Yangian characters and classical $\mathcal{W}$-algebras

A. I. Molev and E. E. Mukhin

Abstract

The Yangian characters (or q-characters) are known to be closely related to the classical $\mathcal{W}$-algebras and to the centers of the affine vertex algebras at the critical level. We make this relationship more explicit by producing families of generators of the $\mathcal{W}$-algebras from the characters of the Kirillov–Reshetikhin modules associated with multiples of the first fundamental weight in types B and D and of the fundamental modules in type C. We also give an independent derivation of the character formulas for these representations in the context of the RTT presentation of the Yangians. In all cases the generators of the $\mathcal{W}$-algebras correspond to the recently constructed elements of the Feigin–Frenkel centers via an affine version of the Harish-Chandra isomorphism.

School of Mathematics and Statistics
University of Sydney, NSW 2006, Australia
alexander.molev@sydney.edu.au

Department of Mathematical Sciences
Indiana University – Purdue University Indianapolis
402 North Blackford St, Indianapolis, IN 46202-3216, USA
mukhin@math.iupui.edu

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1 Introduction

1.1. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Recall that the *Harish-Chandra homomorphism*

$$\mathrm{U}(\mathfrak{g})^{\mathfrak{h}} \to \mathrm{U}(\mathfrak{h}) \tag{1.1}$$

is the projection of the $\mathfrak{h}$-centralizer $\mathrm{U}(\mathfrak{g})^{\mathfrak{h}}$ in the universal enveloping algebra to $\mathrm{U}(\mathfrak{h})$ whose kernel is the two-sided ideal $\mathrm{U}(\mathfrak{g})^{\mathfrak{h}} \cap \mathrm{U}(\mathfrak{g})\mathfrak{n}_+$. The restriction of the homomorphism (1.1) to the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$ yields an isomorphism

$$\mathrm{Z}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{h})^W \tag{1.2}$$

called the *Harish-Chandra isomorphism*, where $\mathrm{U}(\mathfrak{h})^W$ denotes the subalgebra of invariants in $\mathrm{U}(\mathfrak{h})$ with respect to an action of the Weyl group W of $\mathfrak{g}$; see e.g. [7, Sec. 7.4].

In this paper we will be concerned with an affine version of the isomorphism (1.2). Consider the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ which is the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}\,[t, t^{-1}] \oplus \mathbb{C}K,$$

where $\mathfrak{g}[t, t^{-1}]$ is the Lie algebra of Laurent polynomials in t with coefficients in $\mathfrak{g}$. We have a natural analogue of the homomorphism (1.1),

$$\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{h}} \to \mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big). \tag{1.3}$$

The vacuum module $V_{-h^\vee}(\mathfrak{g})$ at the critical level over $\widehat{\mathfrak{g}}$ is defined as the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}})$ by the left ideal generated by $\mathfrak{g}[t]$ and $K + h^\vee$, where $h^\vee$ denotes the dual Coxeter number for $\mathfrak{g}$. The vacuum module $V_{-h^\vee}(\mathfrak{g})$ possesses a vertex algebra structure; see e.g. [11, Ch. 2]. The *center* of this vertex algebra is defined by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{S \in V_{-h^\vee}(\mathfrak{g}) \mid \mathfrak{g}[t]\, S = 0\},$$

its elements are called *Segal–Sugawara vectors*. The center is a commutative associative algebra which can be regarded as a commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{h}}$. By the results of Feigin and Frenkel [10], $\mathfrak{z}(\widehat{\mathfrak{g}})$ is an algebra of polynomials in infinitely many variables and the restriction of the homomorphism (1.3) to the subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ yields an isomorphism

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \to \mathcal{W}({}^L\mathfrak{g}), \tag{1.4}$$

where $\mathcal{W}({}^L\mathfrak{g})$ is the *classical $\mathcal{W}$-algebra* associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$; see [11] for a detailed exposition of these results. The $\mathcal{W}$-algebra $\mathcal{W}({}^L\mathfrak{g})$ can be defined as a subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big)$ which consists of the elements annihilated by the *screening operators*; see Sec. 4 below.

Recently, explicit generators of the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ were constructed for the Lie algebras $\mathfrak{g}$ of all classical types A, B, C and D; see [5], [6] and [24]. Our aim in this paper is to describe the Harish-Chandra images of these generators in types B, C and D. The corresponding results in type A are given in [5]; we also reproduce them below in a slightly different form (as in [5], we work with the reductive Lie algebra $\mathfrak{gl}_N$ rather than the simple Lie algebra $\mathfrak{sl}_N$ of type A). The images of the generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$ under the isomorphism (1.4) turn out to be elements of the $\mathcal{W}$-algebra $\mathcal{W}({}^L\mathfrak{g})$ written in terms of noncommutative analogues of the complete and elementary symmetric functions.

In more detail, for any $X \in \mathfrak{g}$ and $r \in \mathbb{Z}$ introduce the corresponding elements of the loop algebra $\mathfrak{g}[t, t^{-1}]$ by $X[r] = X t^r$. The extended Lie algebra $\widehat{\mathfrak{g}} \oplus \mathbb{C}\tau$ with $\tau = -d/dt$ is defined by the commutation relations

$$\big[\tau, X[r]\big] = -r\, X[r-1], \qquad \big[\tau, K\big] = 0. \tag{1.5}$$

Consider the natural extension of (1.4) to the isomorphism

$$\chi : \mathfrak{z}(\widehat{\mathfrak{g}}) \otimes \mathbb{C}[\tau] \to \mathcal{W}({}^L\mathfrak{g}) \otimes \mathbb{C}[\tau], \tag{1.6}$$

which is identical on $\mathbb{C}[\tau]$; see Sec. 5 for the definition of χ.

1.2. First let $\mathfrak{g} = \mathfrak{gl}_N$ be the general linear Lie algebra with the standard basis elements E_{ij}, $1 \leqslant i, j \leqslant N$. For each $a \in \{1, \dots, m\}$ introduce the element $E[r]_a$ of the algebra

$$\underbrace{\operatorname{End} \mathbb{C}^N \otimes \ldots \otimes \operatorname{End} \mathbb{C}^N}_{m} \otimes \mathrm{U} \tag{1.7}$$

by

$$E[r]_a = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes E_{ij}[r], \tag{1.8}$$

where the e_{ij} are the standard matrix units and U stands for the universal enveloping algebra of $\widehat{\mathfrak{gl}}_N \oplus \mathbb{C}\tau$. Let $H^{(m)}$ and $A^{(m)}$ denote the respective images of the symmetrizer and anti-symmetrizer in the group algebra for the symmetric group $\mathfrak{S}_m$ under its natural action on $(\mathbb{C}^N)^{\otimes m}$; see (2.7). We will identify $H^{(m)}$ and $A^{(m)}$ with the elements $H^{(m)} \otimes 1$ and $A^{(m)} \otimes 1$ of the algebra (1.7). Define the elements $\phi_{m\, a}, \psi_{m\, a} \in \mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)$ by the expansions

$$\operatorname{tr} A^{(m)} \big(\tau + E[-1]_1\big) \ldots \big(\tau + E[-1]_m\big) = \phi_{m\,0}\, \tau^m + \phi_{m\,1}\, \tau^{m-1} + \cdots + \phi_{m\,m}, \tag{1.9}$$

$$\operatorname{tr} H^{(m)} \big(\tau + E[-1]_1\big) \ldots \big(\tau + E[-1]_m\big) = \psi_{m\,0}\, \tau^m + \psi_{m\,1}\, \tau^{m-1} + \cdots + \psi_{m\,m}, \tag{1.10}$$

where the traces are taken over all m copies of $\operatorname{End} \mathbb{C}^N$. The results of [5] and [6] imply that all elements $\phi_{m\,a}$ and $\psi_{m\,a}$, as well as the coefficients of the polynomials $\operatorname{tr}\,(\tau + E[-1])^m$

belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$; see also [25] for a simpler proof. Moreover, each of the families

$$\phi_{11}, \dots, \phi_{NN} \qquad \text{and} \qquad \psi_{11}, \dots, \psi_{NN}$$

is a *complete set of Segal–Sugawara vectors* in the sense that the elements of each family together with their images under all positive powers of the translation operator $T = \operatorname{ad} \tau$ are algebraically independent and generate $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$.

The elements $\mu_i = E_{ii}$ with $i = 1, \dots, N$ span a Cartan subalgebra of $\mathfrak{gl}_N$. Elements of the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{gl}_N)$ are regarded as polynomials in the $\mu_i[r]$ with $r < 0$. A calculation of the images of the polynomials (1.9) with $m = N$ and $\operatorname{tr} (\tau + E[-1])^m$ under the isomorphism (1.6) was given in [5]. The same method applies to all polynomials (1.9) and (1.10) to yield the formulas

$$\chi : \operatorname{tr} A^{(m)} \big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_m\big) \mapsto e_m\big(\tau + \mu_1[-1], \dots, \tau + \mu_N[-1]\big), \qquad (1.11)$$

$$\chi : \operatorname{tr} H^{(m)} \big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_m\big) \mapsto h_m\big(\tau + \mu_1[-1], \dots, \tau + \mu_N[-1]\big), \qquad (1.12)$$

where we use standard noncommutative versions of the complete and elementary symmetric functions in the ordered variables $x_1, \dots, x_p$ defined by the respective formulas

$$h_m(x_1, \dots, x_p) = \sum_{i_1 \leqslant \dots \leqslant i_m} x_{i_1} \dots x_{i_m}, \qquad (1.13)$$

$$e_m(x_1, \dots, x_p) = \sum_{i_1 > \dots > i_m} x_{i_1} \dots x_{i_m}. \qquad (1.14)$$

Relations (1.11) and (1.12) can also be derived from the Yangian character formulas as we indicate below; see Secs 3.1 and 5.

1.3. Now turn to the Lie algebras of types B, C and D and let $\mathfrak{g} = \mathfrak{g}_N$ be the orthogonal Lie algebra $\mathfrak{o}_N$ (with $N = 2n$ or $N = 2n+1$) or the symplectic Lie algebra $\mathfrak{sp}_N$ (with $N = 2n$). We will use the elements $F_{ij}[r]$ of the loop algebra $\mathfrak{g}_N[t, t^{-1}]$, where the F_{ij} are standard generators of $\mathfrak{g}_N$; see Sec. 2.2 for the definitions. For each $a \in \{1, \dots, m\}$ introduce the element $F[r]_a$ of the algebra (1.7) by

$$F[r]_a = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes F_{ij}[r], \qquad (1.15)$$

where U in (1.7) now stands for the universal enveloping algebra of $\widehat{\mathfrak{g}}_N \oplus \mathbb{C}\tau$. We let $S^{(m)}$ denote the element of the algebra (1.7) which is the image of the symmetrizer of the Brauer algebra $\mathcal{B}_m(\omega)$ under its natural action on $(\mathbb{C}^N)^{\otimes m}$, where the parameter ω should be specialized to N or $-N$ in the orthogonal and symplectic case, respectively. The component of $S^{(m)}$ in U is the identity; see (2.18) and (2.19) below for explicit formulas. We will use the notation

$$\gamma_m(\omega) = \frac{\omega + m - 2}{\omega + 2m - 2} \tag{1.16}$$

and define the elements $\phi_{m\,a} \in \mathrm{U}\big(t^{-1}\mathfrak{g}_N[t^{-1}]\big)$ by the expansion

$$\gamma_m(\omega)\,\mathrm{tr}\, S^{(m)}\big(\tau + F[-1]_1\big)\dots\big(\tau + F[-1]_m\big) = \phi_{m\,0}\,\tau^m + \phi_{m\,1}\,\tau^{m-1} + \dots + \phi_{m\,m}, \tag{1.17}$$

where the trace is taken over all m copies of $\mathrm{End}\,\mathbb{C}^N$ (we included the constant factor (1.16) to get a uniform expression in all cases). By the main result of [24], all coefficients $\phi_{m\,a}$ belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}}_N)$. Note that in the symplectic case $\mathfrak{g}_N = \mathfrak{sp}_{2n}$ the values of m were restricted to $1 \leqslant m \leqslant 2n$, but the result and arguments also extend to $m = 2n+1$; see [24, Sec. 3.3]. In the even orthogonal case $\mathfrak{g}_N = \mathfrak{o}_{2n}$ there is an additional element $\phi'_n = \mathrm{Pf}\,\widetilde{F}[-1]$ of the center defined as the (noncommutative) Pfaffian of the skew-symmetric matrix $\widetilde{F}[-1] = [\widetilde{F}_{ij}[-1]]$,

$$\mathrm{Pf}\,\widetilde{F}[-1] = \frac{1}{2^n n!} \sum_{\sigma\in\mathfrak{S}_{2n}} \mathrm{sgn}\,\sigma \cdot \widetilde{F}_{\sigma(1)\,\sigma(2)}[-1]\dots\widetilde{F}_{\sigma(2n-1)\,\sigma(2n)}[-1], \tag{1.18}$$

where $\widetilde{F}_{ij}[-1] = F_{ij'}[-1]$ with $i' = 2n - i + 1$. The family $\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n\,2n}$ is a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n+1}$ and $\mathfrak{sp}_{2n}$, while $\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n-2\,2n-2}, \phi'_n$ is a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n}$.

The Lie algebras $\mathfrak{o}_{2n+1}$ and $\mathfrak{sp}_{2n}$ are Langlands dual to each other, while $\mathfrak{o}_{2n}$ is self-dual. In all the cases we denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}_N$ spanned by the elements $\mu_i = F_{ii}$ with $i = 1, \dots, n$ and identify it with the Cartan subalgebra of ${}^L\mathfrak{g}_N$ spanned by the elements with the same names. We let $\mu_i[r] = \mu_i\, t^r$ with $r < 0$ and $i = 1, \dots, n$ denote the basis elements of the vector space $t^{-1}\mathfrak{h}[t^{-1}]$ so that the elements of the classical $\mathcal{W}$-algebra $\mathcal{W}({}^L\mathfrak{g}_N)$ are regarded as polynomials in the $\mu_i[r]$.

Main Theorem. *The image of the polynomial* (1.17) *under the isomorphism* (1.6) *is given by the formula*:

type B_n: $$h_m\big(\tau + \mu_1[-1], \dots, \tau + \mu_n[-1], \tau - \mu_n[-1], \dots \tau - \mu_1[-1]\big),$$

type D_n: $$\tfrac{1}{2}\, h_m\big(\tau + \mu_1[-1], \dots, \tau + \mu_{n-1}[-1], \tau - \mu_n[-1], \dots \tau - \mu_1[-1]\big)$$
$$+ \tfrac{1}{2}\, h_m\big(\tau + \mu_1[-1], \dots, \tau + \mu_n[-1], \tau - \mu_{n-1}[-1], \dots \tau - \mu_1[-1]\big),$$

type C_n: $$e_m\big(\tau + \mu_1[-1], \dots, \tau + \mu_n[-1], \tau, \tau - \mu_n[-1], \dots \tau - \mu_1[-1]\big).$$

Moreover, the image of the element ϕ'_n in type D_n is given by

$$\big(\mu_1[-1] - \tau\big)\dots\big(\mu_n[-1] - \tau\big)\,1. \tag{1.19}$$

In the last relation τ is understood as the differentiation operator so that $\tau\,1 = 0$.

1.4. The Fourier coefficients of the image of any element of the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ under the state-field correspondence map are well-defined operators (called the *Sugawara operators*) on the Wakimoto modules over $\widehat{\mathfrak{g}}$. These operators act by multiplication by scalars which are determined by the Harish-Chandra image under the isomorphism (1.4); see [11, Ch. 8]. Therefore, the Main Theorem yields explicit formulas for the eigenvalues of a family of the (higher) Sugawara operators in the Wakimoto modules.

Our approach is based on the theory of characters originated in [18] in the Yangian context and in [14] in the context of quantum affine algebras (the latter are commonly known as the q-characters). The theory was further developed in [12] where an algorithm for the calculation of the q-characters was proposed, while conjectures for functional relations satisfied by the q-characters were proved in [15] and [32]. An extensive review of the role of the q-characters in classical and quantum integrable systems is given in [19]; see also earlier papers [20], [22], [30] and [31] where some formulas concerning the representations we dealing with in this paper had been conjectured and studied. In recent work [28], [29] the q-characters have been calculated for a wide class of representations in type B, and associated *extended T-systems* have been introduced.

Due to the general results on the connection of the q-characters with the Feigin–Frenkel center and the classical $\mathcal{W}$-algebras described in [14, Sec. 8.5], one could expect that the character formulas would be useful for the calculation of the Harish-Chandra images of the coefficients of the polynomial (1.17). Indeed, as we demonstrate below, the images in the classical $\mathcal{W}$-algebra are closely related with the top degree components of some linear combinations of the q-characters.

