with distinguished quotient Q_4.

Proof. It follows from the relations of (41) that x^3 and z^3 are central in G_{10}. By Lemma 2.7, we have that $|G_{10}| = 243$ and (42) holds. As before, $\mu(G_{10}) \geq 18$. Consider the following subgroups of G_{10}:

$$H = \langle x,y\rangle \qquad \text{and} \qquad K = \langle y, xz\rangle\;.$$

As before, H is a subgroup of order 27, and $\operatorname{core}(H) = \langle x^3\rangle$. Observe that, in G_{10}, by (6) and (7) of Lemma 2.6, with x, y, z in place of a, b and c respectively,

$$(xz)^3 \;=\; x^3z^{-3} \qquad \text{and} \qquad [xz,y] \;=\; x^{-3}z^3 \;=\; (xz)^{-3}\;,$$

from which it follows that

$$K \;=\; \langle xz\rangle \rtimes \langle y\rangle \;\cong\; C_9 \rtimes C_3$$

is also a subgroup of G_{10} of order 27, and $\operatorname{core}(K) = \langle x^3z^{-3}\rangle$. Hence

$$\operatorname{core}(H\cap K) \;=\; \langle x^3\rangle \cap \langle x^3z^{-3}\rangle \;=\; \{1\}\;,$$

so that $\{H,K\}$ affords a faithful representation of G_{10} of degree $|G_{10} : H| + |G_{10} : K| = 18$. Hence $\mu(G_{10}) \leq 18$ so that $\mu(G_{10}) = 18$. Put $C = \langle x^3z^3\rangle$, which is a central subgroup of G_{10} of order 3. Using the mapping $x \mapsto a$, $y \mapsto b$, $z \mapsto c$, and comparing relations, it follows quickly that

$$G_{10}/C \;\cong\; Q_4\;.$$

But $\mu(Q_4) = 27$, by (31), so that G_{10} is exceptional with distinguished subgroup C and distinguished quotient Q_4, completing the proof. □

5. Nonexceptional preimages

In this section, we document a sequence of lemmas, providing proofs that certain groups of order 243 are not exceptional. These results are then applied in the proofs in Sections 6 and 7 below.

Lemma 5.1. *Let G be the group defined by the following presentation:*

$$G \;=\; \langle x,y,z \mid x^9 = y^9 = z^3 = 1\,,\; [x,y] = y^{3k}\,,\; [x,z] = y\,,\; [y,z] = x^{-3}y^{3m}\rangle \tag{43}$$

where $0 \leq k,m \leq 2$. Then $|G| = 243$, $\mu(G) = 27$,

$$Z(G) = \langle y^3\rangle \;\cong\; C_3\;, \tag{44}$$

and G is not exceptional.

Remark 5.2. The following table documents the correspondence of groups arising in Lemma 5.1 to groups in Table 3 (in Section 8):

TABLE 1. Correspondence of groups from Lemma 5.1 with groups in Table 3.

Cases (k,m)	**Group ID in Table 3**
(0,0), (0,1)	243.25
(0,2)	243.26
(1,0), (1,1), (2,0), (2,1)	243.28
(1,2), (2,2)	243.30

Proof of Lemma 5.1. It follows quickly from (43) that G has the semidirect product decomposition

$$G = \big(\langle y\rangle \rtimes \langle x\rangle\big) \rtimes \langle z\rangle \cong (C_9 \rtimes C_9) \rtimes C_3 ,$$

so that, in particular, $|G| = 3^5 = 243$. It follows from the relations, manipulations of commutators and Lemma 2.2 that y^3 generates the centre of G, so that (44) holds. In particular, the centre of G is cyclic, so, by Theorem 1.2, any minimal faithful representation of G is transitive, so $\mu(G)$ is a power of 3. But G contains the subgroup

$$\langle y, x^3\rangle = \langle y\rangle \times \langle x^3\rangle \cong C_9 \times C_3 ,$$

so that $\mu(G) \geq \mu(C_9 \times C_3) = 12$. Hence $\mu(G) \geq 27$. Put $H = \langle x\rangle$. Then $|H| = 9$ and H has trivial core, so H affords a faithful representation of G of degree 27. Hence also $\mu(G) \leq 27$, so that $\mu(G) = 27$. By Lemma 2.10, G is not exceptional, completing the proof of the lemma. ☐

Lemma 5.3. *Let G be the group defined by the following presentation:*

$$G = \langle x,y,z \mid x^9 = y^9 = 1 ,\ z^3 = y^{3\varepsilon} ,\ [x,y] = y^{3k} ,\ [x,z] = y ,\ [y,z] = x^{-3}y^{3m}\rangle \tag{45}$$

where $0 \leq k,m \leq 2$ *and* $\varepsilon \in \{1,-1\}$*. Then* $|G| = 243$*,* $\mu(G) = 27$*,*

$$Z(G) = \langle y^3\rangle \cong C_3 , \tag{46}$$

and G is not exceptional.

Remark 5.4. The following table documents the correspondence of groups arising in Lemma 5.3 to groups in Table 3 (in Section 8):

TABLE 2. Correspondence of groups from Lemma 5.3 with groups in Table 3.

Cases (k,m)	**Group ID in Table 3 when** $z^3 = y^3$	**Group ID in Table 3 when** $z^3 = y^{-1}$
(0,0), (0,1)	243.25	243.25
(0,2)	243.27	243.27
(1,0),(1,1)	243.30	243.29
(1,2)	243.29	243.28
(2,0),(2,1)	243.29	243.30
(2,2)	243.28	243.29

Proof of Lemma 5.3. To see that G has order 243, first consider the group

$$H = \langle x,y,z \mid x^9 = y^9 = z^9 = 1\,,\ [x,y] = y^{3k}\,,\ [x,z] = y,\ [y,z] = x^{-3}y^{3m}\,\rangle\,.$$

Clearly,

$$H = (\langle x\rangle \rtimes \langle y\rangle) \rtimes \langle z\rangle \cong (C_9 \rtimes C_9) \rtimes C_9\,,$$

which has order 3^6. Further, it follows from the relations that

$$Z(H) = \langle y^3, z^3\rangle\,.$$

Then $K = \langle y^{3\varepsilon}z^{-3}\rangle$ is a central subgroup of H of order 3. By comparing relations, we see that $H/K \cong G$. It follows that $|G| = 3^5 = 243$ and $Z(G) = \langle y^3\rangle$, so that (46) holds. In particular, the centre is cyclic, so, by Theorem 1.2, any minimal faithful representation of G is transitive, so $\mu(G)$ is a power of 3. But G contains the subgroup

$$\langle y,x^3\rangle = \langle y\rangle \times \langle x^3\rangle \cong C_9 \times C_3\,,$$

so that $\mu(G) \geq \mu(C_9 \times C_3) = 12$. Hence $\mu(G) \geq 27$. Put $L = \langle x\rangle$. Then $|L| = 9$ and L has trivial core, so L affords a faithful representation of G of degree 27. Hence also $\mu(G) \leq 27$, so that $\mu(G) = 27$. By Lemma 2.10, G is not exceptional, completing the proof of the lemma. □

Remark 5.5. Let G be any group of order 243 defined by (43) or (45) of the previous two lemmas, so that $\mu(G) = 27$. Let $N = Z(G)$, so that $N = \langle y^3\rangle$, by (44) and (46) respectively, and $|G/N| = 81$. By adding the relation $y^3 = 1$, the presentations (43) and (45) quickly reduce to the equivalent presentation (25) of Q_2, so that $G/N \cong Q_2$. By (31), we have

$$\mu(G) = 27 = \mu(Q_2) = \mu(G/N)\,,$$

so that G is an almost exceptional group with almost distinguished quotient Q_2.

Lemma 5.6. *Let G be the group defined by either of the following presentations:*

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = [y,z] = 1\,,\ [x,y] = z^3\,,\ [x,z] = y^{-1}\rangle \tag{47}$$

or

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = [y,z] = 1\,,\ [x,y] = z^{-3}\,,\ [x,z] = y^{-1}\rangle\,. \tag{48}$$

Then $|G| = 243$, $\mu(G) = 36$,

$$Z(G) = \langle x^3, z^3\rangle \cong C_3 \times C_3\,, \tag{49}$$

and G is not exceptional.

Proof. We will handle both groups simultaneously, using the relation $[x,y] = z^{\pm 3}$, interpreted as (47) or (48) alternatively. By Lemma 2.7, $|G| = 243$ and (49) holds. Further, by part (c) of Lemma 2.7, any subgroup of G of order 27 contains z^3. By Theorem 1.2, a minimal faithful representation of G is afforded by two subgroups H and K, say. If both H and K have orders at least 27 then their core intersection contains z^3, contradicting faithfulness. Hence, without loss of generality $|H| \leq 9$. If $|K| > 27$ then K is a subgroup of G of index at most 3, so that K contains both x^3 and z^3, so that K contains $Z(G)$, contradicting Lemma 2.9. Hence $|K| \leq 27$, and so $\mu(G) = |G:H| + |G:K| \geq 9 + 27 = 36$. Put

$$S = \langle z,y\rangle \cong C_9 \times C_3 \qquad \text{and} \qquad T = \langle x\rangle \cong C_9\,.$$

Then $|S| = 27$, $|T| = 9$ and

$$\mathrm{core}(S \cap T) \; = \; \mathrm{core}(S) \cap \mathrm{core}(T) \; = \; \langle z^3 \rangle \cap \langle x^3 \rangle \; = \; \{1\} \; ,$$

so that $\{S,T\}$ affords a faithful permutation representation of G of degree $|G:S|+|G:T| = 36$. Thus $\mu(G) \leq 36$. Hence, $\mu(G) = 36$, so that G is not exceptional by Lemma 2.10. □

Remark 5.7. The groups defined by (47) and (48) of Lemma 5.6 are groups 243.15, 243.14 respectively in Table 3. Consider the following group:

$$H \; = \; \langle x,y,z \mid x^9 = y^3 = z^9 = [x,y] = 1 \, , \; [x,z] = y \, , \; [y,z] = x^{-3} \rangle \; . \tag{50}$$

The group G defined by (47) is isomorphic to H, which can be verified quickly by applying the transformation $x' = x$, $y' = y$, $z' = x$ to (47), followed by dropping dashes. Thus H also is not exceptional, by Lemma 5.6. However, H appears as the second group labelled as G_6 in [2, Table 1], which is claimed in that paper to be exceptional. It should be noted, however, that the first group labelled as G_6 in [2, Table 1] is indeed exceptional (and of course is not isomorphic to the second group in that table with the same label).

Lemma 5.8. *Let G be the group defined by the following presentation:*

$$G \; = \; \langle x,y,z \mid x^9 = y^3 = z^9 = 1 \, , \; z^3 \text{ central} \, , \; [x,y] = z^3 \, , \; [x,z] = y^{-1} \, , \; [y,z] = x^{-3}z^3 \rangle \; . \tag{51}$$

Then $|G| = 243$*,* $\mu(G) = 36$*,*

$$Z(G) = \langle x^3, z^3 \rangle \; \cong \; C_3 \times C_3 \; , \tag{52}$$

and G is not exceptional.

