Representations of the super Yangians of types A and C

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Abstract

We classify the finite-dimensional irreducible representations of the super Yangian associated with the orthosymplectic Lie superalgebra $\mathfrak{osp}_{2|2n}$. The classification is given in terms of the highest weights and Drinfeld polynomials. We also include an R-matrix construction of the polynomial evaluation modules over the Yangian associated with the Lie superalgebra $\mathfrak{gl}_{m|n}$, as an appendix. This is a super-version of the well-known construction for the $\mathfrak{gl}_n$ Yangian and it relies on the Schur–Sergeev duality.

1 Introduction

The *Yangian* $\mathrm{Y}(\mathfrak{osp}_{M|2n})$ associated with the orthosymplectic Lie superalgebra $\mathfrak{osp}_{M|2n}$ is a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{osp}_{M|2n}[u])$ in the class of Hopf algebras. The original definition is due to Arnaudon *et al.* [1], where some basic properties of the Yangian were also described. Its $\mathfrak{gl}_{m|n}$ counterpart was introduced earlier by Nazarov [14], and the finite-dimensional irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ were classified by Zhang [19] in a way similar to the Drinfeld Yangians [6]. Apart from the case of the Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ considered in [11], the general classification problem for the orthosymplectic Yangians remains open. We show in this paper that the representations of the Yangian $\mathrm{Y}(\mathfrak{osp}_{2|2n})$ are described in essentially the same way as those of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$, thus extending the similarity between the modules over the Lie superalgebras of types A and C as given by Kac [9].

In more detail, we will regard the Yangian $\mathrm{Y}(\mathfrak{osp}_{2|2n})$ as a quotient of the *extended Yangian* $\mathrm{X}(\mathfrak{osp}_{2|2n})$ by an ideal generated by central elements. The highest weight representation $L(\lambda(u))$ of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ is defined as the irreducible quotient of the Verma module $M(\lambda(u))$ associated with an $(n+2)$-tuple $\lambda(u) = (\lambda_1(u), \dots, \lambda_{n+2}(u))$ of formal series in u^{-1}. The tuple is called the *highest weight* of the representation.

A standard argument shows that every finite-dimensional irreducible representation of the extended Yangian is of the form $L(\lambda(u))$ for a certain highest weight $\lambda(u)$. The key step in the classification is to determine the necessary and sufficient conditions on the highest weight for the representation $L(\lambda(u))$ to be finite-dimensional. We will rely on reduction properties of the representations to establish necessary conditions, and their sufficiency will be verified by constructing a family of *fundamental representations* of the Yangian $\mathrm{X}(\mathfrak{osp}_{2|2n})$. This will prove the following theorem.

Main Theorem. *Every finite-dimensional irreducible representation of the algebra* $\mathrm{X}(\mathfrak{osp}_{2|2n})$ *is isomorphic to* $L(\lambda(u))$ *for a certain highest weight* $\lambda(u)$*. The representation* $L(\lambda(u))$ *is finite-dimensional if and only if there exist monic polynomials* $\overline{Q}(u), Q(u), P_2(u), \dots, P_{n+1}(u)$ *in* u *such that*

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{\overline{Q}(u)}{Q(u)}, \tag{1.1}$$

$$\frac{\lambda_{i+1}(u)}{\lambda_i(u)} = \frac{P_i(u+1)}{P_i(u)} \qquad \text{for} \quad i = 2, \dots, n, \tag{1.2}$$

and

$$\frac{\lambda_{n+2}(u)}{\lambda_{n+1}(u)} = \frac{P_{n+1}(u+2)}{P_{n+1}(u)}. \tag{1.3}$$

The finite-dimensional irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{osp}_{2|2n})$ *are in a one-to-one correspondence with the tuples* $\big(\overline{Q}(u), Q(u), P_2(u), \dots, P_{n+1}(u)\big)$ *of monic polynomials in* u*, where the polynomials* $\overline{Q}(u)$ *and* $Q(u)$ *are of the same degree and have no common roots.*

We will refer to the monic polynomials occurring in these conditions as the *Drinfeld polynomials* of the representation; cf. [6].

The Appendix is devoted to the polynomial evaluation modules over the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$. Such modules over the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ proved to be useful in the analysis of the representations of the orthosymplectic Yangians; see Proposition 2.9 and also [2], [11]. Many applications of the evaluation homomorphism rely on its R-matrix interpretation going back to Cherednik [4]. In this interpretation, the polynomial evaluation modules arise from the Schur–Weyl duality and are constructed as submodules of the tensor product of the vector representation of the general linear Lie algebra. As another ingredient, the construction uses some key steps of the *fusion procedure* for the symmetric group originated in the work of Jucys [7]. It was rediscovered in [4], with detailed arguments given in [15]; see also [10, Sec. 6.4].

A super-version of the Schur–Weyl duality involving the general linear Lie superalgebra and the symmetric group is due to Sergeev [16] and Berele and Regev [3]. It is called the *Schur–Sergeev duality* in [5, Sec. 3.2]. We will use this duality to give an R-matrix construction of the polynomial evaluation modules over the Yangian associated with the Lie superalgebra $\mathfrak{gl}_{m|n}$. More precisely, for any Young diagram λ contained in the (m,n)-hook we consider a standard λ-tableau $\mathcal{U}$ and the associated primitive idempotent $e_{\mathcal{U}} \in \mathbb{C}\mathfrak{S}_d$, where d is the number of boxes in λ. By using the action of the symmetric group $\mathfrak{S}_d$ on the tensor product of the $\mathbb{Z}_2$-graded vector spaces $\mathbb{C}^{m|n}$, we show that the subspace $e_{\mathcal{U}}(\mathbb{C}^{m|n})^{\otimes d}$ is a representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$. We derive that this representation and its twisted version are isomorphic to evaluation modules over $\mathfrak{gl}_{m|n}$ and identify them by calculating the highest weights.

2 Representations of the orthosymplectic Yangian

2.1 Definition and basic properties of the Yangian

Fix an integer $n \geqslant 1$ and introduce the involution $i \mapsto i' = 2n-i+3$ on the set $\{1, 2, \dots, 2n+2\}$. Consider the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{2|2n}$ over $\mathbb{C}$ with the canonical basis $e_1, e_2, \dots, e_{2n+2}$, where the vector e_i has the parity $\bar{\imath} \mod 2$ and

$$\bar{\imath} = \begin{cases} 0 & \text{for} \quad i = 1, 1', \\ 1 & \text{for} \quad i = 2, 3, \dots, 2'. \end{cases}$$

The endomorphism algebra $\operatorname{End} \mathbb{C}^{2|2n}$ gets a $\mathbb{Z}_2$-gradation with the parity of the matrix unit e_{ij} found by $\bar{\imath} + \bar{\jmath} \mod 2$. We will identify the algebra of even matrices over a superalgebra $\mathcal{A}$ with the tensor product algebra $\operatorname{End} \mathbb{C}^{2|2n} \otimes \mathcal{A}$, so that a matrix $A = [a_{ij}]$ is regarded as the element

$$A = \sum_{i,j=1}^{2n+2} e_{ij} \otimes a_{ij} (-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}} \in \operatorname{End} \mathbb{C}^{2|2n} \otimes \mathcal{A}.$$

The involutive matrix *super-transposition* t is defined by $(A^t)_{ij} = A_{j'i'}(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}} \theta_i \theta_j$, where we set

$$\theta_i = \begin{cases} 1 & \text{for} \quad i = 1, \dots, n+1, 1', \\ -1 & \text{for} \quad i = n+2, \dots, 2'. \end{cases}$$

This super-transposition is associated with the bilinear form on the space $\mathbb{C}^{2|2n}$ defined by the anti-diagonal matrix $G = [\delta_{ij'} \theta_i]$. We will also regard t as the linear map

$$t : \operatorname{End} \mathbb{C}^{2|2n} \to \operatorname{End} \mathbb{C}^{2|2n}, \qquad e_{ij} \mapsto e_{j'i'} (-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}} \theta_i \theta_j. \tag{2.1}$$

A standard basis of the general linear Lie superalgebra $\mathfrak{gl}_{2|2n}$ is formed by elements E_{ij} of the parity $\bar{\imath} + \bar{\jmath} \mod 2$ for $1 \leqslant i, j \leqslant 2n+2$ with the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj} (-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})}.$$

We will regard the orthosymplectic Lie superalgebra $\mathfrak{osp}_{2|2n}$ associated with the bilinear from defined by G as the subalgebra of $\mathfrak{gl}_{2|2n}$ spanned by the elements

$$F_{ij} = E_{ij} - E_{j'i'} (-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}} \theta_i \theta_j.$$

The symplectic Lie algebra $\mathfrak{osp}_{0|2n} \cong \mathfrak{sp}_{2n}$ will be considered as the subalgebra of $\mathfrak{osp}_{2|2n}$ spanned by the elements F_{ij} with $2 \leqslant i, j \leqslant 2'$.