We now briefly describe the contents of the paper. We start by proving the character formulas for some classes of representations of the Yangian $\mathrm{Y}(\mathfrak{g}_N)$ associated with the Lie algebra $\mathfrak{g}_N$ (Sec. 2). To this end we employ realizations of the representations in harmonic tensors and construct special bases of the representation spaces. The main calculation is given in Sec. 3, where we consider particular linear combinations of the Yangian characters and calculate their top degree terms as elements of the associated graded algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{g}_N) \cong \mathrm{U}(\mathfrak{g}_N[t])$. In Sec. 4 we recall the definition of the classical $\mathcal{W}$-algebras and write explicit screening operators in all classical types. By translating the results of Sec. 3 to the universal enveloping algebra $\mathrm{U}(t^{-1}\mathfrak{g}_N[t^{-1}])$ we will be able to get them in the form provided by the Main Theorem (Sec. 5). Finally, in Sec 6 we apply our results to get the Harish-Chandra images of the Casimir elements for the Lie algebras $\mathfrak{g}_N$ arising from the Brauer–Schur–Weyl duality. We show that our formulas are equivalent to those previously found in [16].

2 Characters of Yangian representations

2.1 Yangian for $\mathfrak{gl}_N$

Denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{gl}_N$ spanned by the basis elements $E_{11}, \dots, E_{NN}$. The highest weights of representations of $\mathfrak{gl}_N$ will be considered with respect to this basis, and the highest vectors will be assumed to be annihilated by the action of the elements E_{ij} with $1 \leqslant i < j \leqslant N$, unless stated otherwise.

Recall the RTT-presentation of the Yangian associated with the Lie algebra $\mathfrak{gl}_N$; see e.g. [23, Ch. 1]. For $1 \leqslant a < b \leqslant m$ introduce the elements P_{ab} of the tensor product algebra

$$\underbrace{\operatorname{End}\mathbb{C}^N \otimes \dots \otimes \operatorname{End}\mathbb{C}^N}_{m} \tag{2.1}$$

by

$$P_{ab} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{ji} \otimes 1^{\otimes(m-b)}. \tag{2.2}$$

The Yang R-matrix $R_{12}(u)$ is a rational function in a complex parameter u with values in the tensor product algebra $\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N$ defined by

$$R_{12}(u) = 1 - \frac{P_{12}}{u}.$$

This function satisfies the Yang–Baxter equation

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u), \tag{2.3}$$

where the subscripts indicate the copies of $\operatorname{End}\mathbb{C}^N$ in the algebra (2.1) with $m = 3$. The *Yangian* $\mathrm{Y}(\mathfrak{gl}_N)$ is an associative algebra with generators $t_{ij}^{(r)}$, where $1 \leqslant i, j \leqslant N$ and $r = 1, 2, \dots$, satisfying certain quadratic relations. To write them down, introduce the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)}\, u^{-r} \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$$

and set

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]].$$

Consider the algebra $\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$ and introduce its elements $T_1(u)$ and $T_2(u)$ by

$$T_1(u) = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes t_{ij}(u), \qquad T_2(u) = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes t_{ij}(u). \tag{2.4}$$

The defining relations for the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ can then be written in the form

$$R_{12}(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R_{12}(u-v). \tag{2.5}$$

We identify the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$ with a subalgebra of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ via the embedding $E_{ij} \mapsto t_{ij}^{(1)}$. Then $\mathrm{Y}(\mathfrak{gl}_N)$ can be regarded as a $\mathfrak{gl}_N$-module with the adjoint action. Denote by $\mathrm{Y}(\mathfrak{gl}_N)^{\mathfrak{h}}$ the subalgebra of $\mathfrak{h}$-invariants under this action. Consider the left ideal I of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ generated by all elements $t_{ij}^{(r)}$ with the conditions $1 \leqslant i < j \leqslant N$ and $r \geqslant 1$. By the Poincaré–Birkhoff–Witt theorem for the Yangian [23, Sec. 1.4], the intersection $\mathrm{Y}(\mathfrak{gl}_N)^{\mathfrak{h}} \cap I$ is a two-sided ideal of $\mathrm{Y}(\mathfrak{gl}_N)^{\mathfrak{h}}$. Moreover, the quotient of $\mathrm{Y}(\mathfrak{gl}_N)^{\mathfrak{h}}$ by this ideal is isomorphic to the commutative algebra freely generated by the images of the elements $t_{ii}^{(r)}$ with $i = 1, \dots, N$ and $r \geqslant 1$ in the quotient. We will use the notation $\lambda_i^{(r)}$ for this image of $t_{ii}^{(r)}$. Thus, we get an analogue of the Harish-Chandra homomorphism (1.1),

$$\mathrm{Y}(\mathfrak{gl}_N)^{\mathfrak{h}} \to \mathbb{C}[\lambda_i^{(r)} \,|\, i = 1, \dots, N,\ r \geqslant 1]. \tag{2.6}$$

We combine the elements $\lambda_i^{(r)}$ into the formal series

$$\lambda_i(u) = 1 + \sum_{r=1}^{\infty} \lambda_i^{(r)}\, u^{-r}, \qquad i = 1, \dots, N,$$

which can be understood as the images of the series $t_{ii}(u)$ under the homomorphism (2.6).

The *symmetrizer* $H^{(m)}$ and *anti-symmetrizer* $A^{(m)}$ in the algebra (2.1) are the operators in the tensor product space $(\mathbb{C}^N)^{\otimes m}$ associated with the corresponding idempotents in the group algebra of the symmetric group $\mathfrak{S}_m$ via its natural action on the tensor product space $(\mathbb{C}^N)^{\otimes m}$. That is,

$$H^{(m)} = \frac{1}{m!} \sum_{s \in \mathfrak{S}_m} P_s \qquad \text{and} \qquad A^{(m)} = \frac{1}{m!} \sum_{s \in \mathfrak{S}_m} \operatorname{sgn} s \cdot P_s, \tag{2.7}$$

where P_s is the element of the algebra (2.1) corresponding to $s \in \mathfrak{S}_m$. Both the symmetrizer and anti-symmetrizer admit multiplicative expressions in terms of the values of the Yang R-matrix,

$$H^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \Big(1 + \frac{P_{a\,b}}{b-a}\Big) \qquad \text{and} \qquad A^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \Big(1 - \frac{P_{a\,b}}{b-a}\Big), \tag{2.8}$$

where the products are taken in the lexicographic order on the pairs (a, b); see e.g. [23, Sec. 6.4]. The operators $H^{(m)}$ and $A^{(m)}$ project $(\mathbb{C}^N)^{\otimes m}$ to the subspaces of symmetric and skew-symmetric tensors, respectively. Both subspaces carry irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$. Consider the tensor product algebra

$$\underbrace{\operatorname{End}\mathbb{C}^N \otimes \dots \otimes \operatorname{End}\mathbb{C}^N}_{m} \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]] \tag{2.9}$$

and extend the notation (2.4) to elements of (2.9). All coefficients of the formal series

$$\operatorname{tr} H^{(m)} T_1(u)\, T_2(u+1) \dots T_m(u+m-1) \tag{2.10}$$

and

$$\operatorname{tr} A^{(m)} T_1(u)\, T_2(u-1) \dots T_m(u-m+1) \tag{2.11}$$

belong to a commutative subalgebra of the Yangian. This subalgebra is contained in $\mathrm{Y}(\mathfrak{gl}_N)^{\mathfrak{h}}$. The next proposition is well-known and easy to prove; see also [3, Sec. 7.4], [13, Sec. 4.5] and [23, Sec. 8.5] for derivations of more general formulas for the characters of the evaluation modules over $\mathrm{Y}(\mathfrak{gl}_N)$. We give a proof of the proposition to stress the similarity of the approaches for all classical types.

Proposition 2.1. *The images of the series* (2.10) *and* (2.11) *under the homomorphism* (2.6) *are given by*

$$\sum_{1\leqslant i_1\leqslant\dots\leqslant i_m\leqslant N} \lambda_{i_1}(u)\, \lambda_{i_2}(u+1) \dots \lambda_{i_m}(u+m-1) \tag{2.12}$$

and

$$\sum_{1\leqslant i_1<\dots<i_m\leqslant N} \lambda_{i_1}(u)\, \lambda_{i_2}(u-1) \dots \lambda_{i_m}(u-m+1), \tag{2.13}$$

respectively.

Proof. By relations (2.5) and (2.8) we can write the product occurring in (2.10) as

$$H^{(m)} T_1(u) \dots T_m(u+m-1) = T_m(u+m-1) \dots T_1(u)\, H^{(m)}. \tag{2.14}$$

This relation shows that the product on each side can be regarded as an operator on $(\mathbb{C}^N)^{\otimes m}$ with coefficients in the algebra $\mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$ such that the subspace $H^{(m)}(\mathbb{C}^N)^{\otimes m}$ is invariant under this operator. A basis of this subspace is comprised by vectors of the form $v_{i_1,\dots,i_m} = H^{(m)}(e_{i_1}\otimes\dots\otimes e_{i_m})$, where $i_1 \leqslant \dots \leqslant i_m$ and $e_1,\dots,e_N$ denote the canonical basis vectors of $\mathbb{C}^N$. To calculate the trace of the operator, we will find the diagonal matrix elements corresponding to the basis vectors. Applying the operator which occurs on the right hand side of (2.14) to a basis vector $v_{i_1,\dots,i_m}$ we get

$$T_m(u+m-1) \dots T_1(u)\, H^{(m)} v_{i_1,\dots,i_m} = T_m(u+m-1) \dots T_1(u) v_{i_1,\dots,i_m}.$$

The coefficient of $v_{i_1,\dots,i_m}$ in the expansion of this expression as a linear combination of the basis vectors is determined by the coefficient of the tensor $e_{i_1}\otimes\dots\otimes e_{i_m}$. Hence, a nonzero contribution to the image of the diagonal matrix element corresponding to $v_{i_1,\dots,i_m}$ under the homomorphism (2.6) only comes from the term $t_{i_m i_m}(u+m-1)\dots t_{i_1 i_1}(u)$. The sum over all basis vectors yields the resulting formula for the image of the element (2.10).

The calculation of the image of the series (2.11) is quite similar. It relies on the identity

$$A^{(m)}T_1(u)\dots T_m(u-m+1) = T_m(u-m+1)\dots T_1(u)\,A^{(m)}$$

and a calculation of the diagonal matrix elements of the operator which occurs on the right hand side on the basis vectors $A^{(m)}(e_{i_1}\otimes\dots\otimes e_{i_m})$, where $i_1<\dots<i_m$. □

2.2 Yangians for $\mathfrak{o}_N$ and $\mathfrak{sp}_N$

Throughout the paper we use the involution on the set $\{1,\dots,N\}$ defined by $i' = N-i+1$. The Lie subalgebra of $\mathfrak{gl}_N$ spanned by the elements $F_{ij} = E_{ij} - E_{j'i'}$ with $i,j\in\{1,\dots,N\}$ is isomorphic to the orthogonal Lie algebra $\mathfrak{o}_N$. Similarly, the Lie subalgebra of $\mathfrak{gl}_{2n}$ spanned by the elements $F_{ij} = E_{ij} - \varepsilon_i\varepsilon_j\,E_{j'i'}$ with $i,j\in\{1,\dots,2n\}$ is isomorphic to the symplectic Lie algebra $\mathfrak{sp}_{2n}$, where $\varepsilon_i = 1$ for $i=1,\dots,n$ and $\varepsilon_i=-1$ for $i=n+1,\dots,2n$. We will keep the notation $\mathfrak{g}_N$ for the Lie algebra $\mathfrak{o}_N$ (with $N=2n$ or $N=2n+1$) or $\mathfrak{sp}_N$ (with $N=2n$). Denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}_N$ spanned by the basis elements $F_{11},\dots,F_{nn}$. The highest weights of representations of $\mathfrak{g}_N$ will be considered with respect to this basis, and the highest vectors will be assumed to be annihilated by the action of the elements F_{ij} with $1\leqslant i<j\leqslant N$, unless stated otherwise.

Recall the RTT-presentation of the Yangian associated with the Lie algebra $\mathfrak{g}_N$ following the general approach of [8] and [34]; see also [1] and [2].

For $1\leqslant a<b\leqslant m$ consider the elements P_{ab} of the tensor product algebra (2.1) defined by (2.2). Introduce also the elements Q_{ab} of (2.1) which are defined by different formulas in the orthogonal and symplectic cases. In the orthogonal case we set

$$Q_{ab} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)}\otimes e_{ij}\otimes 1^{\otimes(b-a-1)}\otimes e_{i'j'}\otimes 1^{\otimes(m-b)},$$

and in the symplectic case

$$Q_{ab} = \sum_{i,j=1}^{N} \varepsilon_i\varepsilon_j\,1^{\otimes(a-1)}\otimes e_{ij}\otimes 1^{\otimes(b-a-1)}\otimes e_{i'j'}\otimes 1^{\otimes(m-b)}.$$

Set $\kappa = N/2-1$ in the orthogonal case and $\kappa = N/2+1$ in the symplectic case. The R-matrix $R_{12}(u)$ is a rational function in a complex parameter u with values in the tensor product algebra $\operatorname{End}\mathbb{C}^N\otimes\operatorname{End}\mathbb{C}^N$ defined by

$$R_{12}(u) = 1 - \frac{P_{12}}{u} + \frac{Q_{12}}{u-\kappa}.$$

It is well known by [36] that this function satisfies the Yang–Baxter equation (2.3).

The *Yangian* $\mathrm{Y}(\mathfrak{g}_N)$ is an associative algebra with generators $t_{ij}^{(r)}$, where $1 \leqslant i, j \leqslant N$ and $r = 1, 2, \dots$, satisfying certain quadratic relations. Introduce the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)}\, u^{-r} \in \mathrm{Y}(\mathfrak{g}_N)[[u^{-1}]]$$

and set

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{g}_N)[[u^{-1}]].$$

Consider the algebra $\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{g}_N)[[u^{-1}]]$ and introduce its elements $T_1(u)$ and $T_2(u)$ by the same formulas (2.4) as in the case of $\mathfrak{gl}_N$. The defining relations for the algebra $\mathrm{Y}(\mathfrak{g}_N)$ can then be written in the form

$$R_{12}(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R_{12}(u-v) \qquad (2.15)$$

together with the relation

$$T'(u+\kappa)\, T(u) = 1,$$

where the prime denotes the matrix transposition defined for an $N \times N$ matrix $A = [A_{ij}]$ by

$$(A')_{ij} = A_{j'i'} \qquad \text{and} \qquad (A')_{ij} = \varepsilon_i \varepsilon_j\, A_{j'i'}$$

in the orthogonal and symplectic case, respectively.

We identify the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$ with a subalgebra of the Yangian $\mathrm{Y}(\mathfrak{g}_N)$ via the embedding

$$F_{ij} \mapsto t_{ij}^{(1)}, \qquad i, j = 1, \dots, N.$$

Then $\mathrm{Y}(\mathfrak{g}_N)$ can be regarded as a $\mathfrak{g}_N$-module with the adjoint action. Denote by $\mathrm{Y}(\mathfrak{g}_N)^{\mathfrak{h}}$ the subalgebra of $\mathfrak{h}$-invariants under this action.

Consider the left ideal I of the algebra $\mathrm{Y}(\mathfrak{g}_N)$ generated by all elements $t_{ij}^{(r)}$ with the conditions $1 \leqslant i < j \leqslant N$ and $r \geqslant 1$. It follows from the Poincaré–Birkhoff–Witt theorem for the Yangian [2, Sec. 3] that the intersection $\mathrm{Y}(\mathfrak{g}_N)^{\mathfrak{h}} \cap I$ is a two-sided ideal of $\mathrm{Y}(\mathfrak{g}_N)^{\mathfrak{h}}$. Moreover, the quotient of $\mathrm{Y}(\mathfrak{g}_N)^{\mathfrak{h}}$ by this ideal is isomorphic to the commutative algebra freely generated by the images of the elements $t_{ii}^{(r)}$ with $i = 1, \dots, n$ and $r \geqslant 1$ in the quotient. We will use the notation $\lambda_i^{(r)}$ for this image of $t_{ii}^{(r)}$ and extend this notation to all values $i = 1, \dots, N$. Thus, we get an analogue of the Harish-Chandra homomorphism (1.1),

$$\mathrm{Y}(\mathfrak{g}_N)^{\mathfrak{h}} \to \mathbb{C}[\lambda_i^{(r)} \,|\, i = 1, \dots, n,\ r \geqslant 1]. \qquad (2.16)$$

We combine the elements $\lambda_i^{(r)}$ into the formal series

$$\lambda_i(u) = 1 + \sum_{r=1}^{\infty} \lambda_i^{(r)} \, u^{-r}, \qquad i = 1, \dots, N$$

which can be understood as the image of the series $t_{ii}(u)$ under the homomorphism (2.16).

