VERONESEAN VARIETIES IN EXCEPTIONAL GEOMETRIES OF TYPE E_6

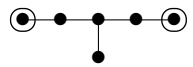
YANNICK NEYT, JAMES PARKINSON, HENDRIK VAN MALDEGHEM, AND MAGALI VICTOOR

Abstract. We classify the nontrivial collineations of the minuscule exceptional geometry $\mathsf{E}_{6,1}(\mathbb{K})$ that fix at least one point but map no point to a distinct collinear point. We prove that their fixed-point sets are ovoidal subplanes realised as quaternion or octonion Veronesean varieties. We determine the corresponding pointwise stabilisers and establish uniqueness up to conjugacy. In the quaternion case, these stabilisers are norm-one groups of quaternion division algebras, or certain additive subgroups of inseparable field extensions in characteristic two. These results provide new examples of linear descent in exceptional buildings and reveal geometric connections with ovoids of hyperbolic polar spaces and line spreads of projective spaces.

1. Introduction

A *uniclass automorphism* of a spherical building is an automorphism that maps each chamber C to a chamber whose Weyl distance to C is contained in a fixed twisted conjugacy class of the underlying Coxeter group. In [11] the first three authors characterised such automorphisms by classifying, type-by-type, the fixed structures of uniclass automorphisms for irreducible (thick) spherical buildings, and in [15] these fixed structures are characterised building theoretically across all types. In [12] uniclass automorphisms are, in the simply laced case, characterised as the (unique) collineations in the corresponding long root subgroup geometry that do not map any point to a point at distance 1 or codistance 1. Such collineations were called *kangaroos* of the long root geometry because they skip certain distances in the displacement spectrum.

In the present paper we turn our attention to collineations of the minuscule Lie geometry $\mathsf{E}_{6,1}(\mathbb{K})$ with similar spectral gap properties. In this geometry, two points are at distance 0, 1, or 2, and by a *kangaroo* of $\mathsf{E}_{6,1}(\mathbb{K})$ we shall mean a collineation of $\mathsf{E}_{6,1}(\mathbb{K})$ mapping no point to distance 1, but fixing at least one point and mapping at least one point to distance 2. Our main theorem stated below gives a complete classification of these collineations, for $|\mathbb{K}| > 2$. Recall that the *fix diagram* of an automorphism is given by encircling the nodes of the Dynkin diagram corresponding to the types of the minimal simplices fixed by the automorphism. In particular, the $\mathsf{E}_{6;2}$ fix diagram is

$\mathsf{E}_{6;2} =$ 

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We refer to Section 2.4 for the definitions of quadratic and octonion division algebras, and Veronesean varieties.

Theorem A. *Let $\Delta = \mathsf{E}_{6,1}(\mathbb{K})$ with $|\mathbb{K}| > 2$.*

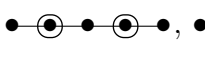
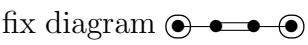
- (*i*) *For each quadratic division algebra $\mathbb{H}$ of degree 4 over $\mathbb{K}$, there exists, up to conjugacy, a unique subgroup H of the automorphism group of Δ enjoying the following properties.*
  - (*a*) *Each nontrivial element of H is a kangaroo collineation with fix diagram $\mathsf{E}_{6;2}$.*
  - (*b*) *The fixed structure of each nontrivial element of H is a (common) ovoidal subplane $\mathscr{V}$ of Δ isomorphic to the Veronesean variety $\mathscr{V}(\mathbb{K}, \mathbb{H})$ when we represent $\mathsf{E}_{6,1}$ in $\mathsf{PG}(26, \mathbb{K})$ in the standard way.*
  - (*c*) *The pointwise stabiliser of $\mathscr{V}$ in $\mathsf{Aut}(\Delta)$ is H.*
  - (*d*) *If $\mathbb{H}$ is non-commutative, then, as an abstract group, H is isomorphic to the subgroup of the multiplicative group $\mathbb{H}^\times$ consisting of the elements with norm 1.*
  - (*e*) *If $\mathbb{H}$ is commutative, say $\mathbb{H} = \mathbb{K}(t_1, t_2)$ is an inseparable extension of degree 4 of $\mathbb{K}$, then H is the subgroup of the additive group of $\mathbb{H}$ consisting of 0 and the multiplicative inverses of the elements $1 + a_1t_1 + a_2t_2 + a_{12}t_1t_2$, $a_1, a_2, a_{12} \in \mathbb{K}$.*
- (*ii*) *For each octonion (Cayley) division algebra $\mathbb{O}$ with centre $\mathbb{F}$, such that $\mathbb{K}/\mathbb{F}$ is a quadratic Galois extension and $\mathbb{K}$ is a subfield of $\mathbb{O}$, stabilised under the standard involution of $\mathbb{O}$, there exists, up to conjugacy, a unique involution θ of Δ enjoying the following properties.*
  - (*a*) *θ is a kangaroo collineation with fix diagram $\mathsf{E}_{6;2}$.*
  - (*b*) *The fixed structure of θ is an ovoidal subplane $\mathscr{V}$ of Δ isomorphic to the octonion Veronesean variety $\mathscr{V}(\mathbb{K}, \mathbb{O})$ when we represent $\mathsf{E}_{6,1}$ in $\mathsf{PG}(26, \mathbb{K})$ in the standard way.*
  - (*c*) *The pointwise stabiliser of $\mathscr{V}$ in $\mathsf{Aut}(\Delta)$ is $\langle\theta\rangle$.*
- (*iii*) *Conversely, every kangaroo collineation of Δ is conjugate to some collineation as in (i) or (ii) above.*

Combining the above theorem with the main theorem of [11] it follows that a type preserving automorphism of the building $\mathsf{E}_6(\mathbb{K})$ is uniclass if and only if the associated collineation of the geometry $\mathsf{E}_{6,1}(\mathbb{K})$ is a kangaroo.

Motivation—We now provide some further motivation for this paper, beyond the connections to uniclass automorphisms.

Firstly, our investigations fit into the bigger framework of understanding so-called *linear descent of spherical buildings*. Linear descent is the counterpart of *Galois descent* by using only linear automorphisms as opposed to Galois groups (using field automorphisms). Galois descent produces subbuildings over subfields as fixed subbuildings, while linear descent produces subbuildings mainly over the same field. The subbuildings in linear descent are rather large, especially if one uses a cyclic automorphism group, and recognising them illuminates features of the ambient building. Moreover, the full linear descent groups (the groups pointwise fixing the subbuildings) are generally much larger than their Galois counterparts. In particular, Theorem A(*i*) implies that the linear descent group corresponding to the fix diagram $\mathsf{E}_{6;2}$ is a sphere of radius 1 in a quaternion division algebra, which is a maximal division algebra inside a split Cayley algebra. This, in turn, is known to describe the triality quadric, that is the building of type D_4 corresponding to

the anisotropic part in the diagram $\mathsf{E}_{6;2}$ (the *anisotropic part* consists of the nodes not encircled in the diagram). Our results also verify that Galois descent with Tits indices ${}^{28}_{6,2}\mathsf{E}$, ${}^{28}_{7,3}\mathsf{E}$ and ${}^{28}_{8,4}\mathsf{E}$ (using the notation of [20]) have linear analogues, which is certainly not a priori obvious.

Secondly, the type preserving case for type E_6 is an exceptionally interesting case, for the following reasons. To begin with, it is a case with a rather peculiar anisotropic part. The latter being of type D_4, it has rank 4 and the diagram contains a branching vertex. Up to now, to the best of our knowledge, full linear descent groups have only been explicitly determined for anisotropic kernels being a product of type A_1 (for the fix diagrams , , and in [4]), and B_2 (in characteristic 2) for the fix diagram  in [17]. Furthermore, it follows from the classification of nontrivial (and non-anisotropic) uniclass automorphisms of thick irreducible spherical buildings of rank at least 3 that in almost all cases such an automorphism stabilises simplices of polar type. In the simply laced case—which is the case where the polar type is most unambiguously defined—there are only three classes of counterexamples, where the fixed structures are as follows: (a) line spreads in projective spaces, (b) ovoids in hyperbolic polar spaces, and (c) so-called *ovoidal subplanes* in parapolar spaces of type $\mathsf{E}_{6,1}$. Now, the E_6 example restricts via its residues of type D_5 to an example of class (b) for hyperbolic polar spaces of type $\mathsf{D}_{5,1}$, and via its equator geometries of type A_5 to an example of class (a) in projective 5-space (see Section 6). Hence, the E_6 case can be seen as the ultimate overarching universal example of this phenomenon.