Introduce the permutation operator P by

$$P = \sum_{i,j=1}^{2n+2} e_{ij} \otimes e_{ji} (-1)^{\bar{\jmath}} \in \operatorname{End} \mathbb{C}^{2|2n} \otimes \operatorname{End} \mathbb{C}^{2|2n}$$

and set

$$Q = \sum_{i,j=1}^{2n+2} e_{ij} \otimes e_{i'j'} (-1)^{\bar\imath\bar\jmath} \theta_i \theta_j \in \operatorname{End} \mathbb{C}^{2|2n} \otimes \operatorname{End} \mathbb{C}^{2|2n}.$$

The R*-matrix* associated with $\mathfrak{osp}_{2|2n}$ is the rational function in u given by

$$R(u) = 1 - \frac{P}{u} + \frac{Q}{u-\kappa}, \qquad \kappa = -n. \tag{2.2}$$

This is a super-version of the R-matrix originally found in [17]. Following [1], we define the *extended Yangian* $\mathrm{X}(\mathfrak{osp}_{2|2n})$ as a $\mathbb{Z}_2$-graded algebra with generators $t_{ij}^{(r)}$ of parity $\bar\imath + \bar\jmath \mod 2$, where $1 \leqslant i, j \leqslant 2n+2$ and $r = 1, 2, \dots$, satisfying certain quadratic relations. To write them down, introduce the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)} u^{-r} \in \mathrm{X}(\mathfrak{osp}_{2|2n})[[u^{-1}]] \tag{2.3}$$

and combine them into the matrix $T(u) = [t_{ij}(u)]$. Consider the elements of the tensor product algebra $\operatorname{End} \mathbb{C}^{2|2n} \otimes \operatorname{End} \mathbb{C}^{2|2n} \otimes \mathrm{X}(\mathfrak{osp}_{2|2n})[[u^{-1}]]$ given by

$$T_1(u) = \sum_{i,j=1}^{2n+2} e_{ij} \otimes 1 \otimes t_{ij}(u)(-1)^{\bar\imath\bar\jmath+\bar\jmath} \qquad \text{and} \qquad T_2(u) = \sum_{i,j=1}^{2n+2} 1 \otimes e_{ij} \otimes t_{ij}(u)(-1)^{\bar\imath\bar\jmath+\bar\jmath}.$$

The defining relations for the algebra $\mathrm{X}(\mathfrak{osp}_{2|2n})$ take the form of the RTT*-relation*

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v). \tag{2.4}$$

As shown in [1], the product $T(u-\kappa)\, T^t(u)$ is a scalar matrix with

$$T(u-\kappa)\, T^t(u) = c(u) 1, \tag{2.5}$$

where $c(u)$ is a series in u^{-1}. As with the Lie algebra case considered in [2], all its coefficients belong to the center $\mathrm{ZX}(\mathfrak{osp}_{2|2n})$ of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ and generate the center.

We will also use the *extended Yangian* $\mathrm{X}(\mathfrak{osp}_{0|2n})$ which is defined in the same way as $\mathrm{X}(\mathfrak{osp}_{2|2n})$, by using the subspace $\mathbb{C}^{0|2n} \subset \mathbb{C}^{2|2n}$ with odd basis vectors $e_2, e_3, \dots, e_{2'}$, instead of $\mathbb{C}^{2|2n}$ and the corresponding RTT-relation (2.4). The R-matrix is defined by (2.2) with the modified expressions

$$P = -\sum_{i,j=2}^{2n+1} e_{ij} \otimes e_{ji} \qquad \text{and} \qquad Q = -\sum_{i,j=2}^{2n+1} e_{ij} \otimes e_{i'j'}\, \theta_i \theta_j$$

and with the value $\kappa = -n-1$. It will be convenient to use the index set $\{2, 3, \dots, 2'\}$ to label the corresponding generating series which we will denote by $\bar t_{ij}(u)$. We have an isomorphism $\mathrm{X}(\mathfrak{osp}_{0|2n}) \cong \mathrm{X}(\mathfrak{sp}_{2n})$ so that the series $\bar t_{ij}(-u)$ satisfy the defining relations for the extended Yangian $\mathrm{X}(\mathfrak{sp}_{2n})$; see [1], [2].

The *Yangian* $\mathrm{Y}(\mathfrak{osp}_{2|2n})$ is defined as the subalgebra of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ which consists of the elements stable under the automorphisms

$$t_{ij}(u) \mapsto f(u)\, t_{ij}(u) \tag{2.6}$$

for all series $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. We have the tensor product decomposition

$$\mathrm{X}(\mathfrak{osp}_{2|2n}) = \mathrm{ZX}(\mathfrak{osp}_{2|2n}) \otimes \mathrm{Y}(\mathfrak{osp}_{2|2n}). \tag{2.7}$$

The Yangian $\mathrm{Y}(\mathfrak{osp}_{2|2n})$ is isomorphic to the quotient of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ by the relation $c(u) = 1$.

An explicit form of the defining relations (2.4) can be written with the use of super-commutator in terms of the series (2.3) as follows:

$$\begin{aligned} \big[t_{ij}(u), t_{kl}(v)\big] &= \frac{1}{u-v}\Big(t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u)\Big)(-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}} \\ &- \frac{1}{u-v-\kappa}\Big(\delta_{ki'} \sum_{p=1}^{2n+2} t_{pj}(u)\, t_{p'l}(v)(-1)^{\bar{\imath}+\bar{\imath}\bar{\jmath}+\bar{\jmath}\bar{p}}\, \theta_i\theta_p \\ &\qquad - \delta_{lj'} \sum_{p=1}^{2n+2} t_{kp'}(v)\, t_{ip}(u)(-1)^{\bar{\jmath}+\bar{p}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}+\bar{\imath}\bar{p}}\, \theta_j\theta_p\Big). \end{aligned} \tag{2.8}$$

Note that the mapping

$$t_{ij}(u) \mapsto t_{ij}(u+a), \quad a \in \mathbb{C}, \tag{2.9}$$

defines an automorphism of $\mathrm{X}(\mathfrak{osp}_{2|2n})$.

The universal enveloping algebra $\mathrm{U}(\mathfrak{osp}_{2|2n})$ can be regarded as a subalgebra of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ via the embedding

$$F_{ij} \mapsto \frac{1}{2}\Big(t_{ij}^{(1)} - t_{j'i'}^{(1)}(-1)^{\bar{\jmath}+\bar{\imath}\bar{\jmath}}\, \theta_i\theta_j\Big)(-1)^{\bar{\imath}}. \tag{2.10}$$

This fact relies on the Poincaré–Birkhoff–Witt theorem for the orthosymplectic Yangian which was pointed out in [1]. It states that the associated graded algebra for $\mathrm{Y}(\mathfrak{osp}_{2|2n})$ is isomorphic to $\mathrm{U}(\mathfrak{osp}_{2|2n}[u])$. This implies that the algebra $\mathrm{X}(\mathfrak{osp}_{2|2n})$ is generated by the coefficients of the series $c(u)$ and $t_{ij}(u)$ with $i+j \leqslant 2n+3$, excluding $t_{11'}(u)$ and $t_{1'1}(u)$. Moreover, given any total ordering on the set of the generators, the ordered monomials with the powers of odd generators not exceeding 1, form a basis of the algebra. The theorem follows by using essentially the same arguments as in the non-super case; see [2, Sec. 3].

The extended Yangian $\mathrm{X}(\mathfrak{osp}_{2|2n})$ is a Hopf algebra with the coproduct defined by

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{2n+2} t_{ik}(u) \otimes t_{kj}(u). \tag{2.11}$$

The coproduct on the algebra $\mathrm{X}(\mathfrak{osp}_{0|2n})$ is defined by the same formula for the series $\bar{t}_{ij}(u)$ instead of $t_{ij}(u)$, with the sum taken over $k = 2, 3, \dots, 2n+1$.

2.2 Highest weight representations

The proof of the necessity of the conditions of the Main Theorem will rely on the following reduction property for representations of the extended Yangians $\mathrm{X}(\mathfrak{osp}_{2|2n})$ which is verified in the same way as for the case of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ in [11, Prop. 4.1]; cf. [2, Lemma 5.13]. For an arbitrary $\mathrm{X}(\mathfrak{osp}_{2|2n})$-module V set

$$V^+ = \{\eta \in V \mid t_{1j}(u)\,\eta = 0 \quad \text{for} \quad j > 1 \quad \text{and} \quad t_{i\,1'}(u)\,\eta = 0 \quad \text{for} \quad i < 1'\}. \tag{2.12}$$

Proposition 2.1. *The subspace V^+ is stable under the action of the operators $t_{ij}(u)$ subject to $2 \leqslant i, j \leqslant 2'$. Moreover, the assignment $\bar{t}_{ij}(u) \mapsto t_{ij}(u)$ defines a representation of the extended Yangian $\mathrm{X}(\mathfrak{osp}_{0|2n})$ on V^+.* □

A representation V of the algebra $\mathrm{X}(\mathfrak{osp}_{2|2n})$ is called a *highest weight representation* if there exists a nonzero vector $\xi \in V$ such that V is generated by ξ,

$$\begin{aligned} t_{ij}(u)\,\xi &= 0 \qquad && \text{for} \quad 1 \leqslant i < j \leqslant 2n+2, \qquad \text{and} \\ t_{ii}(u)\,\xi &= \lambda_i(u)\,\xi \qquad && \text{for} \quad i = 1, \dots, 2n+2, \end{aligned} \tag{2.13}$$

for some formal series

$$\lambda_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]. \tag{2.14}$$

The vector ξ is called the *highest vector* of V.

Proposition 2.2. *The series $\lambda_i(u)$ associated with a highest weight representation V satisfy the consistency conditions*

$$\lambda_i(u)\,\lambda_{i'}(u+n-i+2) = \lambda_{i+1}(u)\,\lambda_{(i+1)'}(u+n-i+2) \tag{2.15}$$

for $i = 1, \dots, n$. Moreover, the coefficients of the series $c(u)$ act in the representation V as the multiplications by scalars determined by $c(u) \mapsto \lambda_1(u)\,\lambda_{1'}(u+n)$.