It follows from [2, Prop. 5.2 and 5.14], that the series $\lambda_i(u)$ satisfy the relations

$$\lambda_i(u + \kappa - i) \, \lambda_{i'}(u) = \lambda_{i+1}(u + \kappa - i) \, \lambda_{(i+1)'}(u), \tag{2.17}$$

for $i = 0, 1, \dots, n-1$ if $\mathfrak{g}_N = \mathfrak{o}_{2n}$ or $\mathfrak{sp}_{2n}$, and for $i = 0, 1, \dots, n$ if $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$, where $\lambda_0(u) = \lambda_{0'}(u) := 1$. Under an appropriate identification, the relations (2.17) coincide with those for the q-characters, as the $\lambda_i(u)$ correspond to the "single box variables"; see for instance [19, Sec. 7] and [30, Sec. 2]. This coincidence is consistent with the general result which establishes the equivalence of the definitions of q-characters in [14] and [18]; see [12, Prop. 2.4] for a proof. The q-characters have been extensively studied; see [13], [14] and [18]. In particular, formulas for the q-characters of some classes of modules were conjectured in [20], [30] and [31] and later proved in [15] and [32]. However, this was done in the context of the new realization of the quantum affine algebras. In what follows we compute some q-characters independently in our setting of the RTT realization of the Yangians.

Introduce the element $S^{(m)}$ of the algebra (2.1) by setting $S^{(1)} = 1$ and for $m \geqslant 2$ define it by the respective formulas in the orthogonal and symplectic cases:

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \Big(1 + \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{N/2 + b - a - 1} \Big) \tag{2.18}$$

and

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \Big(1 - \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{n - b + a + 1} \Big), \tag{2.19}$$

where the products are taken in the lexicographic order on the pairs (a, b) and the condition $m \leqslant n + 1$ is assumed in (2.19). The elements (2.18) and (2.19) are the images of the symmetrizers in the corresponding Brauer algebras $\mathcal{B}_m(N)$ and $\mathcal{B}_m(-N)$ under their actions on the vector space $(\mathbb{C}^N)^{\otimes m}$. In particular, for any $1 \leqslant a < b \leqslant m$ for the operator $S^{(m)}$ we have

$$S^{(m)} \, Q_{ab} = Q_{ab} \, S^{(m)} = 0 \qquad \text{and} \qquad S^{(m)} \, P_{ab} = P_{ab} \, S^{(m)} = \pm S^{(m)} \tag{2.20}$$

with the plus and minus signs taken in the orthogonal and symplectic case, respectively. The symmetrizer admits a few other equivalent expressions which are reproduced in [24].

In the orthogonal case the operator $S^{(m)}$ projects $(\mathbb{C}^N)^{\otimes m}$ to the irreducible representation of the Lie algebra $\mathfrak{o}_N$ with the highest weight $(m, 0, \dots, 0)$. The dimension of this representation equals

$$\frac{N+2m-2}{N+m-2}\binom{N+m-2}{m}.$$

This representation is extended to the Yangian $\mathrm{Y}(\mathfrak{o}_N)$ and it is one of the Kirillov–Reshetikhin modules. In the symplectic case with $m \leqslant n$ the operator $S^{(m)}$ projects $(\mathbb{C}^{2n})^{\otimes m}$ to the subspace of skew-symmetric harmonic tensors which carries an irreducible representation of $\mathfrak{sp}_{2n}$ with the highest weight $(1, \dots, 1, 0, \dots, 0)$ (with m copies of 1). Its dimension equals

$$\frac{2n-2m+2}{2n-m+2}\binom{2n+1}{m} = \binom{2n}{m} - \binom{2n}{m-2}. \tag{2.21}$$

This representation is extended to the m-th fundamental representation of the Yangian $\mathrm{Y}(\mathfrak{sp}_{2n})$ which is also a Kirillov–Reshetikhin module. It is well-known that if $m = n+1$ then the subspace of tensors is zero so that $S^{(n+1)} = 0$.

The existence of the Yangian action on the Lie algebra modules here can be explained by the fact that the projections (2.18) and (2.19) are the products of the evaluated R-matrices

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} R_{ab}(u_a - u_b), \tag{2.22}$$

where $u_a = u + a - 1$ and $u_a = u - a + 1$ for $a = 1, \dots, m$ in the orthogonal and symplectic case, respectively; see [17] for a proof in the context of a fusion procedure for the Brauer algebra. The same fact leads to a construction of a commutative subalgebra of the Yangian $\mathrm{Y}(\mathfrak{g}_N)$; see [24]. We will calculate the images of the elements of this subalgebra under the homomorphism (2.16) and thus reproduce the character formulas for the respective classes of Yangian representations; cf. [19, Sec. 7]. Consider the tensor product algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \dots \otimes \mathrm{End}\,\mathbb{C}^N}_{m} \otimes \mathrm{Y}(\mathfrak{g}_N)[[u^{-1}]] \tag{2.23}$$

and extend the notation (2.4) to elements of (2.23).

2.2.1 Series B_n

The commutative subalgebra of the Yangian $\mathrm{Y}(\mathfrak{o}_N)$ with $N = 2n+1$ is generated by the coefficients of the formal series

$$\mathrm{tr}\, S^{(m)} T_1(u)\, T_2(u+1) \dots T_m(u+m-1) \tag{2.24}$$

with the trace taken over all m copies of $\mathrm{End}\,\mathbb{C}^N$ in (2.23), where $\mathfrak{g}_N = \mathfrak{o}_N$ and $S^{(m)}$ is defined in (2.18). It follows easily from the defining relations (2.15) that all elements of this subalgebra belong to $\mathrm{Y}(\mathfrak{o}_N)^{\mathfrak{h}}$.

Proposition 2.2. *The image of the series* (2.24) *under the homomorphism* (2.16) *is given by*

$$\sum_{1\leqslant i_1\leqslant\dots\leqslant i_m\leqslant N} \lambda_{i_1}(u)\,\lambda_{i_2}(u+1)\dots\lambda_{i_m}(u+m-1)$$

with the condition that $n+1$ *occurs among the summation indices* $i_1,\dots,i_m$ *at most once.*

Proof. By [24, Prop. 3.1] the operator $S^{(m)}$ can be given by the formula

$$S^{(m)} = H^{(m)} \sum_{r=0}^{\lfloor m/2\rfloor} \frac{(-1)^r}{2^r\, r!} \binom{N/2+m-2}{r}^{-1} \sum_{a_i<b_i} Q_{a_1b_1}Q_{a_2b_2}\dots Q_{a_rb_r} \tag{2.25}$$

with the second sum taken over the (unordered) sets of disjoint pairs $\{(a_1,b_1),\dots,(a_r,b_r)\}$ of indices from $\{1,\dots,m\}$. Here $H^{(m)}$ is the symmetrization operator defined in (2.7). Note that for each r the second sum in (2.25) commutes with any element P_s and hence commutes with $H^{(m)}$.

Recall that the subspace of harmonic tensors in $(\mathbb{C}^N)^{\otimes m}$ is spanned by the tensors v with the property $Q_{a\,b}\, v=0$ for all $1\leqslant a<b\leqslant m$. By (2.20) the operator $S^{(m)}$ projects $(\mathbb{C}^N)^{\otimes m}$ to a subspace of symmetric harmonic tensors which we denote by $\mathcal{H}_m$. This subspace carries an irreducible representation of $\mathfrak{o}_N$ with the highest weight $(m,0,\dots,0)$. Therefore, the trace in (2.24) can be calculated over the subspace $\mathcal{H}_m$. We will introduce a special basis of this subspace. We identify the image of the symmetrizer $H^{(m)}$ with the space of homogeneous polynomials of degree m in variables $z_1,\dots,z_N$ via the isomorphism

$$H^{(m)}(e_{i_1}\otimes\dots\otimes e_{i_m}) \mapsto z_{i_1}\dots z_{i_m}. \tag{2.26}$$

The subspace $\mathcal{H}_m$ is then identified with the subspace of harmonic homogeneous polynomials of degree m; they belong to the kernel of the Laplace operator

$$\Delta = \sum_{i=1}^{n} \partial_{z_i}\partial_{z_{i'}} + \frac{1}{2}\,\partial^2_{z_{n+1}}.$$

The basis vectors of $\mathcal{H}_m$ will be parameterized by the N-tuples $(k_1,\dots,k_n,\delta,l_n,\dots,l_1)$, where the k_i and l_i are arbitrary nonnegative integers, $\delta\in\{0,1\}$ and the sum of all entries is m. Given such a tuple, the corresponding harmonic polynomial is defined by

$$\sum_{a_1,\dots,a_n} \frac{(-2)^{a_1+\dots+a_n}(a_1+\dots+a_n)!\, z_{n+1}^{2a_1+\dots+2a_n+\delta}}{a_1!\dots a_n!\,(2a_1+\dots+2a_n+\delta)!} \prod_{i=1}^{n} \frac{z_i^{k_i-a_i}\, z_{i'}^{l_i-a_i}}{(k_i-a_i)!\,(l_i-a_i)!}, \tag{2.27}$$

summed over nonnegative integers a_i satisfying $a_i\leqslant\min\{k_i,l_i\}$. Each polynomial contains a unique monomial (which we call the *leading monomial*) where the variable z_{n+1} occurs with the power not exceeding 1. It is straightforward to see that these polynomials are all harmonic and linearly independent. Furthermore, a simple calculation shows that the number of the polynomials coincides with the dimension of the irreducible representation of $\mathfrak{o}_N$ with the highest weight $(m, 0, \dots, 0)$ and so they form a basis of the subspace $\mathcal{H}_m$.

By relations (2.15) and (2.22) we can write the product occurring in (2.24) as

$$S^{(m)} T_1(u) \dots T_m(u+m-1) = T_m(u+m-1) \dots T_1(u)\, S^{(m)}. \tag{2.28}$$

This relation together with (2.20) shows that the product on each side can be regarded as an operator on $(\mathbb{C}^N)^{\otimes m}$ with coefficients in the algebra $\mathrm{Y}(\mathfrak{o}_N)[[u^{-1}]]$ such that the subspace $\mathcal{H}_m$ is invariant under this operator. Now fix a basis vector $v \in \mathcal{H}_m$ of the form (2.27). Denote the operator on the right hand side of (2.28) by A and consider the coefficient of v in the expansion of Av as a linear combination of the basis vectors. Use the isomorphism (2.26) to write the vector v as a linear combination of the tensors $e_{j_1} \otimes \dots \otimes e_{j_m}$. We have $S^{(m)} v = v$, while the matrix elements of the remaining product are found from the expansion

$$\begin{aligned} T_m(u+m-1) \dots T_1(u)(e_{j_1} \otimes \dots \otimes e_{j_m}) \\ = \sum_{i_1, \dots, i_m} t_{i_m j_m}(u+m-1) \dots t_{i_1 j_1}(u)(e_{i_1} \otimes \dots \otimes e_{i_m}). \end{aligned}$$

The coefficient of v in the expansion of Av is uniquely determined by the coefficient of the tensor $e_{i_1} \otimes \dots \otimes e_{i_m}$ with $i_1 \leqslant \dots \leqslant i_m$ which corresponds to the leading monomial of v under the isomorphism (2.26). It is clear from formula (2.27) that if a tensor of the form $e_{j_1} \otimes \dots \otimes e_{j_m}$ corresponds to a non-leading monomial, then the matrix element $t_{i_m j_m}(u+m-1) \dots t_{i_1 j_1}(u)$ vanishes under the homomorphism (2.16). Therefore, a nonzero contribution to the image of the diagonal matrix element of the operator A corresponding to v under the homomorphism (2.16) only comes from the term $t_{i_m i_m}(u+m-1) \dots t_{i_1 i_1}(u)$. Taking the sum over all basis vectors (2.27) yields the resulting formula for the image of the element (2.24). □

2.2.2 Series D_n

The commutative subalgebra of the Yangian $\mathrm{Y}(\mathfrak{o}_N)$ with $N = 2n$ is generated by the coefficients of the formal series defined by the same formula (2.24), where the parameter N now takes an even value $2n$.

Proposition 2.3. *The image of the series* (2.24) *under the homomorphism* (2.16) *is given by*

$$\sum_{1 \leqslant i_1 \leqslant \dots \leqslant i_m \leqslant N} \lambda_{i_1}(u)\, \lambda_{i_2}(u+1) \dots \lambda_{i_m}(u+m-1)$$

with the condition that n *and* n' *do not occur simultaneously among the summation indices* $i_1, \dots, i_m$.

Proof. As in the proof of Proposition 2.2, we use the formula (2.25) for the symmetrizer $S^{(m)}$ and its properties (2.20). Following the argument of that proof we identify the image $S^{(m)}(\mathbb{C}^N)^{\otimes m}$ with the space $\mathcal{H}_m$ of homogeneous harmonic polynomials of degree m in variables $z_1, \dots, z_N$ via the isomorphism (2.26). This time the harmonic polynomials are annihilated by the Laplace operator of the form

$$\Delta = \sum_{i=1}^{n} \partial_{z_i} \partial_{z_{i'}}.$$

The basis vectors of $\mathcal{H}_m$ will be parameterized by the N-tuples $(k_1, \dots, k_n, l_n, \dots, l_1)$, where the k_i and l_i are arbitrary nonnegative integers, the sum of all entries is m and at least one of k_n and l_n is zero. Given such a tuple, the corresponding harmonic polynomial is now defined by

$$\sum_{a_1,\dots,a_{n-1}} \frac{(-1)^{a_1+\dots+a_{n-1}}(a_1+\dots+a_{n-1})!\, z_n^{a_1+\dots+a_{n-1}+k_n}\, z_{n'}^{a_1+\dots+a_{n-1}+l_n}}{a_1!\dots a_{n-1}!\,(a_1+\dots+a_{n-1}+k_n)!\,(a_1+\dots+a_{n-1}+l_n)!} \times \prod_{i=1}^{n-1} \frac{z_i^{k_i-a_i}\, z_{i'}^{l_i-a_i}}{(k_i-a_i)!\,(l_i-a_i)!}, \tag{2.29}$$

summed over nonnegative integers $a_1, \dots, a_{n-1}$ satisfying $a_i \leqslant \min\{k_i, l_i\}$. A unique *leading monomial* corresponds to the values $a_1 = \dots = a_{n-1} = 0$. The argument is now completed in the same way as for Proposition 2.2 by considering the diagonal matrix elements of the operator on right hand side of (2.28) corresponding to the basis vectors (2.29). These coefficients are determined by those of the leading monomials and their images under the homomorphism (2.16) are straightforward to calculate. □

2.2.3 Series C_n

The commutative subalgebra of the Yangian $\mathrm{Y}(\mathfrak{sp}_N)$ with $N = 2n$ is generated by the coefficients of the formal series

$$\operatorname{tr} S^{(m)} T_1(u)\, T_2(u-1) \dots T_m(u-m+1), \tag{2.30}$$

with the trace taken over all m copies of $\operatorname{End}\mathbb{C}^N$ in (2.23) with $\mathfrak{g}_N = \mathfrak{sp}_N$ and $S^{(m)}$ defined in (2.19) with $m \leqslant n$.

Proposition 2.4. *The image of the series* (2.30) *with* $m \leqslant n$ *under the homomorphism* (2.16) *is given by*

$$\sum_{1\leqslant i_1<\dots<i_m\leqslant 2n} \lambda_{i_1}(u)\, \lambda_{i_2}(u-1) \dots \lambda_{i_m}(u-m+1) \tag{2.31}$$

with the condition that if for any i both i and i' occur among the summation indices as $i = i_r$ and $i' = i_s$ for some $1 \leqslant r < s \leqslant m$, then $s - r \leqslant n - i$.

Proof. Using again [24, Prop. 3.1] we find that the operator $S^{(m)}$ can be given by the formula

$$S^{(m)} = A^{(m)} \sum_{r=0}^{\lfloor m/2 \rfloor} \frac{1}{2^r\, r!} \binom{-n+m-2}{r}^{-1} \sum_{a_i < b_i} Q_{a_1 b_1} Q_{a_2 b_2} \dots Q_{a_r b_r} \tag{2.32}$$

with the second sum taken over the (unordered) sets of disjoint pairs $\{(a_1, b_1), \dots, (a_r, b_r)\}$ of indices from $\{1, \dots, m\}$. Here $A^{(m)}$ is the anti-symmetrization operator defined in (2.7). For each r the second sum in (2.32) commutes with any element P_s and hence commutes with $A^{(m)}$.