Proof. By Lemma 2.7, $|G| = 243$ and (52) holds. Further, by part (d) of Lemma 2.7, any subgroup of G of order 27 contains x^3z^3. By Theorem 1.2, a minimal faithful representation of G is afforded by two subgroups H and K, say. If both H and K have orders at least 27 then their core intersection contains x^3z^3, contradicting faithfulness. Hence, without loss of generality $|H| \leq 9$. If $|K| > 27$ then K is a subgroup of G of index at most 3, so that K contains both x^3 and z^3, so that K contains $Z(G)$, contradicting Lemma 2.9. Hence $|K| \leq 27$, and so $\mu(G) = |G:H| + |G:K| \geq 9 + 27 = 36$. Observe that, by (6) and (7) of Lemma 2.6, putting $\alpha = 1$ and $\gamma = 2$, we have

$$(xz^2)^3 \; = \; x^{-3}z^{-3} \; = \; [xz^2, y] \; .$$

Now put

$$S \; = \; \langle xz^2, y \rangle \; = \; \langle xz^2 \rangle \rtimes \langle y \rangle \; \cong \; C_9 \rtimes C_3 \qquad \text{and} \qquad T \; = \; \langle x \rangle \; \cong \; C_9 \; .$$

Then $|S| = 27$, $|T| = 9$ and

$$\mathrm{core}(S \cap T) \; = \; \mathrm{core}(S) \cap \mathrm{core}(T) \; = \; \langle x^3z^3 \rangle \cap \langle x^3 \rangle \; = \; \{1\} \; ,$$

so that $\{S,T\}$ affords a faithful permutation representation of G of degree $|G:S|+|G:T| = 36$. Thus $\mu(G) \leq 36$. Hence, $\mu(G) = 36$, so that G is not exceptional by Lemma 2.10. □

Remark 5.9. The group defined by (51) of Lemma 5.8 is group 243.5 in Table 3.

Lemma 5.10. *Let G be the group defined by the following presentation:*

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^3z^{-3}\rangle\,. \tag{53}$$

Then $|G| = 243$*,* $\mu(G) = 54$*,*

$$Z(G) = \langle x^3, z^3\rangle \cong C_3 \times C_3\,, \tag{54}$$

and G is not exceptional.

Proof. By Lemma 2.7, $|G| = 243$ and (54) holds. Further, by part (b) of Lemma 2.7, any subgroup of G of order 27 contains $Z(G)$. By Theorem 1.2, a minimal faithful representation of G is afforded by two subgroups H and K, say. If $|H| \geq 27$ then $Z(G) \subseteq H$, contradicting Lemma 2.9. Hence $|H| \leq 9$. Similarly $|K| \leq 9$, and so $\mu(G) = |G:H| + |G:K| \geq 27 + 27 = 54$. Now put

$$S = \langle x\rangle \cong C_9 \qquad \text{and} \qquad T = \langle z\rangle \cong C_9\,.$$

Then $|S| = |T| = 9$ and

$$\mathrm{core}(S \cap T) = \mathrm{core}(S) \cap \mathrm{core}(T) = \langle x^3\rangle \cap \langle z^3\rangle = \{1\}\,,$$

so that $\{S,T\}$ affords a faithful permutation representation of G of degree $|G:S| + |G:T| = 54$. Thus $\mu(G) \leq 54$. Hence, $\mu(G) = 54$, so that G is not exceptional by Lemma 2.10. □

Remark 5.11. The group defined by (53) of Lemma 5.10 is group 243.8 in Table 3. Consider the following group:

$$H = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^3\,,\ [x,z] = y\,,\ [y,z] = x^{-3}\rangle\,. \tag{55}$$

The group G defined by (53) is isomorphic to H. To see this, one may apply the following transformation to (53):

$$x' = x\,, \qquad y' = yx^{-3}z^3\,, \qquad z' = xz^{-1}\,.$$

Using the relations of (53), we have $(x')^9 = (y')^3 = 1$. By (6) of Lemma 2.6 when $\alpha = 1$ and $\gamma = -1$, we have

$$(z')^3 = (xz^{-1})^3 = z^3\,,$$

so that also $(z')^9 = 1$. Further, we have

$$[x',y'] = [x,y] = z^3 = (z')^3\,, \quad [x',z'] = [x,xz^{-1}] = x^{-1}zxz^{-1} = zyz^{-1} = yx^{-3}z^3 = y'\,,$$

$$[y',z'] = [yx^{-3}z^3, xz^{-1}] = [x,y]^{-1}[y,z]^{-1} = z^{-3}x^{-3}z^3 = x^{-3} = (x')^3\,.$$

Dropping the dashes, we obtain the presentation (55), so that G and H are isomorphic. Thus H also is not exceptional, by Lemma 5.6. Though the presentation of G is more complicated than the presentation of H, it is worth comparing (53) with the presentation (51) appearing in Lemma 5.8: the only difference is that the commutator $[y,z]$ has been inverted, with the consequence that the minimal degree jumps from 54 (in Lemma 5.10) to 36 (in Lemma 5.8).

Lemma 5.12. *Let G be the group defined by the following presentation:*

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = x^{-3}z^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^3z^3\rangle\,. \tag{56}$$

Then $|G| = 243$*,* $\mu(G) = 54$*,*

$$Z(G) = \langle x^3, z^3\rangle \cong C_3 \times C_3\,, \tag{57}$$

and G is not exceptional.

Proof. By Lemma 2.7, $|G| = 243$ and (57) holds. Further, by part (a) of Lemma 2.7, any subgroup of G of order 27 contains $Z(G)$. By Theorem 1.2, a minimal faithful representation of G is afforded by two subgroups H and K, say. If $|H| \geq 27$ then $Z(G) \subseteq H$, contradicting Lemma 2.9. Hence $|H| \leq 9$. Similarly $|K| \leq 9$, and so $\mu(G) = |G:H| + |G:K| \geq 27+27 = 54$. Now put

$$S \;=\; \langle x \rangle \;\cong\; C_9 \qquad\text{and}\qquad T \;=\; \langle z \rangle \;\cong\; C_9 \;.$$

Then $|S| = |T| = 9$ and

$$\mathrm{core}(S \cap T) \;=\; \mathrm{core}(S) \cap \mathrm{core}(T) \;=\; \langle x^3 \rangle \cap \langle z^3 \rangle \;=\; \{1\} \;,$$

so that $\{S, T\}$ affords a faithful permutation representation of G of degree $|G:S| + |G:T| = 54$. Thus $\mu(G) \leq 54$. Hence, $\mu(G) = 54$, so that G is not exceptional by Lemma 2.10. □

Remark 5.13. The group defined by (56) of Lemma 5.12 is group 243.9 in Table 3.

Lemma 5.14. *Let G be the group defined by the following presentation:*

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = [z^3,x] = 1\,,\; [x,y] = z^3\,,\; [x,z] = y\,,\; [y,z] = x^{-3}z^3 \rangle \;. \tag{58}$$

Then $|G| = 243$, $\mu(G) = 36$,

$$Z(G) = \langle x^3, z^3 \rangle \;\cong\; C_3 \times C_3 \;, \tag{59}$$

and G is not exceptional.

Proof. It follows from the relations of (58) that x^3 and z^3 are central in G_7. By Lemma 2.7, interchanging the roles of x and z, with corresponding adjustments to the commutator relations, we have that $|G| = 243$ and (59) holds. Elements w of G have the form

$$w \;=\; x^{\alpha} y^{\beta} z^{\gamma} \tag{60}$$

for some α, β, γ such that $0 \leq \alpha, \gamma \leq 8$, and $0 \leq \beta \leq 2$. By (6) of Lemma 2.6, with x, y, z in place of a, b and c respectively, and taking $\varepsilon = 1$, we have

$$w^3 \;=\; x^{3\alpha(1-\gamma^2)} z^{3\gamma(1+\alpha\gamma+\alpha^2)} \;. \tag{61}$$

It follows from (61), by inspection, that $w^3 = 1$ if and only if α and γ are multiples of 3 or α and γ are not multiples of 3 and $\alpha = \gamma$. Let L be the set consisting of elements of G of order 1 or 3. It follows that L is a subgroup of G and

$$L \;=\; \langle xz, x^3, z^3 \rangle \rtimes \langle y \rangle \;\cong\; (C_3 \times C_3 \times C_3) \rtimes C_3 \;.$$

Suppose that H is a subgroup of G of order at least 27. We will show that

$$x^3 \;\in\; H \;. \tag{62}$$

If $H \subseteq L$ then, by comparing orders, either $H = L$ or H is a maximal subgroup of L so contains the commutator

$$[xz, y] \;=\; [x,z][y,z]^{-1} \;=\; z^3 x^3 z^{-1} \;=\; x^3 \;,$$

and (62) holds. Hence we may suppose that there exists $w \in H$ of order 9, given by (60) where α and γ are not both multiples of 3, and if α and γ are both not multiples of 3 then $\alpha = -\gamma$. Note, in particular, this guarantees that

$$\alpha - \gamma \neq 0 \mod 3 \;. \tag{63}$$

Observe that

$$[z^\gamma,x] \;=\; z^{-\gamma}x^{-1}z^\gamma x \;=\; y^{-\gamma} \qquad \text{and} \qquad [x^\alpha,z] \;=\; x^{-\alpha}z^{-1}x^\alpha z \;=\; y^\alpha \;,$$

so that

$$[x^\alpha,z]^{y^\beta z^\gamma} \;=\; (y^\alpha)^{y^\beta z^\gamma} \;=\; (y^\alpha)^{z^\gamma} \;=\; y^\alpha c \;,$$

for some $c \in Z(G)$. Observe also that

$$[x^\alpha y^\beta,x] \in Z(G) \qquad \text{and} \qquad [y^\beta z^\gamma,z] \in Z(G) \;.$$

Hence

$$[w,x] \;=\; [x^\alpha y^\beta z^\gamma,x] \;=\; [x^\alpha y^\beta,x]^{z^\gamma}[z^\gamma,x] \;=\; c_1 y^{-\gamma} \;,$$

for some element $c_1 \in Z(G)$, and

$$[w,z] \;=\; [x^\alpha y^\beta z^\gamma,z] \;=\; [x^\alpha,z]^{y^\beta z^\gamma}[y^\beta z^\gamma,z] \;=\; y^\alpha c[y^\beta z^\gamma,z] \;=\; c_2 y^\alpha \;,$$

for some $c_2 \in Z(G)$. Hence

$$[w,xz] \;=\; [w,z][w,x]^z \;=\; c_2 y^\alpha (c_1 y^{-\gamma})^z \;=\; c_1 c_2 y^\alpha (y^z)^{-\gamma} \;=\; c_1 c_2 y^\alpha (yx^{-3}z^3)^{-\gamma} \;,$$

from which it follows that

$$[w,xz] \;=\; c_3 y^{\alpha-\gamma} \;, \tag{64}$$

for some $c_3 \in Z(G)$. Note also that

$$[w,y^\sigma]^{xz} \;=\; [x^\alpha y^\beta z^\gamma,y^\sigma]^{xz} \;=\; [x,y]^{\alpha\sigma}[z,y]^{\gamma\sigma} \;=\; c_4 \;, \tag{65}$$

for some $c_4 \in Z(G)$. Observe, by properties of groups of order 27, that $H \cap L$ must have order at least 9. If $H \cap L \subseteq Z(G)$ then, comparing sizes, we have $H \cap L = Z(G)$, so that, in particular, $x^3 \in H$, and (62) holds. Hence we may suppose that $H \cap L$ has an element v that is not central, which therefore has the form

$$v \;=\; x^{3\delta}z^{3\varepsilon}y^\sigma(xz)^\tau \tag{66}$$

for some δ, ε, σ, τ, such that $0 \le \delta,\varepsilon,\sigma,\tau \le 2$ and σ and τ are not both zero. We will show that we can guarantee the existence of $v \in H \cap L$ described by (66) but such that $\sigma = 1$ and $\tau = 0$, so that

$$v \;=\; x^{3\delta}z^{3\varepsilon}y \;. \tag{67}$$

Suppose that $\tau \neq 0$, By replacing v by v^2 in (66), if necessary, we may suppose that

$$v \;=\; x^{3\delta}z^{3\varepsilon}y^\sigma xz \;.$$

But then, by (64) and (65), we have

$$[w,v] \;=\; [w,x^{3\delta}z^{3\varepsilon}y^\sigma xz] \;=\; [w,y^\sigma xz] \;=\; [w,xz][w,y^\sigma]^{xz} \;=\; c_3c_4y^{\alpha-\gamma} \;.$$

