Representations of the Yangians associated with Lie superalgebras $\mathfrak{osp}(1|2n)$

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Abstract

We classify the finite-dimensional irreducible representations of the Yangians associated with the orthosymplectic Lie superalgebras $\mathfrak{osp}_{1|2n}$ in terms of the Drinfeld polynomials. The arguments rely on the description of the representations in the particular case $n = 1$ obtained in our previous work.

1 Introduction

The finite-dimensional irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{g})$ associated with a simple Lie algebra $\mathfrak{g}$ were classified by Drinfeld [5]. The arguments rely on the work of Tarasov [9] on the particular case of $\mathrm{Y}(\mathfrak{sl}_2)$, where the classification was carried over in the language of monodromy matrices within the quantum inverse scattering method; see [7, Sec. 3.3] for a detailed adapted exposition of these results. This description of the representations of the Yangian $\mathrm{Y}(\mathfrak{sl}_2)$, along with some other low rank cases, should also play an essential role in the classification of the finite-dimensional irreducible representations of the Yangians associated with simple Lie superalgebras. One of these cases was considered in our previous work [8], where the representations of the Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2})$ were described.

These two basic cases turn out to be sufficient to complete the classification in the case of the Yangians associated with the orthosymplectic Lie superalgebras $\mathfrak{osp}_{1|2n}$. We prove in this paper that, similar to the classification results of [5], the finite-dimensional irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ are in one-to-one correspondence with the n-tuples of monic polynomials $(P_1(u), \dots, P_n(u))$, and so we call them the *Drinfeld polynomials*.

To describe the results in more detail, recall that the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|2n})$, as introduced by Arnaudon *et al.* [1], can be considered as a quotient of the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|2n})$ defined via an RTT relation. A standard argument shows that every finite-dimensional irreducible representation of $\mathrm{X}(\mathfrak{osp}_{M|2n})$ is a highest weight representation. It is isomorphic to the irreducible quotient $L(\lambda(u))$ of the Verma module $M(\lambda(u))$ associated with an $(n + 1)$-tuple $\lambda(u) = (\lambda_1(u), \dots, \lambda_{n+1}(u))$ of formal series $\lambda_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. The tuple is called the *highest weight* of the representation. The key step in the classification is to find the conditions on the highest weight for the representation $L(\lambda(u))$ to be finite-dimensional. The required necessary conditions are derived by induction from those for the associated actions of the Yangians

$\mathrm{Y}(\mathfrak{gl}_2)$ and $\mathrm{X}(\mathfrak{osp}_{1|2})$ on the respective cyclic spans of the highest vector of $L(\lambda(u))$. The sufficiency of these conditions is verified by constructing the *fundamental representations* of the Yangian $\mathrm{X}(\mathfrak{osp}_{M|2n})$; cf. [3], [4]. The following is our main result.

Main Theorem. *Every finite-dimensional irreducible representation of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ is isomorphic to $L(\lambda(u))$ for a certain highest weight $\lambda(u)$. The representation $L(\lambda(u))$ is finite-dimensional if and only if*

$$\frac{\lambda_{i+1}(u)}{\lambda_i(u)} = \frac{P_i(u+1)}{P_i(u)}, \qquad i = 1, \dots, n, \tag{1.1}$$

for some monic polynomials $P_i(u)$ in u. The finite-dimensional irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ are in a one-to-one correspondence with the n-tuples of monic polynomials $(P_1(u), \dots, P_n(u))$.

2 Definitions and preliminaries

For any integer $n \geqslant 1$ introduce the involution $i \mapsto i' = 2n - i + 2$ on the set $\{1, 2, \dots, 2n+1\}$. Consider the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{1|2n}$ over $\mathbb{C}$ with the basis $e_1, e_2, \dots, e_{2n+1}$, where the vectors e_i and $e_{i'}$ with $i = 1, \dots, n$ are odd and the vector e_{n+1} is even. We set

$$\bar{\imath} = \begin{cases} 1 & \text{for} \quad i = 1, \dots, n, n', \dots, 1', \\ 0 & \text{for} \quad i = n+1. \end{cases}$$

The endomorphism algebra $\mathrm{End}\,\mathbb{C}^{1|2n}$ gets a $\mathbb{Z}_2$-gradation with the parity of the matrix unit e_{ij} found by $\bar{\imath} + \bar{\jmath} \mod 2$.