Proof. Introduce the subspace V^+ by (2.12) and note that the vector ξ belongs to V^+. By applying Proposition 2.1 we find that the cyclic span $\mathrm{X}(\mathfrak{osp}_{0|2n})\,\xi$ is a highest weight submodule with the highest weight $(\lambda_2(u), \dots, \lambda_{2'}(u))$. Due to the isomorphism $\mathrm{X}(\mathfrak{osp}_{0|2n}) \cong \mathrm{X}(\mathfrak{sp}_{2n})$, conditions (2.15) for $i = 2, \dots, n$ follow from the consistency conditions for the highest weight modules over $\mathrm{X}(\mathfrak{sp}_{2n})$; see [2, Prop. 5.14]. Furthermore, using the defining relations (2.8), we get

$$t_{12}(u)\,t_{1'2'}(v)\,\xi = -\frac{1}{u-v-\kappa}\Big(t_{12}(u)\,t_{1'2'}(v) + \lambda_1(u)\,\lambda_{1'}(v) - \lambda_2(u)\,\lambda_{2'}(v)\Big)\,\xi$$

and so

$$(u-v-\kappa+1)\,t_{12}(u)\,t_{1'2'}(v)\,\xi = \Big(-\lambda_1(u)\,\lambda_{1'}(v) + \lambda_2(u)\,\lambda_{2'}(v)\Big)\,\xi.$$

Setting $v = u - \kappa + 1 = u + n + 1$, we obtain (2.15) for $i = 1$. Finally, the last part of the proposition is obtained by using the expression for $c(u)$ implied by taking the $(1', 1')$ entry in the matrix relation (2.5). □

By Proposition 2.2, the series $\lambda_i(u)$ in (2.13) with $i > n+2$ are uniquely determined by the first $n+2$ series. The corresponding $(n+2)$-tuple $\lambda(u) = (\lambda_1(u), \dots, \lambda_{n+2}(u))$ will be called the *highest weight* of V.

For an arbitrary $(n+2)$-tuple $\lambda(u) = (\lambda_1(u), \dots, \lambda_{n+2}(u))$ of formal series of the form (2.14) define the *Verma module* $M(\lambda(u))$ as the quotient of the algebra $\mathrm{X}(\mathfrak{osp}_{2|2n})$ by the left ideal generated by all coefficients of the series $t_{ij}(u)$ with $1 \leqslant i < j \leqslant 2n+2$, and $t_{ii}(u) - \lambda_i(u)$ for $i = 1, \dots, 2n+2$, assuming that the series $\lambda_i(u)$ with $i = n+3, \dots, 2n+2$ are defined to satisfy the consistency conditions (2.15).

The Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{X}(\mathfrak{osp}_{2|2n})$ implies that the Verma module $M(\lambda(u))$ is nonzero, and we denote by $L(\lambda(u))$ its irreducible quotient. It is clear that the isomorphism class of $L(\lambda(u))$ is determined by $\lambda(u)$.

The first part of the Main Theorem is implied by the following proposition whose proof is the same as in the non-super case; see [2, Thm 5.1].

Proposition 2.3. *Every finite-dimensional irreducible representation of the algebra $\mathrm{X}(\mathfrak{osp}_{2|2n})$ is a highest weight representation. Moreover, it contains a unique, up to a constant factor, highest vector.* □

We will now suppose that the representation $L(\lambda(u))$ of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ is finite-dimensional and derive the conditions on the highest weight $\lambda(u)$ given in the Main Theorem. First consider the subalgebra Y_0 of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ generated by the coefficients of the series $t_{11}(u)$, $t_{12}(u)$, $t_{21}(u)$ and $t_{22}(u)$. This subalgebra is isomorphic to the Yangian $\mathrm{Y}(\mathfrak{gl}_{1|1})$, and the cyclic span $\mathrm{Y}_0\xi$ of the highest vector ξ is a finite-dimensional module with the highest weight $(\lambda_1(u), \lambda_2(u))$. This implies that condition (1.1) must hold by [18, Thm. 4]; see also [11, Prop. 5.1].

Furthermore, Proposition 2.1 implies that the subspace $L(\lambda(u))^+$ is a module over the extended Yangian $\mathrm{X}(\mathfrak{osp}_{0|2n}) \cong \mathrm{X}(\mathfrak{sp}_{2n})$. The vector ξ generates a highest weight $\mathrm{X}(\mathfrak{sp}_{2n})$-module with the highest weight $(\lambda_2(-u), \dots, \lambda_{n+2}(-u))$. Since this module is finite-dimensional, conditions (1.2) and (1.3) now follow by [2, Thm. 5.16].

2.3 Fundamental representations

Our next step in the proof of the Main Theorem is to show that the conditions (1.1), (1.2) and (1.3) are sufficient for the representation $L(\lambda(u))$ of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ to be finite-dimensional. The tuple

$$\big(\overline{Q}(u), Q(u), P_2(u), \dots, P_{n+1}(u)\big) \tag{2.16}$$

of Drinfeld polynomials determines the highest weight $\lambda(u)$ of the representation up to a simultaneous multiplication of all components $\lambda_i(u)$ by a formal series $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. This operation corresponds to twisting the action of the algebra $\mathrm{X}(\mathfrak{osp}_{2|2n})$ on $L(\lambda(u))$ by the automorphism (2.6). Hence, it suffices to prove that a particular module $L(\lambda(u))$ corresponding to a given tuple (2.16) is finite-dimensional.

Suppose that $L(\nu(u))$ and $L(\mu(u))$ are the irreducible highest weight modules with the highest weights

$$\nu(u) = \big(\nu_1(u), \dots, \nu_{n+2}(u)\big) \qquad \text{and} \qquad \mu(u) = \big(\mu_1(u), \dots, \mu_{n+2}(u)\big). \tag{2.17}$$

By the coproduct rule (2.11), the cyclic span $\mathrm{X}(\mathfrak{osp}_{2|2n})(\xi\otimes\xi')$ of the tensor product of the respective highest vectors $\xi\in L(\nu(u))$ and $\xi'\in L(\mu(u))$ is a highest weight module with the highest weight

$$\big(\nu_1(u)\,\mu_1(u),\dots,\nu_{n+2}(u)\,\mu_{n+2}(u)\big). \tag{2.18}$$

This observation implies the corresponding transition rule for the associated tuples of Drinfeld polynomials. Namely, if the highest weights in (2.17) are associated with the respective tuples

$$\big(\overline{Q}(u),Q(u),P_2(u),\dots,P_{n+1}(u)\big)\qquad\text{and}\qquad\big(\overline{Q}^{\circ}(u),Q^{\circ}(u),P^{\circ}_2(u),\dots,P^{\circ}_{n+1}(u)\big),$$

then the highest weight (2.18) is associated with the tuple

$$\Big(\frac{\overline{Q}(u)\,\overline{Q}^{\circ}(u)}{d(u)},\frac{Q(u)\,Q^{\circ}(u)}{d(u)},P_2(u)P^{\circ}_2(u),\dots,P_{n+1}(u)P^{\circ}_{n+1}(u)\Big), \tag{2.19}$$

where $d(u)$ is the monic polynomial in u defined as the greatest common divisor of the polynomials $\overline{Q}(u)\overline{Q}^{\circ}(u)$ and $Q(u)Q^{\circ}(u)$. Therefore, it will be enough to show that the *fundamental representations* of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ are finite-dimensional. They correspond to the tuples of Drinfeld polynomials of the form

$$\big(u+\alpha,u+\beta,1,\dots,1\big)\qquad\text{and}\qquad\big(1,\dots,1,u+\gamma,1,\dots,1\big), \tag{2.20}$$

for $\alpha\neq\beta$, where $u+\gamma$ represents the polynomial $P_d(u)$ and d runs over the values $2,\dots,n+1$. Note that by applying the shift automorphism (2.9), we can assume that $\beta=0$ in the first tuple and use a particular value of γ in the second tuple.

We will begin with the first tuple in (2.20). The required property is implied by the following.

Proposition 2.4. *The representation $L(\lambda(u))$ of the extended Yangian $\mathrm{X}(\mathfrak{osp}_{2|2n})$ with the components of the highest weight given by $\lambda_1(u)=1+\alpha\,u^{-1}$ and $\lambda_i(u)=1$ for $i=2,3,\dots,n+2$, is finite-dimensional.*

Proof. Let ξ denote the highest vector of $L(\lambda(u))$. We will show first that all vectors $t^{(r)}_{kl}\xi$ with $k,l\in\{2,3,\dots,2'\}$ and $r\geqslant 1$ are zero. This holds by definition for $k\leqslant l$, while the defining relations (2.8) imply that for $k>l$ we have the properties $t_{1j}(u)t^{(r)}_{kl}\xi=0$ for $j>1$ and $t_{i1'}(u)t^{(r)}_{kl}\xi=0$ for $i<1'$. This means that the vector $t^{(r)}_{kl}\xi$ belongs to the subspace $L(\lambda(u))^{+}$ defined in (2.12). By Proposition 2.1, the subspace $L(\lambda(u))^{+}$ is a representation of the algebra $\mathrm{X}(\mathfrak{osp}_{0|2n})$, and ξ generates a highest weight module over this algebra with the highest weight $(1,\dots,1)$. The irreducible quotient of this module is one-dimensional, which implies that the vector $t^{(r)}_{kl}\xi\in L(\lambda(u))$ is annihilated by all coefficients of the series $t_{ij}(u)$ with $1\leqslant i<j\leqslant 1'$ and therefore must be zero.

We now prove two lemmas providing the formulas for the action of some other elements of the extended Yangian on the highest vector ξ.

Lemma 2.5. *We have $t^{(r)}_{k1}\xi=0$ for all $k=2,3,\dots,2'$ and $r\geqslant 2$.*

Proof. As in the above argument, we will check that the vectors $t_{k1}^{(r)}\xi$ are annihilated by all coefficients of the series $t_{ij}(u)$ with $1 \leqslant i < j \leqslant 1'$. The Poincaré–Birkhoff–Witt theorem and the defining relations imply that it is sufficient to check this property for the coefficients of the series $t_{i,i+1}(u)$ with $i = 1, \dots, n+1$. Suppose first that $2 \leqslant k \leqslant n+1$. Then (2.8) gives

$$\Big[t_{k1}(v), t_{i,i+1}(u)\Big] = \frac{1}{v-u}\Big(t_{i1}(v)\, t_{k,i+1}(u) - t_{i1}(u)\, t_{k,i+1}(v)\Big)(-1)^{\bar{\imath}}.$$

Since $t_{k,i+1}(u)\xi = \delta_{k,i+1}\xi$, we derive

$$t_{i,i+1}(u)t_{k1}(v)\xi = \frac{1}{v-u}\,\delta_{k,i+1}\Big(t_{i1}(v) - t_{i1}(u)\Big)\xi. \tag{2.21}$$

Now proceed by induction on k. For $k = 2$, by (2.21) we have

$$t_{12}(u)t_{21}(v)\xi = \frac{1}{v-u}\Big(t_{11}(v) - t_{11}(u)\Big)\xi = -\alpha u^{-1}v^{-1}\xi$$

which implies that $t_{21}(v)\xi = v^{-1}t_{21}^{(1)}\xi$. Using (2.21) for the induction step, we can conclude that $t_{k1}(v)\xi = v^{-1}t_{k1}^{(1)}\xi$ for all $2 \leqslant k \leqslant n+1$.

