Cofull Embeddings in Coset Monoids

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Abstract

Easdown, East and FitzGerald (2004) gave a sufficient condition for a (factorizable inverse) monoid to embed as a cofull submonoid of the coset monoid of its group of units. We show that this condition is also *necessary*. This yields a simple description of the class of finite monoids which embed in the coset monoids of their group of units. We apply our results to give a short proof of the result of McAlister (1980) that the symmetric inverse semigroup on a finite set X does not embed in the coset monoid of the symmetric group on X. We also present examples which show that the word "cofull" may not be removed.

Keywords: Factorizable inverse monoid, coset monoid, symmetric inverse semigroup.

MSC: Primary 20M18; Secondary 20M30.

1 Factorizable Inverse Monoids and Coset Monoids

If M is a monoid then we denote by G_M the group of units of M. We say that a submonoid of M is *cofull* if it contains G_M. If N is another monoid and $\psi : M \to N$ an embedding, then we say that ψ is *cofull* if $M\psi$ is a cofull submonoid of N.

If M is an inverse monoid then we denote by E_M the semilattice of idempotents of M. An inverse monoid M is *factorizable* if $M = E_M G_M$. The study of factorizable inverse monoids (henceforth FIMs) was initiated in [2]; for related studies see [4, 5, 6, 10] and references therein.

Let G be a group and denote by $\mathcal{S}(G)$ the join semilattice of all subgroups of G. The join $H \vee K$ of two subgroups $H, K \in \mathcal{S}(G)$ is defined to be $\langle H \cup K \rangle$, the smallest subgroup of G containing HK. Now let

$$\mathcal{C}(G) = \left\{ Hg \,\middle|\, H \in \mathcal{S}(G)\,,\ g \in G \right\}$$

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be the set of all cosets of all subgroups of G. An associative product $*$ is defined on $\mathcal{C}(G)$, for $H, K \in \mathcal{S}(G)$ and $g, l \in G$, by

$$(Hg) * (Kl) = (H \vee gKg^{-1})gl,$$

the smallest coset of G containing $HgKl$. The set $\mathcal{C}(G)$ is a FIM under $*$ with identity $\{1\}$, known as the *coset monoid* of G. The coset monoid was introduced in [8, 9]; see also [7]. Following is a collection of some elementary properties of coset monoids; these, along with other properties, are stated in [7].

Lemma 1 *Let G be a group. Then*

(i) $E_{\mathcal{C}(G)} = \mathcal{S}(G)$*;*

(ii) $G_{\mathcal{C}(G)} = \big\{\{g\} \,\big|\, g \in G\big\} \cong G$*; and*

(iii) *the subgroups of $\mathcal{C}(G)$ are precisely the sections of G (a section of G is a quotient of a subgroup of G).* □

The following was proved in [7].

Theorem 2 (McAlister) *Let X be a set. Then*

(i) *the symmetric inverse semigroup $\mathcal{I}_X$ embeds in the coset monoid $\mathcal{C}(\mathcal{G}_Y)$ of the symmetric group $\mathcal{G}_Y$ on a set Y with $|Y| = |X| + 1$; and*

(ii) *$\mathcal{I}_X$ does not embed in $\mathcal{C}(\mathcal{G}_X)$ if X is finite and nonempty.* □

It follows by Theorem 2(i) and the Wagner-Preston Theorem that any inverse monoid (indeed any inverse semigroup) embeds in the coset monoid of some group. An interesting question that arises is *"given an inverse monoid M, what is the minimum cardinality of a group G for which an embedding $M \to \mathcal{C}(G)$ exists?"* Now suppose that $\Psi : M \to \mathcal{C}(G)$ is an embedding of an inverse monoid M in the coset monoid of a group G. By Lemma 1(iii), the image of G_M under Ψ is a section of G, showing that the cardinality of G is bounded below by the cardinality of G_M. Thus, another natural question arises: *"Which inverse monoids M embed in $\mathcal{C}(G_M)$?"* The goal of this article is to give necessary and sufficient conditions for an inverse monoid M to embed as a *cofull* submonoid of $\mathcal{C}(G_M)$.