We will consider even square matrices with entries in $\mathbb{Z}_2$-graded algebras, their (i,j) entries will have the parity $\bar{\imath} + \bar{\jmath} \mod 2$. The algebra of even matrices over a superalgebra $\mathcal{A}$ will be identified with the tensor product algebra $\mathrm{End}\,\mathbb{C}^{1|2n} \otimes \mathcal{A}$, so that a matrix $A = [a_{ij}]$ is regarded as the element

$$A = \sum_{i,j=1}^{2n+1} e_{ij} \otimes a_{ij} (-1)^{\bar{\imath}\bar{\jmath} + \bar{\jmath}} \in \mathrm{End}\,\mathbb{C}^{1|2n} \otimes \mathcal{A}.$$

We will use the involutive matrix *super-transposition* t defined by $(A^t)_{ij} = A_{j'i'}(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}}\theta_i\theta_j$, where we set

$$\theta_i = \begin{cases} 1 & \text{for} \quad i = 1, \dots, n+1, \\ -1 & \text{for} \quad i = n+2, \dots, 2n+1. \end{cases}$$

This super-transposition is associated with the bilinear form on the space $\mathbb{C}^{1|2n}$ defined by the anti-diagonal matrix $G = [\delta_{ij'}\,\theta_i]$. We will also regard t as the linear map

$$t : \mathrm{End}\,\mathbb{C}^{1|2n} \to \mathrm{End}\,\mathbb{C}^{1|2n}, \qquad e_{ij} \mapsto e_{j'i'}(-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}}\theta_i\theta_j. \tag{2.1}$$

In the case of multiple tensor products of the endomorphism algebras, we will indicate by t_a the map (2.1) acting on the a-th copy of $\mathrm{End}\,\mathbb{C}^{1|2n}$.

A standard basis of the general linear Lie superalgebra $\mathfrak{gl}_{1|2n}$ is formed by elements E_{ij} of the parity $\bar{\imath}+\bar{\jmath} \mod 2$ for $1 \leqslant i,j \leqslant 2n+1$ with the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{kj}\, E_{il} - \delta_{il}\, E_{kj}(-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})}.$$

We will regard the orthosymplectic Lie superalgebra $\mathfrak{osp}_{1|2n}$ associated with the bilinear from defined by G as the subalgebra of $\mathfrak{gl}_{1|2n}$ spanned by the elements

$$F_{ij} = E_{ij} - E_{j'i'}(-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}}\,\theta_i\theta_j.$$

Introduce the permutation operator P by

$$P = \sum_{i,j=1}^{2n+1} e_{ij} \otimes e_{ji}(-1)^{\bar{\jmath}} \in \operatorname{End}\mathbb{C}^{1|2n} \otimes \operatorname{End}\mathbb{C}^{1|2n}$$

and set

$$Q = P^{t_1} = P^{t_2} = \sum_{i,j=1}^{2n+1} e_{ij} \otimes e_{i'j'}(-1)^{\bar{\imath}\bar{\jmath}}\,\theta_i\theta_j \in \operatorname{End}\mathbb{C}^{1|2n} \otimes \operatorname{End}\mathbb{C}^{1|2n}.$$

The R-*matrix* associated with $\mathfrak{osp}_{1|2n}$ is the rational function in u given by

$$R(u) = 1 - \frac{P}{u} + \frac{Q}{u-\kappa}, \qquad \kappa = -n - 1/2.$$

This is a super-version of the R-matrix originally found in [10]. Following [1], we define the *extended Yangian* $\mathrm{X}(\mathfrak{osp}_{1|2n})$ as a $\mathbb{Z}_2$-graded algebra with generators $t_{ij}^{(r)}$ of parity $\bar{\imath}+\bar{\jmath} \mod 2$, where $1 \leqslant i,j \leqslant 2n+1$ and $r = 1, 2, \dots$, satisfying certain quadratic relations. In order to write them down, introduce the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)}\, u^{-r} \in \mathrm{X}(\mathfrak{osp}_{1|2n})[[u^{-1}]] \tag{2.2}$$

and combine them into the matrix $T(u) = [t_{ij}(u)]$ so that

$$T(u) = \sum_{i,j=1}^{2n+1} e_{ij} \otimes t_{ij}(u)(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}} \in \operatorname{End}\mathbb{C}^{1|2n} \otimes \mathrm{X}(\mathfrak{osp}_{1|2n})[[u^{-1}]].$$