Now let $(n+1)' \leqslant k \leqslant 2'$ and use induction on k starting from $k = (n+1)'$. Note that (2.21) holds for these values of k unless $i = k'$. By applying (2.8) to the super-commutator $\Big[t_{(n+1)',1}(v), t_{n+1,(n+1)'}(u)\Big]$ and using the assumptions $\lambda_i(u) = 1$ for $i = 2, 3, \dots, n+2$, we get

$$\begin{aligned} t_{n+1,(n+1)'}(u)\, t_{(n+1)',1}(v)\,\xi = {} & \frac{1}{v-u}\Big(t_{n+1,1}(v) - t_{n+1,1}(u)\Big)\xi \\ & + \frac{1}{v-u+n}\, t_{11}(v)\, t_{1',(n+1)'}(u)\xi + \frac{1}{v-u+n}\, t_{n+1,1}(v)\xi. \end{aligned}$$

By another application of (2.8) we derive

$$\frac{v-u+n+1}{v-u+n}\, t_{11}(v)\, t_{1',(n+1)'}(u)\xi = \frac{v+\alpha}{v}\, t_{1',(n+1)'}(u)\xi - \frac{1}{v-u+n}\, t_{n+1,1}(v)\xi.$$

Therefore, the previous expression can be written in the form

$$\begin{aligned} t_{n+1,(n+1)'}(u)\, t_{(n+1)',1}(v)\,\xi = {} & \frac{1}{v-u}\Big(t_{n+1,1}(v) - t_{n+1,1}(u)\Big)\xi \\ & + \frac{v+\alpha}{v(v-u+n+1)}\, t_{1',(n+1)'}(u)\xi + \frac{1}{v-u+n+1}\, t_{n+1,1}(v)\xi. \end{aligned} \tag{2.22}$$

On the other hand, by calculating the super-commutator $[t_{n+1,1}(v), t_{1'1'}(u)]$ we obtain

$$t_{1'1'}(u)\, t_{n+1,1}(v)\,\xi = \frac{v-u+n}{v-u+n+1}\, t_{n+1,1}(v)\, t_{1'1'}(u)\,\xi - \frac{1}{v-u+n+1}\, t_{1',(n+1)'}(u)\xi.$$

By the consistency conditions of Proposition 2.2, we find that

$$t_{1'1'}(u)\,\xi = \frac{u-n-1}{u+\alpha-n-1}\,\xi,$$

and so the residue at $v=u-n-1$ gives

$$t_{1',(n+1)'}(u)\,\xi=-\frac{u-n-1}{u+\alpha-n-1}\,t_{n+1,1}(u-n-1)\,\xi.$$

As we showed above, $t_{n+1,1}(v)\,\xi=v^{-1}t^{(1)}_{n+1,1}\,\xi$, so that (2.22) simplifies to

$$t_{n+1,(n+1)'}(u)\,t_{(n+1)',1}(v)\,\xi=-\Big(\frac{1}{uv}+\frac{1}{(u+\alpha-n-1)\,v}\Big)\,t^{(1)}_{n+1,1}\,\xi.$$

Hence we may conclude that the vectors $t^{(r)}_{(n+1)',1}\xi$ with $r\geqslant 2$ are annihilated by all coefficients of the series $t_{i,i+1}(u)$ with $i=1,\dots,n+1$ and so $t^{(r)}_{(n+1)',1}\xi=0$.

Now suppose that $k=l'$ with $2\leqslant l\leqslant n$. Calculating as above, we find that

$$t_{l,l+1}(u)t_{l',1}(v)\,\xi=\frac{1}{v-u+n}\,\Big(t_{11}(v)\,t_{1',l+1}(u)\,\xi-t_{(l+1)',1}(v)\,\xi\Big). \tag{2.23}$$

Furthermore, using the defining relations again, we derive the relation

$$t_{11}(v)\,t_{1',l+1}(u)\,\xi=\frac{1}{v-u+n+1}\,t_{(l+1)',1}(v)\,\xi+\frac{(v+\alpha)(v-u+n)}{v(v-u+n+1)}\,t_{1',l+1}(u)\,\xi,$$

which by taking the residue at $v=u-n-1$ also implies

$$t_{(l+1)',1}(u-n-1)\,\xi=\frac{u+\alpha-n-1}{u-n-1}\,t_{1',l+1}(u)\,\xi.$$

Since $t_{(l+1)',1}(v)\,\xi=v^{-1}\,t^{(1)}_{(l+1)',1}\,\xi$ by the induction hypothesis, relation (2.23) simplifies to

$$t_{l,l+1}(u)t_{l',1}(v)\,\xi=\frac{1}{(u+\alpha-n-1)\,v}\,t^{(1)}_{(l+1)',1}\,\xi,$$

thus proving that the vectors $t^{(r)}_{l',1}\xi$ with $r\geqslant 2$ are annihilated by the coefficients of the series $t_{l,l+1}(u)$. The proof is competed by taking into account (2.21) for the remaining values $i\neq l$. □

Lemma 2.6. *We have*

$$t_{1'1}(v)\,\xi=\frac{1}{v\,(v+\alpha-n-1)}\sum_{k=2}^{n+1}t^{(1)}_{k'1}t^{(1)}_{k1}\,\xi.$$

Proof. Calculate the commutator $[t_{11}(v),t_{1'1}(u)]$ by (2.8) and rearrange the terms to get

$$\frac{v-u+n+1}{v-u+n}\,t_{11}(v)\,t_{1'1}(u)=\frac{n}{(v-u)(v-u+n)}\,t_{1'1}(v)\,t_{11}(u)$$
$$+\frac{v-u-1}{v-u}\,t_{1'1}(u)\,t_{11}(v)-\frac{1}{v-u+n}\sum_{k=2}^{2'}t_{k1}(v)\,t_{k'1}(u)\,\theta_k.$$

By setting $u = v + n + 1$ we come to the relation

$$\frac{n}{n+1}\, t_{1'1}(v)\, t_{11}(v+n+1) + \frac{n+2}{n+1}\, t_{1'1}(v+n+1)\, t_{11}(v) = -\sum_{k=2}^{2'} t_{k1}(v)\, t_{k'1}(v+n+1)\, \theta_k.$$

Apply both sides of the relation to the highest vector ξ. By Lemma 2.4 we have

$$t_{k'1}(v+n+1)\, \xi = (v+n+1)^{-1}\, t_{k'1}^{(1)}\, \xi.$$

Hence, using the super-commutator $[t_{k1}(v), t_{k'1}^{(1)}] = -t_{1'1}(v)\, \theta_k$ we obtain

$$\frac{n(v+\alpha-n-1)}{(n+1)(v+n+1)}\, t_{1'1}(v)\, \xi + \frac{(n+2)(v+\alpha)}{(n+1)v}\, t_{1'1}(v+n+1)\, \xi \\ = \frac{1}{v\,(v+n+1)} \sum_{k=2}^{2'} \theta_k\, t_{k'1}^{(1)} t_{k1}^{(1)}\, \xi.$$

This relation uniquely determines the series $t_{1'1}(v)\, \xi$. It is easily seen to be satisfied by

$$t_{1'1}(v)\, \xi = \frac{1}{2v\,(v+\alpha-n-1)} \sum_{k=2}^{2'} \theta_k\, t_{k'1}^{(1)} t_{k1}^{(1)}\, \xi.$$

In particular, $t_{1'1}^{(1)}\, \xi = 0$ and so $[t_{k'1}^{(1)}, t_{k1}^{(1)}]\, \xi = 0$ yielding the required relation. □

Returning to the proof of the proposition, we can now derive that the representation $L(\lambda(u))$ is spanned by the vectors of the form

$$t_{k_1 1}^{(1)} \dots t_{k_s 1}^{(1)}\, \xi, \qquad 2 \leqslant k_1 < \dots < k_s \leqslant 2', \tag{2.24}$$

with $s \geqslant 0$. Indeed, by the first part of the proof, and the Poincaré–Birkhoff–Witt theorem applied to a suitable ordering of the generators, it is enough to verify that the span of these vectors is stable under the action of all generators of the form $t_{k1}^{(r)}$ with $k = 2, \dots, 2'$. Note that by taking the $(1', 1)$ entry in the matrix relation (2.5), we find that $t_{1'1}^{(1)} = 0$ in the algebra $\mathrm{X}(\mathfrak{osp}_{2|2n})$. This implies that the elements $t_{k1}^{(1)}$ with $k = 2, \dots, 2'$ pairwise anticommute. They also have the property $[t_{1'1}(v), t_{k1}^{(1)}] = 0$. Hence, applying the series $t_{k1}(v)$ to a basis vector (2.24), we get

$$t_{k1}(v)\, t_{k_1 1}^{(1)} \dots t_{k_s 1}^{(1)}\, \xi = \sum_{i=1}^{s} (-1)^{i-1}\, t_{k_1 1}^{(1)} \dots [t_{k1}(v), t_{k_i 1}^{(1)}] \dots t_{k_s 1}^{(1)}\, \xi + (-1)^s\, t_{k_1 1}^{(1)} \dots t_{k_s 1}^{(1)}\, t_{k1}(v)\, \xi.$$

Since $[t_{k1}(v), t_{k_i 1}^{(1)}] = -\delta_{k' k_i} t_{1'1}(v)\, \theta_k$, the sum on the right hand side equals

$$\theta_k \sum_{i=1}^{s} (-1)^i\, \delta_{k' k_i}\, t_{k_1 1}^{(1)} \dots \widehat{t}_{k_i 1}^{(1)} \dots t_{k_s 1}^{(1)}\, t_{1'1}(v)\, \xi.$$

By Lemma 2.5 we have $t_{k1}(v)\, \xi = v^{-1}\, t_{k1}^{(1)}\, \xi$, and together with the relation of Lemma 2.6 this proves that the span of the vectors (2.24) is stable under the action of all operators $t_{k1}^{(r)}$ with $k = 2, \dots, 2'$. Since the representation $L(\lambda(u))$ is spanned by these vectors, its dimension does not exceed 4^n. □

Let $V(\mu)$ denote the irreducible highest weight representation of the Lie superalgebra $\mathfrak{osp}_{2|2n}$ with the highest weight $\mu = (\mu_1, \dots, \mu_{n+1})$ with respect to the upper-triangular Borel subalgebra. This means that $F_{ii}\,\zeta = \mu_i\,\zeta$ for $i = 1, \dots, n+1$ and $F_{ij}\,\zeta = 0$ for $i < j$, where ζ is the highest vector of $V(\mu)$. The modules of the form $V(\alpha, 0, \dots, 0)$ are known to be *typical*, if and only if α does not belong to the set $\{0, 1, \dots, n-1\} \cup \{n+1, \dots, 2n\}$; see [9].