Finally, we note that the results of the present paper will be used in a crucial way to classify so-called *domestic automorphisms* (that is, automorphisms mapping no chamber to an opposite one) of buildings of type E_7 and E_8, see [13] and [14].

Outline of the paper — We conclude this introduction with a brief outline of the structure and strategy of the paper. The necessary background material on polar and parapolar spaces is given in Section 2. In particular, Section 2.3 gives precise and specific information required on the parapolar space $\mathsf{E}_{6,1}(\mathbb{K})$. To prove Theorem A we must first introduce and classify collineations in buildings of types D_n whose displacement spectrum skips certain values (we also call these *kangaroo collineations*). This analysis is given in Section 3. In Section 4 we use this analysis to determine the fix structure of an arbitrary kangaroo collineation of $\mathsf{E}_{6,1}(\mathbb{K})$, for any field $\mathbb{K}$. We also handle Theorem A(*ii*). In Section 5 we then prove our main result Theorem A.

A note on the history of this paper — The results of the present paper were originally contained in a manuscript entitled *Automorphisms and opposition in spherical buildings of exceptional type, IV. The* E_7 *case*. This long manuscript was split into two parts, with the present paper dealing with results on the minuscule geometry $\mathsf{E}_{6,1}(\mathbb{K})$, and the second part dealing with results on domestic automorphisms of E_7 buildings. The second part [13] retained the original title. In particular, the results of the present paper largely precede the recent acceleration of AI usage in mathematics, and were not obtained with any assistance from AI.

2. Preliminaries on polar and parapolar spaces

Our approach to buildings of type E_6 is via their related geometries known as parapolar spaces. Everything we say below can be found in the standard books [2] and [18]; for polar spaces see also [24]. Although we assume a certain familiarity with this theory, we recall some basic definitions in order to settle notation. We shall use the words "automorphism" and "collineation" interchangeably (with the former more connected to the building language, and the latter more connected to the incidence geometry language).

2.1. **Abstract definitions.** Let $\Gamma = (X, \mathscr{L})$ be a point-line geometry (if the incidence relation is not mentioned, we assume it is induced by containment). We will always assume that $\mathscr{L}$ is a set of subsets of X. Points $x, y \in X$ contained in a common line are called *collinear*, denoted as $x \perp y$; the set of all points collinear to x is denoted by $x^\perp$, the *perp* of x. We will always deal with situations where every point is contained in at least one line, so $x \in x^\perp$. Also, for $S \subseteq X$, we denote $S^\perp := \{x \in X \mid x \perp s \text{ for all } s \in S\}$. Likewise we write $S \perp T$ for subsets $S, T \subseteq X$ if $s \perp t$ for each $s \in S$ and each $t \in T$. A *collineation* of Γ is a permutation of X that induces a permutation of $\mathscr{L}$.

The *point graph* of Γ is the graph on X with collinearity as adjacency relation. The *distance* δ between two points $p, q \in X$ (denoted $\delta_\Gamma(p, q)$, or $\delta(p, q)$ if no confusion is possible) is the distance between p and q in the collinearity graph, where $\delta(p, q) = \infty$ if p and q are contained in distinct connected components of the point graph; If $\delta := \delta(p, q)$ is finite, then a *geodesic path* or a *shortest path* between p and q is a path between them in the point graph of length δ. The *diameter* of Γ (denoted $\mathsf{diam}\,\Gamma$) is the diameter of the point graph. We say that Γ is *connected* if every pair of vertices is at finite distance from one another. The point-line geometry Γ is called a *partial linear space* if each pair of distinct points is contained in at most one line. In this case we usually denote the unique line containing two distinct collinear points x and y by xy.

A *subspace* of Γ is a subset A of X such that, if $x, y \in A$ are collinear and distinct, then all lines containing both x and y are contained in A. A subspace A is called *convex* if, for any pair of points $\{p, q\} \subseteq A$, every point occurring in a geodesic between p and q is contained in A; it is *singular* if $\delta(p, q) \leq 1$ for all $p, q \in A$. The intersection of all convex subspaces of Γ containing a given subset $B \subseteq X$ is called the *convex subspace closure* of B. A proper subspace H is called a *geometric hyperplane* if each line of Γ has either one or all its points contained in H.

A *full subgeometry* $\Gamma' = (X', \mathscr{L}')$ of Γ is a geometry with $X' \subseteq X$ and $\mathscr{L}' \subseteq \mathscr{L}$. This implies that all points of Γ on a line of Γ' are points of Γ' and explains the adjective 'full'. Full subgeometries need not be subspaces. When Γ' is considered as an abstract geometry, we sometimes call its inclusion in Γ as a full subgeometry a *full embedding in* Γ.

A *projective plane* is a point-line geometry $(X, \mathscr{L})$ with properties that each line has at least three points, each pair of different points is contained in a unique line, and each pair of distinct lines intersect in a (necessarily unique) point. A *projective space of dimension* d is either a projective plane (if $d = 2$), or is the point-line geometry $\mathsf{PG}(d, \mathbb{K})$, $\mathbb{K}$ any skew field, with point set the 1-dimensional subspaces of a $(d+1)$-dimensional vector space over $\mathbb{K}$ and line set the set of 2-dimensional subspaces of that same vector space. Here, $d \geq 2$, and when $d = 2$, $\mathsf{PG}(2, \mathbb{K})$ is a projective plane as defined earlier. The *dimension* of $\mathsf{PG}(d, \mathbb{K})$ is by definition equal to the integer d. Every subspace of a projective space

is either empty, a point, a single line, or again a projective space. If its dimension is e, then its *codimension in* $\mathsf{PG}(d, \mathbb{K})$ is $d - e$. The smallest subspace of a projective space containing a given set S of points will be denoted as $\langle S \rangle$. Sometimes this is written as $\langle S_1, S_2 \rangle$, where $S_1 \cup S_2 = S$.

Now a *polar space* is a thick point-line geometry in which the perp of every point is a geometric hyperplane. This forces all singular subspaces to be empty, points, lines or projective spaces. In our case the polar spaces will have finite rank, that is, there is a natural number $r \geq 2$ such all singular subspaces (which are projective spaces, as noted above) have dimension $\leq r - 1$, and there exist singular subspaces of dimension $r - 1$. A prominent notion in polar geometry is *opposition*. Two singular subspaces U, W are *opposite* if no point of $U \cup W$ is collinear to all points of $U \cup W$. Opposite subspaces automatically have the same dimension. Opposite points are just non-collinear ones. The singular subspaces of dimension $r - 2$ are called *submaximal*. Maximal subspaces sharing a submaximal subspace are called *adjacent*. An *ovoid* of a polar space is a set of points with the property that each maximal singular subspace has exactly one point in common with the ovoid. Polar spaces of rank 2 are called *generalised quadrangles*. A subspace of a polar space is called *nondegenerate* if it is a polar space itself. This is equivalent to the property that no point of the subspace is collinear to all points of the subspace. The polar spaces that we will encounter in this paper are all full subgeometries of projective spaces and arise from certain "forms" (like bilinear, quadratic, Hermitian forms, see [21, 24]). If such a form produces a set of points without lines, then we call that set sometimes a *polar space of rank* 1.

Now a *parapolar space* is a point-line geometry satisfying the following four axioms:

(PPS1) There is a line L and a point p such that no point of L is collinear to p.
(PPS2) The geometry is connected.
(PPS3) Let x, y be two points at distance 2. Then either there is a unique point collinear to both—and then the pair $\{x, y\}$ is called *special*—or the convex subspace closure of $\{x, y\}$ is a polar space $\xi(x, y)$—and then the pair $\{x, y\}$ is called a *symplectic pair*. Such polar spaces are called *symplecta*, or *symps* for short.
(PPS4) Each line is contained in a symplecton.

The parapolar spaces we will encounter all have the rather peculiar property that all symps have the same rank, which is then called the (uniform) *symplectic rank* of the parapolar space. In contrast, the maximal singular subspaces (which will be projective spaces) will not all have the same dimension. The *singular ranks* of a parapolar space with only projective spaces as singular subspaces (which is automatic if the symplectic rank is at least 3) are the dimensions of the maximal singular subspaces.

A parapolar space without special pairs is called *strong*.