Thus $[w,v]$ is an element of $H \cap L$, which can be written in the form of the right-hand side of (66) where $\tau = 0$ and $\sigma = \alpha - \gamma$, noting that $\alpha - \gamma$ is nonzero by (63). Taking this element or its square as v, there is no loss in generality in assuming that $\sigma = 1$ and (67) holds. But now, using (67) for v, we have

$$[w,v] \;=\; [x^\alpha y^\beta z^\gamma, x^{3\delta}z^{3\varepsilon}y] \;=\; [x,y]^\alpha[y,z]^{-\gamma} \;=\; x^{3\gamma}z^{3(\alpha-\gamma)} \;. \tag{68}$$

If α is a multiple of 3 then γ is not a multiple of 3 and, by (61) and (68), we have

$$\langle w^3,[w,v]\rangle \;=\; \langle z^{3\gamma},x^{3\gamma}z^{-3\gamma}\rangle \;=\; \langle x^3,z^3\rangle \;=\; Z(G) \;,$$

whence $x^3 \in H$, and (62) holds. Hence we may suppose that α is not a multiple of 3. If γ is a multiple of 3 then α is not a multiple of 3 and, by (61) and (68), we have

$$\langle w^3, [w,v] \rangle \;=\; \langle x^{3\alpha}, z^{3\alpha} \rangle \;=\; Z(G)\,,$$

whence $x^3 \in H$, and (62) holds. Hence we may suppose also that γ is not a multiple of 3 so that $\alpha = -\gamma$. But then, by (61) and(68),

$$\langle w^3, [w,v] \rangle \;=\; \langle z^{3\gamma}, x^{3\gamma} z^{3\gamma} \rangle \;=\; \langle x^3, z^3 \rangle \;=\; Z(G)\,,$$

whence $x^3 \in H$, and (62) holds. This completes the proof that (62) always holds. By Theorem 1.2, a minimal faithful representation of G is afforded by two subgroups H and K, say. If both H and K have orders at least 27 then their core intersection contains x^3, by (62), contradicting faithfulness. Hence, without loss of generality $|H| \leq 9$. If $|K| > 27$ then K is a subgroup of G of index at most 3, so that K contains both x^3 and z^3, so that K contains $Z(G)$, contradicting Lemma 2.9. Hence $|K| \leq 27$, and so $\mu(G) = |G:H| + |G:K| \geq 9+27 = 36$. Observe that $(xz)^y = xzx^3$, so we may consider

$$S \;=\; \langle x^3, xz, y \rangle \;=\; \big(\langle x^3 \rangle \times \langle xz \rangle\big) \rtimes \langle y \rangle \;\cong\; (C_3 \times C_3) \rtimes C_3 \quad \text{and} \quad T \;=\; \langle z \rangle \;\cong\; C_9\,.$$

Then $|S| = 27$, $|T| = 9$ and $\mathrm{core}(S \cap T) = \langle x^3 \rangle \cap \langle z^3 \rangle = \{1\}$, so that $\{S,T\}$ affords a faithful permutation representation of G of degree $|G:S| + |G:T| = 36$. Thus $\mu(G) \leq 36$. Hence, $\mu(G) = 36$, so that G is not exceptional by Lemma 2.10. □

Remark 5.15. The group defined by (58) of Lemma 5.14 is group 243.6 in Table 3.

6. Exceptional preimages of second distinguished quotient

We now prove the following theorem, stated above as Theorem 3.3, classifying exceptional preimages of Q_2 of order 243, up to isomorphism.

Theorem 6.1. *The following two groups have order* $3^5 = 243$ *and have* Q_2 *defined by* (25) *as a distinguished quotient:*

(i) $G_6 \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = [x,y] = 1\,, [x,z] = y\,,\; [y,z] = x^{-3}z^3 \rangle$,
(ii) $G_7 \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\; z^3 \text{ central}\,,\; [x,y] = z^{-3}\,,\; [x,z] = y\,,\; [y,z] = x^{-3} \rangle$.

Suppose that G is an exceptional group of order p^5 with distinguished quotient Q_2. Then there is a distinguished normal subgroup N, generated by a central element of G, such that $G/N \cong Q$, and G is isomorphic to G_6 or G_7. Moreover, $\mu(G) = 18$.

Proof. The first claim follows by Propositions 4.1 and 4.2. Let G be a group of order 243, which is exceptional with distinguished normal subgroup N and distinguished quotient $G/N \cong Q_2$. Hence $\mu(G) < \mu(Q_2) = 27$, by (31). Since $|G| = 243$ and $|Q| = 81$, we have $|N| = 3$, so that $N = \langle n \rangle$ must be generated by a central element n, say, of order 3, since the centre of a 3-group intersects each nontrivial normal subgroup nontrivially. Let x,y,z be preimages of a,b,c respectively, with respect to an epimorphism from G onto Q_3, which must exist, with kernel N. Certainly $|x| \geq |a| = 9$. If $|x| \geq 27$, then $\mu(G) \geq \mu(\langle x \rangle) \geq 27$, which is a contradiction. Hence $|x| = 9$. Because of the respective relations that hold in the presentation (25) of Q_2, we have, in G,

$$y^3\,,\; z^3\,,\; [x,y]\,,\; [x,z]y^{-1}\,,\; [y,z]x^3 \;\in\; N\,.$$

Thus there exist i,j,k,ℓ,m such that $0 \le i,j,k,\ell,m \le 2$ and the following equations hold in G:

$$y^3 = n^i, \quad z^3 = n^j, \quad [x,y] = n^k, \quad [x,z] = yn^\ell, \quad [y,z] = x^{-3}n^m .$$

We then get the following presentation, which we may identify with G:

$$G = \langle x,y,z,n \mid x^9 = n^3 = 1\,,\ n \text{ central}\,,\ y^3 = n^i\,,\ z^3 = n^j\,,\ [x,y] = n^k\,,\ [x,z] = yn^\ell\,,\ [y,z] = x^{-3}n^m \rangle. \tag{69}$$

By considering the transformation $x' = x$, $y' = yn^\ell$, $z' = z$, $n' = n$, and then dropping the dashes, we simplify one of the relations, so that $\ell = 0$, giving the following presentation of G:

$$G = \langle x,y,z,n \mid x^9 = n^3 = 1\,,\ n \text{ central}\,,\ y^3 = n^i\,,\ z^3 = n^j\,,\ [x,y] = n^k\,,\ [x,z] = y\,,\ [y,z] = x^{-3}n^m \rangle\,. \tag{70}$$

Suppose first that $i \ne 0$. We will aim for a contradiction by showing that G is not exceptional. Consider the case that $j = 0$, and then (70) becomes

$$G = \langle x,y,z,n \mid x^9 = z^3 = n^3 = 1\,,\ n \text{ central}\,,\ y^3 = n^i\,,\ [x,y] = n^k\,,\ [x,z] = y\,,\ [y,z] = x^{-3}n^m \rangle\,. \tag{71}$$

Since i is nonzero, we may delete n, add the relation $y^9 = 1$, and express each power of n as a power of y^3, renaming the exponents, to transform (71) into the following:

$$G = \langle x,y,z \mid x^9 = y^9 = z^3 = 1\,,\ [x,y] = y^{3k}\,,\ [x,z] = y\,,\ [y,z] = x^{-3}y^{3m} \rangle \tag{72}$$

for some k,m such that $0 \le k,m \le 2$. Note that (72) and (71) are equivalent, because it follows from the relations in (72) that y^3 commutes with both x and z, so that y^3 is central. By Lemma 5.1, the group defined by (72) is not exceptional. Consider now the case that j is nonzero, so that $z^3 = y^{3\varepsilon}$ where $\varepsilon = \pm 1$. Again, we may delete n, add the relation $y^9 = 1$, and express each power of n as a power of y^3, renaming the exponents, to transform (71) into the following:

$$G = \langle x,y,z \mid x^9 = y^9 = 1\,,\ z^3 = y^{3\varepsilon}\,,\ [x,y] = y^{3k}\,,\ [x,z] = y\,,\ [y,z] = x^{-3}y^{3m} \rangle \tag{73}$$

for some k,m such that $0 \le k,m \le 2$. Again the presentations are equivalent, because it follows from the relations in (73) that y^3 is central. By Lemma 5.3, the group defined by (73) is not exceptional. Both cases contradict that G is exceptional.

Hence $i = 0$ and (70) simplifies to the following:

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n \text{ central}\,,\ z^3 = n^j\,,\ [x,y] = n^k\,,\ [x,z] = y\,,\ [y,z] = x^{-3}n^m \rangle. \tag{74}$$

Suppose now that j is nonzero. We may delete n, add the relation $z^9 = 1$, express each nonzero power of n as a power of z^3, and rename the exponents, to transform (74) into the following:

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^{3k}\,,\ [x,z] = y\,,\ [y,z] = x^{-3}z^{3m} \rangle \tag{75}$$

for some k,m such that $0 \le k,m \le 2$. Suppose first that $k = 0$, yielding the following:

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = [x,y] = 1\,,\ [x,z] = y\,,\ [y,z] = x^{-3}z^{3m} \rangle\,, \tag{76}$$

noting that centrality of z^3 follows easily from these relations. If $m = 0$ then this becomes

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = [x,y] = 1\,,\ [x,z] = y\,,\ [y,z] = x^{-3} \rangle\,,$$

and then G is not exceptional by Remark 5.7, which is a contradiction. Hence $m \ne 0$, so $m = 1$ or $m = 2$. Suppose first that $m = 2$, so that (76) is equivalent to

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = [x,y] = 1\,,\ [x,z] = y\,,\ [y,z] = x^{-3}z^{-3} \rangle\,. \tag{77}$$

Put $x' = x$, $y' = y^{-1}x^{-3}z^{-3}$ and $z' = z^{-1}$. After removing dashes, (77) becomes equivalent to

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = [x,y] = 1\,,\ [x,z] = y\,,\ [y,z] = x^{-3}z^3\rangle\ . \tag{78}$$

But (78) becomes the case $m = 1$, and also the presentation of G_6 in Lemma 4.1, which is exceptional. Hence, for both alternatives, $m = 1$ or $m = 2$, we have $G \cong G_6$.