Consider the algebra $\operatorname{End}\mathbb{C}^{1|2n} \otimes \operatorname{End}\mathbb{C}^{1|2n} \otimes \mathrm{X}(\mathfrak{osp}_{1|2n})[[u^{-1}]]$ and introduce its elements $T_1(u)$ and $T_2(u)$ by

$$T_1(u) = \sum_{i,j=1}^{2n+1} e_{ij} \otimes 1 \otimes t_{ij}(u)(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}}, \qquad T_2(u) = \sum_{i,j=1}^{2n+1} 1 \otimes e_{ij} \otimes t_{ij}(u)(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}}.$$

The defining relations for the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ take the form of the RTT-*relation*

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v). \tag{2.3}$$

As shown in [1], the product $T(u)\,T^t(u-\kappa)$ is a scalar matrix with

$$T(u-\kappa)\,T^t(u) = c(u)1, \tag{2.4}$$

where $c(u)$ is a series in u^{-1}. All its coefficients belong to the center $\mathrm{ZX}(\mathfrak{osp}_{1|2n})$ of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ and generate the center.

The *Yangian* $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ is defined as the subalgebra of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ which consists of the elements stable under the automorphisms

$$t_{ij}(u) \mapsto f(u)\,t_{ij}(u) \tag{2.5}$$

for all series $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. We have the tensor product decomposition

$$\mathrm{X}(\mathfrak{osp}_{1|2n}) = \mathrm{ZX}(\mathfrak{osp}_{1|2n}) \otimes \mathrm{Y}(\mathfrak{osp}_{1|2n}). \tag{2.6}$$

The Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ can be equivalently defined as the quotient of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ by the relation

$$T(u-\kappa)\,T^t(u) = 1.$$

We will also use a more explicit form of the defining relations (2.3) written in terms of the series (2.2) as follows:

$$\begin{aligned}[][t_{ij}(u), t_{kl}(v)] = \frac{1}{u-v}\Big(t_{kj}(u)\,t_{il}(v) - t_{kj}(v)\,t_{il}(u)\Big)(-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}} \\ - \frac{1}{u-v-\kappa}\Big(\delta_{ki'}\sum_{p=1}^{2n+1} t_{pj}(u)\,t_{p'l}(v)(-1)^{\bar{\imath}+\bar{\imath}\bar{\jmath}+\bar{\jmath}\bar{p}}\,\theta_i\theta_p \\ - \delta_{lj'}\sum_{p=1}^{2n+1} t_{kp'}(v)\,t_{ip}(u)(-1)^{\bar{\jmath}+\bar{p}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}+\bar{\imath}\bar{p}}\,\theta_j\theta_p\Big).\end{aligned} \tag{2.7}$$

For any $a \in \mathbb{C}$ the mapping

$$t_{ij}(u) \mapsto t_{ij}(u+a) \tag{2.8}$$

defines an automorphism of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$.

The universal enveloping algebra $\mathrm{U}(\mathfrak{osp}_{1|2n})$ can be regarded as a subalgebra of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ via the embedding

$$F_{ij} \mapsto \frac{1}{2}\Big(t_{ij}^{(1)} - t_{j'i'}^{(1)}(-1)^{\bar{\jmath}+\bar{\imath}\bar{\jmath}}\,\theta_i\theta_j\Big)(-1)^{\bar{\imath}}. \tag{2.9}$$

This fact relies on the Poincaré–Birkhoff–Witt theorem for the orthosymplectic Yangian which was pointed out in [1] and [2]. It states that the associated graded algebra for $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ is isomorphic to $\mathrm{U}(\mathfrak{osp}_{1|2n}[u])$. A detailed proof of the theorem can be given by extending the arguments of [3, Sec. 3] to the super case with the use of the vector representation recalled below in (3.6).