Now let $\Gamma = (X, \mathscr{L})$ be a parapolar space all of whose symps have rank at least 3. Let $x \in X$. Then we define the geometry $\mathsf{Res}_\Gamma(x) = (\mathscr{L}_x, \Pi_x)$ as the geometry with point set the set of lines $\mathscr{L}_x$ through x and set of lines the set Π_x of planar line pencils with vertex x and call it the *residue at* x, or the *residual geometry at* x.

In the present paper we will mainly deal with buildings of type D_n and E_6. The parapolar spaces we will be concerned with are half spin geometries and the exceptional geometries of type $\mathsf{E}_{6,1}$, which we now briefly introduce.

2.2. **Lie incidence geometries.** Let Δ be an irreducible thick spherical building. Let n be its rank, let S be its type set and let $s \in S$. Then we define a point-line geometry $\Gamma = (X, \mathscr{L}, *)$ as follows. The point set X is just the set of vertices of Δ of type s; the set $\mathscr{L}$ of lines are the flags of type $s^\sim$, where $s^\sim$ is the set of types adjacent to s in the Coxeter diagram of Δ. If x is a vertex of type s and F a flag of type $s^\sim$, then $x * F$ if $F \cup \{x\}$ is a flag. The geometry Γ is called a *Lie incidence geometry*. For instance, if Δ has type A_n, and $s = 1$ (remember we use Bourbaki labelling), then Γ is a projective space. If X_n is the Coxeter type of Δ and Γ is defined using $s \in S$ as above, then we say that Γ has *type* $\mathsf{X}_{n,s}$. Another example: Geometries of type $\mathsf{B}_{n,1}$ and $\mathsf{D}_{n,1}$ are polar spaces.

For the classical diagrams $\mathsf{X} \in \{\mathsf{A}, \mathsf{B}, \mathsf{D}\}$, the geometries of type $\mathsf{X}_{n,k}$ are the projective and polar Grassmannians. Geometries of type $\mathsf{B}_{n,n}$ are more specifically called *dual polar spaces* and there is a huge literature about them. Geometries of type $\mathsf{D}_{n,n}$ are more specifically called *half spin geometries*. They are in fact Grassmannians of the corresponding oriflamme geometries (see the beginning of Section 3). Many properties of dual polar spaces and half spin geometries can be deduced from the underlying polar spaces.

Buildings of type $\mathsf{A}, \mathsf{D}, \mathsf{E}$ are uniquely defined by their underlying field $\mathbb{K}$ (or skew field in the case of A), provided the rank is at least 3. We denote the corresponding building of type X_n by $\mathsf{X}_n(\mathbb{K})$, and the corresponding Lie incidence geometries of type $\mathsf{X}_{n,k}$ by $\mathsf{X}_{n,k}(\mathbb{K})$.

If an automorphism of a building of type X_n preserves the vertices of type k, then it induces a collineation in the corresponding Lie incidence geometry of type $\mathsf{X}_{n,k}$. Note that an automorphism of a spherical building preserves the type function, but possibly interchanges types. The corresponding map on the types is an automorphism of the corresponding Coxeter diagram.

We now describe the Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$, for an arbitrary field $\mathbb{K}$. This is best done by displaying the possible mutual positions of the symps and the points.

2.3. **The parapolar spaces of exceptional type $\mathsf{E}_{6,1}$.** First let $\Delta = (X, \mathscr{L})$ be the parapolar space $\mathsf{E}_{6,1}(\mathbb{K})$, for some field $\mathbb{K}$. The elements of the corresponding building of types $1, 2, 3, 4, 5, 6$, are the *points,* 5*-spaces, lines, planes,* 4*-spaces* and *symps*, respectively. The symps are isomorphic to polar spaces $\mathsf{D}_{5,1}(\mathbb{K})$. The hyperplanes of the 5-spaces are called $4'$*-spaces* to distinguish them from the 4-spaces. The $4'$-spaces correspond in the building to flags of type $\{2, 6\}$. Hence every $4'$-space is contained in a unique 5-space, and also in a unique symp. All singular 3-spaces form one orbit (of $\mathsf{Aut}\,\Delta$) and correspond to flags of type $\{2, 5, 6\}$; they all arise as the intersection of a 4-space and a $4'$-space, or of two $4'$-spaces.

The mutual positions of the elements can be deduced from a model of an apartment of the corresponding building, more precisely the graph on the vertices of type 1 of a Coxeter complex of type E_6, adjacent when contained in adjacent chambers. This is exactly the complement of the point graph of the generalised quadrangle of order $(2, 4)$ (three points per line, five lines through each point). A concrete model is the following: Let V be the set of all pairs of the set $\{1, 2, 3, 4, 5, 6\}$, plus two copies of the latter, denoted $\{1, 2, 3, 4, 5, 6\}$ and $\{1', 2', 3', 4', 5', 6'\}$. Adjacency is given by $\{i, j\} \sim \{k, \ell\}$ if $|\{i, j, k, \ell\}| = 3$; $i \sim \{j, k\} \sim i'$ if $i \notin \{j, k\}$; $i \sim i'$; $i \sim j$ and $i' \sim j'$ if $i \neq j$. As an example how to use such a model, we deduce the mutual positions of points and 5-space.

We may fix a 5-space $\{1,2,3,4,5,6\}$. Then a point is either of the form i, or of the form (i,j) or of the form i'. In the first case the point belongs to the 5-space; in the second case it is collinear to a 3-space (given by the four vertices of $\{1,2,3,4,5,6\}$ distinct from i,j); in the third case it is collinear to a unique point of the 5-space.

We now mention all such relevant possible mutual positions, introducing some more terminology.

Fact 2.1. *Let $\Delta = (X, \mathscr{L})$ be the parapolar space $\mathsf{E}_{6,1}(\mathbb{K})$. Then the following holds.*

- (*i*) *Δ is strong and has diameter 2, that is, two distinct points either lie in a unique symplecton, or are collinear (and then lie in many symplecta).*
- (*ii*) *The geometry with point set the set of symps of Δ, and where the lines are the sets of symps containing a fixed 4-space, is isomorphic to Δ itself (and we denote it by $\mathsf{E}_{6,6}(\mathbb{K})$); this is the duality principle, which implies that two distinct symps in Δ either intersect in a unique point, or in a unique 4-space (and in the latter case we call the symps adjacent).*
- (*iii*) *A point not contained in a symp is collinear to either no points of that symp—then we say that the point and the symp are far—or to all points of a unique $4'$-space of that symp—the point and the symp are close.*
- (*iv*) *A point not contained in a 5-space is collinear to either a unique point of that 5-space, or to a 3-space contained in that 5-space.*
- (*v*) *A symp and a 5-space intersect in either a $4'$-space, a line, or the empty set.*
- (*vi*) *Two 5-spaces are either disjoint or intersect in a point or plane.*

For each point $x \in X$, the residual geometry $\mathsf{Res}_\Delta(x)$ is isomorphic to the half spin geometry $\mathsf{D}_{5,5}(\mathbb{K})$.

We will need the following specific property.

Lemma 2.2. *Let ξ and ξ' be two distinct symps of $\mathsf{E}_{6,1}(\mathbb{K})$. Then there exists a point x far from ξ and close to ξ' (and hence not contained in either).*

Proof. The set of points close to ξ' is precisely the union of all lines having at least one point in common with ξ'. It follows from Section 3.2 of [5] that each point x of that set not contained in ξ' is contained in a line containing a point far from ξ', and all such lines are not contained in one single symp. The assertion now follows from applying this to a point x of ξ' close to (so not contained in) ξ, which exists since $\xi \cap \xi' \neq \varnothing$, by Fact 2.1(*ii*). □

Finally, recall from the introduction the following definition.

Definition 2.3. An *ovoidal subplane* P of $\mathsf{E}_{6,1}(\mathbb{K})$ is a set of points of $\mathsf{E}_{6,1}(\mathbb{K})$ such that each symp ξ containing at least two points of P intersects P in an ovoid of ξ and the set of ovoids thus obtained defines the structure of a projective plane on P.