Suppose, secondly, with respect to (75), that k is nonzero, so $k = 1$ or $k = 2$. First consider the case that $k = 1$. If $m = 0$ then (75) becomes the following

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^3\,,\ [x,z] = y\,,\ [y,z] = x^{-3}\rangle\ ,$$

which coincides with (55), so that G is not exceptional, by Remark 5.11, which is a contradiction. Hence m is nonzero, so $m = 1$ or $m = 2$. Suppose first that $m = 2$, so that (75) is equivalent to

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^3\,,\ [x,z] = y\,,\ [y,z] = x^{-3}z^{-3}\rangle\ . \tag{79}$$

Put $x' = x$, $y' = y^{-1}x^{-3}$ and $z' = z^{-1}$. After removing dashes, (79) becomes equivalent to

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^3\,,\ [x,z] = y\,,\ [y,z] = x^{-3}z^3\rangle\ . \tag{80}$$

But (80) becomes the case $m = 1$, and G is not exceptional, by Lemma 5.14. Hence, both alternatives, $m = 1$ and $m = 2$, lead to a contradiction.

Thus $k = 2$ and (75) becomes equivalent to the following:

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^{-3}\,,\ [x,z] = y\,,\ [y,z] = x^{-3}z^{3m}\rangle \tag{81}$$

where $0 \le m \le 2$. Suppose first that $m = 2$, so (81) becomes equivalent to

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^{-3}\,,\ [x,z] = y\,,\ [y,z] = x^{-3}z^{-3}\rangle\ . \tag{82}$$

By (6) of Lemma 2.6, with x, y, z in place of a, b and c respectively, and $\alpha = \gamma = \varepsilon = 1$, we have

$$(xz)^3 \ = \ z^{-3}\ .$$

Put $x' = x$, $y' = y$ and $z' = xz$. After removing dashes, (82) becomes equivalent to

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^3\,,\ [x,z] = y\,,\ [y,z] = x^{-3}\rangle$$

which is not exceptional, again by Remark 5.11, which is a contradiction. Suppose, secondly, that $m = 1$, so that so that (81) becomes equivalent to

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^{-3}\,,\ [x,z] = y\,,\ [y,z] = x^{-3}z^3\rangle\ . \tag{83}$$

Again, by (6) of Lemma 2.6, with $\alpha = \gamma = \varepsilon = 1$, we have

$$(xz)^3 \ = \ z^3\ .$$

Now putting $x' = x$, $y' = y$ and $z' = xz$, and dropping dashes, transforms (83) back into (82), which we saw leads to a contradiction. Hence, in fact, $m = 0$, so that (81) becomes equivalent to

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^{-3}\,,\ [x,z] = y\,,\ [y,z] = x^{-3}\rangle\ ,$$

which is the presentation (34) of G_7 in Lemma 4.2, which is exceptional. Hence $G \cong G_7$.

It remains to consider the case that $j = 0$ in (74), which then becomes the following:

$$G = \langle x,y,z,n \mid x^9 = y^3 = z^3 = n^3 = 1\,,\ n \text{ central}\,,\ [x,y] = n^k\,,\ [x,z] = y\,,\ [y,z] = x^{-3}n^m\rangle\ . \tag{84}$$

It follows from these relations that

$$(xz)^3 \ = \ x(zx)z(xz) \ = \ x(xzy^2)z(zxy) \ = \ x^2y^2z^3xyx^6n^{-2m} \ = \ x^2y^3xx^6n^{-2m}n^k \ = \ n^{k-2m}\ .$$

If $k-2m \neq 0$, then the transformation $x' = x, y' = y, z' = xz, n' = n$, followed by dropping dashes, converts (84) back into (74) with $j \neq 0$, in which case we proved that $G \cong G_6$ or $G \cong G_7$.

Hence we may suppose that $k = 2m$. If $k = 0$ then $m = 0$ and $G \cong Q_2 \times C_3$, so that $\mu(G) \geq \mu(Q_2)$, so that G is not exceptional, which is a contradiction. Hence $k = 1$ or $k = 2$. In the latter case, we may replace n by n^{-1} to transform (84), so there is no loss of generality in assuming $k = 1$, and then (84) becomes

$$G = \langle x,y,z,n \mid x^9 = y^3 = z^3 = n^3 = 1\,,\ n \text{ central}\,,\ [x,y] = n\,,\ [x,z] = y\,,\ [y,z] = x^{-3}n^{-1} \rangle\,. \qquad (85)$$

It follows from these relations that $zx^2 = x^2zyn^{-2}$ and $x^2z = zx^2y^2n^{-1}$. Hence

$$(x^2z)^3 \;=\; x^2(x^2zyn^{-2})z(zx^2y^2n^{-1}) \;=\; x^4(yzx^3n)z^2x^2y^2 \;=\; x^4(x^2yn^{-2})y^2x^3n \;=\; n^{-1}$$

Put $x' = x$, $y' = y$, $z' = x^2z$, $n' = n$. After dropping dashes, (85) becomes

$$G \;=\; \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n \text{ central}\,,\ z^3 = n^{-1}\,,\ [x,y] = n\,,\ [x,z] = y\,,\ [y,z] = x^{-3} \rangle\,.$$

Deleting n, this simplifies to

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^{-3}\,,\ [x,z] = y\,,\ [y,z] = x^{-3} \rangle\,,$$

which is again the presentation (34) of G_7 in Lemma 4.2, which is exceptional. Hence $G \cong G_7$.

This shows that in all cases $G \cong G_6$ or $G \cong G_7$. By Lemmas 4.1 and 4.2, $\mu(G) = 18$, and the proof of Theorem 3.3 is complete. □

7. Exceptional preimages of third and fourth distinguished quotients

Theorem 7.1 below (stated above as Theorem 3.5) classifies exceptional preimages of Q_3 and Q_4 of order 243, up to isomorphism.

Theorem 7.1. *The following groups have order* $3^5 = 243$*:*

(i) $\widetilde{G}_6 \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = [y,z] = 1\,,\ [x,y] = x^{-3}z^3\,,\ [x,z] = y^{-1} \rangle\,,$
(ii) $G_8 \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = [y,z] = 1\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1} \rangle\,,$
(iii) $G_9 \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^{-3}z^3 \rangle\,,$
(iv) $G_{10} \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^{-3}z^{-3} \rangle\,.$

The groups G_8 and G_9 are exceptional with distinguished quotient Q_3, and the groups $\widetilde{G}_6$, G_8 and G_{10} are exceptional with distinguished quotient Q_4. Let G be an exceptional group of order 243*. If G has distinguished quotient Q_3 then G is isomorphic to G_8 or G_9. If G has distinguished quotient Q_4 then G is isomorphic to $\widetilde{G}_6$, G_8 or G_{10}. In all cases, $\mu(G) = 18$.*

Proof. By Lemmas 4.3, 4.4 and 4.6, each of the groups G_8, G_9 and G_{10} has order 243 and minimal degree 18. That the groups G_8 and G_9 are exceptional with distinguished quotient Q_3 follows from Lemmas 4.3 and 4.4. That the groups G_8 and G_{10} are exceptional with distinguished quotient Q_4 follows from Lemmas 4.3 and 4.6. By Remark 3.6, the groups $\widetilde{G}_6$ and G_6 are isomorphic, and therefore, by Lemma 4.1, we have $|\widetilde{G}_6| = 243$ and $\mu(\widetilde{G}_6) = 18$. It follows from (33) that x^3z^3 is a central element of $\widetilde{G}_6$. Put $N = \langle x^3z^3 \rangle$. Then

$$\widetilde{G}_6/N \;\cong\; \langle x,y,z \mid x^9 = y^3 = [y,z] = 1\,,\ z^3 = x^{-3}\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1} \rangle\,,$$

which is isomorphic to Q_4, under the isomorphism induced by the map $x \mapsto a$, $y \mapsto b$, $z \mapsto c$. But $\mu(Q_4) = 27 > 18 = \mu(\widetilde{G}_6)$, so that $\widetilde{G}_6$ is exceptional with distinguished quotient Q_4.

Suppose now that G is an exceptional group of order 243 with distinguished quotient Q_3. Hence there is a central subgroup $N = \langle n\rangle$ of G of order 3 such that $G/N \cong Q_3$ and $\mu(G) < \mu(Q_3) = 27$. Let x,y,z be preimages of a,b,c respectively, with respect to an epimorphism from G onto Q_3, which must exist, with kernel N. As before, $|x| = 9$. Because of the respective relations that hold in the presentation (26) of Q_3, we have, in G,

$$y^3\,,\ z^3x^{-3}\,,\ [y,z]\,,\ [x,y]x^{-3}\,,\ [x,z]y\ \in\ N\,.$$

Thus there exist $i, j, k, \ell, m \in \mathbb{Z}_3$ such that the following equations hold in G:

$$y^3\ =\ n^i,\quad z^3\ =\ x^3n^j,\quad [y,z]\ =\ n^k,\quad [x,y]\ =\ x^3n^\ell,\quad [x,z]\ =\ y^{-1}n^m\,.$$

We then get the following presentation, which we may identify with G:

$$G = \langle x,y,z,n \mid x^9 = n^3 = 1\,,\ n \text{ central}\,,\ y^3 = n^i\,,\ z^3 = x^3n^j\,,\ [y,z] = n^k\,,\ [x,y] = x^3n^\ell\,,\ [x,z] = y^{-1}n^m\rangle.$$

Using the transformation $x' = x$, $y' = yn^{-m}$, $z' = z$, $n' = n$, and then removing the dashes, we may rewrite the presentation so that it becomes the following, where $m = 0$:

$$G = \langle x,y,z,n \mid x^9 = n^3 = 1\,,\ n \text{ central}\,,\ y^3 = n^i\,,\ z^3 = x^3n^j\,,\ [y,z] = n^k\ [x,y] = x^3n^\ell\,,\ [x,z] = y^{-1}\rangle. \tag{86}$$

It follows from the relations that z^3 and x^3 are central in G. In particular,

$$x\ =\ x^{z^3}\ =\ (xy^{-1})^{z^2}\ =\ (xy^{-1})^z(y^{-1}n^{-k})^z\ =\ x^z(y^{-2})^zn^{-k}\ =\ xy^{-1}y^{-2}n^{-2k}n^{-k}\ =\ xy^{-3}\,,$$

so that $y^3 = 1$. Hence $i = 0$, so that (86) simplifies to become

$$G\ =\ \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n \text{ central}\,,\ z^3 = x^3n^j\,,\ [y,z] = n^k\,,\ [x,y] = x^3n^\ell\,,\ [x,z] = y^{-1}\rangle. \tag{87}$$

Note that at least one of j, k, ℓ is nonzero, for otherwise $G \cong Q_3 \times C_3$ would not be exceptional.