Corollary 2.7. *The typical representations $V(\alpha, 0, \dots, 0)$ of the Lie superalgebra $\mathfrak{osp}_{2|2n}$ extend to modules over the Yangian $\mathrm{X}(\mathfrak{osp}_{2|2n})$.*

Proof. Equip the representation $L(1 + \alpha\, u^{-1}, 1, \dots, 1)$ of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ with the action of $\mathfrak{osp}_{2|2n}$ via the embedding (2.10). By taking the (i, j) entry in the matrix relation (2.5) for $i \neq j$ we find

$$t^{(1)}_{ij} + t^{(1)}_{j'i'}(-1)^{\bar{j}+\bar{i}\bar{j}}\,\theta_i\theta_j = 0.$$

Therefore, under the embedding we have $F_{ij} \mapsto t^{(1)}_{ij}(-1)^{\bar{i}}$.

On the other hand, any typical representation $V(\alpha, 0, \dots, 0)$ is isomorphic to the corresponding Kac module and so is equipped with a basis of the form

$$F_{k_1 1} \dots F_{k_s 1}\,\zeta, \qquad 2 \leqslant k_1 < \dots < k_s \leqslant 2',$$

with $s \geqslant 0$. The proof of Proposition 2.4 shows that the mapping $\zeta \to \xi$ extends to an $\mathfrak{osp}_{2|2n}$-module isomorphism $V(\alpha, 0, \dots, 0) \to L(1 + \alpha\, u^{-1}, 1, \dots, 1)$. □

We will now turn to the second family of fundamental representations in (2.20). We will show that they can be constructed as subquotients of tensor products of the vector representations of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ and representations of the first family in (2.20).

The *vector representation* of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ on $\mathbb{C}^{2|2n}$ is defined by

$$t_{ij}(u) \mapsto \delta_{ij} + u^{-1} e_{ij}(-1)^{\bar{i}} - (u-n)^{-1} e_{j'i'}(-1)^{\bar{i}\bar{j}}\,\theta_i\theta_j. \tag{2.25}$$

The homomorphism property follows from the RTT-relation (2.4) and the Yang–Baxter equation satisfied by $R(u)$ as in (A.8); cf. [1]. The mapping $T(u) \mapsto R(u)$ defines an algebra homomorphism $\mathrm{X}(\mathfrak{osp}_{2|2n}) \to \mathrm{End}\,\mathbb{C}^{2|2n}$, and to get (2.25) we take its composition with the automorphism $t_{ij}(u) \mapsto t_{i'j'}(u+n)\,\theta_i\theta_j(-1)^{\bar{i}+\bar{j}}$.

Now use the coproduct (2.11) and the shift automorphism (2.9) to equip the tensor square $V = (\mathbb{C}^{2|2n})^{\otimes 2}$ with the action of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ by setting

$$t_{ij}(u) \mapsto \sum_{l=1}^{2n+2} t_{il}(u-1) \otimes t_{lj}(u), \tag{2.26}$$

where the generators act in the respective copies of the vector space $\mathbb{C}^{2|2n}$ via the rule (2.25). Introduce the vectors $w_k \in V$ by $w_k = e_1 \otimes e_k - e_k \otimes e_1$ for $k = 2, 3, \dots, 2'$ and denote by W the subspace of V spanned by these vectors.

The vector representation $\mathbb{C}^{0|2n}$ of the extended Yangian $\mathrm{X}(\mathfrak{osp}_{0|2n})$ is defined by the rule similar to (2.25); see [2]:

$$\bar{t}_{ij}(u) \mapsto \delta_{ij} - u^{-1} e_{ij} + (u-n-1)^{-1} e_{j'i'}\,\theta_i\theta_j, \qquad 2 \leqslant i, j \leqslant 2'. \tag{2.27}$$

Proposition 2.8. (i) *The subspace W is invariant with respect to the action of the operators $t_{ij}(u)$ with $2 \leqslant i,j \leqslant 2'$. Moreover, the assignment $\bar{t}_{ij}(u) \mapsto t_{ij}(u)$ defines a representation of the algebra $\mathrm{X}(\mathfrak{osp}_{0|2n})$ on W isomorphic to the vector representation.*

(ii) *The subspace W is annihilated by the operators $t_{1j}(u)$ with $j > 1$ and $t_{i\,1'}(u)$ with $i < 1'$. Moreover, the coefficients of the series $t_{11}(u)$ act on W as multiplications by scalars determined by $t_{11}(u) \mapsto 1 + u^{-1}$.*

Proof. Assuming that $2 \leqslant i,j \leqslant 2'$, for the action of the operator $t_{ij}(u)$ we have

$$\begin{aligned} t_{ij}(u)\, w_k &= t_{ij}(u)\,(e_1 \otimes e_k - e_k \otimes e_1) \\ &= \sum_{l=1}^{2n+2} t_{il}(u-1)\, e_1 \otimes t_{lj}(u)\, e_k - \sum_{l=1}^{2n+2} (-1)^{l+j}\, t_{il}(u-1)\, e_k \otimes t_{lj}(u)\, e_1. \end{aligned}$$

Due to (2.25), a nonzero contribution can only come from the terms

$$\begin{aligned} t_{i1}(u-1)\, e_1 \otimes t_{1j}(u)\, e_k + t_{ii}(u-1)\, e_1 \otimes t_{ij}(u)\, e_k \\ - t_{ij}(u-1)\, e_k \otimes t_{jj}(u)\, e_1 + t_{i1'}(u-1)\, e_k \otimes t_{1'j}(u)\, e_1 \end{aligned}$$

and the calculation yields the formula

$$t_{ij}(u)\, w_k = \delta_{ij}\, w_k - \delta_{kj}\, u^{-1}\, w_i + \delta_{ki'}\, \theta_i \theta_j\, (u-n-1)^{-1}\, w_{j'}.$$

Taking onto account (2.27), we thus prove the first part of the proposition. The second part is verified by a similar calculation. □

For any $a \in \mathbb{C}$ we will denote by V_a the representation of the algebra $\mathrm{X}(\mathfrak{osp}_{2|2n})$ on the space $V = (\mathbb{C}^{2|2n})^{\otimes 2}$ obtained by twisting the action (2.26) with the automorphism (2.9). Similarly, we let W_a denote the representation of $\mathrm{X}(\mathfrak{osp}_{0|2n})$ on W twisted by the same shift automorphism on $\mathrm{X}(\mathfrak{osp}_{0|2n})$. For $d = 1, \dots, n$ introduce the vectors

$$\xi_d = \sum_{\sigma \in \mathfrak{S}'_d} \operatorname{sgn} \sigma \cdot w_{\sigma(2)} \otimes w_{\sigma(3)} \otimes \dots \otimes w_{\sigma(d+1)} \in V_0 \otimes V_{-1} \otimes \dots \otimes V_{-d+1},$$

where $\mathfrak{S}'_d$ denotes the group of permutations of the set $\{2, \dots, d+1\}$.

Proposition 2.9. *The cyclic span $\mathrm{X}(\mathfrak{osp}_{2|2n})\, \xi_d$ is a highest weight module whose tuple of Drinfeld polynomials (2.16) has the form*

$$(u+1, u-d, 1, \dots, 1, u-d, 1, \dots, 1) \quad \textit{with} \quad P_{d+1}(u) = u - d, \tag{2.28}$$

for $d = 1, \dots, n-1$, while for $d = n$ it has the form

$$(u+1, u-n, 1, \dots, 1, u-n-1). \tag{2.29}$$

Proof. It follows from Proposition 2.8 and the coproduct rule (2.11) that for all $2 \leqslant i, j \leqslant 2'$ the image $t_{ij}(u)\, \xi_d$ coincides with $\bar{t}_{ij}(u)\, \xi_d$ in the representation $W_0 \otimes W_{-1} \otimes \ldots \otimes W_{-d+1}$ of the algebra $\mathrm{X}(\mathfrak{osp}_{0|2n})$. Since the series $\bar{t}_{ij}(-u)$ satisfy the defining relations of the extended Yangian $\mathrm{X}(\mathfrak{sp}_{2n})$, we derive from the proof of [2, Thm. 5.16] (see also Corollary A.3 below), that the vector ξ_d has the properties

$$t_{ij}(u)\, \xi_d = 0 \qquad \text{for} \quad 2 \leqslant i < j \leqslant 2'$$

and

$$t_{ii}(u)\, \xi_d = \begin{cases} \dfrac{u-d}{u-d+1}\, \xi_d & \text{for} \quad i = 2, \ldots, d+1, \\ \xi_d & \text{for} \quad i = d+2, \ldots, n+2, \end{cases} \tag{2.30}$$

where $d = 1, \ldots, n-1$. Moreover, the same relations hold for $d = n$, except for (2.30) with $i = n+2$, which is replaced by

$$t_{n+2,n+2}(u)\, \xi_n = \frac{u-n}{u-n-1}\, \xi_n.$$

Finally, Proposition 2.8(ii) and (2.11) imply that ξ_d is annihilated by the action of $t_{1j}(u)$ with $j > 1$, while

$$t_{11}(u)\, \xi_d = \frac{u+1}{u-d+1}\, \xi_d.$$

Thus, the vector ξ_d generates a highest weight module over the algebra $\mathrm{X}(\mathfrak{osp}_{2|2n})$, whose highest weight is found from the action of the series $t_{ii}(u)$. The formulas for the Drinfeld polynomials easily follow. □