2.4. **Quaternion and octonion Veronese varieties.** We will also have to deal with Veronese varieties. Let us briefly introduce these. Let $\mathbb{K}$ be a field and $\mathbb{A} \supseteq \mathbb{K}$ a quadratic alternative algebra over $\mathbb{K}$. Recall that such an algebra admits a unique involution $\mathbb{A} \to \mathbb{A} : x \mapsto \overline{x}$ such that $x\overline{x} \in \mathbb{K}$ and $x + \overline{x} \in \mathbb{K}$ (and then x satisfies the quadratic equation $x^2 - (x+\overline{x})x + x\overline{x} = 0$ with coefficients in $\mathbb{K}$). This involution is called the *standard involution*. It is trivial if and only if $\mathbb{A} = \mathbb{K}$ or $\mathbb{A}$ is an inseparable field extension of $\mathbb{K}$ in characteristic 2. The dimension of $\mathbb{A}$ over $\mathbb{K}$ is sometimes also called the *degree* of the algebra.

Example 2.4. (*i*) Each quadratic extension $\mathbb{L}$ of $\mathbb{K}$ is a quadratic division algebra over $\mathbb{K}$. If it is separable, then the standard involution is the unique corresponding Galois involution; in the inseparable case, the standard involution is trivial (as noted above) and $\mathsf{char}\,\mathbb{K} = 2$.

(*ii*) If $\mathbb{B}$ is an associative quadratic algebra (both necessarily division), then, choosing an arbitrary nonzero element $k \in \mathbb{K}$, the set $\mathbb{A} = \mathbb{B} \times \mathbb{B}$ with multiplication

$$(a, b) \cdot (c, d) = (ac - kd\bar{b}, \bar{a}d + cb)$$

is again a quadratic algebra with standard involution $\overline{(a, b)} = (\bar{a}, -b)$. It is a division algebra if, and only if, $\mathbb{B}$ is a division algebra and k cannot be written as $x\bar{x}$, for $x \in \mathbb{B}$. If it is not a division algebra, it is *split*, that is, there exists a basis $\{e_i\}_{i\in I}$ of $\mathbb{A}$ over $\mathbb{K}$ and constants $\ell_i \in \mathbb{K}$, $i \in I$, such that, for $x = \sum_{i\in I} a_i e_i$ we have $x\bar{x} = \sum \ell_i a_i^2$. (We can always choose $e_1 = 1$ and then $\ell_1 = 1$; in our case $|I| = 4$.) This "doubling" process is called the *Cayley-Dickson process (with primitive element k)*.

(*iii*) A *quaternion algebra* is a quadratic algebra of degree 4 which is the doubling under the Cayley-Dickson process of either a separable quadratic extension of $\mathbb{K}$, or the direct product algebra $\mathbb{K} \times \mathbb{K}$ (with componentwise addition and multiplication; it is quadratic with standard involution $\overline{(a, b)} = (b, a)$). If it is split, it is isomorphic to the algebra of 2×2 matrices over $\mathbb{K}$.

(*iv*) An *octonion algebra* is a quadratic algebra of degree 8 which is the doubling under the Cayley-Dickson process of a quaternion algebra. It is either a non-associative (alternative) division algebra, or it is split. In the latter case, it is unique and one can represent it with Zorn's matrices as we shall do in Section 5.2.2.

In any case, the dimension of $\mathbb{A}$ over $\mathbb{K}$ is either infinite or a power of 2. If the standard involution is not the identity, then $\mathbb{A}$ is either a quadratic separable field extension of $\mathbb{K}$, a quaternion algebra, or an octonion algebra. The dimensions are then 2,4 and 8, respectively. Set $d = \dim_{\mathbb{K}} \mathbb{A}$.

Now let $\mathbb{A}$ be a quadratic division algebra over $\mathbb{K}$. The *Veronese variety* $\mathscr{V}(\mathbb{K}, \mathbb{A})$ is the following set of points of $\mathsf{PG}(2 + 3d, \mathbb{K})$:

$$\{(x\bar{x}, y\bar{y}, 1, y, \bar{x}, x\bar{y}) \mid x, y \in \mathbb{A}\} \cup \{(x\bar{x}, 1, 0, 0, 0, x) \mid x \in \mathbb{A}\} \cup \{(1, 0, 0, 0, 0, 0)\}.$$

If $\mathbb{A}$ is associative, this coincides with the set $\{(x\bar{x}, y\bar{y}, z\bar{z}, y\bar{z}, z\bar{x}, x\bar{y}) \mid x, y, z \in \mathbb{A}\}$.

The set of points of $\mathscr{V}(\mathbb{K}, \mathbb{A})$ can be identified with the points of the projective plane $\mathsf{PG}(2, \mathbb{A})$ in such a way that lines bijectively correspond to the quadrics of Witt index 1 contained in subspaces of dimension $d + 1$ of $\mathsf{PG}(2 + 3d, \mathbb{K})$. Such subspaces are called *host spaces* of $\mathscr{V}(\mathbb{K}, \mathbb{A})$.

If $\mathbb{A}$ is a quaternion division algebra, then $\mathscr{V}(\mathbb{K}, \mathbb{A})$ lives in $\mathsf{PG}(14, \mathbb{K})$. We call this Veronese variety a *(separable) quaternion Veronesean*, or *(separable) quaternion Veronese variety*. We add the adjective "separable" in order to distinguish it from the other case in which the variety spans a 14-dimensional projective space, namely, the case where $\mathbb{A}$ is an inseparable field extension of degree 4 of $\mathbb{K}$. In this case, $\mathbb{K}$ has characteristic 2 and the corresponding Veronese variety will be—slightly abusively—called an *inseparable quaternion Veronesean* or *inseparable quaternion Veronese variety*. The class of inseparable field extensions of $\mathbb{K}$ of degree 4 in characteristic 2 together with quaternion division

algebras over $\mathbb{K}$ will be referred to as the class of *quadratic division algebras of degree* 4 *over* $\mathbb{K}$.

Similar considerations hold for the octonion case, but we will never need the inseparable case here.

2.5. **Opposition and domesticity.** For a given automorphism θ of a spherical building Ω, we say that a vertex v is *non-domestic* (*domestic*) if θ maps (does not map) v to an opposite. If all vertices of type t are domestic, we say that θ is t-domestic. We sometimes make t explicit by calling it by its geometric name, like *point-domestic* if we are dealing with a certain point-line geometry related to Ω (like in Section 2.3). We say that θ is *domestic* if it does not map any chamber to an opposite. It is obvious that t-domestic automorphisms are domestic, for any type t.

Opposition of vertices defines an automorphism of the corresponding Coxeter diagram, called the *opposition map*.

2.6. **Baer involutions and Baer subspaces.** Let $\mathbb{K}$ and $\mathbb{F}$ be (commutative) fields such that $\mathbb{K}$ is a quadratic extension of $\mathbb{F}$ (separable or inseparable). Let $d \geq 2$ and consider the projective space $\mathsf{PG}(d, \mathbb{K})$. Since the latter arises from a vector space we can introduce coordinates for points, after choosing a basis in the vector space. Each point gets a $(d+1)$-tuple, determined up to a multiplicative nonzero constant, which describes any nonzero vector on the 1-dimensional subspace of the vector space corresponding to that point of the projective space. Restricting these coordinates to values in $\mathbb{F}$ produces $\mathsf{PG}(d, \mathbb{F})$, viewed as a *sub projective space* of $\mathsf{PG}(d, \mathbb{K})$. Note that we do not write "projective subspace", as it is not a subspace (the lines of $\mathsf{PG}(d, \mathbb{F})$ do not contain all points of $\mathsf{PG}(d, \mathbb{K})$ on the corresponding line of $\mathsf{PG}(d, \mathbb{K})$). We call $\mathsf{PG}(d, \mathbb{F})$ a *Baer subgeometry* of $\mathsf{PG}(d, \mathbb{K})$. If $\mathbb{K}/\mathbb{F}$ is separable, then it is also obtained by taking the fixed points of the collineation defined by applying the Galois involution of the field extension $\mathbb{K}/\mathbb{F}$ on every coordinate. Such a collineation is sometimes called a *Baer involution*.

3. Kangaroos in oriflamme geometries

Oriflamme geometries are related to the polar spaces with the property that each submaximal singular subspace is contained in exactly two maximal subspaces; it follows that there are two natural classes of maximal singular spaces such that adjacent ones are in a different class. The oriflamme geometry treats these two classes as elements of different type, and ignores the submaximal subspaces, just like the corresponding building of type D_n. The polar spaces themselves are sometimes called *hyperbolic*, or *of hyperbolic type*. Disjoint maximal singular subspaces are *opposite* in the building theoretical sense. Opposite maximal singular subspaces have the same type if and only if n is even. Maximal subspaces are also sometimes called *generators*. Oriflamme geometries arise either as line Grassmannians of projective spaces over non-commutative fields (which we will not need nor consider in the paper), or as hyperbolic quadrics, that is, quadrics of maximal Witt index in odd-dimensional projective space. With respect to a well-chosen basis, a hyperbolic quadric of Witt index n has equations $x_{-n}x_n + \cdots + x_{-2}x_2 + x_{-1}x_1 = 0$. Ovoids of hyperbolic quadrics that arise as intersections with subspaces of the ambient projective space will be called *linear*.