The extended Yangian $\mathrm{X}(\mathfrak{osp}_{1|2n})$ is a Hopf algebra with the coproduct defined by

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{2n+1} t_{ik}(u) \otimes t_{kj}(u). \tag{2.10}$$

For the image of the series $c(u)$ we have $\Delta : c(u) \mapsto c(u) \otimes c(u)$ and so the Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ inherits the Hopf algebra structure from $\mathrm{X}(\mathfrak{osp}_{1|2n})$.

3 Highest weight representations

We will start by deriving a general reduction property for representations of the extended Yangians $\mathrm{X}(\mathfrak{osp}_{1|2n})$ analogous to [3, Lemma 5.13]. For an $\mathrm{X}(\mathfrak{osp}_{1|2n})$-module V set

$$V^{+} = \{\eta \in V \mid t_{1j}(u)\,\eta = 0 \quad \text{for} \quad j > 1 \quad \text{and} \quad t_{i1'}(u)\,\eta = 0 \quad \text{for} \quad i < 1'\}. \tag{3.1}$$

Proposition 3.1. *The subspace V^{+} is stable under the action of the operators $t_{ij}(u)$ subject to $2 \leqslant i,j \leqslant 2n$. Moreover, the assignment $\bar{t}_{ij}(u) \mapsto t_{i+1,j+1}(u)$ for $1 \leqslant i,j \leqslant 2n-1$ defines a representation of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n-2})$ on V^{+}, where the $\bar{t}_{ij}(u)$ denote the respective generating series for $\mathrm{X}(\mathfrak{osp}_{1|2n-2})$.*

Proof. Suppose that $2 \leqslant k,l \leqslant 2n$ and $j > 1$. For any $\eta \in V^{+}$ apply (2.7) to get

$$t_{1j}(u)t_{kl}(u)\eta = \frac{1}{u-v-\kappa}\,\delta_{lj'}(-1)^{\bar{j}+\bar{k}+\bar{j}\bar{k}}\theta_j\, t_{k1'}(v)\, t_{11}(u)\eta.$$

Another application of (2.7) yields

$$t_{k1'}(v)\, t_{11}(u)\eta = -[t_{11}(u), t_{k1'}(v)]\,\eta = \frac{1}{u-v-\kappa}\, t_{k1'}(v)\, t_{11}(u)\eta,$$

implying $t_{1j}(u)t_{kl}(u)\eta = 0$. A similar calculation shows that $t_{i1'}(u)t_{kl}(u)\eta = 0$ for $i < 1'$ thus proving the first part of the proposition.

Now suppose that $2 \leqslant i,j,k,l \leqslant 2n$. By (2.7) the super-commutator $[t_{ij}(u), t_{kl}(v)]$ of the operators in V^{+} equals

$$\begin{aligned}
&\frac{1}{u-v}\Big(t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u)\Big)(-1)^{\bar{\imath}\bar{j}+\bar{\imath}\bar{k}+\bar{j}\bar{k}}\\
&-\frac{1}{u-v-\kappa}\Big(\delta_{ki'}\sum_{p=2}^{2n} t_{pj}(u)\, t_{p'l}(v)(-1)^{\bar{\imath}+\bar{\imath}\bar{j}+\bar{j}\bar{p}}\,\theta_i\theta_p\\
&\qquad\qquad - \delta_{lj'}\sum_{p=2}^{2n} t_{kp'}(v)\, t_{ip}(u)(-1)^{\bar{j}+\bar{p}+\bar{\imath}\bar{k}+\bar{j}\bar{k}+\bar{\imath}\bar{p}}\,\theta_j\theta_p\Big)
\end{aligned}$$

plus the additional terms

$$-\frac{1}{u-v-\kappa}\Big(\delta_{ki'}\, t_{1j}(u)\, t_{1'l}(v)(-1)^{\bar{\imath}+\bar{\imath}\bar{j}+\bar{j}}\,\theta_i + \delta_{lj'}\, t_{k1'}(v)\, t_{i1}(u)(-1)^{\bar{j}+\bar{\imath}\bar{k}+\bar{j}\bar{k}+\bar{\imath}}\,\theta_j\Big).$$