Case (i): Suppose that $k = 0$. Then y and z commute, and (87) simplifies further to become

$$G\ =\ \langle x,y,z,n \mid x^9 = y^3 = n^3 = [y,z] = 1\,,\ n \text{ central}\,,\ z^3 = x^3n^j\,,\ [x,y] = x^3n^\ell\,,\ [x,z] = y^{-1}\rangle. \tag{88}$$

Our aim is to show that $G \cong G_8$. Suppose first that j is nonzero. Without loss of generality (replacing n by n^2 if necessary), we may assume that $j = 1$ and (88) becomes

$$G\ =\ \langle x,y,z,n \mid x^9 = y^3 = n^3 = [y,z] = 1\,,\ n \text{ central}\,,\ z^3 = x^3n\,,\ [x,y] = x^3n^\ell\,,\ [x,z] = y^{-1}\rangle. \tag{89}$$

We may introduce the relation $z^9 = 1$, which is a consequence of the other relations, and use the relation $z^3 = x^3n$ to delete the generator n, and rewrite (89) to become

$$G\ =\ \langle x,y,z \mid x^9 = y^3 = z^9 = [y,z] = 1\,,\ [x,y] = x^3(z^3x^{-3})^\ell\,,\ [x,z] = y^{-1}\rangle\,, \tag{90}$$

noting that these relations imply that z^3 and x^3 are both central. If $\ell = 1$, then (90) becomes

$$G\ =\ \langle x,y,z \mid x^9 = y^3 = z^9 = [y,z] = 1\,,\ [x,y] = z^3\,,\ [x,z] = y^{-1}\rangle\,,$$

which is not exceptional, by Lemma 5.6 and (47). Hence $\ell \neq 1$. If $\ell = 0$ then (90) becomes

$$G\ =\ \langle x,y,z \mid x^9 = y^3 = z^9 = [y,z] = 1\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1}\rangle\,,$$

which is (36), so that $G \cong G_8$, and we are done. Suppose then that $\ell = 2$. Then (90) may be rewritten to become

$$G\ =\ \langle x,y,z \mid x^9 = y^3 = z^9 = [y,z] = 1\,,\ [x,y] = x^{-3}z^{-3}\,,\ [x,z] = y^{-1}\rangle\,. \tag{91}$$

Observe that, by (6) of Lemma 2.6, with $\alpha = \gamma = 1$ and $\varepsilon = -1$, we have

$$(xz)^3 \;=\; x^{-3}z^{-3}\,.$$

Using the transformation $x' = xz$, $y' = y$, $z' = z$, and then removing the dashes, we have that (91) becomes (36), so that again $G \cong G_8$, and again we are done.

Suppose now that $j = 0$, so that $\ell \neq 0$. Without loss of generality (replacing n by n^2 if necessary), we may assume that $\ell = 1$ and (88) simplifies to become

$$G \;=\; \langle x,y,z,n \mid x^9 = y^3 = n^3 = [y,z] = 1\,,\; n \text{ central}\,,\; z^3 = x^3\,,\; [x,y] = x^3n\,,\; [x,z] = y^{-1}\rangle\,. \tag{92}$$

Since x^3 is central we have

$$(xz^2)^3 \;=\; x(xz^2y^2)z^2(z^2xy^{-2}) \;=\; x^2y^2xy^{-2}z^6 \;=\; x^3y^2x^{-6}n^{-2}y^{-2}z^6 \;=\; x^3n\,.$$

We may then apply the transformation

$$x' \;=\; xz^2\,,\quad y' \;=\; y\,,\quad z' \;=\; z\,,\quad n' \;=\; n^{-1}\,,$$

and then remove the dashes, to get the following presentation:

$$G \;=\; \langle x,y,z,n \mid x^9 = y^3 = n^3 = [y,z] = 1\,,\; n \text{ central}\,,\; z^3 = x^3n\,,\; [x,y] = x^3\,,\; [x,z] = y^{-1}\rangle\,.$$

But this becomes the case $j = 1$ and $\ell = 0$ of (89) considered earlier, so that $G \cong G_8$, and we are done, completing the analysis of **Case (i)**.

Case (ii): Suppose that $k \neq 0$. Without loss of generality (replacing n with n^2 if necessary), we may assume $k = 1$, and then (87) becomes

$$G \;=\; \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\; n \text{ central}\,,\; z^3 = x^3n^j\,,\; [y,z] = n\,,\; [x,y] = x^3n^\ell\,,\; [x,z] = y^{-1}\rangle\,. \tag{93}$$

Our aim is to show that $G \cong G_9$. Suppose first that $j = 1$, so that (93) becomes

$$G \;=\; \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\; n \text{ central}\,,\; z^3 = x^3n\,,\; [y,z] = n\,,\; [x,y] = x^3n^\ell\,,\; [x,z] = y^{-1}\rangle\,. \tag{94}$$

We may introduce the relation $z^9 = 1$, a consequence of the other relations, and use the relation $z^3 = x^3n$ to delete the generator n, and rewrite (94) to become

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\; z^3 \text{ central}\,,\; [x,y] = x^3(x^{-3}z^3)^\ell\,,\; [x,z] = y^{-1}\,,\; [y,z] = x^{-3}z^3\rangle\,, \tag{95}$$

noting that these relations imply also that x^3 is central. If $\ell = 1$, then (95) becomes

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\; z^3 \text{ central}\,,\; [x,y] = z^3\,,\; [x,z] = y^{-1}\,,\; [y,z] = x^{-3}z^3\rangle\,,$$

which is (51), so that G is not exceptional, by Lemma 5.8. Hence $\ell \neq 1$. If $\ell = 0$ then (95) becomes

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\; [x,y] = x^3\,,\; [x,z] = y^{-1}\,,\; [y,z] = x^{-3}z^3\rangle\,,$$

noting that these relations imply that z^3 is central, which is (39), so that $G \cong G_9$, and we are done. Suppose then that $\ell = 2$. Then (95) may be rewritten to become

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\; z^3 \text{ central}\,,\; [x,y] = x^{-3}z^{-3}\,,\; [x,z] = y^{-1}\,,\; [y,z] = x^{-3}z^3\rangle\,. \tag{96}$$

Observe that, by (6) of Lemma 2.6, with $\alpha = \gamma = 1$ and $\varepsilon = -1$, we have

$$(xz)^3 \;=\; z^3\,.$$

Using the transformation $x' = xz$, $y' = yx^{-3}z^{-3}$, $z' = x^{-1}$, and then removing the dashes, noting that the centrality relations become superfluous, we have that (96) becomes (39), so that again $G \cong G_9$, and we are done.

Suppose now that $j = 2$, so that (93) can be rewritten as

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n\text{ central}\,,\ z^3 = x^3 n^{-1}\,,\ [y,z] = n\,,\ [x,y] = x^3 n^{\ell}\,,\ [x,z] = y^{-1}\rangle. \quad (97)$$

We may introduce the relation $z^9 = 1$, a consequence of the other relations, and use the relation $z^3 = x^3 n^{-1}$ to delete the generator n, and rewrite (97) to become

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3\text{ central}\,,\ [x,y] = x^3(x^3z^{-3})^{\ell}\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^3z^{-3}\rangle\,. \quad (98)$$

Consider the case that $\ell = 0$. Then (98) becomes

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3\text{ central}\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^3z^{-3}\rangle\,. \quad (99)$$

Observe that, by (6) of Lemma 2.6, with $\alpha = \gamma = \varepsilon = -1$, we have

$$(x^{-1}z^{-1})^3 \;=\; x^3z^3\,.$$

Using the transformation $x' = x^{-1}z^{-1}$, $y' = yx^{-3}z^{-3}$, $z' = z^{-1}$, and then removing dashes, (99) becomes (51), so that G is not exceptional, by Lemma 5.8. Hence $\ell \neq 0$. Consider the case that $\ell = 1$. Then (98) becomes

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3\text{ central}\,,\ [x,y] = x^{-3}z^{-3}\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^3z^{-3}\rangle\,. \quad (100)$$

Observe that, by (6) of Lemma 2.6, with $\alpha = \varepsilon = -1$ and $\gamma = 1$, we have

$$(x^{-1}z)^3 \;=\; x^3z^3\,.$$

Using the transformation $x' = x^{-1}z$, $y' = yx^{-3}z^{-3}$, $z' = z^{-1}$, and then removing dashes, (100) becomes (51), so that G is not exceptional, by Lemma 5.8. Hence $\ell \neq 1$, and so $\ell = 2$. We may now rewrite (98) to become

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3\text{ central}\,,\ [x,y] = z^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^3z^{-3}\rangle\,,$$

which is (53), so that G is not exceptional, by Lemma 5.10. This shows that $j \neq 2$.

Suppose finally that $j = 0$, so that (93) becomes

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n\text{ central}\,,\ z^3 = x^3\,,\ [y,z] = n\,,\ [x,y] = x^3n^{\ell}\,,\ [x,z] = y^{-1}\rangle. \quad (101)$$

Suppose first that $\ell = 0$, so that (101) becomes

$$G \;=\; \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n\text{ central}\,,\ z^3 = x^3\,,\ [y,z] = n\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1}\rangle\,. \quad (102)$$

Since $x^3 = z^3$ and n are central we have

$$(xz)^3 \;=\; x(xzy)z(zxy^{-1}) \;=\; x^2(yzn^{-1})z^2xy^{-1} \;=\; x^2(yxy^{-1})n^{-1}z^3 \;=\; x^2(xx^6)n^{-1}z^3 \;=\; x^3n^{-1}\,.$$

We may then apply the following transformation to (102):

$$x' \;=\; xz\,,\quad y' \;=\; yn\,,\quad z' \;=\; z\,,\quad n' \;=\; n\,, \quad (103)$$

and then remove the dashes, so that (101) becomes

$$G \;=\; \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n\text{ central}\,,\ z^3 = x^3n\,,\ [y,z] = n\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1}\rangle\,.$$

But this is the case $\ell = 0$ of (94) considered earlier (when $j = 1$), so that $G \cong G_9$ is exceptional. Suppose next that $\ell = 1$, so that (101) becomes

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n\text{ central}\,,\ z^3 = x^3\,,\ [y,z] = n\,,\ [x,y] = x^3n\,,\ [x,z] = y^{-1}\rangle. \quad (104)$$

Now we have

$$(xz)^3 \;=\; x^2(yxy^{-1})n^{-1}z^3 \;=\; x^2(xx^6n^2)n^{-1}z^3 \;=\; nz^3 \;=\; x^3n\,,$$

and again apply the transformation (103), and then remove the dashes, so that (104) becomes

$$G \;=\; \langle x,y,z,n \mid x^9=y^3=n^3=1\,,\; n \text{ central}\,,\; z^3=x^3n^{-1}\,,\; [y,z]=n\,,\; [x,y]=x^3n^{-1}\,,\; [x,z]=y^{-1}\rangle.$$

But this becomes the case $\ell=2$ of (97) considered earlier (when $j=2$), where G is not exceptional, which is impossible. Finally, suppose that $\ell=2$, so that (101) may be rewritten as

$$G \;=\; \langle x,y,z,n \mid x^9=y^3=n^3=1\,,\; n \text{ central}\,,\; z^3=x^3\,,\; [y,z]=n\,,\; [x,y]=x^3n^{-1}\,,\; [x,z]=y^{-1}\rangle. \quad (105)$$

Now we have

$$(xz)^3 \;=\; x^2(yxy^{-1})n^{-1}z^3 \;=\; x^2(xx^6n^{-2})n^{-1}z^3 \;=\; z^3 \;=\; x^3\,,$$

and again apply the transformation (103), and then remove the dashes, so that (105) becomes (104), which was seen earlier not be exceptional. Thus $\ell=2$ also does not arise, completing the analysis of **Case (ii)**.