As with the orthogonal case, the subspace of harmonic tensors in $(\mathbb{C}^N)^{\otimes m}$ is spanned by the tensors v with the property $Q_{a\,b}\, v = 0$ for all $1 \leqslant a < b \leqslant m$. The operator $S^{(m)}$ projects $(\mathbb{C}^N)^{\otimes m}$ to a subspace of skew-symmetric harmonic tensors which we denote by $\mathcal{H}_m$. Hence, the trace in (2.30) can be calculated over the subspace $\mathcal{H}_m$. We introduce a special basis of this subspace by identifying the image of the anti-symmetrizer $A^{(m)}$ with the space of homogeneous polynomials of degree m in the anti-commuting variables $\zeta_1, \dots, \zeta_{2n}$ via the isomorphism

$$A^{(m)}(e_{i_1} \otimes \dots \otimes e_{i_m}) \mapsto \zeta_{i_1} \wedge \dots \wedge \zeta_{i_m}. \tag{2.33}$$

The subspace $\mathcal{H}_m$ is then identified with the subspace of harmonic homogeneous polynomials of degree m; they belong to the kernel of the Laplace operator

$$\Delta = \sum_{i=1}^{n} \partial_i \wedge \partial_{i'},$$

where ∂_i denotes the (left) partial derivative over ζ_i.

The basis vectors of $\mathcal{H}_m$ will be parameterized by the subsets $\{i_1, \dots, i_m\}$ of the set $\{1, \dots, 2n\}$ satisfying the condition as stated in the proposition, when the elements $i_1, \dots, i_m$ are written in the increasing order. We will call such subsets *admissible*. The number of admissible subsets can be shown to be given by the formula (2.21), which coincides with the dimension of $\mathcal{H}_m$. Consider monomials of the form

$$\zeta_{a_1} \wedge \zeta_{a'_1} \wedge \dots \wedge \zeta_{a_k} \wedge \zeta_{a'_k} \wedge \zeta_{b_1} \wedge \dots \wedge \zeta_{b_l} \tag{2.34}$$

with $1 \leqslant a_1 < \dots < a_k \leqslant n$ and $1 \leqslant b_1 < \dots < b_l \leqslant 2n$, associated with subsets $\{a_1, a'_1, \dots, a_k, a'_k, b_1, \dots, b_l\}$ of $\{1, \dots, 2n\}$ of cardinality $m = 2k + l$, where $b_i \neq b'_j$ for all i and j. We will suppose that the parameters b_i are fixed and label the monomial (2.34) by the k-tuple $(a_1, \dots, a_k)$. Furthermore, we order the k-tuples and the corresponding monomials lexicographically.

Now let the subset $\{a_1, a'_1, \dots, a_k, a'_k, b_1, \dots, b_l\}$ be admissible and suppose that the parameters $a_1, \dots, a_k$ are fixed too. We will call the corresponding monomial (2.34) *admissible*. Fix $i \in \{1, \dots, k\}$. Let s be the number of the elements b_j of the subset satisfying $a_i < b_j < a'_i$. By the admissibility condition applied to a_i and a'_i, we have the inequality $2(k-i)+s<n-a_i$. Therefore, there exist elements $c_i,\dots,c_k$ satisfying $a_i<c_i<\dots<c_k\leqslant n$ so that none of c_j or c_j' with $j=i,\dots,k$ belongs to the subset $\{a_1,a_1',\dots,a_k,a_k',b_1,\dots,b_l\}$. Taking the consecutive values $i=k,k-1,\dots,1$ choose the maximum possible element c_i at each step. Thus, we get a family of elements $c_1<\dots<c_k$ uniquely determined by the admissible subset. In particular, $c_i>a_i$ for all i.

Note that our condition on the parameters b_i implies that the monomial $\zeta_{b_1}\wedge\dots\wedge\zeta_{b_l}$ is annihilated by the operator Δ. We denote this monomial by y and set $x_a=\zeta_a\wedge\zeta_{a'}$ for $a=1,\dots,n$. The vector

$$\sum_{p=0}^{k}(-1)^p\sum_{1\leqslant d_1<\dots<d_p\leqslant k}x_{a_1}\wedge\dots\wedge\widehat{x}_{a_{d_1}}\wedge\dots\wedge\widehat{x}_{a_{d_p}}\wedge\dots\wedge x_{a_k}\wedge x_{c_{d_1}}\wedge\dots\wedge x_{c_{d_p}}\wedge y,$$

where the hats indicate the factors to be skipped, is easily seen to belong to the kernel of the operator Δ so it is an element of the subspace $\mathcal{H}_m$. Furthermore, these vectors parameterized by all admissible subsets form a basis of $\mathcal{H}_m$. Indeed, the vectors are linearly independent because the linear combination defining each vector is uniquely determined by the admissible monomial $x_{a_1}\wedge\dots\wedge x_{a_k}\wedge y$ which precedes all the other monomials occurring in the linear combination with respect to the lexicographic order.

Note that apart from the minimal admissible monomial $x_{a_1}\wedge\dots\wedge x_{a_k}\wedge y$, the linear combination defining a basis vector may contain some other admissible monomials. By eliminating such additional admissible monomials with the use of an obvious induction on the lexicographic order, we can produce another basis of the space $\mathcal{H}_m$ parameterized by all admissible subsets with the property that each basis vector is given by a linear combination of monomials of the same form as above, containing a unique admissible monomial.

By relations (2.15) and (2.22) we can write the product occurring in (2.30) as

$$S^{(m)}T_1(u)\dots T_m(u-m+1)=T_m(u-m+1)\dots T_1(u)\,S^{(m)} \tag{2.35}$$

and complete the argument exactly as in the proof of Proposition 2.2. Indeed, relations (2.20) and (2.35) show that the product on each side can be regarded as an operator on $(\mathbb{C}^N)^{\otimes m}$ with coefficients in the algebra $\mathrm{Y}(\mathfrak{sp}_N)[[u^{-1}]]$ such that the subspace $\mathcal{H}_m$ is invariant under this operator. Denote the operator on the right hand side of (2.35) by A and let v denote the basis vector of $\mathcal{H}_m$ corresponding to an admissible subset $\{i_1,\dots,i_m\}$ with $i_1<\dots<i_m$. The properties of the basis vectors imply that a nonzero contribution to the image of the diagonal matrix element of the operator A corresponding to v under the homomorphism (2.16) only comes from the term $t_{i_mi_m}(u-m+1)\dots t_{i_1i_1}(u)$. □

We will be using an equivalent formula for the expression (2.31) given in [21, Prop. 2.4]. The argument there is combinatorial and relies only on the identities (2.17). To state the formula from [21] introduce new parameters $\varkappa_i(u)$ for $i=1,\dots,2n+2$ by

$$\varkappa_i(u)=\lambda_i(u),\qquad \varkappa_{2n-i+3}(u)=\lambda_{2n-i+1}(u)\qquad \text{for}\quad i=1,\dots,n,$$

and $\varkappa_{n+2}(u) = -\varkappa_{n+1}(u)$, where $\varkappa_{n+1}(u)$ is a formal series in u^{-1} with constant term 1 defined by

$$\varkappa_{n+1}(u)\,\varkappa_{n+1}(u-1) = \lambda_n(u)\,\lambda_{n'}(u-1).$$

Corollary 2.5. *The image of the series* (2.30) *with* $m \leqslant n$ *under the homomorphism* (2.16) *can be written as*

$$\sum_{1\leqslant i_1<\dots<i_m\leqslant 2n+2} \varkappa_{i_1}(u)\,\varkappa_{i_2}(u-1)\dots\varkappa_{i_m}(u-m+1). \tag{2.36}$$

Moreover, the expression (2.36) *is zero for* $m = n+1$. □

3 Harish-Chandra images for the current algebras

We will use the character formulas obtained in Sec. 2 to calculate the Harish-Chandra images of elements of certain commutative subalgebras of $\mathrm{U}\big(\mathfrak{g}[t]\big)$ for the simple Lie algebras $\mathfrak{g}$ of all classical types. The results in the case of $\mathfrak{gl}_N$ are well-known, the commutative subalgebras were constructed explicitly in [35]; see also [5], [6], [25], [26] and [27].

3.1 Case of $\mathfrak{gl}_N$

Identify the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$ with a subalgebra of $\mathrm{U}\big(\mathfrak{gl}_N[t]\big)$ via the embedding $E_{ij} \mapsto E_{ij}[0]$. Then $\mathrm{U}\big(\mathfrak{gl}_N[t]\big)$ can be regarded as a $\mathfrak{gl}_N$-module with the adjoint action. Denote by $\mathrm{U}\big(\mathfrak{gl}_N[t]\big)^{\mathfrak{h}}$ the subalgebra of $\mathfrak{h}$-invariants under this action. Consider the left ideal I of the algebra $\mathrm{U}\big(\mathfrak{gl}_N[t]\big)$ generated by all elements $E_{ij}[r]$ with the conditions $1 \leqslant i < j \leqslant N$ and $r \geqslant 0$. By the Poincaré–Birkhoff–Witt theorem, the intersection $\mathrm{U}\big(\mathfrak{gl}_N[t]\big)^{\mathfrak{h}} \cap I$ is a two-sided ideal of $\mathrm{U}\big(\mathfrak{gl}_N[t]\big)^{\mathfrak{h}}$. Moreover, the quotient of $\mathrm{U}\big(\mathfrak{gl}_N[t]\big)^{\mathfrak{h}}$ by this ideal is isomorphic to the commutative algebra freely generated by the images of the elements $E_{ii}[r]$ with $i = 1,\dots,N$ and $r \geqslant 0$ in the quotient. We will denote by $\mu_i[r]$ this image of $E_{ii}[r]$. We get an analogue of the Harish-Chandra homomorphism (1.1),

$$\mathrm{U}\big(\mathfrak{gl}_N[t]\big)^{\mathfrak{h}} \to \mathbb{C}[\mu_i[r]\,|\, i = 1,\dots,N,\ r \geqslant 0]. \tag{3.1}$$

Combine the elements $E_{ij}[r]$ and $\mu_i[r]$ into the formal series

$$E_{ij}(u) = \sum_{r=0}^{\infty} E_{ij}[r]\,u^{-r-1} \qquad \text{and} \qquad \mu_i(u) = \sum_{r=0}^{\infty} \mu_i[r]\,u^{-r-1}.$$

Then $\mu_i(u)$ is understood as the image of the series $E_{ii}(u)$ under the homomorphism (3.1). Consider tensor product algebras

$$\underbrace{\operatorname{End}\mathbb{C}^N \otimes \dots \otimes \operatorname{End}\mathbb{C}^N}_{m} \otimes \mathrm{U}\big(\mathfrak{gl}_N[t]\big)[[u^{-1},\partial_u]]$$

and use matrix notation as in (1.8).

Proposition 3.1. *For the images under the Harish-Chandra homomorphism* (3.1) *we have*

$$\operatorname{tr} A^{(m)}\big(\partial_u + E_1(u)\big)\dots\big(\partial_u + E_m(u)\big) \mapsto e_m\big(\partial_u + \mu_1(u),\dots,\partial_u + \mu_N(u)\big), \tag{3.2}$$

$$\operatorname{tr} H^{(m)}\big(\partial_u + E_1(u)\big)\dots\big(\partial_u + E_m(u)\big) \mapsto h_m\big(\partial_u + \mu_1(u),\dots,\partial_u + \mu_N(u)\big). \tag{3.3}$$

Proof. The argument is essentially the same as in the proof of Proposition 2.1. Both relations are immediate from the cyclic property of trace and the identities

$$\big(\partial_u + E_1(u)\big)\dots\big(\partial_u + E_m(u)\big)A^{(m)} = A^{(m)}\big(\partial_u + E_1(u)\big)\dots\big(\partial_u + E_m(u)\big)A^{(m)},$$

$$H^{(m)}\big(\partial_u + E_1(u)\big)\dots\big(\partial_u + E_m(u)\big) = H^{(m)}\big(\partial_u + E_1(u)\big)\dots\big(\partial_u + E_m(u)\big)H^{(m)},$$

implied by the fact that $\partial_u + E(u)$ is a *left Manin matrix*; see [4, Prop. 18]. □

An alternative (longer) way to proof Proposition 3.1 is to derive it from the character formulas of Proposition 2.1. Indeed, $\partial_u + E(u)$ coincides with the image of the matrix $T(u)e^{\partial_u} - 1$ in the component of degree -1 of the graded algebra associated with the Yangian. Here we extend the filtration on the Yangian to the algebra of formal series $\mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}, \partial_u]]$ by setting $\deg u^{-1} = \deg \partial_u = -1$ so that the associated graded algebra is isomorphic to $\mathrm{U}\big(\mathfrak{gl}_N[t]\big)[[u^{-1}, \partial_u]]$. Hence, for instance, the element on the left hand side of (3.2) can be found as the image of the component of degree $-m$ of the expression

$$\operatorname{tr} A^{(m)}\big(T_1(u)e^{\partial_u} - 1\big)\dots\big(T_m(u)e^{\partial_u} - 1\big).$$

The image of this expression under the homomorphism (2.6) can be found from (2.13).

There is no known analogue of the argument which we used in the proof of Proposition 3.1 for the B, C and D types. Therefore to prove its counterparts for these types we have to resort to the argument making use of the character formulas of Sec. 2.2.

3.2 Types B, C and D

Recall that $F_{ij}[r] = F_{ij}\, t^r$ with $r \in \mathbb{Z}$ denote elements of the loop algebra $\mathfrak{g}_N[t, t^{-1}]$, where the F_{ij} are standard generators of $\mathfrak{g}_N$; see Sec. 2.

Consider the ascending filtration on the Yangian $\mathrm{Y}(\mathfrak{g}_N)$ defined by

$$\deg t_{ij}^{(r)} = r - 1.$$

Denote by $\bar{t}_{ij}^{(r)}$ the image of the generator $t_{ij}^{(r)}$ in the $(r-1)$-th component of the associated graded algebra $\operatorname{gr}\mathrm{Y}(\mathfrak{g}_N)$. By [2, Theorem 3.6] the mapping

$$F_{ij}[r] \mapsto \bar{t}_{ij}^{(r+1)}, \qquad r \geqslant 0,$$

defines an algebra isomorphism $\mathrm{U}\big(\mathfrak{g}_N[t]\big) \to \mathrm{gr}\,\mathrm{Y}(\mathfrak{g}_N)$. Our goal here is to use this isomorphism and Propositions 2.2, 2.3 and 2.4 to calculate the Harish-Chandra images of certain elements of $\mathrm{U}\big(\mathfrak{g}_N[t]\big)$ defined with the use of the corresponding operators (2.18) and (2.19). These elements generate a commutative subalgebra of $\mathrm{U}\big(\mathfrak{g}_N[t]\big)$ and they can be obtained from the generators (1.17) of the Feigin–Frenkel center by an application of the vertex algebra structure on the vacuum module $V_{-h^\vee}(\mathfrak{g}_N)$; see [24, Sec. 5].

We identify the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$ with a subalgebra of $\mathrm{U}\big(\mathfrak{g}_N[t]\big)$ via the embedding $F_{ij} \mapsto F_{ij}[0]$. Then $\mathrm{U}\big(\mathfrak{g}_N[t]\big)$ can be regarded as a $\mathfrak{g}_N$-module with the adjoint action. Denote by $\mathrm{U}\big(\mathfrak{g}_N[t]\big)^{\mathfrak{h}}$ the subalgebra of $\mathfrak{h}$-invariants under this action. Consider the left ideal I of the algebra $\mathrm{U}\big(\mathfrak{g}_N[t]\big)$ generated by all elements $F_{ij}[r]$ with the conditions $1 \leqslant i < j \leqslant N$ and $r \geqslant 0$. By the Poincaré–Birkhoff–Witt theorem, the intersection $\mathrm{U}\big(\mathfrak{g}_N[t]\big)^{\mathfrak{h}} \cap I$ is a two-sided ideal of $\mathrm{U}\big(\mathfrak{g}_N[t]\big)^{\mathfrak{h}}$. Moreover, the quotient of $\mathrm{U}\big(\mathfrak{g}_N[t]\big)^{\mathfrak{h}}$ by this ideal is isomorphic to the commutative algebra freely generated by the images of the elements $F_{ii}[r]$ with $i = 1, \dots, n$ and $r \geqslant 0$ in the quotient. We will write $\mu_i[r]$ for this image of $F_{ii}[r]$ and extend this notation to all values $i = 1, \dots, N$ so that $\mu_{i'}[r] = -\mu_i[r]$ for all i. We get an analogue of the Harish-Chandra homomorphism (1.1),

$$\mathrm{U}\big(\mathfrak{g}_N[t]\big)^{\mathfrak{h}} \to \mathbb{C}[\mu_i[r] \,|\, i = 1, \dots, n,\ r \geqslant 0]. \tag{3.4}$$

We will combine the elements $F_{ij}[r]$ into the formal series

$$F_{ij}(u) = \sum_{r=0}^{\infty} F_{ij}[r]\, u^{-r-1}$$

and write

$$\mu_i(u) = \sum_{r=0}^{\infty} \mu_i[r]\, u^{-r-1}, \qquad i = 1, \dots, N.$$

Then $\mu_i(u)$ is understood as the image of the series $F_{ii}(u)$ under the homomorphism (3.4).