To transform these terms, use (2.7) again to get the relations

$$\begin{aligned}
t_{1j}(u)\, t_{1'l}(v) = &\frac{1}{u-v-\kappa-1}\sum_{p=2}^{2n} t_{pj}(u)\, t_{p'l}(v)(-1)^{\bar{j}+\bar{j}\bar{p}}\,\theta_p\\
&-\frac{1}{u-v-\kappa-1}\,\delta_{lj'}\, t_{1'1'}(v)t_{11}(u)\,\theta_j
\end{aligned}$$

and

$$t_{k\,1'}(v)\, t_{i1}(u) = [t_{i1}(u)\, t_{k\,1'}(v)](-1)^{\bar{\imath}+\bar{k}+\bar{\imath}\bar{k}} = \frac{1}{u-v-\kappa-1}\, \delta_{ki'}\, t_{11}(u)\, t_{1'1'}(v)(-1)^{\bar{\imath}}\, \theta_i$$
$$- \frac{1}{u-v-\kappa-1} \sum_{p=2}^{2n} t_{k\,p'}(v)\, t_{ip}(u)(-1)^{\bar{\imath}+\bar{p}+\bar{\imath}\bar{p}}\, \theta_p.$$

Now combine the expressions together and observe that the actions of the operators $t_{11}(u)$ and $t_{1'1'}(v)$ in V^+ commute. Taking into account the change of the value $\kappa \mapsto \kappa + 1$ for the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n-2})$, we find that the formula for the super-commutator $[t_{ij}(u), t_{kl}(v)]$ agrees with the defining relations of $\mathrm{X}(\mathfrak{osp}_{1|2n-2})$. □

Remark 3.2. The reduction property of Proposition 3.1 should be related to a super-version of the embedding theorem for the orthogonal and symplectic Yangians proven in [6, Thm 3.1]. The arguments of that paper should apply to the super-case to lead to a Drinfeld-type presentation of the Yangians $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ extending the work [2]. □

A representation V of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ is called a *highest weight representation* if there exists a nonzero vector $\xi \in V$ such that V is generated by ξ,

$$\begin{aligned} t_{ij}(u)\, \xi &= 0 \qquad &&\text{for} \quad 1 \leqslant i < j \leqslant 2n+1, \qquad \text{and} \\ t_{ii}(u)\, \xi &= \lambda_i(u)\, \xi \qquad &&\text{for} \quad i = 1, \dots, 2n+1, \end{aligned} \tag{3.2}$$

for some formal series

$$\lambda_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]. \tag{3.3}$$

The vector ξ is called the *highest vector* of V.

Proposition 3.3. *The series $\lambda_i(u)$ associated with a highest weight representation V satisfy the consistency conditions*

$$\lambda_i(u)\, \lambda_{i'}(u + n - i + 1/2) = \lambda_{i+1}(u)\, \lambda_{(i+1)'}(u + n - i + 1/2) \tag{3.4}$$

for $i = 1, \dots, n$. Moreover, the coefficients of the series $c(u)$ act in the representation V as the multiplications by scalars determined by

$$c(u) \mapsto \lambda_1(u)\, \lambda_{1'}(u + n + 1/2).$$

Proof. To derive the consistency conditions, we will use the induction on n with the base case $n = 1$ already considered in [8]. Suppose that $n \geqslant 2$ and introduce the subspace V^+ by (3.1). The vector ξ belongs to V^+, and applying Proposition 3.1 we find that the cyclic span $\mathrm{X}(\mathfrak{osp}_{1|2n-2})\, \xi$ is a highest weight submodule with the highest weight $(\lambda_2(u), \dots, \lambda_{2'}(u))$. By the induction hypothesis, this implies conditions (3.4) with $i = 2, \dots, n$. Furthermore, using the defining relations (2.7), we get

$$t_{12}(u)\, t_{1'2'}(v)\, \xi = \frac{1}{u-v-\kappa} \Big(t_{12}(u)\, t_{1'2'}(v) - \lambda_1(u)\, \lambda_{1'}(v) + \lambda_2(u)\, \lambda_{2'}(v) \Big)\, \xi$$

and so

$$(u-v-\kappa-1)\, t_{12}(u)\, t_{1'2'}(v)\, \xi = \Big(-\lambda_1(u)\, \lambda_{1'}(v) + \lambda_2(u)\, \lambda_{2'}(v)\Big)\, \xi.$$