An automorphism of an oriflamme geometry is called an *oppomorphism* if its induced action on the Coxeter diagram agrees with the opposition map. Otherwise, it is called an *anti-oppomorphism*.

A collineation of an oriflamme geometry, or hyperbolic polar space, is called *type-preserving* if it stabilises the two natural systems of generators; hence if it induces a type-preserving automorphism of the corresponding spherical building. A collineation which is not type-preserving is called *type-interchanging*.

As (anti-)oppomorphisms of polar spaces act faithfully on the point set, they are collineations of the polar space. For hyperbolic polar spaces we have:

- oppomorphisms are type-preserving if n is even and type-interchanging if n is odd;
- anti-oppomorphisms are type-preserving if n is odd and type-interchanging if n is even.

We now introduce the notion of a kangaroo.

3.1. **Kangaroos and general properties.**

Definition 3.1. Let θ be a non-trivial collineation of a polar space $\Gamma = (X, \mathscr{L})$ of type D_n, $n \geq 2$. If θ maps no point of X to another collinear point, and θ has at least one fixed point, then we call θ a *kangaroo (collineation)* of Γ. If θ has at least one pointwise fixed line, we call it *lazy*. If θ has no fixed lines, we call it *diligent*.

Although we are mainly interested in diligent kangaroos, we prove some general properties of azy ones, as this helps us recognise diligent kangaroos more easily.

Our first aim is to show a characterisation of kangaroos, providing alternative definitions. Therefore, we prove the following lemmas.

We assume throughout that $\Gamma = (X, \mathscr{L})$ is a polar space of type D_n, $n \geq 2$.

Lemma 3.2. *Let θ be a kangaroo of Γ and let U be any singular subspace of Γ. Then $U \cap U^\theta$ is fixed pointwise.*

Proof. Suppose $p \in U \cap U^\theta$ is mapped to $p^\theta \neq p$. As $p \in U$, we know that $p^\theta \in U^\theta \ni p$, but then $p \perp p^\theta$, a contradiction. Hence $p = p^\theta$ and the lemma follows. □

Corollary 3.3. *The fixed point set of a kangaroo in a generator of Γ is a singular subspace.*

Lemma 3.4. *The parity of the dimension of the fixed point set of a kangaroo in a generator is the same as*

- *n for type-interchanging kangaroos;*
- *$n - 1$ for type-preserving kangaroos.*

Proof. The dimension of the intersection of two generators has the same parity as n if, and only if, they are of different type. The assertion now follows from Lemma 3.2. □

Lemma 3.5. *The dimension of the fixed point set of a kangaroo in a generator is the same for every generator.*

Proof. Suppose U is a generator of Γ with a fixed k-space K, where k is chosen maximal, $0 \leq k < n$. Each generator S adjacent to U also contains a fixed k-space, as $S \cap S^\theta$ contains the fixed $(k-1)$-space $S \cap K$, and due to parity and Lemma 3.4, it has to be a k-space. A connectivity argument completes the proof. □

Since kangaroos have, by definition, at least one fixed point, it follows that kangaroos do not map generators to disjoint ones, hence they are domestic. The following characterisation shows which domestic collineations are kangaroos. To state it, we need first some definitions. Recall from [19] that line-domestic collineations of generalised quadrangles are characterised by the shape of their fixed point set: either the perp of a point is pointwise fixed, or an ovoid is pointwise fixed, or a large full subquadrangle is pointwise fixed. The latter is a full subquadrangle that has a non-empty intersection with each line. There is some similarity with the following definition.

Definition 3.6. A subspace $\Gamma' = (X', \mathscr{L}')$ of Γ is called i-large, if every i-dimensional singular subspace intersects X' nontrivially, and there exists an $(i-1)$-dimensional singular subspace disjoint from X'.

We are interested in a special class of i-large subspaces, namely, the non-degenerate ones of minimal rank. It will turn out that those can also be characterised by the counterpart for polar subspaces of the so-called *ideal subpolygons*.

Definition 3.7. A subspace $\Gamma' = (X', \mathscr{L}')$ of Γ is called *ideal* if the residue of each singular subspace S of Γ' in Γ', where

$$\dim S = \max\{\dim U \mid U \subseteq X' \text{ is a singular subspace of } \Gamma'\} - 1,$$

is an ovoid in the corresponding residue in Γ.

We now have the following equivalences.

Proposition 3.8. *Let Γ have rank at least 3. The following are equivalent.*

- (*i*) *θ is a kangaroo or the identity.*
- (*ii*) *There exists k, $0 \leq k \leq n-1$ such that for every generator M, the subspace $M \cap M^\theta$ is k-dimensional.*
- (*iii*) *There exists k, $0 \leq k \leq n-1$ such that for every generator M, the subspace $M \cap M^\theta$ is k-dimensional and globally fixed by θ.*
- (*iv*) *There exists k, $0 \leq k \leq n-1$ such that for every generator M, the subspace $M \cap M^\theta$ is k-dimensional and pointwise fixed by θ.*
- (*v*) *θ has at least one fixed point, maps no line to a different but coplanar line, and no line to a disjoint but non-opposite and non-collinear one.*
- (*vi*) *There exists k, $0 \leq k \leq n-1$ such that the fixed point set of θ is an $(n-k-1)$-large nondegenerate polar subspace of rank $k+1$.*
- (*vii*) *The fixed point set of θ is a nonempty ideal subspace.*

Proof. (*iii*) $\Rightarrow$ (*iv*). If $k = 0$, the intersections are points themselves and are thus fixed pointwise.

Now let $k > 0$, suppose M is a generator and $U = M \cap M^\theta$ is a k-dimensional subspace, fixed by θ. Let V be a $(k-1)$-dimensional subspace of U. Suppose for a contradiction that $V \neq V^\theta$. Thus $V \cap V^\theta$ is $(k-2)$-dimensional.

Pick a complement W of U in M; then W is $(n-k-2)$-dimensional. Take a generator $N \neq M$ through $\langle W, V\rangle$. We know that $N \cap N^\theta$ is k-dimensional and $V \cap V^\theta \subset N \cap N^\theta$. So there exists a line L in $N \cap N^\theta$ disjoint from $V \cap V^\theta$. But L is collinear with V and V^θ, so L is collinear with U. Since L is also collinear with $W \subseteq M$ and $W^\theta \subseteq M^\theta$, we infer that L is collinear with M and M^θ. So it is contained in M and M^θ, and hence in $M \cap M^\theta = U$. But then $N \cap N^\theta = \langle V \cap V^\theta, L\rangle = U$, a contradiction. Hence every $(k-1)$-dimensional subspace of U is stabilised, implying that U is pointwise fixed.

$(i) \Rightarrow (iv)$. This follows from Lemma 3.5.

$(iv) \Rightarrow (i)$. Obvious.

$(iv) \Rightarrow (iii) \Rightarrow (ii)$. Trivial.

$(ii) \Rightarrow (iii)$. See [12, Proposition 2.5].

$(i) \Rightarrow (v)$. If a line is mapped to a different but coplanar line, then clearly some points are mapped onto collinear points, a contradiction. Similar argument for a line L mapped onto a disjoint but non-opposite line L^θ. Indeed, a point x on L collinear to all points of L^θ (which exists since L and L^θ are not opposite) is mapped onto a collinear point.

$(v) \Rightarrow (i)$. Let x_0 be a fixed point of θ. We first show that no point collinear to x_0 is mapped onto a collinear point. Suppose for a contradiction that some point $x \perp x_0$ is mapped onto a collinear point $x' \neq x$. If x' were not on the line xx_0, then the line xx_0 would be mapped onto a coplanar but distinct line. Hence $x' \in xx_0$. Since the rank of Γ is at least 3, there exists a point $y \perp xx_0$. If y is mapped onto a collinear (or fixed) point y', then again $y' \in yx_0$. But then the line xy is distinct and coplanar with $x'y'$, a contradiction. Hence $y' = y^\theta$ is not collinear to y. But then the line xy is mapped onto the disjoint but non-opposite and non-collinear line $x'y'$ ($x' \perp xy$), the final contradiction. Hence either $x = x^\theta$ or x is mapped onto an opposite point. If there exists some point $x_1 \perp x_0$, $x_1 \neq x_0$, fixed under θ, then the whole line x_0x_1 is fixed pointwise, every point is collinear to a fixed point and the previous argument yields (i).