It is clear from the definitions of the homomorphisms (2.16) and (3.4), that the graded version of (2.16) coincides with (3.4) in the sense that the following diagram commutes

$$\begin{array}{ccc} \mathrm{U}\big(\mathfrak{g}_N[t]\big)^{\mathfrak{h}} & \longrightarrow & \mathbb{C}\big[\mu_i[r]\big] \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{gr}\,\mathrm{Y}(\mathfrak{g}_N)^{\mathfrak{h}} & \longrightarrow & \mathrm{gr}\,\mathbb{C}\big[\lambda_i^{(r+1)}\big], \end{array} \tag{3.5}$$

where i ranges over the set $\{1, \dots, n\}$ while $r \geqslant 0$ and the second vertical arrow indicates the isomorphism which takes $\mu_i[r]$ to the image of $\lambda_i^{(r+1)}$ in the graded polynomial algebra with the grading defined by the assignment $\deg \lambda_i^{(r+1)} = r$.

In what follows we extend the filtration on the Yangian to the algebra of formal series $\mathrm{Y}(\mathfrak{g}_N)[[u^{-1}, \partial_u]]$ by setting $\deg u^{-1} = \deg \partial_u = -1$. The associated graded algebra will then be isomorphic to $\mathrm{U}\big(\mathfrak{g}_N[t]\big)[[u^{-1},\partial_u]]$. We consider tensor product algebras

$$\underbrace{\mathrm{End}\,\mathbb{C}^N\otimes\ldots\otimes\mathrm{End}\,\mathbb{C}^N}_{m}\otimes\mathrm{U}\big(\mathfrak{g}_N[t]\big)[[u^{-1},\partial_u]] \tag{3.6}$$

and use matrix notation as in (1.15).

3.2.1 Series B_n

Take $\mathfrak{g}_N=\mathfrak{o}_N$ with $N=2n+1$ and consider the operator $S^{(m)}$ defined in (2.18). We also use notation (1.16) with $\omega=N$ and (1.13). The trace is understood to be taken over all copies of the endomorphism algebra $\mathrm{End}\,\mathbb{C}^N$ in (3.6).

Theorem 3.2. *For the image under the Harish-Chandra homomorphism* (3.4) *we have*

$$\gamma_m(N)\,\mathrm{tr}\,S^{(m)}\big(\partial_u+F_1(u)\big)\ldots\big(\partial_u+F_m(u)\big)$$
$$\mapsto h_m\big(\partial_u+\mu_1(u),\ldots,\partial_u+\mu_n(u),\partial_u+\mu_{n'}(u),\ldots,\partial_u+\mu_{1'}(u)\big). \tag{3.7}$$

Proof. The element $\partial_u+F(u)$ coincides with the image of the matrix $T(u)e^{\partial_u}-1$ in the component of degree -1 of the graded algebra associated with the Yangian. Therefore the element on the left hand side of (3.7) can be found as the image of the component of degree $-m$ of the expression

$$\gamma_m(N)\,\mathrm{tr}\,S^{(m)}\big(T_1(u)e^{\partial_u}-1\big)\ldots\big(T_m(u)e^{\partial_u}-1\big). \tag{3.8}$$

Hence, the theorem can be proved by making use of the commutative diagram (3.5) and the Harish-Chandra image of (3.8) implied by Proposition 2.2. We have

$$\mathrm{tr}\,S^{(m)}\,\big(T_1(u)e^{\partial_u}-1\big)\ldots\big(T_m(u)e^{\partial_u}-1\big)$$
$$=\sum_{k=0}^{m}(-1)^{m-k}\sum_{1\leqslant a_1<\cdots<a_k\leqslant m}\mathrm{tr}\,S^{(m)}\,T_{a_1}(u)e^{\partial_u}\ldots T_{a_k}(u)e^{\partial_u}.$$

Each product $T_{a_1}(u)e^{\partial_u}\ldots T_{a_k}(u)e^{\partial_u}$ can be written as $P\,T_1(u)e^{\partial_u}\ldots T_k(u)e^{\partial_u}\,P^{-1}$, where P is the image in (3.6) (with the identity component in the last tensor factor) of a permutation $p\in\mathfrak{S}_m$ such that $p(r)=a_r$ for $r=1,\ldots,k$. By the second property in (2.20) and the cyclic property of trace, we can bring the above expression to the form

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\,\mathrm{tr}\,S^{(m)}\,T_1(u)e^{\partial_u}\ldots T_k(u)e^{\partial_u}.$$

Now apply [24, Lemma 4.1] to calculate the partial traces of the symmetrizer $S^{(m)}$ over the copies $k+1,\dots,m$ of the algebra $\operatorname{End}\mathbb{C}^N$ in (2.1) to get

$$\operatorname{tr}_{k+1,\dots,m} S^{(m)} = \frac{\gamma_k(N)}{\gamma_m(N)} \binom{N+m-2}{m-k} \binom{m}{k}^{-1} S^{(k)}.$$

Thus, by Proposition 2.2, the Harish-Chandra image of the expression (3.8) is found by

$$\sum_{k=0}^{m} (-1)^{m-k}\, \gamma_k(N) \binom{N+m-2}{m-k} \sum_{1\leqslant i_1\leqslant\dots\leqslant i_k\leqslant N} \lambda_{i_1}(u)\, e^{\partial_u} \dots \lambda_{i_k}(u)\, e^{\partial_u} \tag{3.9}$$

with the condition that $n+1$ occurs among the summation indices $i_1,\dots,i_k$ at most once.

The next step is to express (3.9) in terms of the new variables

$$\sigma_i(u) = \lambda_i(u)\, e^{\partial_u} - 1, \qquad i = 1,\dots,N. \tag{3.10}$$

This is done by a combinatorial argument as shown in the following lemma.

Lemma 3.3. *The expression* (3.9) *multiplied by* $-2\binom{N/2-2}{N+m-2}$ *equals*

$$\sum_{r=0}^{m} \binom{N/2-2}{N+r-3} \sum_{a_1+\dots+a_{1'}=r} \sigma_1(u)^{a_1} \dots \sigma_n(u)^{a_n} \sigma_{n'}(u)^{a_{n'}} \dots \sigma_{1'}(u)^{a_{1'}}$$

$$+ \sum_{r=1}^{m} \binom{N/2-2}{N+r-3} \sum_{a_1+\dots+a_{1'}=r-1} \sigma_1(u)^{a_1} \dots \sigma_n(u)^{a_n} \big(\sigma_{n+1}(u)+2\big)\, \sigma_{n'}(u)^{a_{n'}} \dots \sigma_{1'}(u)^{a_{1'}},$$

where $a_1,\dots,a_{1'}$ *run over nonnegative integers.*

Proof. The statement is verified by substituting (3.10) into both terms and calculating the coefficients of the sum

$$\sum_{1\leqslant i_1\leqslant\dots\leqslant i_k\leqslant N} \lambda_{i_1}(u)\, e^{\partial_u} \dots \lambda_{i_k}(u)\, e^{\partial_u} \tag{3.11}$$

for all $0 \leqslant k \leqslant m$, where $n+1$ occurs among the summation indices $i_1,\dots,i_k$ at most once. Note the following expansion formula for the noncommutative complete symmetric functions (1.13),

$$h_r(x_1-1,\dots,x_p-1) = \sum_{k=0}^{r} (-1)^{r-k} \binom{p+r-1}{r-k} h_k(x_1,\dots,x_p). \tag{3.12}$$

Take $x_i = \lambda_i(u)\, e^{\partial_u}$ with $i = 1,\dots,n,n'\dots,1'$ and apply (3.12) with $p = 2n$ to the first term in the expression of the lemma. Using a similar expansion for the second term we find that the coefficient of the sum (3.11) in the entire expression will be found as

$$\sum_{r=k}^{m} \binom{N/2-2}{N+r-3} \binom{N+r-3}{r-k} = \binom{N/2-2}{N+k-3} \binom{N/2+m-1}{m-k},$$

which coincides with

$$-2\,(-1)^{m-k}\,\gamma_k(N)\binom{N/2-2}{N+m-2}\binom{N+m-2}{m-k},$$

as claimed. □

Denote the expression in Lemma 3.3 by A_m. Since the degree of the element (3.8) is $-m$, its Harish-Chandra image (3.9) and the expression A_m also have degree $-m$. Observe that the terms in the both sums of A_m are independent of m so that $A_{m+1} = A_m + B_{m+1}$, where

$$B_{m+1} = \binom{N/2-2}{N+m-2}\sum_{a_1+\dots+a_{1'}=m+1}\sigma_1(u)^{a_1}\dots\sigma_n(u)^{a_n}\sigma_{n'}(u)^{a_{n'}}\dots\sigma_{1'}(u)^{a_{1'}}$$

$$+\binom{N/2-2}{N+m-2}\sum_{a_1+\dots+a_{1'}=m}\sigma_1(u)^{a_1}\dots\sigma_n(u)^{a_n}\big(\sigma_{n+1}(u)+2\big)\,\sigma_{n'}(u)^{a_{n'}}\dots\sigma_{1'}(u)^{a_{1'}}.$$

Since A_{m+1} has degree $-m-1$, its component of degree $-m$ is zero, and so the sum of the homogeneous components of degree $-m$ of A_m and B_{m+1} is zero. However, each element $\sigma_i(u)$ has degree -1 with the top degree component equal to $\partial_u + \mu_i(u)$. This implies that the component of A_m of degree $-m$ equals the component of degree $-m$ of the term

$$-2\binom{N/2-2}{N+m-2}\sum_{a_1+\dots+a_{1'}=m}\sigma_1(u)^{a_1}\dots\sigma_n(u)^{a_n}\sigma_{n'}(u)^{a_{n'}}\dots\sigma_{1'}(u)^{a_{1'}}.$$

Taking into account the constant factor used in Lemma 3.3, we can conclude that the component in question coincides with the noncommutative complete symmetric function as given in (3.7). □

As we have seen in the proof of the theorem, all components of the expression in Lemma 3.3 of degrees exceeding $-m$ are equal to zero. Since the summands do not depend on m, we derive the following corollary.

Corollary 3.4. *The series*

$$\sum_{r=0}^{\infty}\binom{N/2-2}{N+r-3}\sum_{a_1+\dots+a_{1'}=r}\sigma_1(u)^{a_1}\dots\sigma_n(u)^{a_n}\sigma_{n'}(u)^{a_{n'}}\dots\sigma_{1'}(u)^{a_{1'}}$$

$$+\sum_{r=1}^{\infty}\binom{N/2-2}{N+r-3}\sum_{a_1+\dots+a_{1'}=r-1}\sigma_1(u)^{a_1}\dots\sigma_n(u)^{a_n}\big(\sigma_{n+1}(u)+2\big)\,\sigma_{n'}(u)^{a_{n'}}\dots\sigma_{1'}(u)^{a_{1'}}$$

is equal to zero. □

3.2.2 Series D_n

Now take $\mathfrak{g}_N = \mathfrak{o}_N$ with $N = 2n$ and consider the operator $S^{(m)}$ defined in (2.18). We keep using notation (1.16) with $\omega = N$ and (1.13).

Theorem 3.5. *For the image under the Harish-Chandra homomorphism* (3.4) *we have*

$$\gamma_m(N)\,\mathrm{tr}\, S^{(m)}\big(\partial_u + F_1(u)\big)\dots\big(\partial_u + F_m(u)\big)$$

$$\mapsto \tfrac{1}{2}\, h_m\big(\partial_u + \mu_1(u), \dots, \partial_u + \mu_{n-1}(u), \partial_u + \mu_{n'}(u), \dots, \partial_u + \mu_{1'}(u)\big)$$

$$+ \tfrac{1}{2}\, h_m\big(\partial_u + \mu_1(u), \dots, \partial_u + \mu_n(u), \partial_u + \mu_{(n-1)'}(u), \dots, \partial_u + \mu_{1'}(u)\big).$$

Proof. We repeat the beginning of the proof of Theorem 3.5 with N now taking the even value $2n$ up to the application of the formula for Yangian characters. This time we apply Proposition 2.3 to conclude that the Harish-Chandra image of the expression (3.8) is found by

$$\sum_{k=0}^{m} (-1)^{m-k}\,\gamma_k(2n)\binom{2n+m-2}{m-k}\sum_{1\leqslant i_1\leqslant\dots\leqslant i_k\leqslant 2n}\lambda_{i_1}(u)\,e^{\partial_u}\dots\lambda_{i_k}(u)\,e^{\partial_u} \tag{3.13}$$

with the condition that n and n' do not occur simultaneously among the summation indices $i_1, \dots, i_k$. Introducing new variables by the same formulas (3.10) we come to the D_n series counterpart of Lemma 3.3, where we use the notation

$$c_r = (-1)^{r-1}\binom{2n+r-2}{n-1}^{-1}.$$

Lemma 3.6. *The expression* (3.13) *multiplied by* $2\,c_m$ *equals*

$$2\,c_m \sum_{\substack{a_1+\dots+a_{1'}=m\\ a_n=a_{n'}=0}} \sigma_1(u)^{a_1}\dots\sigma_{1'}(u)^{a_{1'}} + c_m \sum_{\substack{a_1+\dots+a_{1'}=m\\ \text{only one of } a_n \text{ and } a_{n'} \text{ is zero}}} \sigma_1(u)^{a_1}\dots\sigma_{1'}(u)^{a_{1'}}$$

$$-\sum_{r=1}^{m}\frac{r\,c_r}{n+r-1}\sum_{\substack{a_1+\dots+a_{1'}=r\\ a_n=a_{n'}=0}}\sigma_1(u)^{a_1}\dots\sigma_{1'}(u)^{a_{1'}}$$

$$+\sum_{r=1}^{m}\frac{(n-1)\,c_r}{n+r-1}\sum_{\substack{a_1+\dots+a_{1'}=r\\ \text{only one of } a_n \text{ and } a_{n'} \text{ is zero}}}\sigma_1(u)^{a_1}\dots\sigma_{1'}(u)^{a_{1'}},$$

where $a_1, \dots, a_{1'}$ *run over nonnegative integers.*

Proof. Substitute (3.10) into the expression and calculate the coefficients of the sum

$$\sum_{1\leqslant i_1\leqslant\dots\leqslant i_k\leqslant 2n} \lambda_{i_1}(u)\, e^{\partial_u}\dots\lambda_{i_k}(u)\, e^{\partial_u}. \tag{3.14}$$

The argument splits into two cases, depending on whether neither of n and n' occurs among the summation indices $i_1,\dots,i_k$ in (3.14) or only one of them occurs. The application of the expansion formula (3.12) brings this to a straightforward calculation with the binomial coefficients in both cases. □

Let A_m denote the four-term expression in Lemma 3.6. This expression equals $2\,c_m$ times the Harish-Chandra image of (3.8) and so A_m has degree $-m$. Hence, the component of degree $-m$ of the expression A_{m+1} is zero. On the other hand, each element $\sigma_i(u)$ has degree -1 with the top degree component equal to $\partial_u+\mu_i(u)$. This implies that the component of degree $-m$ in the sum of the third and fourth terms in A_m is zero. Therefore, the component of A_m of degree $-m$ equals the component of degree $-m$ in the sum of the first and the second terms. Taking into account the constant factor $2\,c_m$, we conclude that the component takes the desired form. □

The following corollary is implied by the proof of the theorem.

Corollary 3.7. *The series*

$$-\sum_{r=1}^{\infty}\frac{r\,c_r}{n+r-1}\sum_{\substack{a_1+\dots+a_{1'}=r\\ a_n=a_{n'}=0}}\sigma_1(u)^{a_1}\dots\sigma_{1'}(u)^{a_{1'}}$$

$$+\sum_{r=1}^{\infty}\frac{(n-1)\,c_r}{n+r-1}\sum_{\substack{a_1+\dots+a_{1'}=r\\ \text{only one of } a_n \text{ and } a_{n'} \text{ is zero}}}\sigma_1(u)^{a_1}\dots\sigma_{1'}(u)^{a_{1'}}$$

is equal to zero. □

3.2.3 Series C_n

Now we let $\mathfrak{g}_N=\mathfrak{sp}_N$ with $N=2n$ and consider the operator $S^{(m)}$ defined in (2.19). We also use notation (1.16) with $\omega=-2n$ and (1.14). Although the operator $S^{(m)}$ is defined only for $m\leqslant n+1$, it is possible to extend the values of expressions of the form (1.17) and those which are used in the next theorem to all m with $m\leqslant 2n+1$; see [24, Sec. 3.3]. The Harish-Chandra images turn out to be given by the same expression for all these values of m. We postpone the proof to Corollary 5.2 below, and assume first that $m\leqslant n$.