Setting $v = u-\kappa-1 = u+n-1/2$ we obtain (3.4) for $i=1$. Finally, the last part of the proposition is obtained by using the expression for $c(u)$ implied by taking the $(1',1')$ entry in the matrix relation (2.4). □

As Proposition 3.3 shows, the series $\lambda_i(u)$ in (3.2) with $i > n+1$ are uniquely determined by the first $n+1$ series. The corresponding $(n+1)$-tuple $\lambda(u) = (\lambda_1(u),\dots,\lambda_{n+1}(u))$ is called the *highest weight* of V.

Given an arbitrary $(n+1)$-tuple $\lambda(u) = (\lambda_1(u),\dots,\lambda_{n+1}(u))$ of formal series of the form (3.3), introduce the series $\lambda_i(u)$ with $i = n+2,\dots,2n+1$ to satisfy the consistency conditions (3.4). Define the *Verma module* $M(\lambda(u))$ as the quotient of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ by the left ideal generated by all coefficients of the series $t_{ij}(u)$ with $1 \leqslant i < j \leqslant 2n+1$, and $t_{ii}(u) - \lambda_i(u)$ for $i = 1,\dots,2n+1$. As in [3, Prop. 5.14], the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ implies that the Verma module $M(\lambda(u))$ is nonzero, and we denote by $L(\lambda(u))$ its irreducible quotient.

Proposition 3.4. *Every finite-dimensional irreducible representation of the algebra* $\mathrm{X}(\mathfrak{osp}_{1|2n})$ *is isomorphic to* $L(\lambda(u))$ *for a certain highest weight* $\lambda(u) = (\lambda_1(u),\dots,\lambda_{n+1}(u))$.

Proof. The argument is essentially the same as for the proof of the corresponding counterparts of the property for the Yangians associated with Lie algebras; cf. [3, Thm 5.1], [7, Sec. 3.2]. We online some key steps.

Suppose that V is a finite-dimensional irreducible representation of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ and introduce its subspace V^0 by

$$V^0 = \{\eta \in V \mid t_{ij}(u)\, \eta = 0, \qquad 1 \leqslant i < j \leqslant 2n+1\}.$$

First we note that V^0 is nonzero, which follows by considering the set of weights of V, regarded as an $\mathfrak{osp}_{1|2n}$-module defined via the embedding (2.9). This set is finite and hence contains a maximal weight with respect to the standard partial ordering on the set of weights of V. A weight vector with this weight belongs to V^0.

Furthermore, we show that V^0 is stable under the action of all operators $t_{ii}(u)$. This follows by straightforward calculations similar to those used in the proof of Proposition 3.1, relying on the defining relations (2.7). In a similar way, we verify that all the operators $t_{ii}(u)$ with $i = 1,\dots,2n+1$ form a commuting family of operators on V^0. Hence they have a simultaneous eigenvector $\xi \in V^0$. Since the representation V is irreducible, the submodule $\mathrm{X}(\mathfrak{osp}_{1|2n})\xi$ must coincide with V thus proving that V is a highest weight module.

By considering the $\mathfrak{osp}_{1|2n}$-weights of V we can also conclude that the highest vector ξ of V is determined uniquely, up to a constant factor. □

Proposition 3.4 yields the first part of the Main Theorem. Our next step is to show that the conditions in the theorem are necessary for the representation $L(\lambda(u))$ to be finite-dimensional. So we now suppose that $\dim L(\lambda(u)) < \infty$ and argue by induction on n. The conditions (1.1) in the base case $n = 1$ are implied by the main result of [8]. Suppose further that $n \geqslant 2$.