Hence we may assume that θ maps every point collinear to x_0 to an opposite. It follows that each generator M through x_0 is mapped onto a generator M^θ with $|M \cap M^\theta| = 1$. By parity, the intersection S of any generator N with its image is never empty. By the previous paragraph, and since each point of S is mapped onto a collinear point, S is a singleton $\{s\}$. Now suppose for a contradiction that $s \neq s^\theta$. Let $p \in N \cap x_0^\perp$ be arbitrary. Then s is collinear to $ps \cup p^\theta s^\theta$, so ps is not opposite $(ps)^\theta$. Since p is opposite p^θ by our assumption, ps is not collinear to $(ps)^\theta$ either, a contradiction. Hence $s = s^\theta$ and the fixed point set is an ovoid. In any case, each point is collinear to a fixed point and so by the first paragraph, θ is a kangaroo.

$(iv) \Rightarrow (vi)$. Let Γ' be the fixed point set. Every $(n-k-1)$-dimensional singular subspace lies in some generator M and hence has at least one point x in common with $M \cap M^\theta$, which has dimension k by assumption. But x is fixed by assumption, hence Γ' is at most $n-k-1$-large. It is exactly $n-k-1$-large since a dimension argument yields in each generator an $(n-k-2)$-dimensional subspace disjoint from $M \cap M^\theta$, and hence disjoint from Γ'. Since Γ' obviously has rank $k+1$, it remains to show that Γ' is non-degenerate. Suppose, for a contradiction, that $x \in \Gamma'$ is collinear to all points of Γ'. An arbitrary generator not through x contains a k-dimensional subspace S pointwise fixed under θ. Then $\langle x, S\rangle$ has dimension $k+1$ and is fixed, a contradiction.

$(vi) \Rightarrow (vii)$. Let S be a submaximal singular subspace of the fixed point set Γ', hence of dimension $k-1$. Let M be a maximal singular subspace of Γ containing S. If M did not contain a singular subspace U of Γ' of dimension k, then it would contain a subspace W of M complementary to S, hence of dimension $n-k-1$ and disjoint from Γ'. This contradicts Γ' being $(n-k-1)$-large.

$(vii) \Rightarrow (vi)$. Let Γ' again be the fixed point set. We first show that it is a non-degenerate subspace. Indeed, suppose for a contradiction that some point $x \in \Gamma'$ is collinear to all points of Γ'. Let $k \geq 1$ be the maximal dimension of singular subspaces in Γ', and consider a subspace $U \subseteq \Gamma'$ of dimension k. Then $x \in U$. Let $W \subseteq U$ be a subspace of dimension $k-1$ not containing x. By maximality of k, the singular subspace W is contained in a unique singular subspace of dimension k of Γ' (namely, U), contradicting the fact that these should form an ovoid in the residue. Hence Γ' is non-degenerate of rank $k+1$. We claim that every generator of Γ contains a k-dimensional singular subspace of Γ'. Indeed, assume that some generator M of Γ intersects Γ' in a subspace U of dimension $i < k$ (possibly $i = -1$). By the non-degeneracy of Γ', we find a singular subspace S of dimension $k-1$ in Γ' containing U. Then there is a unique generator M' containing S and intersecting M in a subspace S' of dimension $n-k-i$ containing U. By assumption M' contains a unique subspace K of Γ' of dimension k containing S. A dimension argument yields $(M \cap K) \setminus S \neq \emptyset$, a contradiction. The claim follows. Since every singular subspace of dimension $n-k-1$ is contained in a generator, we see that Γ' is $(n-k-1)$-large.

$(vii) \Rightarrow (i)$. Suppose for a contradiction that some point x is mapped onto a collinear point $x' \neq x$. Let M be a generator of Γ containing x and x'. We know from the previous paragraph that the fixed point set Γ' is an $(n-k-1)$-large (nondegenerate) polar subspace of rank $k+1$, and hence that M contains a k-dimensional subspace S of Γ'. Let $R \subseteq S$ be a subspace of dimension $k-1$. Considering the residue of R, it suffices to show the assertion for Γ' an ovoid.

Now $x^\perp \cap \Gamma' = x'^\perp \cap \Gamma' = (xx')^\perp \cap \Gamma'$. But $x^\perp \cap \Gamma'$ defines an ovoid in the residue of x, whereas $(xx')^\perp \cap \Gamma'$ defines an ovoid in a singular hyperplane of that residue. Hence the assertion follows if we show that no ovoid of any polar space is contained in the perp of some point p. But this is obviously true as there exists at least one maximal singular subspace M containing a given submaximal singular subspace S in a maximal singular subspace through p, with S and M both disjoint from both the ovoid and $\{p\}$. □

Note that (v) expresses kangaroo behaviour of θ in the long root subgroup geometry of Γ, that is, the line Grassmannian geometry. In this geometry, there are five distinct mutual positions of two points: They can be equal, collinear, symplectic, special and opposite. Now (v) says that points can be mapped only to itself, symplectic and opposite ones, skipping the possibilities of collinear and special.

Whenever θ is lazy, the fixed point set is a polar space and, by the main result of [3], automatically arises as the intersection of the corresponding hyperbolic quadric with a subspace of the ambient projective space. We now investigate this behaviour for diligent kangaroos.

3.2. **Diligent kangaroos.** Suppose θ is a diligent kangaroo of a polar space $\Gamma = (X, \mathscr{L})$ of type D_n. We assume that Γ is defined over a field $\mathbb{K}$ (hence it corresponds to a hyperbolic quadric in $\mathsf{PG}(2n-1, \mathbb{K})$ and hence it can be viewed as a full subgeometry of this projective space). Note that θ extends to a collineation of $\mathsf{PG}(2n-1, \mathbb{K})$ and

as such is either *linear* or *semi-linear*, according to whether or not the companion field automorphism of the underlying vector space automorphism is trivial.

Lemma 3.9. *The fixed point set of θ is an ovoid.*

Proof. This follows from Proposition 3.8(*vii*). □

Lemma 3.10. *Let $\Gamma = (X, \mathscr{L})$ be a full subgeometry of $\mathsf{PG}(2n-1, \mathbb{K})$ as above, and let S be a subspace of $\mathsf{PG}(2n-1, \mathbb{K})$ with $\dim S < n$. Then there exists a generator M of Γ disjoint from S.*

Proof. We prove this by induction on n.

For $n = 2$, we may assume that S is a line. If S belongs to Γ, then we find a disjoint line of Γ; if S does not belong to Γ, then it intersects Γ in at most two points and we find a line of Γ containing neither.

Now suppose $n \geq 3$. We may obviously assume that $S \cap X$ is nonempty. For an arbitrary point $q \in S \cap X$, we select a point $p \in X \setminus S$ opposite q. Then $S \cap \langle p^\perp \rangle$ is a hyperplane of S and has hence dimension at most $n-2$. Let H be a hyperplane of $\langle p^\perp \rangle$ not containing p. Projecting $p^\perp$ and $S \cap \langle p^\perp \rangle$ onto H we obtain a polar space Γ' of type D_{n-1} in H and a subspace S' of H with $\dim S' < n-1$. Applying induction we find a generator M' of Γ' disjoint from S'. It follows that $M = \langle p, M' \rangle$ is a generator of Γ disjoint from S. □

We now prove a certain "rigidity" for ovoids, and we do this in general for all polar spaces that are full subgeometries of projective spaces. We show that the linear pointwise stabiliser of any ovoid pointwise fixes the subspace of the ambient projective space generated by the ovoid.

Lemma 3.11. *Let O be an ovoid of a polar space Γ with point set X of rank at least 2, which is a full subgeometry of $\mathsf{PG}(d, \mathbb{K})$. Then every collineation of Γ that extends to a linear projective collineation of $\mathsf{PG}(d, \mathbb{K})$ and pointwise fixes O, also pointwise fixes the subspace $\langle O \rangle$.*

Proof. Suppose for a contradiction that some collineation pointwise fixing O does not pointwise fix $\langle O \rangle$. Then, by linearity, O is contained in two complementary subspaces U and W of $\langle U, W \rangle$, that is, $U \cap W = \varnothing$ and $\langle U, W \rangle = \langle O \rangle$ (and $O \subseteq U \cup W$).