Suppose now that G is an exceptional group of order 243 with distinguished quotient Q_4. As before, but making an adjustment to the one relation that differs between Q_3 and Q_4, we may assume that there exist $j,k,\ell \in \mathbb{Z}_3$ such that

$$G \;=\; \langle x,y,z,n \mid x^9=y^3=n^3=1\,,\; n \text{ central}\,,\; z^3=x^{-3}n^j\,,\; [y,z]=n^k\,,[x,y]=x^3n^\ell\,,\; [x,z]=y^{-1}\rangle. \quad (106)$$

Again, at least one of j,k,ℓ is nonzero, for otherwise $G\cong Q_4\times C_3$ would not be exceptional.

Case (iii): Suppose that $k=0$. Then (106) simplifies to become

$$G \;=\; \langle x,y,z,n \mid x^9=y^3=n^3=[y,z]=1\,,\; n \text{ central}\,,\; z^3=x^{-3}n^j\,,\; [x,y]=x^3n^\ell\,,\; [x,z]=y^{-1}\rangle. \quad (107)$$

We show that G is isomorphic to $\widetilde{G}_6$ or G_8. Suppose first that j is nonzero. Without loss of generality (replacing n by n^2 if necessary), we may assume that $j=1$ and (107) becomes

$$G \;=\; \langle x,y,z,n \mid x^9=y^3=n^3=[y,z]=1\,,\; n \text{ central}\,,\; z^3=x^{-3}n\,,\; [x,y]=x^3n^\ell\,,\; [x,z]=y^{-1}\rangle. \quad (108)$$

We may introduce the relation $z^9=1$ and use the relation $z^3=x^{-3}n$ to delete n, and rewrite (108) to become

$$G \;=\; \langle x,y,z \mid x^9=y^3=z^9=[y,z]=1\,,\; [x,y]=x^3(z^3x^3)^\ell\,,\; [x,z]=y^{-1}\rangle\,, \quad (109)$$

noting that these relations imply z^3 and x^3 are central. Suppose first that $\ell=2$. Then (109) may be rewritten to become

$$G \;=\; \langle x,y,z \mid x^9=y^3=z^9=[y,z]=1\,,\; [x,y]=z^{-3}\,,\; [x,z]=y^{-1}\rangle\,,$$

which is (48), so that G is not exceptional by Lemma 5.6, which is impossible. Hence $\ell\neq 2$. If $\ell=1$, then (109) becomes

$$G \;=\; \langle x,y,z \mid x^9=y^3=z^9=[y,z]=1\,,\; [x,y]=x^{-3}z^3\,,\; [x,z]=y^{-1}\rangle\,,$$

which is the presentation for $\widetilde{G}_6$, which proves that $G\cong\widetilde{G}_6$. If $\ell=0$ then (109) becomes

$$G \;=\; \langle x,y,z \mid x^9=y^3=z^9=[y,z]=1\,,\; [x,y]=x^3\,,\; [x,z]=y^{-1}\rangle\,,$$

which is (36), so that $G\cong G_8$.

Suppose now that $j=0$, so that $\ell \neq 0$. Without loss of generality (replacing n by n^2 if necessary), we may assume that $\ell = 1$ and (107) simplifies to become

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = [y,z] = 1\,,\ n \text{ central}\,,\ z^3 = x^{-3}\,,\ [x,y] = x^3 n\,,\ [x,z] = y^{-1} \rangle. \quad (110)$$

We have

$$(xz)^3 \;=\; x(xzy)z(zxy^{-1}) \;=\; x^2yxy^{-1}z^3 \;=\; x^3x^6n^2z^3 \;=\; x^{-3}n^{-1}\,,$$

and may apply the transformation $x' = xz,\ y' = y^{-1},\ z' = z^{-1},\ n' = n^{-1}$, and then remove the dashes, to get the following presentation:

$$G \;=\; \langle x,y,z,n \mid x^9 = y^3 = n^3 = [y,z] = 1\,,\ n \text{ central}\,,\ z^3 = x^{-3}n\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1} \rangle\,.$$

But this becomes the case $j=1$ and $\ell = 0$ of (108) considered earlier, so that $G \cong G_8$, and we are done, completing the analysis of **Case (iii)**.

Case (iv): Suppose that $k \neq 0$.

Without loss of generality (replacing n with n^2 if necessary), we may assume $k=1$, and then (106) becomes

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n \text{ central}\,,\ z^3 = x^{-3}n^j\,,\ [y,z] = n\,,\ [x,y] = x^3n^\ell\,,\ [x,z] = y^{-1} \rangle. \quad (111)$$

We show that $G \cong G_{10}$. Suppose first that $j=1$, so that (111) becomes

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n \text{ central}\,,\ z^3 = x^{-3}n\,,\ [y,z] = n\,,\ [x,y] = x^3n^\ell\,,\ [x,z] = y^{-1} \rangle. \quad (112)$$

We may introduce the relation $z^9 = 1$ and use the relation $z^3 = x^{-3}n$ to delete the generator n, and rewrite (112) to become

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = x^3(x^3z^3)^\ell\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^3z^3 \rangle\,. \quad (113)$$

If $\ell = 1$ then (113) becomes

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = x^{-3}z^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^3z^3 \rangle\,, \quad (114)$$

which is not exceptional, by Lemma 5.12. If $\ell = 0$ then (113) becomes

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^3z^3 \rangle\,, \quad (115)$$

noting that these relations imply the centrality of z^3. By (6) of Lemma 2.6, with $\alpha = \varepsilon = -1$ and $\gamma = 1$, we have $(x^{-1}z)^3 = x^{-3}z^{-3}$. We may now apply the transformation $x' = z,\ y' = yx^{-3}z^3,\ z' = x^{-1}z$ to (115), drop the dashes, noting that the centrality relations are all equivalent, in the presence of the other relations, and obtain the following presentation

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^{-3}z^3 \rangle\,.$$

But this is (51), so that G is not exceptional, by Lemma 5.8. If $\ell = 2$ then (113) becomes

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^{-3}\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^3z^3 \rangle\,. \quad (116)$$

Applying the tranformation $x' = x,\ y' = yx^{-3}z^{-3},\ z' = z^{-1}$ to (116), and removing dashes, yields (58), so that G is not exceptional, by Lemma 5.14. This shows that the case $j=1$ does not occur.

Suppose now that $j=2$, so that (111) can be rewritten as

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n \text{ central}\,,\ z^3 = x^{-3}n^{-1}\,,\ [y,z] = n\,,\ [x,y] = x^3n^\ell\,,\ [x,z] = y^{-1} \rangle. \quad (117)$$

We may introduce the relation $z^9 = 1$, use the relation $z^3 = x^{-3}n^{-1}$ to delete the generator n, and rewrite (117) to become

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = x^3(x^{-3}z^{-3})^{\ell}\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^{-3}z^{-3}\rangle\,. \quad (118)$$

If $\ell = 0$ then (118) becomes

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^{-3}z^{-3}\rangle\,,$$

noting that these relations imply the centrality of z^3, which is (41), so that $G \cong G_{10}$. Consider the case that $\ell = 1$. Then (118) becomes

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = z^{-3}\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^{-3}z^{-3}\rangle\,. \quad (119)$$

Applying the transformation $x' = z\,,\ y' = y^{-1}x^{-3}z^{-3}\,,\ z' = xz^{-1}$ to (119), and removing dashes, yields (51), so that G is not exceptional, by Lemma 5.8. Hence $\ell = 1$ does not occur. Now consider the case $\ell = 2$. We may now rewrite (118) to become

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\ z^3 \text{ central}\,,\ [x,y] = x^{-3}z^3\,,\ [x,z] = y^{-1}\,,\ [y,z] = x^{-3}z^{-3}\rangle\,. \quad (120)$$

Applying the transformation $x' = x^{-1}z\,,\ y' = y^{-1}\,,\ z' = x$ to (120), and removing dashes, yields (51), so that again G is not exceptional, by Lemma 5.8. Hence $\ell = 2$ also does not occur.

Suppose finally then that $j = 0$, so that (111) can be rewritten as

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n \text{ central}\,,\ z^3 = x^{-3}\,,\ [y,z] = n\,,\ [x,y] = x^3n^{\ell}\,,\ [x,z] = y^{-1}\rangle. \quad (121)$$

Suppose first that $\ell = 0$, so that (121) becomes

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n \text{ central}\,,\ z^3 = x^{-3}\,,\ [y,z] = n\,,\ [x,y] = x^3\,,\ [x,z] = y^{-1}\rangle. \quad (122)$$

Since $x^3 = z^{-3}$ and n are central we have

$$(xz)^3 = x(xzy)z(zxy^{-1}) = x^2(yzn^{-1})z^2xy^{-1} = x^2(yxy^{-1})n^{-1}z^3 = x^2(xx^6)n^{-1}z^3 = x^{-3}n^{-1}\,.$$

Applying the transformation

$$x' = xz\,,\quad y' = y^{-1}\,,\quad z' = z^{-1}\,,\quad n' = n\,, \quad (123)$$

to (122), and removing dashes, yields

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n \text{ central}\,,\ z^3 = x^{-3}n^{-1}\,,\ [y,z] = n\,,[x,y] = x^3n^{-1}\,,\ [x,z] = y^{-1}\rangle.$$

But this is the case $\ell = 2$ of (117) considered earlier (when $j = 2$), so that G is not exceptional, which is impossible. Suppose next that $\ell = 1$, so that (121) becomes

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n \text{ central}\,,\ z^3 = x^{-3}\,,\ [y,z] = n\,,\ [x,y] = x^3n\,,\ [x,z] = y^{-1}\rangle. \quad (124)$$

Now we have

$$(xz)^3 = x^2(yxy^{-1})n^{-1}z^3 = x^2(xx^6n^2)n^{-1}z^3 = nz^3 = x^{-3}n\,,$$

and again apply the transformation (123), and then remove the dashes, so that (124) becomes

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n \text{ central}\,,\ z^3 = x^{-3}n\,,\ [y,z] = n\,,\ [x,y] = x^3n^{-1}\,,\ [x,z] = y^{-1}\rangle.$$

But this becomes the case $\ell = 2$ of (112) considered earlier (when $j = 1$), where G is not exceptional, which is impossible. Finally, suppose that $\ell = 2$, so that (121) may be rewritten as

$$G = \langle x,y,z,n \mid x^9 = y^3 = n^3 = 1\,,\ n \text{ central}\,,\ z^3 = x^{-3}\,,\ [y,z] = n\,,\ [x,y] = x^3n^{-1}\,,\ [x,z] = y^{-1}\rangle\,. \quad (125)$$

Observe that, using the relations of (125) other that $z^3 = x^{-3}$, we have

$$(x^{-1}z)^3 \;=\; x^{-1}(zx^{-1})z(x^{-1}z) \;=\; x^{-1}(x^{-1}zxy^{-1}x^{-1})z(zyx^{-1}) \;=\; x^{-2}zxy^{-1}x^{-1}z^2yx^{-1}$$
$$=\; x^{-2}z(xy^{-1}x^{-1})z^2yx^{-1} \;=\; x^{-2}z(y^{-1}x^{-3}n)z^2yx^{-1} \;=\; x^{-2}(zy^{-1})z^2yx^{-1}x^{-3}n$$
$$=\; x^{-2}(y^{-1}zn)z^2yx^{-1}x^{-3}n \;=\; x^{-2}z^3x^{-1}x^{-3}n^2 \;=\; x^3z^3n^{-1}\;.$$