Recall that the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ for the general linear Lie algebra $\mathfrak{gl}_n$ is defined as a unital associative algebra with countably many generators $t_{ij}^{(1)\circ},\ t_{ij}^{(2)\circ},\dots$ where $1\leqslant i,j\leqslant n$, and the defining relations

$$(u-v)\,[t^{\circ}_{ij}(u),t^{\circ}_{kl}(v)]=t^{\circ}_{kj}(u)\,t^{\circ}_{il}(v)-t^{\circ}_{kj}(v)\,t^{\circ}_{il}(u)$$

written in terms of the series

$$t^{\circ}_{ij}(u)=\delta_{ij}+t_{ij}^{(1)\circ}u^{-1}+t_{ij}^{(2)\circ}u^{-2}+\dots\in\mathrm{Y}(\mathfrak{gl}_n)[[u^{-1}]];$$

see [7] for a detailed exposition of the algebraic structure and representation of the Yangians associated with $\mathfrak{gl}_n$. The Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ can be regarded as a subalgebra of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ via the embedding

$$\mathrm{Y}(\mathfrak{gl}_n)\hookrightarrow\mathrm{X}(\mathfrak{osp}_{1|2n}),\qquad t^{\circ}_{ij}(u)\mapsto t_{ij}(-u)\quad\text{for}\quad 1\leqslant i,j\leqslant n. \tag{3.5}$$

The cyclic span $\mathrm{Y}(\mathfrak{gl}_n)\xi\subset L(\lambda(u))$ is a highest weight module over $\mathrm{Y}(\mathfrak{gl}_n)$. Its highest weight is the n-tuple $(\lambda_1(-u),\dots,\lambda_n(-u))$. Since $\dim L(\lambda(u))<\infty$, the corresponding conditions for finite-dimensional highest weight representations of $\mathrm{Y}(\mathfrak{gl}_n)$ must be satisfied; see [7, Sec. 3.4]. This implies conditions (1.1) of the Main Theorem for $i=1,\dots,n-1$.

Furthermore, by Proposition 3.1, the subspace $L(\lambda(u))^+$ is a module over the extended Yangian $\mathrm{X}(\mathfrak{osp}_{1|2n-2})$. The vector ξ generates a highest weight $\mathrm{X}(\mathfrak{osp}_{1|2n-2})$-module with the highest weight $(\lambda_2(u),\dots,\lambda_{n+1}(u))$. Since this module is finite-dimensional, conditions (1.1) hold for $i=2,\dots,n$ by the induction hypothesis. This completes the proof of the necessity of the conditions.

Now suppose that conditions (1.1) hold and derive that the corresponding module $L(\lambda(u))$ is finite-dimensional. The n-tuple of Drinfeld polynomials $(P_1(u),\dots,P_n(u))$ determines the highest weight $\lambda(u)$ up to a simultaneous multiplication of all components $\lambda_i(u)$ by a series $f(u)\in 1+u^{-1}\mathbb{C}[[u^{-1}]]$. This operation corresponds to twisting the action of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ on $L(\lambda(u))$ by the automorphism (2.5). Hence, it suffices to prove that a particular module $L(\lambda(u))$ corresponding to a given set of Drinfeld polynomials is finite-dimensional.

Suppose that $L(\lambda(u))$ and $L(\mu(u))$ are the irreducible highest weight modules with the highest weights

$$\lambda(u)=\big(\lambda_1(u),\dots,\lambda_{n+1}(u)\big)\qquad\text{and}\qquad\mu(u)=\big(\mu_1(u),\dots,\mu_{n+1}(u)\big).$$

By the coproduct rule (2.10), the cyclic span $\mathrm{X}(\mathfrak{osp}_{1|2n})(\xi\otimes\xi')$ of the tensor product of the respective highest vectors of $L(\lambda(u))$ and $L(\mu(u))$ is a highest weight module with the highest weight

$$\big(\lambda_1(u)\,\mu_1(u),\dots,\lambda_{n+1}(u)\,\mu_{n+1}(u)\big).$$

This observation implies that the cyclic span corresponds to the set of Drinfeld polynomials $(P_1(u)Q_1(u),\dots,P_n(u)Q_n(u))$, where the $P_i(u)$ and $Q_i(u)$ are the Drinfeld polynomials for $L(\lambda(u))$ and $L(\mu(u))$, respectively. Therefore, we only need to establish the sufficiency of conditions (1.1) for the *fundamental representations* of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ associated with the n-tuples of