We reduce the situation to lower rank. Let L be a line through some point u of $O \cap U$, and not contained in U itself (which clearly exists). Let $w \in O \cap W$ and $u' \in O \cap U \setminus \{u\}$. Since the assertion is immediate for $\mathbb{K} \cong \mathbb{F}_2$, there is some point $y \in L \setminus \{u\}$ not collinear to either w or u'. We want to find M through y, hence we consider the residue Γ' at y. In order to find M we only have to avoid the subspaces U' and W' corresponding to $\langle y^\perp \rangle \cap U$ and $\langle y^\perp \rangle \cap W$, respectively. By our choice of y, they have one dimension less than U and W, respectively. Since the dimension of the ambient projective space is dropped by 2, we see that U' and W' are complementary. We can repeat this process until we obtain a polar space Γ'' of rank 1 or 2. If X'', U'' and W'' correspond to X, U and W, respectively, then $X'' \cap (U'' \cup W'')$ is an ovoid in Γ''. Now suppose Γ was not a hyperbolic quadric. Then we may assume that Γ'' has rank 1. But ovoids of polar spaces of rank 1 clearly coincide with the polar space itself. Hence X'' lies in the union of two complementary subspaces, which we claim is a contradiction for $|\mathbb{K}| > 2$.

Indeed, Γ'' contains at least three points. If X'' is contained in a line, the assertion is easy. So we may suppose that W'' has dimension at least 1. We have $X'' \subseteq U'' \cup W''$. Select $u \in U''$ and let H_u be the tangent hyperplane at u. Then H_u does not contain any point of $X'' \cap W''$, so it intersects W'' in a hyperplane. Pick $w \in W'' \setminus (H_u \cup X'')$ (and note w exists since $\dim W'' \geq 1$ and we may assume $|\mathbb{K}| > 2$). Then the line $\langle u, w\rangle$ is not a tangent line and hence intersects X'' in at least two points, one of which necessarily lies in W''. This yields $w \in X''$, a contradiction.

Now suppose Γ is a hyperbolic quadric. Then we consider the case where Γ'' has rank 2, hence is a hyperbolic quadric in $\mathsf{PG}(3, \mathbb{K})$. Here, the ovoid is either contained in the union of two lines, or, except for one point, in a plane. First suppose U'' and W'' are lines. Since at most two points of O'' can be contained in a line of $\mathsf{PG}(3, \mathbb{K})$, we deduce that $|O''| \leq 4$. Since we may assume $|\mathbb{K}| > 2$, we conclude $|O''| = 4$. But it is readily checked that each ovoid of Γ'' is contained in a plane, in which U'' and W'' cannot be complementary, a contradiction. At last assume that, for some point $x \in O''$, we have $O'' \setminus \{x\} = X'' \cap \pi$, for some plane π of $\mathsf{PG}(3, \mathbb{K})$. Since x belongs to two generators of Γ'', it is collinear to two points of the ovoid $\pi \cap X''$, hence to at least one point of O'', a contradiction.

The assertion is proved. □

We now return to our diligent kangaroo θ.

Lemma 3.12. *If θ is a linear diligent kangaroo, then the fixed point set O spans an n-dimensional subspace S in $\mathsf{PG}(2n-1, \mathbb{K})$, and $O = S \cap X$.*

Proof. By Lemma 3.11, θ pointwise fixes $\langle O\rangle$. Then Lemma 3.10 implies that S is at least n-dimensional. Now suppose $\dim S > n$, then $S \cap X$ contains lines, as generators of Γ intersect S in at least a line, contradicting the fact that O spanning S implies that S is pointwise fixed under the unique extension of θ to $\mathsf{PG}(2n-1, \mathbb{K})$, and θ being diligent. □

The situation in the semilinear case is completely different, in fact, opposite. In the linear case, the span of the ovoid has minimal dimension, in the semi-linear case it has maximal dimension. Also, only involutions qualify!

Lemma 3.13. *If θ is semilinear, then it is an involution and an anti-oppomorphism, and the fixed point ovoid O spans $\mathsf{PG}(2n-1, \mathbb{K})$.*

Proof. We prove this statement by induction on n. First let $n = 2$. Let $x \in X$ be a point that is not fixed under θ. Then $x^\perp \cap (x^\theta)^\perp$ is fixed pointwise by θ as every line L of Γ through x carries a fixed point p_L and $L = \langle x, p_L\rangle$ is mapped onto $\langle x^\sigma, p_L\rangle$. This already implies that θ is an anti-oppomorphism. Hence $(x^\perp \cap (x^\theta)^\perp)^\perp$, which equals $\{x, x^\theta\}$, is preserved under the action of θ, yielding $(x^\theta)^\theta = x$. We conclude that θ is an involution. Now assume, for a contradiction, that all fixed points are contained in some plane π, in which θ then induces a Baer involution. Clearly, π intersects X in a conic C and since O is an ovoid of Γ, we have $C = O$. But no conic in a projective plane $\mathsf{PG}(2, \mathbb{K})$ is pointwise fixed under a Baer involution (since the conic would be a too big conic in the Baer subgeometry). This settles the case $n = 2$.

Now let $n > 2$. We claim that θ is involutive and that, for each point x with $x \neq x^\theta$, and for each line L through x, there exists a point $x' \in L$, $x' \neq x$, contained in $\langle O\rangle$.

Indeed, let x be a point not fixed by θ. Set $X' = x^\perp \cap (x^\theta)^\perp$, and let $\Gamma' = (X', \mathscr{L}')$ be the corresponding hyperbolic polar space. Then X' spans a subspace of dimension $2n-3$. We show first that $O' = O \cap X'$ is an ovoid of Γ'. Let U' be a generator of Γ'. Then $U = \langle x, U' \rangle$ is a generator of Γ and so, by Proposition 3.8(iv), $U \cap U^\theta$ is a fixed point p, which is collinear to both x (as $x, p \in U$) and x^θ (as $x^\theta, p \in U^\theta$). Hence $p \in U'$ (as $U' = U \cap (x^\theta)^\perp$). This shows that O' is an ovoid of Γ'.

Now define the following collineation θ' of Γ'. Let $z \in X'$. Then $z^{\theta'}$ is by definition the unique point on $\langle x^\theta, z^\theta \rangle$ collinear to x. Since θ' is the composition of a semilinear map and a (linear) projection, it is a semilinear collineation of Γ'. Now O' is exactly the set of fixed points of θ', since each fixed point q is collinear to q^θ, which is a contradiction to θ being a kangaroo if $q \neq q^\theta$. By induction, O' generates $\langle X' \rangle$. It follows that $O'^\perp = \{x, x^\theta\}$, and so $\{x, x^\theta\}^\theta = \{x, x^\theta\}$, implying θ is an involution (varying x). Also, every line $L \in \mathscr{L}$ through x has a unique point x' in common with X', and that point lies in $\langle O' \rangle \subseteq \langle O \rangle$. Our claim is proved.

Now suppose for a contradiction that O is contained in some hyperplane H of $\mathsf{PG}(2n-1, \mathbb{K})$. Obviously H contains a point $x \notin O$ with $x^\perp \not\subseteq H$. Hence some line $L \in \mathscr{L}$ through x intersects H in precisely x. But our claim above implies that L contains a point $x' \in \langle O \rangle \subseteq H$, with $x' \neq x$. Hence $L = \langle x, x' \rangle \subseteq H$, a contradiction.

The fact that θ is an anti-oppomorphism follows from the fact that it maps each generator M to a generator intersecting M in precisely a point, hence adjacent to an opposite generator. □

Remark 3.14. Since the fixed point set of a lazy kangaroo contains full lines, a lazy kangaroo is always linear. Similarly as above one shows that, if the fixed point set has rank $k+1$ as a polar space, it arises as the intersection of $X \subseteq \mathsf{PG}(2n-1, \mathbb{K})$ with a subspace of dimension $n+k$. This also follows directly from Lemma 3.1.2 of [9].