We are now in a position to complete the proof of the Main Theorem. Proposition 2.4 implies that the irreducible highest weight representations of $\mathrm{X}(\mathfrak{osp}_{2|2n})$, associated with any tuple of Drinfeld polynomials of the first type in (2.20) is finite-dimensional. Furthermore, Proposition 2.9 shows that the irreducible highest weight representations of $\mathrm{X}(\mathfrak{osp}_{2|2n})$ associated with the tuples of the form (2.28) and (2.29) are also finite-dimensional. On the other hand, by applying the shift automorphism (2.9) and using the transition rule (2.19), we can get all tuples of Drinfeld polynomials of the second type in (2.20) from certain tuples of the first type and those appearing in (2.28) and (2.29). This proves the sufficiency of the conditions of the Main Theorem for the representation $L(\lambda(u))$ to be finite-dimensional. The last part of the theorem is an easy consequence of the decomposition (2.7); cf. [2, Cor. 5.19].

A Polynomial evaluation modules over the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$

A.1 Defining relations and representations

Given nonnegative integers m and n, we will use the notation $\bar{\imath} = 0$ for $i = 1, \ldots, m$ and $\bar{\imath} = 1$ for $i = m+1, \ldots, m+n$. Introduce the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{m|n}$ over $\mathbb{C}$ with the basis $e_1, e_2, \ldots, e_{m+n}$, where the parity of the basis vector e_i is defined to be $\bar{\imath} \mod 2$. Accordingly, equip the endomorphism algebra $\mathrm{End}\,\mathbb{C}^{m|n}$ with the $\mathbb{Z}_2$-gradation, where the parity of the matrix unit e_{ij} is found by $\bar{\imath}+\bar{\jmath} \mod 2$.

A standard basis of the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ is formed by elements E_{ij} of the parity $\bar{\imath}+\bar{\jmath} \mod 2$ for $1 \leqslant i, j \leqslant m+n$ with the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{kj}\, E_{il} - \delta_{il}\, E_{kj}(-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})}.$$

The *Yang R-matrix* associated with $\mathfrak{gl}_{m|n}$ is the rational function in u given by

$$R(u) = 1 - P u^{-1}, \tag{A.1}$$

where P is the permutation operator,

$$P = \sum_{i,j=1}^{m+n} e_{ij} \otimes e_{ji}(-1)^{\bar{\jmath}} \in \mathrm{End}\,\mathbb{C}^{m|n} \otimes \mathrm{End}\,\mathbb{C}^{m|n}.$$

Following [14], define the *Yangian* $\mathrm{Y}(\mathfrak{gl}_{m|n})$ as the $\mathbb{Z}_2$-graded algebra with generators $t_{ij}^{(r)}$ of parity $\bar{\imath}+\bar{\jmath} \mod 2$, where $1 \leqslant i, j \leqslant m+n$ and $r = 1, 2, \dots$, satisfying the quadratic relations

$$\big[t_{ij}(u), t_{kl}(v)\big] = \frac{1}{u-v}\Big(t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u)\Big)(-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}},$$

written in terms of the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)}\, u^{-r} \in \mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]].$$

Combining them into the matrix $T(u) = [t_{ij}(u)]$ and regarding it as the element

$$T(u) = \sum_{i,j=1}^{m+n} e_{ij} \otimes t_{ij}(u)(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}} \in \mathrm{End}\,\mathbb{C}^{m|n} \otimes \mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]],$$

we can write the defining relations in the standard RTT-form (2.4) with the R-matrix (A.1).

The universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_{m|n})$ can be regarded as a subalgebra of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ via the embedding $E_{ij} \mapsto t_{ij}^{(1)}(-1)^{\bar{\imath}}$, while the mapping

$$t_{ij}(u) \mapsto \delta_{ij} + E_{ij}(-1)^{\bar{\imath}}\, u^{-1} \tag{A.2}$$

defines the *evaluation homomorphism* $\mathrm{ev}\colon \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{U}(\mathfrak{gl}_{m|n})$.

The Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is a Hopf algebra with the coproduct defined by

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{m+n} t_{ik}(u) \otimes t_{kj}(u). \tag{A.3}$$

A representation V of the algebra $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is called a *highest weight representation* if there exists a nonzero vector $\xi \in V$ such that V is generated by ξ,

$$\begin{aligned} t_{ij}(u)\,\xi &= 0 \qquad &&\text{for} \quad 1 \leqslant i < j \leqslant m+n, \qquad \text{and}\\ t_{ii}(u)\,\xi &= \pi_i(u)\,\xi \qquad &&\text{for} \quad i = 1, \dots, m+n, \end{aligned}$$

for some formal series

$$\pi_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]. \tag{A.4}$$

The vector ξ is called the *highest vector* of V and the $(m+n)$-tuple $\pi(u) = (\pi_1(u), \dots, \pi_{m+n}(u))$ is called its *highest weight*.

Given an arbitrary tuple $\pi(u) = (\pi_1(u), \dots, \pi_{m+n}(u))$ of formal series of the form (A.4), the *Verma module* $M(\pi(u))$ is defined as the quotient of the algebra $\mathrm{Y}(\mathfrak{gl}_{m|n})$ by the left ideal generated by all coefficients of the series $t_{ij}(u)$ with $1 \leqslant i < j \leqslant m+n$, and $t_{ii}(u) - \pi_i(u)$ for $i = 1, \dots, m+n$. We will denote by $L(\pi(u))$ its irreducible quotient. The isomorphism class of $L(\pi(u))$ is determined by $\pi(u)$. Necessary and sufficient conditions on $\pi(u)$ for the representation $L(\pi(u))$ to be finite-dimensional are known due to [19]. Their extension to arbitrary parity sequences via odd reflections was given in [12].

Consider the irreducible highest weight representation $V(\pi)$ of the Lie superalgebra $\mathfrak{gl}_{m|n}$ with the highest weight $\pi = (\pi_1, \dots, \pi_{m+n})$, associated with the standard Borel subalgebra. This means that $E_{ii}\,\zeta = \pi_i\,\zeta$ for $i = 1, \dots, m+n$ and $E_{ij}\,\zeta = 0$ for $i < j$, where ζ is the highest vector of $V(\pi)$. The representation $V(\pi)$ is finite-dimensional if and only if the highest weight π satisfies the conditions

$$\pi_i - \pi_{i+1} \in \mathbb{Z}_+ \quad \text{for all} \quad i \neq m;$$

see [9]. Use the evaluation homomorphism (A.2) to equip $V(\pi)$ with an $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module structure. The *evaluation module* $V(\pi)$ is isomorphic to the Yangian highest weight module $L(\pi(u))$, where the components of the highest weight $\pi(u)$ are

$$\pi_i(u) = 1 + \pi_i\,(-1)^{\bar{i}}\,u^{-1}, \qquad i = 1, \dots, m+n.$$

A.2 Schur–Sergeev duality and fusion procedure

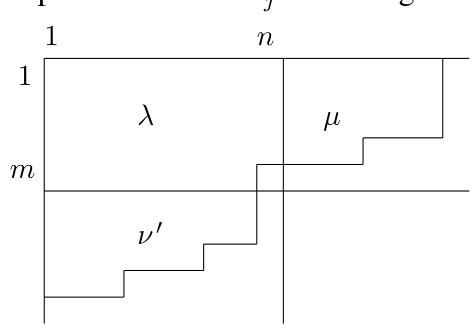
We will follow [5, Sec. 3.2] to recall a version of the *Schur–Sergeev duality* going back to [3] and [16]. An (m,n)-*hook partition* $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition with the property $\lambda_{m+1} \leqslant n$. This means that the Young diagram λ is contained in the (m,n)-hook as depicted below. The figure also illustrates the partitions $\mu = (\mu_1, \dots, \mu_m)$ and $\nu = (\nu_1, \dots, \nu_n)$ associated with λ. They are introduced by setting

$$\mu_i = \max\{\lambda_i - n, 0\}, \qquad i = 1, \dots, m,$$

and

$$\nu_j = \max\{\lambda'_j - m, 0\}, \qquad j = 1, \dots, n,$$

where λ' denotes the conjugate partition so that λ'_j is the length of column j in the diagram λ:



We will associate two $(m+n)$-tuples of nonnegative integers with λ by

$$\lambda^{\sharp} = (\lambda_1, \dots, \lambda_m, \nu_1, \dots, \nu_n) \qquad \text{and} \qquad \lambda^{\flat} = (\mu_1, \dots, \mu_m, \lambda'_1, \dots, \lambda'_n).$$

The tensor product space $(\mathbb{C}^{m|n})^{\otimes d}$ is naturally a module over both $\mathfrak{gl}_{m|n}$ and the symmetric group $\mathfrak{S}_d$. For the action of the basis elements of $\mathfrak{gl}_{m|n}$ we have

$$E_{ij} \mapsto \sum_{a=1}^{d} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(d-a)}, \tag{A.5}$$

while the transposition $(a\,b) \in \mathfrak{S}_d$ with $a < b$ acts by $(a\,b) \mapsto P_{ab}$ with

$$P_{ab} = \sum_{i,j=1}^{m+n} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{ji} \otimes 1^{\otimes(d-b)} (-1)^{\bar{\jmath}}.$$