Theorem 3.8. *For all* $1 \leqslant m \leqslant n$ *for the image under the Harish-Chandra homomorphism* (3.4) *we have*

$$\gamma_m(-2n)\,\mathrm{tr}\, S^{(m)}\big(\partial_u - F_1(u)\big)\dots\big(\partial_u - F_m(u)\big)$$

$$\mapsto e_m\big(\partial_u + \mu_1(u), \dots, \partial_u + \mu_n(u), \partial_u, \partial_u + \mu_{n'}(u), \dots, \partial_u + \mu_{1'}(u)\big). \quad (3.15)$$

Proof. The element $\partial_u - F(u)$ coincides with the image of the matrix $1 - T(u)\,e^{-\partial_u}$ in the component of degree -1 of the graded algebra associated with the Yangian. Hence the left hand side of (3.15) can be found as the image of the component of degree $-m$ of the expression

$$(-1)^m\,\gamma_m(-2n)\,\mathrm{tr}\, S^{(m)}\big(T_1(u)\,e^{-\partial_u} - 1\big)\dots\big(T_m(u)\,e^{-\partial_u} - 1\big). \quad (3.16)$$

Now we use the commutative diagram (3.5) and the Harish-Chandra image of (3.16) implied by Proposition 2.4. We have

$$\mathrm{tr}\, S^{(m)}\,\big(T_1(u)\,e^{-\partial_u} - 1\big)\dots\big(T_m(u)\,e^{-\partial_u} - 1\big)$$

$$= \sum_{k=0}^{m}(-1)^{m-k}\sum_{1\leqslant a_1<\dots<a_k\leqslant m}\mathrm{tr}\, S^{(m)}\,T_{a_1}(u)\,e^{-\partial_u}\dots T_{a_k}(u)\,e^{-\partial_u}. \quad (3.17)$$

As in the proof of Theorem 3.8, we use the second property in (2.20) and the cyclic property of trace to bring the expression to the form

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\,\mathrm{tr}\, S^{(m)}\,T_1(u)\,e^{-\partial_u}\dots T_k(u)\,e^{-\partial_u}.$$

Further, the partial traces of the symmetrizer $S^{(m)}$ over the copies $k+1, \dots, m$ of the algebra $\mathrm{End}\,\mathbb{C}^N$ in (2.1) are found by applying [24, Lemma 4.1] to get

$$\mathrm{tr}_{k+1,\dots,m}\, S^{(m)} = \frac{\gamma_k(-2n)}{\gamma_m(-2n)}\binom{2n-k+1}{m-k}\binom{m}{k}^{-1} S^{(k)}.$$

By Proposition 2.4 and Corollary 2.5, the Harish-Chandra image of the expression (3.16) is found by

$$\sum_{k=0}^{m}(-1)^k\,\gamma_k(-2n)\binom{2n-k+1}{m-k}\sum_{1\leqslant i_1<\dots<i_k\leqslant 2n+2}\varkappa_{i_1}(u)\,e^{-\partial}\dots\varkappa_{i_k}(u)\,e^{-\partial}. \quad (3.18)$$

Introduce new variables by

$$\sigma_i(u) = \varkappa_i(u)\,e^{-\partial} - 1, \qquad i = 1, \dots, 2n+2, \quad i \neq n+2, \quad (3.19)$$

and $\sigma_{n+2}(u) = \varkappa_{n+2}(u)\,e^{-\partial} + 1$.

Lemma 3.9. *For $m \leqslant n$ the expression* (3.18) *multiplied by* $2(-1)^m \binom{2n-m+1}{n+1}$ *equals*

$$\sum_{r=0}^{m} \binom{2n-r+2}{n+1} \sum_{1\leqslant i_1<\dots<i_r\leqslant 2n+2} \sigma_{i_1}(u)\dots\sigma_{i_r}(u)$$
$$- 2\sum_{r=0}^{m-1} \binom{2n-r+1}{n+1} \sum_{\substack{1\leqslant i_1<\dots<i_r\leqslant 2n+2\\ i_s\neq n+2}} \sigma_{i_1}(u)\dots\sigma_{i_r}(u),$$

where $n+2$ does not occur among the summation indices in the last sum.

Proof. Substituting (3.19) into the expression and simplifying gives

$$\sum_{r=0}^{m} \binom{2n-r+2}{n+1} \sum_{1\leqslant i_1<\dots<i_r\leqslant 2n+2} \big(\varkappa_{i_1}(u)e^{-\partial}-1\big)\dots\big(\varkappa_{i_r}(u)e^{-\partial}-1\big). \tag{3.20}$$

Now use the expansion formula for the noncommutative elementary symmetric functions (1.14),

$$e_r(x_1-1,\dots,x_p-1)=\sum_{k=0}^{r}(-1)^{r-k}\binom{p-k}{r-k}e_k(x_1,\dots,x_p).$$

Taking $x_i = \varkappa_i(u)e^{-\partial_u}$ with $i=1,\dots,2n+2$, it is straightforward to verify that the coefficient of the sum

$$\sum_{1\leqslant i_1<\dots<i_k\leqslant N} \varkappa_{i_1}(u)e^{-\partial_u}\dots\varkappa_{i_k}(u)e^{-\partial_u}$$

in (3.20) equals

$$(-1)^{m-k}\binom{n-k}{m-k}\binom{2n-k+2}{n+1}$$

which coincides with

$$2(-1)^{m-k}\gamma_k(-2n)\binom{2n-m+1}{n+1}\binom{2n-k+1}{m-k}$$

as claimed. □

For $m \leqslant n$ let A_m denote the expression in Lemma 3.9. Note that A_m coincides with the Harish-Chandra image of (3.17) multiplied by $\binom{2n-m+2}{n+1}$. The proof of Lemma 3.9 and the second part of Corollary 2.5 show that A_m is also well-defined for the value $m=n+1$ and $A_{n+1}=0$.

Since the degree of the element (3.16) is $-m$, for $m\leqslant n$ the expression A_m also has degree $-m$. Hence, the component of degree $-m$ of the expression A_{m+1} is zero; this holds for $m=n$ as well, because $A_{n+1}=0$. Furthermore, each element $\sigma_i(u)$ has degree -1 and so the component of A_m of degree $-m$ must be equal to the component of degree $-m$ of the expression

$$2\binom{2n-m+1}{n+1}\sum_{\substack{1\leqslant i_1<\dots<i_m\leqslant 2n+2\\ i_s\neq n+2}}\sigma_{i_1}(u)\dots\sigma_{i_m}(u).$$

The component of $\sigma_i(u)$ of degree -1 equals

$$\begin{cases}-\partial_u+\mu_i(u) & \text{for}\quad i=1,\dots,n,\\ -\partial_u & \text{for}\quad i=n+1,\\ -\partial_u+\mu_{i-2}(u) & \text{for}\quad i=n+3,\dots,2n+2.\end{cases}$$

The proof is completed by taking the signs and the constant factor used in Lemma 3.9 into account. □

4 Classical $\mathcal{W}$-algebras

We define the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ associated with a simple Lie algebra $\mathfrak{g}$ following [11, Sec. 8.1], where more details and proofs can be found. We let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$ and let $\mu_1,\dots,\mu_n$ be a basis of $\mathfrak{h}$. The universal enveloping algebra $\mathrm{U}(t^{-1}\mathfrak{h}[t^{-1}])$ will be identified with the algebra of polynomials in the infinitely many variables $\mu_i[r]$ with $i=1,\dots,n$ and $r<0$ and will be denoted by π_0. We will also use the extended algebra with the additional generator τ subject to the relations

$$\big[\tau,\mu_i[r]\big]=-r\,\mu_i[r-1],$$

implied by (1.5). The extended algebra is isomorphic to $\pi_0\otimes\mathbb{C}[\tau]$ as a vector space. Furthermore, we will need the operator $T=\operatorname{ad}\tau$ which is the derivation $T:\pi_0\to\pi_0$ defined on the generators by the relations

$$T\,\mu_i[r]=-r\,\mu_i[r-1].$$

In particular, $T\,1=0$. The *classical $\mathcal{W}$-algebra* is defined as the subspace $\mathcal{W}(\mathfrak{g})\subset\pi_0$ spanned by the elements which are annihilated by the *screening operators*

$$V_i:\pi_0\to\pi_0,\qquad i=1,\dots,n,$$

which we will write down explicitly for each classical type below,[1]

$$\mathcal{W}(\mathfrak{g})=\{P\in\pi_0\mid V_i\,P=0,\quad i=1,\dots,n\}.$$

[1] Our V_i essentially coincides with the operator $\overline{V}_i[1]$ in the notation of [11, Sec. 7.3.4], which is associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$.

The operators V_i are derivations of π_0 so that $\mathcal{W}(\mathfrak{g})$ is a subalgebra of π_0. The subalgebra $\mathcal{W}(\mathfrak{g})$ is T-invariant. Moreover, there exist elements $B_1, \dots, B_n \in \mathcal{W}(\mathfrak{g})$ such that the family of elements $T^r B_i$ with $i = 1, \dots, n$ and $r \geqslant 0$ is algebraically independent and generates the algebra $\mathcal{W}(\mathfrak{g})$. We will call $B_1, \dots, B_n$ a *complete set of generators* of $\mathcal{W}(\mathfrak{g})$. Examples of such sets in the classical types will be given below.

We extend the screening operators to the algebra $\pi_0 \otimes \mathbb{C}[\tau]$ by

$$V_i\big(P \otimes Q(\tau)\big) = V_i(P) \otimes Q(\tau), \qquad P \in \pi_0, \quad Q(\tau) \in \mathbb{C}[\tau].$$

4.1 Screening operators and generators for $\mathcal{W}(\mathfrak{gl}_N)$

Here π_0 is the algebra of polynomials in the variables $\mu_i[r]$ with $i = 1, \dots, N$ and $r < 0$. The screening operators $V_1, \dots, V_{N-1}$ are defined by

$$V_i = \sum_{r=0}^{\infty} V_{i\,[r]} \Big(\frac{\partial}{\partial \mu_i[-r-1]} - \frac{\partial}{\partial \mu_{i+1}[-r-1]} \Big),$$

where the coefficients $V_{i\,[r]}$ are found from the expansion of a formal generating function in a variable z,

$$\sum_{r=0}^{\infty} V_{i\,[r]}\, z^r = \exp \sum_{m=1}^{\infty} \frac{\mu_i[-m] - \mu_{i+1}[-m]}{m}\, z^m.$$

Define elements $\mathcal{E}_1, \dots, \mathcal{E}_N$ of π_0 by the expansion in $\pi_0 \otimes \mathbb{C}[\tau]$,

$$\big(\tau + \mu_N[-1]\big) \dots \big(\tau + \mu_1[-1]\big) = \tau^N + \mathcal{E}_1\, \tau^{N-1} + \dots + \mathcal{E}_N, \tag{4.1}$$

known as the *Miura transformation*. Explicitly, using the notation (1.14) we can write the coefficients as

$$\mathcal{E}_m = e_m\big(T + \mu_1[-1], \dots, T + \mu_N[-1]\big), \tag{4.2}$$

which follows easily from (4.1) by induction. The family $\mathcal{E}_1, \dots, \mathcal{E}_N$ is a complete set of generators of $\mathcal{W}(\mathfrak{gl}_N)$. Verifying that all elements $\mathcal{E}_i$ are annihilated by the screening operators is straightforward. This is implied by the relations for the operators on π_0,

$$V_i\, T = \big(T + \mu_i[-1] - \mu_{i+1}[-1]\big)\, V_i, \qquad i = 1, \dots, N-1. \tag{4.3}$$

They imply the corresponding relations for the operators on $\pi_0 \otimes \mathbb{C}[\tau]$,

$$V_i\, \tau = \big(\tau + \mu_i[-1] - \mu_{i+1}[-1]\big)\, V_i, \qquad i = 1, \dots, N-1, \tag{4.4}$$

where τ is regarded as the operator of left multiplication by τ. For each i the relation

$$V_i\, \big(\tau + \mu_N[-1]\big) \dots \big(\tau + \mu_1[-1]\big) = 0$$

then follows easily. Indeed, it reduces to the particular case $N = 2$ where we have

$$V_1 \big(\tau + \mu_2[-1]\big)\big(\tau + \mu_1[-1]\big) = \Big(\big(\tau + \mu_1[-1] - \mu_2[-1]\big)\, V_1 + \mu_2[-1]\, V_1 - 1\Big)\big(\tau + \mu_1[-1]\big)$$
$$= \big(\tau + \mu_1[-1]\big)\, V_1 \big(\tau + \mu_1[-1]\big) - \big(\tau + \mu_1[-1]\big) = 0.$$

Showing that the elements $T^r \mathcal{E}_i$ are algebraically independent generators requires a comparison of the sizes of graded components of π_0 and $\mathcal{W}(\mathfrak{gl}_N)$.

By the definitions (1.13) and (1.14), we have the relations

$$\sum_{k=0}^{m} (-1)^k \, \mathcal{E}_k \, h_{m-k}\big(T + \mu_1[-1], \dots, T + \mu_N[-1]\big) = 0 \tag{4.5}$$

for $m \geqslant 1$, where $\mathcal{E}_0 = 1$ and $\mathcal{E}_k = 0$ for $k > N$. They imply that all elements

$$h_m\big(T + \mu_1[-1], \dots, T + \mu_N[-1]\big), \qquad m \geqslant 1, \tag{4.6}$$

belong to $\mathcal{W}(\mathfrak{gl}_N)$. Moreover, the family (4.6) with $m = 1, \dots, N$ is a complete set of generators of $\mathcal{W}(\mathfrak{gl}_N)$.

Note that the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{sl}_N)$ associated with the special linear Lie algebra $\mathfrak{sl}_N$ can be obtained as the quotient of $\mathcal{W}(\mathfrak{gl}_N)$ by the relation $\mathcal{E}_1 = 0$.

4.2 Screening operators and generators for $\mathcal{W}(\mathfrak{o}_N)$ and $\mathcal{W}(\mathfrak{sp}_N)$

Now π_0 is the algebra of polynomials in the variables $\mu_i[r]$ with $i = 1, \dots, n$ and $r < 0$. The families of generators of the algebras $\mathcal{W}(\mathfrak{o}_N)$ and $\mathcal{W}(\mathfrak{sp}_N)$ reproduced below were constructed in [9, Sec. 8], where equations of the KdV type were introduced for arbitrary simple Lie algebras. The generators are associated with the Miura transformations of the corresponding equations.

4.2.1 Series B_n

The screening operators $V_1, \dots, V_n$ are defined by

$$V_i = \sum_{r=0}^{\infty} V_{i\,[r]} \Big(\frac{\partial}{\partial \mu_i[-r-1]} - \frac{\partial}{\partial \mu_{i+1}[-r-1]}\Big), \tag{4.7}$$

for $i = 1, \dots, n-1$, and

$$V_n = \sum_{r=0}^{\infty} V_{n\,[r]} \, \frac{\partial}{\partial \mu_n[-r-1]},$$

where the coefficients $V_{i\,[r]}$ are found from the expansions

$$\sum_{r=0}^{\infty} V_{i\,[r]} \, z^r = \exp \sum_{m=1}^{\infty} \frac{\mu_i[-m] - \mu_{i+1}[-m]}{m} \, z^m, \qquad i = 1, \dots, n-1$$

and

$$\sum_{r=0}^{\infty} V_{n\,[r]}\, z^r = \exp \sum_{m=1}^{\infty} \frac{\mu_n[-m]}{m}\, z^m.$$

Define elements $\mathcal{E}_2, \dots, \mathcal{E}_{2n+1}$ of π_0 by the expansion

$$\begin{aligned}\big(\tau - \mu_1[-1]\big) \dots \big(\tau - \mu_n[-1]\big)\, \tau\, \big(\tau + \mu_n[-1]\big) \dots \big(\tau + \mu_1[-1]\big)\\ = \tau^{2n+1} + \mathcal{E}_2\, \tau^{2n-1} + \mathcal{E}_3\, \tau^{2n-2} + \dots + \mathcal{E}_{2n+1}. \qquad (4.8)\end{aligned}$$

All of them belong to $\mathcal{W}(\mathfrak{o}_{2n+1})$. By (4.2) we have

$$\mathcal{E}_m = e_m\big(T + \mu_1[-1], \dots, T + \mu_n[-1], T, T - \mu_n[-1], \dots, T - \mu_1[-1]\big). \qquad (4.9)$$

The family $\mathcal{E}_2, \mathcal{E}_4, \dots, \mathcal{E}_{2n}$ is a complete set of generators of $\mathcal{W}(\mathfrak{o}_{2n+1})$. The relation

$$V_i\, \big(\tau - \mu_1[-1]\big) \dots \big(\tau - \mu_n[-1]\big)\, \tau\, \big(\tau + \mu_n[-1]\big) \dots \big(\tau + \mu_1[-1]\big) = 0 \qquad (4.10)$$

is verified for $i = 1, \dots, n-1$ in the same way as for $\mathfrak{gl}_N$ with the use of (4.4). Furthermore,

$$V_n\, \tau = \big(\tau + \mu_n[-1]\big)\, V_n,$$

so that

$$\begin{aligned}V_n\, \big(\tau - \mu_n[-1]\big)\, \tau\, \big(\tau + \mu_n[-1]\big) &= \big(\tau V_n - 1\big)\, \tau\, \big(\tau + \mu_n[-1]\big)\\ &= \tau\, \big(\tau + \mu_n[-1]\big)\, \big(\tau + 2\mu_n[-1]\big)\, V_n,\end{aligned}$$

which implies that (4.10) holds for $i = n$ as well.