Thus, in the presence of these relations, it follows that the following two relations are equivalent:

$$z^3 \;=\; x^{-3} \qquad \text{and} \qquad (x^{-1}z)^3 \;=\; n^{-1}\;.$$

Hence, we may rewrite the presentation (125) for G as follows:

$$G = \langle x,y,z,n \,|\, x^9 = y^3 = n^3 = 1\,,\, n \text{ central}\,,\, (x^{-1}z)^3 = n^{-1}\,,\, [y,z] = n\,,\, [x,y] = x^3n^{-1}\,,\, [x,z] = y^{-1}\rangle. \tag{126}$$

We may now delete n and the relation $(x^{-1}z)^3 = n^{-1}$, add the relation $z^9 = 1$ and rewrite the other relations to get the following presentation:

$$G = \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\; (x^{-1}z)^3 \text{ central}\,,\; [y,z] = (x^{-1}z)^{-3}\,,\; [x,y] = x^3(x^{-1}z)^3\,,\; [x,z] = y^{-1}\rangle. \tag{127}$$

We may now apply the transformation $x' = x\,,\; y' = y\,,\; z' = x^{-1}z$ to (127), drop the dashes, and obtain the following presentation:

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\; z^3 \text{ central}\,,\; [y,z] = x^3\,,\; [x,y] = x^3z^3\,,\; [x,z] = y^{-1}\rangle\;. \tag{128}$$

By (6) of Lemma 2.6, with $\alpha = \gamma = 1$ and $\varepsilon = -1$, we have $(xz)^3 = x^{-3}$. We may now apply the transformation $x' = z\,,\; y' = yx^3\,,\; z' = xz$ to (128), drop the dashes, noting that centrality of $(z')^3 = x^{-3}$ is equivalent to centrality of z^3, in the presence of the other relations, and obtain the following presentation

$$G \;=\; \langle x,y,z \mid x^9 = y^3 = z^9 = 1\,,\; z^3 \text{ central}\,,\; [x,y] = z^3\,,\; [x,z] = y\,,\; [y,z] = x^{-3}\rangle\;.$$

But this is presentation (55), so that G is not exceptional, again by Remark 5.11, which is impossible. Hence, in fact, $j = 0$ does not arise, completing the analysis of **Case (iv)**. This completes the proof of the theorem. □

Remark 7.2. We have observed (Corollary 3.7) that $G_6 \cong \widetilde{G}_6$ and G_8 are the only exceptional groups of order 243, up to isomorphism, each with two nonisomorphic distinguished quotients. Each of these has a further unique property of being the only groups of order 243 that are simultaneously exceptional and almost exceptional. To see the latter claim, it follows from the presentations (32) and (36) for G_6 and G_8 respectively that they both have the following group of order 81 as a homomorphic image:

$$H \;=\; \langle a,b,c \mid a^9 = b^3 = c^3 = 1\,,\; b \text{ central}\,,\; [a,c] = b\rangle \;\cong\; (C_9 \times C_3) \rtimes C_3\;.$$

The centre of H is elementary abelian of order 9, and it follows (using, say, Proposition 1.3) that

$$\mu(H) \;=\; \mu(G_6) \;=\; \mu(G_8) \;=\; 18\,,$$

whence G_6 and G_8 are almost exceptional with almost distinguished quotient H. The group H arises in Table 3 below as group 81.3, which is an image of groups 243.13, 243.17 and 243.18.

Remark 7.3. Hitherto, the almost exceptional groups mentioned in Remarks 5.5 and 7.2 above have all had minimal degree 18. There is one more almost exceptional group of order 243, namely the group

$$W \;=\; C_3 \times (C_3 \wr C_3)\,,$$

which is group 243.51 in Table 3, with minimal degree 12. That W is almost exceptional follows because W is the direct product of C_3, of degree 3, with a wreath product, of degree 9, which is well-known to be almost exceptional, and a special case of a large class of examples of almost exceptional groups, studied in [1], arising as sections of wreath products. The almost distinguished quotient of W is isomorphic to $C_3 \times \big((C_3 \times C_3) \rtimes C_3\big)$, which arises as group 81.12 in Table 3, highlighted in blue, where it occurs as an image of group 243.51.

8. Table 3: groups of order 243 and quotients of order 81

$\mathscr{A}$: G is abelian

$\mathscr{E}$: G is exceptional, i.e. $\exists\, N \lhd G$ such that $\mu(G) < \mu(G/N)$

$\mathscr{A}\mathscr{E}$: G is almost exceptional, i.e. $\exists$ nontrivial $N \lhd G$ such that $\mu(G) = \mu(G/N)$

Structural Description	ID	$\mu(G)$	Quotients of Order 81	$\mu(Q)$	$\mathscr{E}$	$\mathscr{A}\mathscr{E}$
C_{243}	243.1	243	$\mathscr{A}$	81	×	×
$(C_9 \times C_3) \rtimes_1 C_9$	243.2	27	$81.2 \cong C_9 \times C_9$ $81.3 \cong (C_3 \times C_3) \rtimes C_9$ $81.4 \cong C_9 \rtimes C_9$	18 18 18	×	×
$(C_3 \times ((C_3 \times C_3) \rtimes C_3)) \rtimes_1 C_3$	243.3	18	$81.7 \cong (C_3 \times C_3 \times C_3) \rtimes_1 C_3$ $81.9 \cong (C_9 \times C_3) \rtimes_3 C_3$	9 **27**	✓	×
$(C_3 \times (C_9 \rtimes C_3)) \rtimes_1 C_3$	243.4	18	$81.7 \cong (C_3 \times C_3 \times C_3) \rtimes_1 C_3$ $81.8 \cong (C_9 \times C_3) \rtimes_2 C_3$	9 **27**	✓	×
$(C_3 \times C_3).((C_3 \times C_3) \rtimes_1 C_3) = (C_3 \times C_3 \times C_3).{}_3(C_3 \times C_3)$	243.5	36	$81.7 \cong (C_3 \times C_3 \times C_3) \rtimes_1 C_3$ $81.8 \cong (C_9 \times C_3) \rtimes_2 C_3$ $81.10 \cong C_3.{}_5((C_3 \times C_3) \rtimes C_3) = (C_3 \times C_3).{}_4(C_3 \times C_3)$	9 27 27	×	×
$((C_3 \times C_3) \rtimes C_9) \rtimes_3 C_3$	243.6	36	$81.7 \cong (C_3 \times C_3 \times C_3) \rtimes_1 C_3$ $81.8 \cong (C_9 \times C_3) \rtimes_2 C_3$ $81.9 \cong (C_9 \times C_3) \rtimes_3 C_3$	9 27 27	×	×
$(C_3 \times C_3).{}_{28}((C_3 \times C_3) \rtimes C_3) = (C_3 \times C_3 \times C_3).{}_5(C_3 \times C_3)$	243.7	18	$81.7 \cong (C_3 \times C_3 \times C_3) \rtimes_1 C_3$ $81.10 \cong C_3.{}_5((C_3 \times C_3) \rtimes C_3) = (C_3 \times C_3).{}_4(C_3 \times C_3)$	9 **27**	✓	×
$(C_3 \times C_3).{}_{29}((C_3 \times C_3) \rtimes C_3) = (C_3 \times C_3 \times C_3).{}_6(C_3 \times C_3)$	243.8	54	$81.8 \cong (C_9 \times C_3) \rtimes_2 C_3$ $81.9 \cong (C_9 \times C_3) \rtimes_3 C_3$ $81.10 \cong C_3.{}_5((C_3 \times C_3) \rtimes C_3) = (C_3 \times C_3).{}_4(C_3 \times C_3)$	27 27 27	×	×

Continued on next page

Table 3 – *continued from previous page*

Structural Description	ID	$\mu(G)$	Quotients of Order 81	$\mu(Q)$	$\mathscr{E}$	$\mathscr{A}\mathscr{E}$
$(C_3 \times C_3)._{30}((C_3 \times C_3) \rtimes C_3) = (C_3 \times C_3 \times C_3)._{7}(C_3 \times C_3)$	243.9	54	$81.8 \cong (C_9 \times C_3) \rtimes_2 C_3$	27	×	×
$C_{27} \times C_9$	243.10	36	$\mathscr{A}$	30 18	×	×
$C_{27} \rtimes_2 C_9$	243.11	36	$81.2 \cong C_9 \times C_9$ $81.6 \cong C_{27} \rtimes C_3$	18 27	×	×
$(C_{27} \times C_3) \rtimes_1 C_3$	243.12	36	$81.3 \cong (C_3 \times C_3) \rtimes C_9$ $81.5 \cong C_{27} \times C_3$ $81.6 \cong C_{27} \rtimes C_3$	18 30 27	×	×
$(C_3 \times C_3 \times C_3) \rtimes_1 C_9$	243.13	18	$81.3 \cong (C_3 \times C_3) \rtimes C_9$ $81.7 \cong (C_3 \times C_3 \times C_3) \rtimes_1 C_3$	18 9	×	✓
$(C_9 \times C_3) \rtimes_2 C_9$	243.14	36	$81.3 \cong (C_3 \times C_3) \rtimes C_9$ $81.8 \cong (C_9 \times C_3) \rtimes_2 C_3$	18 27	×	×
$(C_9 \times C_3) \rtimes_3 C_9$	243.15	36	$81.3 \cong (C_3 \times C_3) \rtimes C_9$ $81.9 \cong (C_9 \times C_3) \rtimes_3 C_3$ $81.10 \cong C_3._5((C_3 \times C_3) \rtimes C_3) = (C_3 \times C_3)._4(C_3 \times C_3)$	18 27 27	×	×
$(C_{27} \rtimes C_3) \rtimes_1 C_3$	243.16	27	$81.3 \cong (C_3 \times C_3) \rtimes C_9$	18	×	×
$((C_3 \times C_3) \rtimes C_3) \rtimes C_9$	243.17	18	$81.3 \cong (C_3 \times C_3) \rtimes C_9$ $81.7 \cong (C_3 \times C_3 \times C_3) \rtimes_1 C_3$ $81.8 \cong (C_9 \times C_3) \rtimes_2 C_3$ $81.9 \cong (C_9 \times C_3) \rtimes_3 C_3$	18 9 **27** **27**	✓	✓
$(C_9 \rtimes C_3) \rtimes C_9$	243.18	18	$81.3 \cong (C_3 \times C_3) \rtimes C_9$ $81.7 \cong (C_3 \times C_3 \times C_3) \rtimes_1 C_3$ $81.8 \cong (C_9 \times C_3) \rtimes_2 C_3$ $81.10 \cong C_3._5((C_3 \times C_3) \rtimes C_3) = (C_3 \times C_3)._4(C_3 \times C_3)$	18 9 **27** **27**	✓	✓
$(C_{27} \rtimes C_3) \rtimes_2 C_3$	243.19	81	$81.3 \cong (C_3 \times C_3) \rtimes C_9$	18	×	×
$(C_{27} \rtimes C_3) \rtimes_3 C_3$	243.20	81	$81.3 \cong (C_3 \times C_3) \rtimes C_9$	18	×	×
$C_9 \rtimes C_{27}$	243.21	36	$81.4 \cong C_9 \rtimes C_9$ $81.5 \cong C_{27} \times C_3$ $81.6 \cong C_{27} \rtimes C_3$	18 30 27	×	×
$C_{27} \rtimes C_9$	243.22	27	$81.4 \cong C_9 \rtimes C_9$	18	×	×
$C_{81} \times C_3$	243.23	84	$\mathscr{A}$	30 81	×	×