Drinfeld polynomials such that $P_j(u) = 1$ for all $j \neq i$ and $P_i(u) = u + b$ for a certain $i \in \{1, \dots, n\}$ and $b \in \mathbb{C}$; cf. [4]. Moreover, it is sufficient to take one particular value of $b \in \mathbb{C}$; the general case will then follow by twisting the action of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ in such representations by automorphisms of the form (2.8).

Consider the vector representation of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ on $\mathbb{C}^{1|2n}$ defined by

$$t_{ij}(u) \mapsto \delta_{ij} + u^{-1} e_{ij}(-1)^{\bar{i}} - (u+\kappa)^{-1} e_{j'i'}(-1)^{\bar{i}\bar{j}}\,\theta_i\theta_j. \tag{3.6}$$

The homomorphism property follows from (2.3) by applying the standard transposition to one copy of $\operatorname{End}\mathbb{C}^{1|2n}$ in the Yang–Baxter equation satisfied by $R(u)$. Now use the coproduct (2.10) and suitable automorphisms (2.8) to equip the tensor product space $(\mathbb{C}^{1|2n})^{\otimes k}$ with the action of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ by setting

$$t_{ij}(u) \mapsto \sum_{a_1,\dots,a_{k-1}=1}^{2n+1} t_{ia_1}(u) \otimes t_{a_1a_2}(u-1) \otimes \dots \otimes t_{a_{k-1}j}(u-k+1), \tag{3.7}$$

where the generators act in the respective copies of the vector space $\mathbb{C}^{1|2n}$ via the rule (3.6). For the values $k = 1, \dots, n$ introduce the vectors

$$\xi_k = \sum_{\sigma\in\mathfrak{S}_k} \operatorname{sgn}\sigma \cdot e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(k)} \in (\mathbb{C}^{1|2n})^{\otimes k}.$$

Now verify that each vector ξ_k has the properties

$$t_{ij}(u)\,\xi_k = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n+1 \tag{3.8}$$

and

$$t_{ii}(u)\,\xi_k = \begin{cases} \dfrac{u-k}{u-k+1}\,\xi_k & \text{for} \quad i = 1, \dots, k, \\ \xi_k & \text{for} \quad i = k+1, \dots, n+1. \end{cases} \tag{3.9}$$

The expression for the vector ξ_k involves only tensor products of the basis vectors e_i with $i \leqslant n$. This implies that for the application of the operators $t_{ij}(u)$ with $1 \leqslant i \leqslant j \leqslant n$ to ξ_k we may restrict the sum in formula (3.7) to the values $a_p \in \{1, \dots, n\}$.

By using the embedding (3.5), we may regard the cyclic span $\mathrm{Y}(\mathfrak{gl}_n)\xi_k$ as a $\mathrm{Y}(\mathfrak{gl}_n)$-module. Moreover, this module is isomorphic to $A^{(k)}(\mathbb{C}^n)^{\otimes k}$, where $A^{(k)}$ is the anti-symmetrization operator. It is well-known that this $\mathrm{Y}(\mathfrak{gl}_n)$-module is isomorphic to the evaluation module $L(1, \dots, 1, 0, \dots, 0)$ (with k ones) twisted by a shift automorphism $u \mapsto u + k - 1$; see e.g. [7, Sec. 6.5]. This yields formulas (3.8) and (3.9) with $1 \leqslant i \leqslant j \leqslant n$. They are easily verified directly for the remaining generators.

Formulas (3.9) show that the corresponding set of Drinfeld polynomials for the highest weight module $\mathrm{X}(\mathfrak{osp}_{1|2n})\xi_k$ has the form $P_i(u) = 1$ for $i \neq k$, while $P_k(u) = u - k$. This completes the proof of the second part of the Main Theorem concerning conditions (1.1). The last part is immediate from the decomposition (2.6); cf. [3, Sec. 5.3].

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