By (4.5) all elements

$$h_m\big(T + \mu_1[-1], \dots, T + \mu_n[-1], T, T - \mu_n[-1], \dots, T - \mu_1[-1]\big) \qquad (4.11)$$

belong to $\mathcal{W}(\mathfrak{o}_{2n+1})$. The family of elements (4.11) with $m = 2, 4, \dots, 2n$ forms another complete set of generators of $\mathcal{W}(\mathfrak{o}_{2n+1})$.

4.2.2 Series C_n

The screening operators $V_1, \dots, V_n$ are defined by (4.7) for $i = 1, \dots, n-1$, and

$$V_n = \sum_{r=0}^{\infty} V_{n\,[r]}\, \frac{\partial}{\partial \mu_n[-r-1]},$$

where

$$\sum_{r=0}^{\infty} V_{n\,[r]}\, z^r = \exp \sum_{m=1}^{\infty} \frac{2\mu_n[-m]}{m}\, z^m.$$

Define elements $\mathcal{E}_2, \dots, \mathcal{E}_{2n}$ of π_0 by the expansion

$$\big(\tau - \mu_1[-1]\big) \dots \big(\tau - \mu_n[-1]\big)\big(\tau + \mu_n[-1]\big) \dots \big(\tau + \mu_1[-1]\big) \\ = \tau^{2n} + \mathcal{E}_2\, \tau^{2n-2} + \mathcal{E}_3\, \tau^{2n-3} + \dots + \mathcal{E}_{2n}.$$

All of them belong to $\mathcal{W}(\mathfrak{sp}_{2n})$. By (4.2) we have

$$\mathcal{E}_m = e_m\big(T + \mu_1[-1], \dots, T + \mu_n[-1], T - \mu_n[-1], \dots, T - \mu_1[-1]\big).$$

The family $\mathcal{E}_2, \mathcal{E}_4, \dots, \mathcal{E}_{2n}$ is a complete set of generators of $\mathcal{W}(\mathfrak{sp}_{2n})$. The relation

$$V_i\, \big(\tau - \mu_1[-1]\big) \dots \big(\tau - \mu_n[-1]\big)\big(\tau + \mu_n[-1]\big) \dots \big(\tau + \mu_1[-1]\big) = 0 \qquad (4.12)$$

is verified for $i = 1, \dots, n-1$ in the same way as for $\mathfrak{gl}_N$ with the use of (4.4). In the case $i = n$ we have

$$V_n\, \tau = \big(\tau + 2\mu_n[-1]\big)\, V_n,$$

so that

$$V_n\, \big(\tau - \mu_n[-1]\big)\big(\tau + \mu_n[-1]\big) = \Big(\big(\tau + \mu_n[-1]\big)\, V_n - 1\Big)\big(\tau + \mu_n[-1]\big) \\ = \big(\tau + \mu_n[-1]\big)\big(\tau + 3\mu_n[-1]\big)\, V_n,$$

and (4.12) with $i = n$ also follows.

It follows from (4.5) that the elements

$$h_m\big(T + \mu_1[-1], \dots, T + \mu_n[-1], T - \mu_n[-1], \dots, T - \mu_1[-1]\big)$$

with $m = 2, 4, \dots, 2n$ form another complete set of generators of $\mathcal{W}(\mathfrak{sp}_{2n})$.

4.2.3 Series D_n

The screening operators $V_1, \dots, V_n$ are defined by (4.7) for $i = 1, \dots, n-1$, and

$$V_n = \sum_{r=0}^{\infty} V_{n\,[r]} \Big(\frac{\partial}{\partial \mu_{n-1}[-r-1]} + \frac{\partial}{\partial \mu_n[-r-1]}\Big)$$

where

$$\sum_{r=0}^{\infty} V_{n\,[r]}\, z^r = \exp \sum_{m=1}^{\infty} \frac{\mu_{n-1}[-m] + \mu_n[-m]}{m}\, z^m.$$

Define elements $\mathcal{E}_2, \mathcal{E}_3, \dots$ of π_0 by the expansion of the *pseudo-differential operator*

$$\big(\tau - \mu_1[-1]\big) \dots \big(\tau - \mu_n[-1]\big)\, \tau^{-1}\, \big(\tau + \mu_n[-1]\big) \dots \big(\tau + \mu_1[-1]\big) \\ = \tau^{2n-1} + \sum_{k=2}^{\infty} \mathcal{E}_k\, \tau^{2n-k-1}.$$

The coefficients $\mathcal{E}_k$ are calculated with the use of the relations

$$\tau^{-1}\mu_i[-r-1] = \sum_{k=0}^{\infty} \frac{(-1)^k\,(r+k)!}{r!}\,\mu_i[-r-k-1]\,\tau^{-k-1}.$$

All the elements $\mathcal{E}_k$ belong to $\mathcal{W}(\mathfrak{o}_{2n})$. Moreover, define $\mathcal{E}'_n \in \pi_0$ by

$$\mathcal{E}'_n = \big(\mu_1[-1]-T\big)\dots\big(\mu_n[-1]-T\big), \tag{4.13}$$

so that this element coincides with (1.19). The family $\mathcal{E}_2, \mathcal{E}_4, \dots, \mathcal{E}_{2n-2}, \mathcal{E}'_n$ is a complete set of generators of $\mathcal{W}(\mathfrak{o}_{2n})$. The identity

$$V_i\,\big(\tau-\mu_1[-1]\big)\dots\big(\tau-\mu_n[-1]\big)\,\tau^{-1}\,\big(\tau+\mu_n[-1]\big)\dots\big(\tau+\mu_1[-1]\big)=0 \tag{4.14}$$

is verified with the use of (4.4) and the additional relations

$$V_i\,\tau^{-1} = \big(\tau+\mu_i[-1]-\mu_{i+1}[-1]\big)^{-1}\,V_i, \qquad i=1,\dots,n-1,$$

and

$$V_n\,\tau^{-1} = \big(\tau+\mu_{n-1}[-1]+\mu_n[-1]\big)^{-1}\,V_n. \tag{4.15}$$

In comparison with the types B_n and C_n, an additional calculation is needed for the case $i=n$ in (4.14). It suffices to take $n=2$. We have

$$\begin{aligned}
V_2\,&\big(\tau-\mu_1[-1]\big)\big(\tau-\mu_2[-1]\big)\,\tau^{-1}\,\big(\tau+\mu_2[-1]\big)\big(\tau+\mu_1[-1]\big)\\
&= \Big(\big(\tau+\mu_2[-1]\big)\,V_2-1\Big)\big(\tau-\mu_2[-1]\big)\,\tau^{-1}\,\big(\tau+\mu_2[-1]\big)\big(\tau+\mu_1[-1]\big)\\
&= \Big(\big(\tau+\mu_2[-1]\big)\big(\tau+\mu_1[-1]\big)\,V_2-2\tau\Big)\,\tau^{-1}\,\big(\tau+\mu_2[-1]\big)\big(\tau+\mu_1[-1]\big).
\end{aligned}$$

Furthermore, applying the operator V_2 we find

$$\begin{aligned}
V_2\,\big(\tau+\mu_2[-1]\big)\big(\tau+\mu_1[-1]\big) &= \Big(\big(\tau+\mu_1[-1]+2\,\mu_2[-1]\big)\,V_2+1\Big)\big(\tau+\mu_1[-1]\big)\\
&= 2\,\big(\tau+\mu_1[-1]+\mu_2[-1]\big)
\end{aligned}$$

and so by (4.15),

$$V_2\,\tau^{-1}\,\big(\tau+\mu_2[-1]\big)\big(\tau+\mu_1[-1]\big)=2$$

thus completing the calculation.

The relations

$$V_i\,\big(\mu_1[-1]-T\big)\dots\big(\mu_n[-1]-T\big)=0, \qquad i=1,\dots,n,$$

are verified with the use of (4.3).

5 Generators of the $\mathcal{W}$-algebras

Here we prove the Main Theorem stated in the Introduction by deriving it from Theorems 3.2, 3.5 and 3.8.

Choose a basis $X_1, \dots, X_d$ of the simple Lie algebra $\mathfrak{g}$ and write the commutation relations

$$[X_i, X_j] = \sum_{k=1}^{d} c_{ij}^k\, X_k$$

with structure constants c_{ij}^k. Consider the Lie algebras $\mathfrak{g}[t]$ and $t^{-1}\mathfrak{g}[t^{-1}]$ and combine their generators into formal series in u^{-1} and u,

$$X_i(u) = \sum_{r=0}^{\infty} X_i[r]\, u^{-r-1} \qquad \text{and} \qquad X_i(u)_+ = \sum_{r=0}^{\infty} X_i[-r-1]\, u^{r}.$$

The commutation relations of these Lie algebras written in terms of the formal series take the form

$$(u-v)\,[X_i(u), X_j(v)] = -\sum_{k=1}^{d} c_{ij}^k\, \big(X_k(u) - X_k(v)\big),$$

$$(u-v)\,[X_i(u)_+, X_j(v)_+] = \sum_{k=1}^{d} c_{ij}^k\, \big(X_k(u)_+ - X_k(v)_+\big).$$

Observe that the second family of commutation relations is obtained from the first by replacing $X_i(u)$ with the respective series $-X_i(u)_+$.

On the other hand, in the classical types, the elements of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}[t])$ and their Harish-Chandra images calculated in Proposition 3.1 and Theorems 3.2, 3.5 and 3.8 are all expressed in terms of the series of the form $X_i(u)$. Therefore, the corresponding Harish-Chandra images of the elements of the universal enveloping algebra $\mathrm{U}(t^{-1}\mathfrak{g}[t^{-1}])$ are readily found from those theorems by replacing $X_i(u)$ with the respective series $-X_i(u)_+$.

To be consistent with the definition for the Wakimoto modules in [11], we will write the resulting formulas for the opposite choice of the Borel subalgebra, as compared to the homomorphism (3.4). To this end, in types B, C and D we consider the automorphism σ of the Lie algebra $t^{-1}\mathfrak{g}_N[t^{-1}]$ defined on the generators by

$$\sigma : F_{ij}[r] \mapsto -F_{ji}[r]. \tag{5.1}$$

We get the commutative diagram

$$\begin{array}{ccc} \mathrm{U}\big(t^{-1}\mathfrak{g}_N[t^{-1}]\big)^{\mathfrak{h}} & \longrightarrow & \mathbb{C}\,\big[\mu_i[r]\big] \\ \sigma\big\downarrow & & \big\downarrow\sigma \\ \mathrm{U}\big(t^{-1}\mathfrak{g}_N[t^{-1}]\big)^{\mathfrak{h}} & \xrightarrow{\ \chi\ } & \mathbb{C}\,\big[\mu_i[r]\big], \end{array} \tag{5.2}$$

where i ranges over the set $\{1,\dots,n\}$ while $r<0$. The top and bottom horizontal arrows indicate the versions of the Harish-Chandra homomorphism defined as in (3.4), where the left ideal I is now generated by all elements $F_{ij}[r]$ with the conditions $1\leqslant i<j\leqslant N$ and $r<0$ for the top arrow, and by all elements $F_{ij}[r]$ with the conditions $N\geqslant i>j\geqslant 1$ and $r<0$ for the bottom arrow (which we denote by χ). The second vertical arrow indicates the isomorphism which takes $\mu_i[r]$ to $-\mu_i[r]$.

Note that an automorphism analogous to (5.1) can be used in the case of the Lie algebra $\mathfrak{gl}_N$ to get the corresponding description of the homomorphism χ and to derive the formulas (1.11) and (1.12). However, these formulas follow easily from the observation that $\tau+E[-1]$ is a Manin matrix by the same argument as in the proof of Proposition 3.1.

To state the result in types B, C and D, introduce the formal series

$$\gamma_m(\omega)\,\mathrm{tr}\, S^{(m)}\big(\partial_u+F_1(u)_+\big)\dots\big(\partial_u+F_m(u)_+\big), \tag{5.3}$$

where we use notation (1.16) with $\omega=N$ and $\omega=-N$ in the orthogonal and symplectic case, respectively, and

$$F(u)_+=\sum_{i,j=1}^{N} e_{ij}\otimes F_{ij}(u)_+\in \mathrm{End}\,\mathbb{C}^N\otimes \mathrm{U}\big(t^{-1}\mathfrak{g}_N[t^{-1}]\big)[[u]].$$

We will assume that in the symplectic case the values of m in (5.3) are restricted to $1\leqslant m\leqslant 2n+1$; see [24, Sec. 3.3 and Sec. 4.1]. The trace is taken over all m copies $\mathrm{End}\,\mathbb{C}^N$ in the algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^N\otimes\dots\otimes\mathrm{End}\,\mathbb{C}^N}_{m}\otimes\,\mathrm{U}\big(t^{-1}\mathfrak{g}_N[t^{-1}]\big)[[u,\partial_u]] \tag{5.4}$$

and we use matrix notation as in (1.15). We set

$$\mu_i(u)_+=\sum_{r=0}^{\infty}\mu_i[-r-1]\,u^r,\qquad i=1,\dots,n.$$

Proposition 5.1. *The image of the series* (5.3) *under the homomorphism* χ *is given by the formula*:

type B_n:
$$h_m\big(\partial_u+\mu_1(u)_+,\dots,\partial_u+\mu_n(u)_+,\partial_u-\mu_n(u)_+,\dots\partial_u-\mu_1(u)_+\big),$$

type D_n:
$$\tfrac12\,h_m\big(\partial_u+\mu_1(u)_+,\dots,\partial_u+\mu_{n-1}(u)_+,\partial_u-\mu_n(u)_+,\dots\partial_u-\mu_1(u)_+\big)$$
$$+\,\tfrac12\,h_m\big(\partial_u+\mu_1(u)_+,\dots,\partial_u+\mu_n(u)_+,\partial_u-\mu_{n-1}(u)_+,\dots\partial_u-\mu_1(u)_+\big),$$

type C_n:
$$e_m\big(\partial_u+\mu_1(u)_+,\dots,\partial_u+\mu_n(u)_+,\partial_u,\partial_u-\mu_n(u)_+,\dots\partial_u-\mu_1(u)_+\big).$$

Proof. We start with the orthogonal case $\mathfrak{g}_N = \mathfrak{o}_N$. The argument in the beginning of this section shows that the image of the series

$$\gamma_m(N)\,\mathrm{tr}\,S^{(m)}\big(\partial_u - F_1(u)_+\big)\dots\big(\partial_u - F_m(u)_+\big)$$

under the homomorphism given by the top horizontal arrow in (5.2) is found by Theorems 3.2 and 3.5, where $\mu_i(u)$ should be respectively replaced by $-\mu_i(u)_+$ for $i = 1,\dots,n$. Therefore, using the diagram (5.2) we find that the image of the series

$$\gamma_m(N)\,\mathrm{tr}\,S^{(m)}\big(\partial_u + F^t_1(u)_+\big)\dots\big(\partial_u + F^t_m(u)_+\big) \tag{5.5}$$

under the homomorphism χ is given by the respective B_n and D_n type formulas in the proposition, where we set $F^t(u)_+ = \sum_{i,j} e_{ij} \otimes F_{ji}(u)_+$. It remains to observe that the series (5.5) coincides with (5.3). This follows by applying the simultaneous transpositions $e_{ij} \mapsto e_{ji}$ to all m copies of $\mathrm{End}\,\mathbb{C}^N$ and taking into account the fact that $S^{(m)}$ stays invariant.

In the symplectic case, we suppose first that $m \leqslant n$. Starting with the Harish-Chandra image provided by Theorem 3.8 and applying the same argument as in the orthogonal case, we conclude that the image of the series

$$\gamma_m(-2n)\,\mathrm{tr}\,S^{(m)}\big(\partial_u - F_1(u)_+\big)\dots\big(\partial_u - F_m(u)_+\big) \tag{5.6}$$

under the homomorphism χ agrees with the C_n type formula given by the statement of the proposition. One more step here is to observe that this series coincides with (5.3). Indeed, this follows by applying the simultaneous transpositions $e_{ij} \mapsto \varepsilon_i\varepsilon_j\, e_{j'i'}$ to all m copies of $\mathrm{End}\,\mathbb{C}^N$. On the one hand, this transformation does not affect the trace of any element of (5.4), while on the other hand, each factor $\partial_u - F_i(u)_+$ is taken to $\partial_u + F_i(u)_+$ and the operator $S^{(m)}$ stays invariant.