The images of $\mathrm{U}(\mathfrak{gl}_{m|n})$ and $\mathbb{C}\mathfrak{S}_d$ in the endomorphism algebra $\mathrm{End}\,(\mathbb{C}^{m|n})^{\otimes d}$ satisfy the double centralizer property, which leads to the multiplicity-free decomposition

$$(\mathbb{C}^{m|n})^{\otimes d} = \bigoplus_{\lambda} V(\lambda^{\sharp}) \otimes S^{\lambda},$$

summed over the (m,n)-hook partitions λ with d boxes, where S^{λ} is the Specht module over $\mathfrak{S}_d$ associated with λ. By representing the group algebra $\mathbb{C}\mathfrak{S}_d$ as the direct sum of matrix algebras

$$\mathbb{C}\mathfrak{S}_d \cong \bigoplus_{\lambda \vdash d} \mathrm{Mat}_{f_\lambda}(\mathbb{C}), \qquad f_\lambda = \dim S^{\lambda},$$

we can think of S^{λ} as the canonical irreducible module over $\mathrm{Mat}_{f_\lambda}(\mathbb{C})$ isomorphic to $\mathbb{C}^{f_\lambda}$. The diagonal matrix units $e_{\mathcal{U}} = e^{\lambda}_{\mathcal{U}\mathcal{U}} \in \mathrm{Mat}_{f_\lambda}(\mathbb{C})$ parameterized by standard λ-tableaux $\mathcal{U}$ are primitive idempotents in $\mathbb{C}\mathfrak{S}_d$. We may conclude that if λ is an (m,n)-hook partition with d boxes, then the image $e_{\mathcal{U}}(\mathbb{C}^{m|n})^{\otimes d}$ is isomorphic to the $\mathfrak{gl}_{m|n}$-module $V(\lambda^{\sharp})$. If λ is not contained in the (m,n)-hook, then the image is zero.

Explicit formulas for the idempotents $e_{\mathcal{U}}$ can be derived with the use of the orthonormal Young basis of S^{λ} via the *Jucys–Murphy elements* $x_1, \dots, x_d$ of the group algebra $\mathbb{C}\mathfrak{S}_d$ defined by

$$x_1 = 0 \qquad \text{and} \qquad x_a = (1\,a) + \dots + (a-1\,a) \qquad \text{for} \quad a = 2, \dots, d.$$

Given a standard λ-tableau $\mathcal{U}$, denote by $\mathcal{V}$ the standard tableau obtained from $\mathcal{U}$ by removing the box α occupied by d. Then the shape of $\mathcal{V}$ is a diagram which we denote by λ^{-}. We let $c = j - i$ denote the *content* of the box $\alpha = (i,j)$ and let $a_1, \dots, a_l$ be the contents of all addable boxes of λ^{-} except for α. The Jucys–Murphy formula gives an inductive rule for the calculation of $e_{\mathcal{U}}$:

$$e_{\mathcal{U}} = e_{\mathcal{V}} \frac{(x_d - a_1) \dots (x_d - a_l)}{(c - a_1) \dots (c - a_l)}; \tag{A.6}$$

see [8] and [13]. The idempotents $e_{\mathcal{U}}$ can also be obtained from the *fusion procedure* for the symmetric group; see [4], [7] and [15], and we recall a version following [10, Sec. 6.4], where it was essentially derived from (A.6). Take d complex variables $u_1,\dots,u_d$ and consider the rational function with values in $\mathbb{C}\mathfrak{S}_d$ defined by

$$\phi(u_1,\dots,u_d)=\prod_{1\leqslant a<b\leqslant d}\Big(1-\frac{(a\,b)}{u_a-u_b}\Big),$$

where the product is taken in the lexicographical order on the set of pairs (a,b). Suppose that $\lambda\vdash d$ and let $\mathcal{U}$ be a standard λ-tableau. Let $c_a=c_a(\mathcal{U})$ for $a=1,\dots,d$ be the contents of $\mathcal{U}$ so that $c_a=j-i$ if a occupies the box (i,j) in $\mathcal{U}$. Then the consecutive evaluations of the rational function $\phi(u_1,\dots,u_d)$ are well-defined and the value coincides with the primitive idempotent $e_{\mathcal{U}}$ multiplied by the product of hook lengths $h(\lambda)$ of λ,

$$\phi(u_1,\dots,u_d)\Big|_{u_1=c_1}\Big|_{u_2=c_2}\dots\Big|_{u_d=c_d}=h(\lambda)\,e_{\mathcal{U}}. \tag{A.7}$$

A.3 Yangian action on polynomial modules

The Yang R-matrix (A.1) is a solution of the Yang–Baxter equation

$$R_{12}(u)\,R_{13}(u+v)\,R_{23}(v)=R_{23}(v)\,R_{13}(u+v)\,R_{12}(u) \tag{A.8}$$

in $\mathrm{End}\,(\mathbb{C}^{m|n})^{\otimes 3}$ with $R_{ab}(u)=1-P_{ab}\,u^{-1}$. This implies that the mapping $T(u)\mapsto R(u)$ defines a representation of the algebra $\mathrm{Y}(\mathfrak{gl}_{m|n})$ on the space $\mathbb{C}^{m|n}$, known as the vector representation. In terms of the generating series it has the form

$$t_{ij}(u)\mapsto \delta_{ij}-u^{-1}e_{ji}(-1)^{\bar{i}\bar{j}}. \tag{A.9}$$

For an (m,n)-hook partition $\lambda\vdash d$ fix a standard λ-tableau $\mathcal{U}$. As above, let $c_1,\dots,c_d$ be the contents of the respective entries in $\mathcal{U}$. By using the coproduct (A.3) and the shift automorphism of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ defined by (2.9), we get a representation of the Yangian on the space $(\mathbb{C}^{m|n})^{\otimes d}$ defined by

$$T(u)\mapsto R_{01}(u-c_1)\dots R_{0d}(u-c_d), \tag{A.10}$$

which is written in terms of elements of the algebras

$$\mathrm{End}\,\mathbb{C}^{m|n}\otimes \mathrm{Y}(\mathfrak{gl}_{m|n})\to \mathrm{End}\,\mathbb{C}^{m|n}\otimes \mathrm{End}\,(\mathbb{C}^{m|n})^{\otimes d}$$

with the first copy of the endomorphism algebra labelled by 0.

Another form of the vector representation is related to (A.9) via twisting with the super-transposition automorphism

$$t_{ij}(u)\mapsto t_{ji}(-u)(-1)^{\bar{i}\bar{j}+\bar{i}}. \tag{A.11}$$

We then get the action of the algebra $\mathrm{Y}(\mathfrak{gl}_{m|n})$ on the space $\mathbb{C}^{m|n}$ given by

$$t_{ij}(u)\mapsto \delta_{ij}+u^{-1}e_{ij}(-1)^{\bar{i}}. \tag{A.12}$$

It can be written in a matrix form as $T(u) \mapsto R'(-u)$ with

$$R'(u) = 1 - Q\, u^{-1}, \qquad Q = \sum_{i,j=1}^{m+n} e_{ij} \otimes e_{ij} \, (-1)^{\bar{\imath}+\bar{\jmath}+\bar{\imath}\bar{\jmath}}.$$

Accordingly, the composition of the representation (A.10) with the automorphism (A.11) yields another representation of the Yangian on the space $(\mathbb{C}^{m|n})^{\otimes d}$ given by

$$T(u) \mapsto R'_{0\,d}(-u - c_d) \dots R'_{01}(-u - c_1), \tag{A.13}$$

where $R'_{0\,a}(u) = 1 - Q_{0\,a}\, u^{-1}$ and

$$Q_{0\,a} = \sum_{i,j=1}^{m+n} e_{ij} \otimes 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(d-a)} (-1)^{\bar{\imath}+\bar{\jmath}+\bar{\imath}\bar{\jmath}}.$$

Similar to (A.10), this representation can also be obtained by using the opposite coproduct on the Yangian, which is the composition of (A.3) and the map swapping the tensor factors.

Theorem A.1. *The subspace $L_{\mathcal{U}} = e_{\mathcal{U}}(\mathbb{C}^{m|n})^{\otimes d}$ is invariant under the Yangian actions (A.10) and (A.13). Moreover, the respective representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ on $L_{\mathcal{U}}$ are isomorphic to the highest weight representations $L(\pi^\flat(u))$ and $L(\pi^\sharp(u))$, where*

$$\pi^\flat(u) = \Big(1 - \mu_m u^{-1}, \dots, 1 - \mu_1 u^{-1}, 1 + \lambda'_n u^{-1}, \dots, 1 + \lambda'_1 u^{-1}\Big)$$

and

$$\pi^\sharp(u) = \Big(1 + \lambda_1 u^{-1}, \dots, 1 + \lambda_m u^{-1}, 1 - \nu_1 u^{-1}, \dots, 1 - \nu_n u^{-1}\Big).$$

Proof. Let $E_{\mathcal{U}} \in \mathrm{End}\,(\mathbb{C}^{m|n})^{\otimes d}$ denote the image of the idempotent $e_{\mathcal{U}}$ under the representation of the symmetric group on $(\mathbb{C}^{m|n})^{\otimes d}$. We have the relation

$$R_{01}(u - c_1) \dots R_{0d}(u - c_d)\, E_{\mathcal{U}} = E_{\mathcal{U}} \left(1 - \frac{P_{01} + \dots + P_{0d}}{u}\right) \tag{A.14}$$

which plays a key role in the derivation of the fusion formula (A.7); cf. [15] and [10, Prop. 6.4.4]. Apply the anti-automorphism $e_{ij} \mapsto e_{ji}(-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}}$ to the zeroth copy of the endomorphism algebra $\mathrm{End}\,\mathbb{C}^{m|n}$ to derive

$$R'_{0d}(-u - c_d) \dots R'_{01}(-u - c_1)\, E_{\mathcal{U}} = E_{\mathcal{U}} \left(1 + \frac{Q_{01} + \dots + Q_{0d}}{u}\right). \tag{A.15}$$

Together with (A.14) this proves the first part of the theorem.