Let M be an inverse monoid and write $E = E_M$ and $G = G_M$. For $e \in E$ let

$$G_e = \{g \in G \,|\, eg = e\}.$$

It is easy to check that each G_e is a subgroup of G, and that $G_e \vee G_f \subseteq G_{ef}$ for each $e, f \in E$. Define a map

$$\psi_M : E \to \mathcal{S}(G) : e \mapsto G_e \qquad \text{for each } e \in E.$$

Let $\mathscr{C}$ denote the class of *factorizable* inverse monoids M for which ψ_M is a semilattice embedding. The following was proved in [5].

Theorem 3 (Easdown, East, FitzGerald) *A monoid M embeds as a cofull submonoid of $\mathcal{C}(G_M)$ if $M \in \mathscr{C}$.* $\square$

This theorem was proved by showing that if $M \in \mathscr{C}$, then the map

$$M \to \mathcal{C}(G) : eg \mapsto G_e g \qquad \text{for each } e \in E \text{ and } g \in G$$

is a cofull embedding. Our main goal is to show that the condition $M \in \mathscr{C}$ is also *necessary* for a monoid to embed as a cofull submonoid of $\mathcal{C}(G_M)$. In addition we show that the word "cofull" may be removed within the class of *finite* (but *not* infinite) inverse monoids.

2 Cofull Embeddings

Our goal in this section is to show that a monoid M embeds as a cofull submonoid of $\mathcal{C}(G_M)$ if and only if $M \in \mathscr{C}$.

Lemma 4 *Any cofull submonoid of a factorizable inverse monoid is a factorizable inverse monoid.*

Proof Suppose that N is a cofull submonoid of a FIM M, and choose $m \in N$. Then $m = eg$ for some $e \in E_M$ and $g \in G_M$. Since N is cofull, we have $g^{-1} \in N$ and so $m^{-1} = g^{-1}e = g^{-1}(eg)g^{-1} = g^{-1}mg^{-1} \in N$, showing that N is inverse. We also have $e = mg^{-1} \in N$ so that N is factorizable. $\square$

Theorem 5 *A monoid M embeds as a cofull submonoid of $\mathcal{C}(G_M)$ if and only if $M \in \mathscr{C}$.*

Proof The "if" part of the theorem is true by Theorem 3. To show the converse, it suffices to show that $N \in \mathscr{C}$ for every cofull submonoid N of $\mathcal{C}(G_M)$. Write $G = G_M$, and $\overline{G} = G_{\mathcal{C}(G)} = \big\{\{g\} \,\big|\, g \in G\big\}$. Now N is a FIM by Lemma 4, so it remains only to show that

$$\psi_N : E_N \to \mathcal{S}(G_N)$$

is an embedding. Now $E_N = \mathcal{S}(G) \cap N$, and $G_N = \overline{G}$ since N is cofull. Further, if $H \in E_N$, then $H\psi_N = \overline{H} = \big\{\{h\} \,\big|\, h \in H\big\}$. It follows that ψ_N is an embedding since $\overline{H} \vee \overline{K} = \overline{H \vee K}$ for any subgroups $H, K \in \mathcal{S}(G)$. $\square$

As a corollary, we have the following.

Theorem 6 *A finite monoid M embeds in $\mathcal{C}(G_M)$ if and only if $M \in \mathscr{C}$.*

Proof Write $G = G_M$. Any embedding $\Psi : M \to \mathcal{C}(G)$ gives rise to an embedding $\overline{\Psi} : G \to G_{\mathcal{C}(G)} \cong G$ since the image of G under Ψ, being finite, cannot be contained in a proper section of G. Since G is finite, $\overline{\Psi}$ is an isomorphism whence Ψ is cofull, and we are done by Theorem 5. $\square$

We now apply Theorem 6 to provide an alternative proof of Theorem 2(ii).