Finally, extending the argument of [24, Sec. 3.3] to the case $m = 2n+1$ and using the results of [24, Sec. 5], we find that for all values $1 \leqslant m \leqslant 2n+1$ the coefficients $\Phi^{(s)}_{ma}$ in the expansion

$$\gamma_m(-2n)\,\mathrm{tr}\,S^{(m)}\big(\partial_u + F_1(u)_+\big)\dots\big(\partial_u + F_m(u)_+\big) = \sum_{a=0}^{m}\sum_{s=0}^{\infty} \Phi^{(s)}_{ma}\, u^s\, \partial_u^a$$

belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{sp}}_{2n})$. The image of the element $\Phi^{(s)}_{ma}$ under the isomorphism (1.4) is a polynomial in the generators $T^r\mathcal{E}_{2k}$ of the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{o}_{2n+1})$, where $k = 1,\dots,n$ and $r \geqslant 0$; see (4.9). For a fixed value of m and varying values of n the coefficients of the polynomial are rational functions in n. Therefore, they are uniquely determined by infinitely many values of $n \geqslant m$. This allows us to extend the range of n to all values $n \geqslant (m-1)/2$ for which the expression (5.3) is defined. □

Corollary 5.2. *Theorem 3.8 holds for all values* $1 \leqslant m \leqslant 2n+1$.

Proof. This follows by reversing the argument used in the proof of Proposition 5.1. □

With the exception of the formula (1.19) for the image of the element ϕ'_n in type D_n, all statements of the Main Theorem now follow from Proposition 5.1. It suffices to note that the coefficients of the polynomial (1.17) and the differential operator (5.3) are related via the vertex algebra structure on the vacuum module $V_{-h^\vee}(\mathfrak{g}_N)$. In particular, the evaluation of the coefficients of the differential operator (5.3) at $u = 0$ reproduces the corresponding coefficients of the polynomial (1.17). This implies the desired formulas for the Harish-Chandra images in the Main Theorem; see e.g. [11, Ch. 2] for the relevant properties of vertex algebras.

Now consider the element $\mathcal{E}'_n$ of the algebra $\mathcal{W}(\mathfrak{o}_{2n})$ defined in (4.13) and which coincides with the element (1.19). To prove that the Harish-Chandra image of the element ϕ'_n introduced by (1.18) equals $\mathcal{E}'_n$, use the automorphism of the Lie algebra $t^{-1}\mathfrak{o}_{2n}[t^{-1}]$ defined on the generators by

$$F_{k\,l}[r] \mapsto F_{\tilde{k}\,\tilde{l}}[r], \tag{5.7}$$

where $k \mapsto \tilde{k}$ is the involution on the set $\{1, \dots, 2n\}$ such that $n \mapsto n'$, $n' \mapsto n$ and $k \mapsto k$ for all $k \neq n, n'$. Note that $\phi'_n \mapsto -\phi'_n$ under the automorphism (5.7). Similarly, $\mathcal{E}'_n \mapsto -\mathcal{E}'_n$ with respect to the automorphism of $t^{-1}\mathfrak{h}_{2n}[t^{-1}]$ induced by (5.7).

As a corollary of the Main Theorem and the results of [24] we obtain from the isomorphism (1.4) that the elements

$$\mathcal{F}_m = \tfrac{1}{2}\, h_m\big(T + \mu_1[-1], \dots, T + \mu_{n-1}[-1], T - \mu_n[-1], \dots T - \mu_1[-1]\big)$$
$$+ \tfrac{1}{2}\, h_m\big(T + \mu_1[-1], \dots, T + \mu_n[-1], T - \mu_{n-1}[-1], \dots T - \mu_1[-1]\big),$$

with $m = 2, 4, \dots, 2n-2$ together with $\mathcal{E}'_n$ form a complete set of generators of $\mathcal{W}(\mathfrak{o}_{2n})$ (this fact does not rely on the calculation of the image of the Pfaffian). Observe that all elements $T^r\mathcal{F}_{2k}$ with $k = 1, \dots, n-1$ and $r \geqslant 0$ are stable under the automorphism (5.7). Since the Harish-Chandra image $\chi(\phi'_n)$ is a unique polynomial in the generators of $\mathcal{W}(\mathfrak{o}_{2n})$ and its degree with respect to the variables $\mu_1[-1], \dots, \mu_n[-1]$ does not exceed n, we can conclude that $\chi(\phi'_n)$ must be proportional to $\mathcal{E}'_n$. The coefficient of the product $\mu_1[-1] \dots \mu_n[-1]$ in each of these two polynomials is equal to 1 thus proving that $\chi(\phi'_n) = \mathcal{E}'_n$. This completes the proof of the Main Theorem.

The properties of vertex algebras mentioned above and the relation $\chi(\phi'_n) = \mathcal{E}'_n$ imply the respective formulas for the Harish-Chandra images of the Pfaffians $\operatorname{Pf} \widetilde{F}(u)_+$ and $\operatorname{Pf} \widetilde{F}(u)$ defined by (1.18) with the matrix $\widetilde{F}[-1]$ replaced by the skew-symmetric matrices $\widetilde{F}(u)_+ = [F_{ij'}(u)_+]$ and $\widetilde{F}(u) = [F_{ij'}(u)]$, respectively.

Corollary 5.3. *The Harish-Chandra images of the Pfaffians are found by*

$$\chi : \operatorname{Pf} \widetilde{F}(u)_+ \mapsto \big(\mu_1(u)_+ - \partial_u\big) \dots \big(\mu_n(u)_+ - \partial_u\big)\, 1,$$

$$\operatorname{Pf} \widetilde{F}(u) \mapsto \big(\mu_1(u) - \partial_u\big) \dots \big(\mu_n(u) - \partial_u\big)\, 1,$$

where the second map is defined in (3.4)*.*

Proof. The first relation follows by the application of the state-field correspondence map to the Segal–Sugawara vector (1.18) and using its Harish-Chandra image (1.19). To get the second relation, apply the automorphism (5.1) to the first relation to calculate the image of $\operatorname{Pf} \widetilde{F}(u)_+$ with respect to the homomorphism defined by the top arrow in (5.2),

$$\operatorname{Pf} \widetilde{F}(u)_+ \mapsto \big(\mu_1(u)_+ + \partial_u\big) \dots \big(\mu_n(u)_+ + \partial_u\big)\, 1.$$

Now replace $\widetilde{F}(u)_+$ with $-\widetilde{F}(u)$ and replace $\mu_i(u)_+$ with $-\mu_i(u)$ for $i = 1, \dots, n$. □

The isomorphism (1.4) and the Main Theorem provide complete sets of generators of the classical $\mathcal{W}$-algebras. In types B and C they coincide with those introduced in Sec. 4.2, but different in type D, as pointed out in the above argument.

Corollary 5.4. *The elements $\mathcal{F}_2, \mathcal{F}_4, \dots, \mathcal{F}_{2n-2}, \mathcal{E}'_n$ form a complete set of generators of $\mathcal{W}(\mathfrak{o}_{2n})$.* □

To complete this section, we point out that the application of the state-field correspondence map to the coefficients of the polynomial (1.17) and to the additional element (1.18) in type D_n yields Sugawara operators associated with $\widehat{\mathfrak{g}}_N$. They act as scalars in the Wakimoto modules at the critical level. The eigenvalues are found from the respective formulas of Proposition 5.1 and Corollary 5.3 as follows from the general theory of Wakimoto modules and their connection with the classical $\mathcal{W}$-algebras; see [11, Ch. 8].

6 Casimir elements for $\mathfrak{g}_N$

We apply the theorems of Sec. 3 to calculate the Harish-Chandra images of certain Casimir elements for the orthogonal and symplectic Lie algebras previously considered in [16]. Our formulas for the Harish-Chandra images are equivalent to those in [16], but take a different form. We will work with the isomorphism (1.2), where the Cartan subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g} = \mathfrak{g}_N$ is defined in the beginning of Sec. 2 and the subalgebra $\mathfrak{n}_+$ is spanned by the elements F_{ij} with $1 \leqslant i < j \leqslant N$. We will use the notation $\mu_i = F_{ii}$ for $i = 1, \dots, N$ so that $\mu_i + \mu_{i'} = 0$ for all i.

Consider the evaluation homomorphism

$$\operatorname{ev} : \mathrm{U}(\mathfrak{g}_N[t]) \to \mathrm{U}(\mathfrak{g}_N), \qquad F_{ij}(u) \mapsto F_{ij}\, u^{-1},$$

so that $F_{ij}[0] \mapsto F_{ij}$ and $F_{ij}[r] \mapsto 0$ for $r \geqslant 1$. The image of the series $\mu_i(u)$ then coincides with $\mu_i\, u^{-1}$. Applying the evaluation homomorphism to the series involved in Theorems 3.2, 3.5 and 3.8 we get the corresponding Harish-Chandra images of the elements of the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$. The formulas are obtained by replacing $F_{ij}(u)$ with $F_{ij}\, u^{-1}$ and $\mu_i(u)$ with $\mu_i\, u^{-1}$. Multiply the resulting formulas by u^m from the left. In the case $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$ use the relation

$$u^m\, (\partial_u + F_1 u^{-1}) \dots (\partial_u + F_m u^{-1}) = (u\partial_u + F_1 - m + 1) \dots (u\partial_u + F_m) \tag{6.1}$$

to conclude that the Harish-Chandra image of the polynomial

$$\gamma_m(N)\, \mathrm{tr}\, S^{(m)}(F_1 + v - m + 1) \dots (F_m + v) \tag{6.2}$$

with $v = u\partial_u$ is found by

$$\sum_{1 \leqslant i_1 \leqslant \dots \leqslant i_m \leqslant 1'} (\mu_{i_1} + v - m + 1) \dots (\mu_{i_m} + v),$$

summed over the multisets $\{i_1, \dots, i_m\}$ with entries from $\{1, \dots, n, n', \dots, 1'\}$. By the arguments of [16], the Harish-Chandra image of the polynomial (6.2) is essentially determined by those for the even values $m = 2k$ and a particular value of v.

Corollary 6.1. *For $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$ the image of the Casimir element*

$$\gamma_{2k}(N)\, \mathrm{tr}\, S^{(2k)}\, (F_1 - k) \dots (F_{2k} + k - 1)$$

under the Harish-Chandra isomorphism is given by

$$\sum_{1 \leqslant i_1 \leqslant \dots \leqslant i_{2k} \leqslant 1'} (\mu_{i_1} - k) \dots (\mu_{i_{2k}} + k - 1), \tag{6.3}$$

summed over the multisets $\{i_1, \dots, i_{2k}\}$ with entries from $\{1, \dots, n, n', \dots, 1'\}$. Moreover, the element (6.3) *coincides with the factorial complete symmetric function*

$$\sum_{1 \leqslant j_1 \leqslant \dots \leqslant j_k \leqslant n} \big(l_{j_1}^2 - (j_1 - 1/2)^2\big) \dots \big(l_{j_k}^2 - (j_k + k - 3/2)^2\big), \tag{6.4}$$

where $l_i = \mu_i + n - i + 1/2$ for $i = 1, \dots, n$.

Proof. The coincidence of the elements (6.3) and (6.4) is verified by using the characterization theorem for the factorial symmetric functions [33]; see also [16]. Namely, both elements are symmetric polynomials in $l_1^2, \dots, l_n^2$ of degree k, and their top degree components are both equal to the complete symmetric polynomial $h_k(l_1^2, \dots, l_n^2)$. It remains to verify that each of the elements (6.3) and (6.4) vanishes when $(\mu_1, \dots, \mu_n)$ is specialized to a partition with $\mu_1 + \dots + \mu_n < k$ which is straightforward. □

Similarly, if $\mathfrak{g}_N = \mathfrak{o}_{2n}$ use the same relation (6.1) to conclude from Theorem 3.5 that the Harish-Chandra image of the polynomial

$$2\,\gamma_m(N)\,\mathrm{tr}\,S^{(m)}(F_1+v-m+1)\dots(F_m+v)$$

is found by

$$\sum_{\substack{1\leqslant i_1\leqslant\dots\leqslant i_m\leqslant 2n\\ i_s\neq n}}(\mu_{i_1}+v-m+1)\dots(\mu_{i_m}+v)+\sum_{\substack{1\leqslant i_1\leqslant\dots\leqslant i_m\leqslant 2n\\ i_s\neq n'}}(\mu_{i_1}+v-m+1)\dots(\mu_{i_m}+v),$$

where the summation indices in the first sum do not include n and the summation indices in the second sum do not include n'.

Corollary 6.2. *For $\mathfrak{g}_N = \mathfrak{o}_{2n}$ the image of the Casimir element*

$$\gamma_{2k}(N)\,\mathrm{tr}\,S^{(2k)}\,(F_1-k)\dots(F_{2k}+k-1)$$

under the Harish-Chandra isomorphism is given by

$$\tfrac{1}{2}\sum_{\substack{1\leqslant i_1\leqslant\dots\leqslant i_{2k}\leqslant 2n\\ i_s\neq n}}(\mu_{i_1}-k)\dots(\mu_{i_{2k}}+k-1)+\tfrac{1}{2}\sum_{\substack{1\leqslant i_1\leqslant\dots\leqslant i_{2k}\leqslant 2n\\ i_s\neq n'}}(\mu_{i_1}-k)\dots(\mu_{i_{2k}}+k-1).$$

Moreover, this element coincides with the factorial complete symmetric function

$$\sum_{1\leqslant j_1\leqslant\dots\leqslant j_k\leqslant n}\big(l_{j_1}^2-(j_1-1)^2\big)\dots\big(l_{j_k}^2-(j_k+k-2)^2\big),$$

where $l_i=\mu_i+n-i$ for $i=1,\dots,n$.

Proof. The coincidence of the two expressions for the Harish-Chandra image is verified in the same way as for the case of $\mathfrak{o}_{2n+1}$ outlined above. □

Now suppose that $\mathfrak{g}_N = \mathfrak{sp}_{2n}$ and use the relation

$$u^m\,(-\partial_u+F_1\,u^{-1})\dots(-\partial_u+F_m\,u^{-1})=(-u\partial_u+F_1+m-1)\dots(-u\partial_u+F_m)$$

to conclude from Theorem 3.8 and Corollary 5.2 that the Harish-Chandra image of the polynomial

$$\gamma_m(-2n)\,\mathrm{tr}\,S^{(m)}(F_1+v+m-1)\dots(F_m+v)$$

with $v=-u\partial_u$ is found by

$$\sum_{1\leqslant i_1<\dots<i_m\leqslant 1'}(\mu_{i_1}+v+m-1)\dots(\mu_{i_m}+v),$$

summed over the subsets $\{i_1,\dots,i_m\}$ of the set $\{1,\dots,n,0,n',\dots,1'\}$ with the ordering $1<\dots<n<0<n'<\dots<1'$, where $\mu_0:=0$. Taking $m=2k$ and $v=-k+1$ we get the following.

Corollary 6.3. *For $\mathfrak{g}_N = \mathfrak{sp}_{2n}$ the image of the Casimir element*

$$\gamma_{2k}(-2n)\,\mathrm{tr}\, S^{(2k)}\,(F_1 + k)\dots(F_{2k} - k + 1)$$

under the Harish-Chandra isomorphism is given by

$$\sum_{1\leqslant i_1<\dots<i_{2k}\leqslant 1'} (\mu_{i_1} + k)\dots(\mu_{i_{2k}} - k + 1), \tag{6.5}$$

summed over the subsets $\{i_1,\dots,i_{2k}\} \subset \{1,\dots,n,0,n',\dots,1'\}$. Moreover, the element (6.5) *coincides with the factorial elementary symmetric function*

$$(-1)^k \sum_{1\leqslant j_1<\dots<j_k\leqslant n} \big(l_{j_1}^2 - j_1^2\big)\dots\big(l_{j_k}^2 - (j_k - k + 1)^2\big), \tag{6.6}$$

where $l_i = \mu_i + n - i + 1$ for $i = 1,\dots,n$.

Proof. To verify that the elements (6.5) and (6.6) coincide, use again the characterization theorem for the factorial symmetric functions [33]; see also [16]. Both elements are symmetric polynomials in $l_1^2,\dots,l_n^2$ of degree k, and their top degree components are both equal to the elementary symmetric polynomial $(-1)^k e_k(l_1^2,\dots,l_n^2)$. Furthermore, it is easily seen that each of the elements (6.5) and (6.6) vanishes when $(\mu_1,\dots,\mu_n)$ is specialized to a partition with $\mu_1 + \dots + \mu_n < k$. □

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