Furthermore, relation (A.15) shows that the action (A.13) on $L_{\mathcal{U}}$ is the composition of the evaluation homomorphism (A.2) and the action (A.5) of $\mathfrak{gl}_{m|n}$. Hence this representation is isomorphic to the evaluation module $V(\lambda^\sharp) \cong L(\pi^\sharp(u))$. Similarly, relation (A.14) shows that the Yangian action (A.10) on the subspace $L_{\mathcal{U}}$ is the composition of the evaluation homomorphism (A.2) and the action (A.5) of $\mathfrak{gl}_{m|n}$ twisted by the automorphism

$$E_{ij} \mapsto -E_{ji}(-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}}. \tag{A.16}$$

On the other hand, by [5, Sec. 2.4], an application of a chain of odd reflections shows that the extremal weight of the $\mathfrak{gl}_{m|n}$-module $V(\lambda^\sharp)$ with respect to the opposite Borel subalgebra is $\lambda^\flat$. That is, there is a nonzero vector $\eta \in V(\lambda^\sharp)$ of the weight $\lambda^\flat$ such that $E_{i,i+1}\eta = 0$ for all $i \neq m$ and $E_{m+n,1}\eta = 0$. By taking the lowest vector with respect to the action of $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ we conclude that $V(\lambda^\sharp)$ contains a nonzero vector ζ of the weight $(\mu_m, \dots, \mu_1, \lambda'_n, \dots, \lambda'_1)$ such that $E_{ij}\zeta = 0$ for all $i > j$. Thus, the vector ζ is the highest vector of the Yangian module $L_{\mathcal{U}}$ and its weight is found by taking into account (A.2) and (A.16) so that this module is isomorphic to $L(\pi^\flat(u))$. $\square$

By using the coproduct (A.3) and the vector representation (A.12) instead of (A.9), for any complex parameters z_a we get a representation of the Yangian on the space $(\mathbb{C}^{m|n})^{\otimes d}$ defined by

$$T(u) \mapsto R'_{01}(-u - z_1) \dots R'_{0d}(-u - z_d).$$

For suitable parameters z_a, the subspace $L_{\mathcal{U}} = e_{\mathcal{U}}(\mathbb{C}^{m|n})^{\otimes d}$ associated with the standard tableaux $\mathcal{U}$ of shapes $\lambda = (d)$ and $\lambda = (1^d)$, turns out to be invariant under this Yangian action as well. The primitive idempotents $e_{\mathcal{U}}$ associated with the standard row and column tableaux are, respectively, the symmetrizer and anti-symmetrizer in $\mathbb{C}\mathfrak{S}_d$,

$$h^{(d)} = \frac{1}{d!} \sum_{s \in \mathfrak{S}_d} s \qquad \text{and} \qquad a^{(d)} = \frac{1}{d!} \sum_{s \in \mathfrak{S}_d} \operatorname{sgn} s \cdot s \in \mathbb{C}\mathfrak{S}_d.$$

Note that (A.6) and (A.7) yield multiplicative formulas for $h^{(d)}$ and $a^{(d)}$.

Corollary A.2. (i) *The subspace $h^{(d)}(\mathbb{C}^{m|n})^{\otimes d}$ is invariant under the action*

$$T(u) \mapsto R'_{01}(-u - d + 1) \dots R'_{0d}(-u).$$

Moreover, the representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ on this subspace is isomorphic to the highest weight representation with the highest weight $\big(1 + du^{-1}, 1 \dots, 1\big)$.

(ii) *The subspace $a^{(d)}(\mathbb{C}^{m|n})^{\otimes d}$ is invariant under the action*

$$T(u) \mapsto R'_{01}(-u + d - 1) \dots R'_{0d}(-u).$$

Moreover, the representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ on this subspace is isomorphic to the highest weight representation with the highest weight

$$\big(\underbrace{1 + u^{-1}, \dots, 1 + u^{-1}}_{d}, 1, \dots, 1\big) \qquad \textit{if} \quad d \leqslant m, \tag{A.17}$$

and

$$\big(\underbrace{1 + u^{-1}, \dots, 1 + u^{-1}}_{m}, 1 + (m - d)u^{-1}, 1, \dots, 1\big) \qquad \textit{if} \quad d > m.$$

Proof. Multiplying both sides of (A.15) by the image $P_\omega = P_{1,d}\, P_{2,d-1} \dots$ of the longest permutation $\omega \in \mathfrak{S}_d$ from the left, we get

$$R'_{01}(-u - c_d) \dots R'_{0d}(-u - c_1)\, P_\omega E_{\mathcal{U}} = P_\omega E_{\mathcal{U}} \left(1 + \frac{Q_{01} + \dots + Q_{0d}}{u}\right).$$

Since $\omega\, h^{(d)} = h^{(d)}$ and $\omega\, a^{(d)} = \operatorname{sgn}\omega \cdot a^{(d)}$, both parts of the corollary follow from the particular cases of Theorem A.1 concerning the action (A.13) for the row and column tableaux $\mathcal{U}$. $\square$

The following corollary in the case $n = 0$ was used in the construction of the fundamental modules over the Yangians in types B, C and D in [2, Sec. 5.3]. It was also applied to the orthosymplectic Yangians in [11] and in the proof of Proposition 2.9 in the previous section. Equip the tensor product space $(\mathbb{C}^{m|n})^{\otimes d}$ with the action of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ by setting

$$t_{ij}(u) \mapsto \sum_{a_1,\dots,a_{d-1}=1}^{m+n} t_{ia_1}(u-d+1) \otimes t_{a_1a_2}(u-d+2) \otimes \dots \otimes t_{a_{d-1}j}(u),$$

where the generators act in the respective copies of the vector space $\mathbb{C}^{m|n}$ via the rule (A.12). Set

$$\xi_d = \sum_{\sigma\in\mathfrak{S}_d} \operatorname{sgn}\sigma \cdot e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(d)} \in (\mathbb{C}^{m|n})^{\otimes d}.$$

Corollary A.3. *For any $1 \leqslant d \leqslant m$ the vector ξ_d has the properties*

$$t_{ij}(u)\,\xi_d = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant m+n$$

and

$$t_{ii}(u)\,\xi_d = \begin{cases} \dfrac{u+1}{u}\,\xi_d & \text{for} \quad i = 1,\dots,d,\\ \xi_d & \text{for} \quad i = d+1,\dots,m+n. \end{cases}$$

Proof. This is immediate from Corollary A.2(ii), because the vector ξ_d is the highest vector of the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $a^{(d)}(\mathbb{C}^{m|n})^{\otimes d}$ with the highest weight (A.17). □

References

[1] D. Arnaudon, J. Avan, N. Crampé, L. Frappat, E. Ragoucy, *R-matrix presentation for super-Yangians $Y(\mathrm{osp}(m|2n))$*, J. Math. Phys. **44** (2003), 302–308.

[2] D. Arnaudon, A. Molev and E. Ragoucy, *On the R-matrix realization of Yangians and their representations*, Annales Henri Poincaré **7** (2006), 1269–1325.

[3] A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Adv. Math. **64** (1987), 118–175.

[4] I. V. Cherednik, *A new interpretation of Gelfand–Tzetlin bases*, Duke Math. J. **54** (1987), 563–577.

[5] S.-J. Cheng and W. Wang, *Dualities and representations of Lie superalgebras*, Graduate Studies in Mathematics, 144. AMS, Providence, RI, 2012.

[6] V. G. Drinfeld, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** (1988), 212–216.

[7] A. Jucys, *On the Young operators of the symmetric group*, Lietuvos Fizikos Rinkinys **6** (1966), 163–180.

[8] A. Jucys, *Factorization of Young projection operators for the symmetric group*, Lietuvos Fizikos Rinkinys **11** (1971), 5–10.

[9] V. Kac, *Representations of classical Lie superalgebras*, in "Differential geometrical methods in mathematical physics", II (Proc. Conf., Univ. Bonn, Bonn, 1977), pp. 597–626, Lecture Notes in Math., 676, Springer, Berlin, 1978.

[10] A. Molev, *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, 143. AMS, Providence, RI, 2007.

[11] A. Molev, *Representations of the Yangians associated with Lie superalgebras* $\mathfrak{osp}(1|2n)$, arXiv:2109.02361.

[12] A. Molev, *Odd reflections in the Yangian associated with* $\mathfrak{gl}(m|n)$, arXiv:2109.09462.

[13] G. E. Murphy, *A new construction of Young's seminormal representation of the symmetric group*, J. Algebra **69** (1981), 287–291.

[14] M. L. Nazarov, *Quantum Berezinian and the classical Capelli identity*, Lett. Math. Phys. **21** (1991), 123–131.

[15] M. Nazarov, *Yangians and Capelli identities*, in: "Kirillov's Seminar on Representation Theory" (G. I. Olshanski, Ed.), Amer. Math. Soc. Transl. **181**, Amer. Math. Soc., Providence, RI, 1998, pp. 139–163.

[16] A. N. Sergeev, *Tensor algebra of the identity representation as a module over the Lie superalgebras* $\mathfrak{gl}(n,m)$ *and* $Q(n)$, Mat. Sb. (N.S.) 123(165) (1984), 422–430.

[17] A. B. Zamolodchikov and Al. B. Zamolodchikov, *Factorized* S*-matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, Ann. Phys. **120** (1979), 253–291.

[18] R. B. Zhang, *Representations of super Yangian*, J. Math. Phys. **36** (1995), 3854–3865.

[19] R. B. Zhang, *The* $\mathfrak{gl}(M|N)$ *super Yangian and its finite-dimensional representations*, Lett. Math. Phys. **37** (1996), 419–434.

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