Corollary 7 (McAlister) *Let X be a finite nonempty set. Then $\mathcal{I}_X$ does not embed in $\mathcal{C}(\mathcal{G}_X)$.*

Proof Put $G = \mathcal{G}_X = G_{\mathcal{I}_X}$. For $A \subseteq X$ denote by id_A the identity map on A so that $E_{\mathcal{I}_X} = \{\mathrm{id}_A \mid A \subseteq X\}$. Then for each $A \subseteq X$,

$$G_{\mathrm{id}_A} = \big\{\pi \in G \,\big|\, a\pi = a \;\; (\forall a \in A)\big\}$$

is the pointwise stabilizer of A, which we will denote by $\mathrm{Stab}(A)$. Now if $x \in X$, then $\mathrm{Stab}(X) = \mathrm{Stab}\big(X \setminus \{x\}\big) = \{\mathrm{id}_X\}$ so that $\psi_{\mathcal{I}_X}$ is not injective. We are now done by Theorem 6. □

We remark that if X is any set with $|X| \geq 2$, then the map $\psi_{\mathcal{I}_X}$ is not a semilattice homomorphism since if $x, y \in X$ with $x \neq y$ then, writing $A = X \setminus \{x\}$ and $B = X \setminus \{y\}$, we have

$$G_{\mathrm{id}_A} \vee G_{\mathrm{id}_B} = \mathrm{Stab}(A) \vee \mathrm{Stab}(B) = \mathrm{Stab}(A) = \mathrm{Stab}(B) = \{\mathrm{id}_X\}$$

while the transposition which interchanges x and y is in

$$\mathrm{Stab}(X \setminus \{x, y\}) = \mathrm{Stab}(A \cap B) = G_{\mathrm{id}_{A \cap B}} = G_{\mathrm{id}_A \circ \mathrm{id}_B}.$$

3 Other Embeddings

In this final section we consider examples of FIMs M which embed in $\mathcal{C}(G_M)$ but do not belong to $\mathscr{C}$. These FIMs are necessarily infinite, and the embeddings are not cofull.

Example 8 Let X be an infinite set. Then the symmetric inverse semigroup $\mathcal{I}_X \notin \mathscr{C}$ by the comments after the proof of Corollary 7. On the other hand, $\mathcal{I}_X$ does embed in the coset monoid of the symmetric group $\mathcal{G}_X = G_{\mathcal{I}_X}$ by Theorem 2(i).

Now $\mathcal{I}_X$ (indeed $E_{\mathcal{I}_X}$) is uncountable for any infinite set X. Our second example is a countable FIM M for which $|E_M| = 3$ and $\mathrm{rank}(G_M) = 1$. Here for a group G we have written $\mathrm{rank}(G)$ for the minimal cardinality of a set which generates G (as a group).

Example 9 Let $G = \langle x \rangle$ be the infinite cyclic group generated by x, and let G^y be the semigroup obtained by adjoining a zero y to G. Let $M = (G^y)^z$ be the semigroup obtained by adjoing a new zero z to G^y. It is easy to check that M is a FIM with $G_M = G$ and $E_M = \{1, y, z\}$. We also have $G_y = G_z = G$ so that $M \notin \mathscr{C}$. Now define

$$\Psi : M \to \mathcal{C}(G) : \begin{cases} x & \mapsto & \{x^2\} \\ y & \mapsto & \langle x^2 \rangle \\ z & \mapsto & G. \end{cases}$$

Then one may easily check that Ψ is an embedding.

Our final example is also a countable FIM M although in this case we have $|E_M| = 2$ and $\mathrm{rank}(G_M) = 2$.

Example 10 Let $G = \langle x, y \rangle$ be the free group freely generated by $\{x, y\}$. Define a homomorphism

$$\varphi : G \to G : x \mapsto x^2,\ y \mapsto y^2,$$

and put $K = \langle x^2, y^2 \rangle$, the image of φ. Let $B = G/N$ where N is the normal closure in G of $\{xyxy^{-1}x^{-1}y^{-1}\}$. So B has presentation $\langle x, y \mid xyx = yxy \rangle$ and is isomorphic to the braid group on 3 strings; see [1]. It is well known that Nx^2 and Ny^2 generate a free subgroup of B of rank 2; see for example [3]. It follows that $N \cap K = \{1\}$.