Continued on next page

Table 3 – *continued from previous page*

Structural Description	ID	$\mu(G)$	Quotients of Order 81	$\mu(Q)$	$\mathscr{E}$	$\mathscr{AE}$
$C_{81} \rtimes C_3$	243.24	81	$81.5 \cong C_{27} \times C_3$	30	×	×
$(C_9 \times C_9) \rtimes_1 C_3$	243.25	27	$81.9 \cong (C_9 \times C_3) \rtimes_3 C_3$	27	×	✓
$(C_9 \times C_9) \rtimes_2 C_3$	243.26	27	$81.9 \cong (C_9 \times C_3) \rtimes_3 C_3$	27	×	✓
$(C_9 \times C_9).{}_2C_3$	243.27	27	$81.9 \cong (C_9 \times C_3) \rtimes_3 C_3$	27	×	✓
$(C_9 \rtimes C_9) \rtimes_1 C_3$	243.28	27	$81.9 \cong (C_9 \times C_3) \rtimes_3 C_3$	27	×	✓
$(C_9 \times C_3).{}_3(C_3 \times C_3)$	243.29	27	$81.9 \cong (C_9 \times C_3) \rtimes_3 C_3$	27	×	✓
$(C_9 \rtimes C_9) \rtimes_2 C_3$	243.30	27	$81.9 \cong (C_9 \times C_3) \rtimes_3 C_3$	27	×	✓
$C_9 \times C_9 \times C_3$	243.31	21	$\mathscr{A}$	15 18	×	×
$C_3 \times ((C_3 \times C_3) \rtimes C_9)$	243.32	21	$81.3 \cong (C_3 \times C_3) \rtimes C_9$ $81.11 \cong C_9 \times C_3 \times C_3$ $81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$ $81.13 \cong C_3 \times (C_9 \rtimes C_3)$	18 15 12 12	×	×
$C_3 \times (C_9 \rtimes C_9)$	243.33	21	$81.4 \cong C_9 \rtimes C_9$ $81.11 \cong C_9 \times C_3 \times C_3$ $81.13 \cong C_3 \times (C_9 \rtimes C_3)$	18 15 12	×	×
$(C_9 \times C_9) \rtimes_3 C_3$	243.34	36	$81.11 \cong C_9 \times C_3 \times C_3$ $81.14 \cong (C_9 \times C_3) \rtimes_5 C_3$	15 27	×	×
$C_9 \times ((C_3 \times C_3) \rtimes C_3)$	243.35	18	$81.11 \cong C_9 \times C_3 \times C_3$ $81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$ $81.14 \cong (C_9 \times C_3) \rtimes_5 C_3$	15 12 **27**	✓	×
$C_9 \times (C_9 \rtimes C_3)$	243.36	18	$81.11 \cong C_9 \times C_3 \times C_3$ $81.13 \cong C_3 \times (C_9 \rtimes C_3)$ $81.14 \cong (C_9 \times C_3) \rtimes_5 C_3$	15 12 **27**	✓	×
$(C_3 \times ((C_3 \times C_3) \rtimes C_3)) \rtimes_2 C_3$	243.37	18	$81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$	12	×	×
$(C_9 \times C_3) \rtimes_1 (C_3 \times C_3)$	243.38	18	$81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$ $81.13 \cong C_3 \times (C_9 \rtimes C_3)$	12 12	×	×
$C_9 \rtimes ((C_3 \times C_3) \rtimes C_3)$	243.39	18	$81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$ $81.13 \cong C_3 \times (C_9 \rtimes C_3)$ $81.14 \cong (C_9 \times C_3) \rtimes_5 C_3$	12 12 **27**	✓	×
$(C_3 \times (C_9 \rtimes C_3)) \rtimes_4 C_3$	243.40	36	$81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$ $81.14 \cong (C_9 \times C_3) \rtimes_5 C_3$	12 27	×	×
$(C_9 \rtimes C_9) \rtimes_3 C_3$	243.41	18	$81.13 \cong C_3 \times (C_9 \rtimes C_3)$ $81.14 \cong (C_9 \times C_3) \rtimes_5 C_3$	12 **27**	✓	×

Continued on next page

Table 3 – *continued from previous page*

Structural Description	ID	$\mu(G)$	Quotients of Order 81	$\mu(Q)$	$\mathscr{E}$	$\mathscr{A}\mathscr{E}$
$(C_9 \rtimes C_9) \rtimes_4 C_3$	243.42	36	$81.14 \cong (C_9 \times C_3) \rtimes_5 C_3$	27	×	×
$(C_9 \rtimes C_9) \rtimes_5 C_3$	243.43	18	$81.13 \cong C_3 \times (C_9 \rtimes C_3)$ $81.14 \cong (C_9 \times C_3) \rtimes_5 C_3$	12 **27**	✓	×
$(C_9 \rtimes C_9) \rtimes_6 C_3$	243.44	54	$81.14 \cong (C_9 \times C_3) \rtimes_5 C_3$	27	×	×
$(C_9 \rtimes C_9) \rtimes_7 C_3$	243.45	54	$81.14 \cong (C_9 \times C_3) \rtimes_5 C_3$	27	×	×
$(C_9 \times C_9) \rtimes_8 C_3$	243.46	36	$81.13 \cong C_3 \times (C_9 \rtimes C_3)$ $81.14 \cong (C_9 \times C_3) \rtimes_5 C_3$	12 27	×	×
$(C_9 \times C_9) \rtimes_9 C_3$	243.47	18	$81.13 \cong C_3 \times (C_9 \rtimes C_3)$	12	×	×
$C_{27} \times C_3 \times C_3$	243.48	33	$\mathscr{A}$	15 30	×	×
$C_3 \times (C_{27} \rtimes C_3)$	243.49	30	$81.6 \cong C_{27} \rtimes C_3$ $81.11 \cong C_9 \times C_3 \times C_3$	27 15	×	×
$(C_{27} \times C_3) \rtimes_5 C_3$	243.50	81	$81.11 \cong C_9 \times C_3 \times C_3$	15	×	×
$C_3 \times ((C_3 \times C_3 \times C_3) \rtimes C_3)$	243.51	12	$81.7 \cong (C_3 \times C_3 \times C_3) \rtimes_1 C_3$ $81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$	9 12	×	✓
$C_3 \times ((C_9 \times C_3) \rtimes_2 C_3)$	243.52	30	$81.8 \cong (C_9 \times C_3) \rtimes_2 C_3$ $81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$	27 12	×	×
$C_3 \times ((C_9 \times C_3) \rtimes_3 C_3)$	243.53	30	$81.9 \cong (C_9 \times C_3) \rtimes_3 C_3$ $81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$	27 12	×	×
$C_3 \times (C_3 \cdot_5 (C_3 \times C_3) \rtimes C_3))$	243.54	30	$81.10 \cong C_3 \cdot_5 ((C_3 \times C_3) \rtimes C_3)$ $81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$	27 12	×	×
$((C_9 \times C_3) \rtimes_2 C_3) \rtimes_3 C_3$	243.55	27	$81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$	12	×	×
$((C_3 \times C_3 \times C_3) \rtimes_1 C_3)) \rtimes_1 C_3$	243.56	27	$81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$	12	×	×
$((C_9 \times C_3) \rtimes_2 C_3) \rtimes_1 C_3$	243.57	27	$81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$	12	×	×
$(C_3 \times ((C_3 \times C_3) \rtimes C_3)) \rtimes_6 C_3$	243.58	27	$81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$	12	×	×
$(C_3.((C_3 \times C_3) \rtimes C_3))) \rtimes_4 C_3$	243.59	27	$81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$	12	×	×
$((C_9 \times C_3) \rtimes_2 C_3) \rtimes_2 C_3$	243.60	27	$81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$	12	×	×
$C_9 \times C_3 \times C_3 \times C_3$	243.61	18	$\mathscr{A}$	12 15	×	×
$C_3 \times C_3 \times ((C_3 \times C_3) \rtimes C_3)$	243.62	15	$81.12 \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$ $81.15 \cong C_3 \times C_3 \times C_3 \times C_3$	12 12	×	×

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Table 3 – *continued from previous page*

Structural Description	ID	$\mu(G)$	Quotients of Order 81	$\mu(Q)$	$\mathscr{E}$	$\mathscr{A}\mathscr{E}$
$C_3 \times C_3 \times (C_9 \rtimes C_3)$	243.63	15	$81.13 \cong C_3 \times (C_9 \rtimes C_3)$ $81.15 \cong C_3 \times C_3 \times C_3 \times C_3$	12 12	$\times$	$\times$
$C_3 \times ((C_9 \times C_3) \rtimes_5 C_3)$	243.64	30	$81.14 \cong (C_9 \times C_3) \rtimes_5 C_3$ $81.15 \cong C_3 \times C_3 \times C_3 \times C_3$	27 12	$\times$	$\times$
$(C_3 \times ((C_3 \times C_3) \rtimes C_3)) \rtimes_7 C_3$	243.65	27	$81.15 \cong C_3 \times C_3 \times C_3 \times C_3$	12	$\times$	$\times$
$(C_3 \times (C_9 \rtimes C_3)) \rtimes_{11} C_3$	243.66	27	$81.15 \cong C_3 \times C_3 \times C_3 \times C_3$	12	$\times$	$\times$
$C_3 \times C_3 \times C_3 \times C_3 \times C_3$	243.67	15	$\mathscr{A}$	12	$\times$	$\times$

REFERENCES

[1] I. Alotaibi and D. Easdown. Minimal degrees of groups associated with some wreath products. Preprint, School of Mathematics and Statistics, University of Sydney, August 2023.

[2] J.R. Britnell, N. Saunders and T. Skyner. On exceptional groups of order p^5. *J. Pure App. Algebra*, 221 (2017), 2647–2665.

[3] W. Burnside. *Theory of groups of finite order.* Cambridge University Press, 1911.

[4] R. Chamberlain. Minimal exceptional p-groups. *Bull. Aust. Math. Soc.* 98 (2018), 434–438.

[5] D. Easdown and C.E.. Praeger. On minimal permutation representations of finite groups. *Bull. Aust. Math. Soc.*, 38 (1988), 207–220.

[6] D. Johnson. Minimal permutation representations of finite groups. *Am. J. Math.*, 93 (1971), 857–866.

[7] G.I. Karpilovsky. The least degree of a faithful representation of abelian groups. *Vestni Khar'kov Gos. Univ.*, 53 (1970), 107-115.

[8] S. Lemieux. Finite exceptional p-groups of small order. *Comm. Algebra*, 35 (2007), 1890–1894.

[9] P. Neumann. Some algorithms for computing with finite permutation groups. In E. Robertson and C. Campbell (Eds.), Groups (St Andrews 1985), London Math. Soc. Lecture Note Ser. 121. Cambridge University Press, Cambridge (1987), 59–92.

[10] D. Wright. Degrees of minimal embeddings for some direct products. *Am. J. Math.*, 97 (1976), 897–903.

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