Now let $E = \{0, 1\}$ which we consider as a semilattice under multiplication, and put $M = E \times G$. So M is a FIM with $E_M = (E, 1) \cong E$ and $G_M = (1, G) \cong G$, and $M \notin \mathscr{C}$ since $G_{(1,1)} = G_{(0,1)} = \big\{(1, 1)\big\}$. Define

$$\Psi : M \to \mathcal{C}(G) : \begin{cases} (1, g) & \mapsto & \{g\varphi\} & \text{for each } g \in G \\ (0, g) & \mapsto & N(g\varphi) & \text{for each } g \in G. \end{cases}$$

Then Ψ is a homomorphism since N is normal in G and φ is a homomorphism. To show that Ψ is injective, suppose that $e_1, e_2 \in E$ and $g_1, g_2 \in G$ such that

$$(e_1, g_1)\Psi = (e_2, g_2)\Psi.$$

Then we clearly must have $e_1 = e_2$. Suppose first that $e_1 = e_2 = 1$. Then

$$\{g_1\varphi\} = (e_1, g_1)\Psi = (e_2, g_2)\Psi = \{g_2\varphi\}.$$

It then follows that $g_1 = g_2$ since φ is injective and so $(e_1, g_1) = (e_2, g_2)$. Finally, suppose that $e_1 = e_2 = 0$. Then

$$N(g_1\varphi) = (e_1, g_1)\Psi = (e_2, g_2)\Psi = N(g_2\varphi)$$

from which it follows that $(g_1g_2^{-1})\varphi = (g_1\varphi)(g_2\varphi)^{-1} \in N$. But then $(g_1g_2^{-1})\varphi = 1$ since $N \cap K = \{1\}$, and so $g_1g_2^{-1} = 1$ since φ is injective, whence $g_1 = g_2$ and $(e_1, g_1) = (e_2, g_2)$. This completes the proof that Ψ is injective.

While the monoids M considered in Examples 9 and 10 had different values of $|E_M|$ and $\mathrm{rank}(G_M)$, they shared the property that $|E_M| + \mathrm{rank}(G_M) = 4$. It turns out that 4 is the minimum value of $|E_M| + \mathrm{rank}(G_M)$ for any FIM $M \notin \mathscr{C}$ which embeds in $\mathcal{C}(G_M)$.

Proposition 11 *Suppose that $M \notin \mathscr{C}$ is a FIM for which there exists an embedding $\Psi : M \to \mathcal{C}(G_M)$. Then $|E_M| + \mathrm{rank}(G_M) \geq 4$.*

Proof Write $E = E_M$ and $G = G_M$ and suppose that $|E| + \mathrm{rank}(G) \leq 3$. Since $M \notin \mathscr{C}$, we have $|E| \geq 2$, and since Ψ is injective, we have $\mathrm{rank}(G) \geq 1$. It then follows that $|E| = 2$ and $\mathrm{rank}(G) = 1$. Write $E = \{1, e\}$ where 1 is the identity of M. Since $M \notin \mathscr{C}$ we must have $G_e = \{1\}$. Since M is infinite, G must be an infinite cyclic group generated by x say, and since Ψ is an embedding, we have $x\Psi = \{x^i\}$ and $e\Psi = \langle x^j \rangle$ for some $i, j \in \mathbb{Z} \setminus \{0\}$. But then $ex^j \neq e$ since $G_e = \{1\}$, yet

$$(ex^j)\Psi = (e\Psi) * (x^j\Psi) = \langle x^j \rangle * \{x^{ij}\} = \langle x^j \rangle x^{ij} = \langle x^j \rangle = e\Psi$$

contradicting the injectivity of Ψ. This completes the proof. □

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