Example 3.15. Let θ be a linear lazy kangaroo of $\mathsf{D}_{5,1}(\mathbb{K})$ fixing points, lines and planes, but no 3-spaces. The fix structure is a polar space of rank 3 embedded in a projective space of dimension $5+2=7$. Hence its equation is of the form

$$X_{-3}X_3 + X_{-2}X_2 + X_{-1}X_1 = X_0^2 + aX_0X_0' + bX_0'^2,$$

with $a, b \in \mathbb{K}$. Since the polynomial $x^2 + ax + b$ does not have solutions in $\mathbb{K}$, it defines a quadratic extension $\mathbb{L}$ of $\mathbb{K}$ and we see, using the notation of [9], that the fix structure is a polar space isomorphic to $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{L})$.

4. Kangaroos in $\mathsf{E}_{6,1}(\mathbb{K})$

In this section we let θ be a type preserving automorphism of $\mathsf{E}_6(\mathbb{K})$ mapping no point of $\mathsf{E}_{6,1}(\mathbb{K})$ to a distinct collinear one, but having at least one fixed point and mapping at least one point to a point at distance 2. We call such an automorphism a *kangaroo (collineation) of* $\mathsf{E}_{6,1}(\mathbb{K})$. In order to describe the action of such a kangaroo, it is convenient to consider the standard representation of $\mathsf{E}_{6,1}(\mathbb{K})$ as a full subgeometry in $\mathsf{PG}(26, \mathbb{K})$. It is also the so-called *universal embedding*, see [16], and, as such, every collineation of it extends to $\mathsf{PG}(26, \mathbb{K})$. We denote it by $\mathscr{E}_6(\mathbb{K})$. Our aim is to prove the following classification.

Theorem 4.1. *Let θ be a collineation of $\mathsf{E}_{6,1}(\mathbb{K})$ and consider it as a collineation of $\mathsf{PG}(26, \mathbb{K})$ stabilizing $\mathscr{E}_6(\mathbb{K})$. Then θ is a kangaroo if, and only if, its fixed point structure F is a—separable or inseparable—quaternion Veronese variety in a 14-dimensional subspace*

U of $\mathsf{PG}(26,\mathbb{K})$, with $F = U \cap \mathscr{E}_6(\mathbb{K})$, or an octonion Veronese variety (and then θ is a Galois involution) in a Baer subspace $\Sigma \cong \mathsf{PG}(26,\mathbb{F})$ of $\mathsf{PG}(26,\mathbb{K})$ over a field $\mathbb{F}$ with $\mathbb{K}/\mathbb{F}$ a quadratic Galois extension and $F = \Sigma \cap \mathscr{E}_6(\mathbb{K})$.

We prove some lemmas which ultimately will culminate in a proof of the theorem. In the sequel, $\Delta = (X, \mathscr{L})$ is the parapolar space $\mathsf{E}_{6,1}(\mathbb{K})$. Recall that the symp containing the noncollinear points x, y is denoted by $\xi(x,y)$.

4.1. **General properties of kangaroo collineations.**

Lemma 4.2. *The kangaroo collineation θ is also a dual kangaroo collineation.*

Proof. We have to show that θ does not map any symp to an adjacent one and that θ fixes at least one symp.

Suppose first for a contradiction that θ maps the symp ξ to an adjacent symp ξ^θ. Set $U := \xi \cap \xi^\theta$. Then U^θ is a 4-space of ξ^θ and so $U \cap U^\theta \neq \emptyset$. But any point of $U \cap U^\theta$ is mapped onto a point of U^θ, hence, if not fixed, sent to a collinear one, a contradiction. Consequently $U \cap U^\theta$ is fixed pointwise.

Pick a 3-space $W \subseteq U$ and let V, V' be the unique $4'$-space in ξ, ξ^θ, respectively, which contains W. Fact 2.1(iv) implies that V and V' are contained in a common 5-space and hence all points of V are collinear to all points of V'. Consequently all points of $V' \cap V^\theta$ are fixed. But all fixed points of $\xi \cup \xi^\theta$ lie in U, hence those of $V \cup V'$ lie in W, and consequently those of V^θ in W^θ. Therefore, $V^\theta \cap V' \subseteq W \cap W^\theta$, so $V^\theta \cap V' = W \cap W^\theta$. Hence $W \cap W^\theta$ is a point or a plane (since V' and V^θ are both $4'$-spaces of ξ^θ). But we violate this requirement if we choose W so that it intersects $U \cap U^\theta$ in a subspace of codimension 1. We obtain the desired contradiction.

Now let x be a fixed point of θ and suppose no symp through x is fixed. Then, in $\mathsf{Res}(x)$ viewed as a polar space, every point is mapped onto an opposite, and hence θ induces a type reversing automorphism. This is a contradiction as types in $\mathsf{E}_6(\mathbb{K})$ are preserved. □

Thanks to this lemma we may from now on assume that also the dual of everything we prove, holds.

The next lemma generates a lot of fixed points in a fixed symp.

Lemma 4.3. *If a symp ξ is fixed under θ, then every generator of ξ contains a fixed point. Moreover, θ restricted to ξ is a kangaroo.*

Proof. Let U be a generator of the fixed symp ξ. Then, since θ preserves types, $U \cap U^\theta \neq \emptyset$ and any point in the intersection is mapped onto a point of U^θ and hence, if not fixed, to a collinear one, contradicting the definition of a kangaroo collineation. Since θ does not map points to collinear ones, and since there are fixed points, θ is a kangaroo by definition. □

The next lemma generates a lot of fixed symps, and hence also a lot of fixed points.

Lemma 4.4. *For every point x with $x \neq x^\theta$, the symp $\xi(x, x^\theta)$ is fixed. Moreover, every point is collinear to a fixed point.*

Proof. Let x be a point which is not fixed under the kangaroo collineation θ. Suppose first that x is collinear to some fixed point $p = p^\theta$. Then the symps $\xi(x, x^\theta)$ and $\xi(x^\theta, x^{\theta^2})$ have the line px^θ in common and hence coincide by Lemma 4.2. Hence the symp $\xi(x, x^\theta)$ is fixed.

Now suppose that x is not collinear to any fixed point. Let p be any fixed point of θ and let y be collinear to both p and x. By the previous paragraph we know that $\xi := \xi(y, y^\theta)$ is fixed. Since $x \perp y$, the intersection $x^\perp \cap \xi$ is a $4'$-space by Fact 2.1(iii) and hence contains a fixed point by Lemma 4.3. This contradicts the assumption that x is not collinear to a fixed point. The lemma is proved. □

We note the following direct implications of Lemmas 4.2 and 4.4.

Corollary 4.5. *Each point is contained in a fixed symp and, dually, every symp contains a fixed point.*

Lemma 4.6. *No two fixed points are collinear. In particular, the fixed points contained in a fixed symp form an ovoid of the symp.*

Proof. Assume for a contradiction that x and y are two collinear points with $x = x^\theta$ and $y = y^\theta$. Every symp through the line xy is fixed by Lemma 4.2. Hence $\mathsf{Res}(xy)$ is fixed elementwise. Noting that points of a fixed singular subspace are either mapped to collinear ones or fixed, we see that all points collinear to xy are fixed. By connectivity, all points are fixed and θ is the identity, a contradiction.

Hence θ induces a diligent kangaroo in each fixed symp and the assertion follows from Lemma 3.9. □

We can now define a new incidence structure (F, Φ) as follows. The point set F consists of all points fixed under θ; the block set Φ consists of all symps fixed under θ. Incidence is natural.

Lemma 4.7. *The point-block geometry (F, Φ) is a projective plane. Hence F is an ovoidal subplane of $\mathsf{E}_{6,1}(\mathbb{K})$.*

Proof. Since two distinct fixed points are not collinear, they define a unique fixed symp. Dually, two distinct fixed symps intersect in a unique fixed point. Since blocks contain at least three points, the lemma follows. □

4.2. **Collineations of $\mathsf{E}_{6,1}(\mathbb{K})$ that pointwise fix a quaternion Veronesean.** Now let $Y \subseteq X$ be a subset of points of the exceptional geometry $\mathsf{E}_{6,1}(\mathbb{K})$ with the following properties:

(VV1) Each symp ξ containing at least two points of Y intersects Y in an ovoid of ξ; we call such an ovoid a Y-ovoid.
(VV2) Y is not contained in a symp.

Our main aim is to show that, if some nontrivial collineation θ of $\mathsf{E}_{6,1}(\mathbb{K})$ pointwise fixes Y, then it is a kangaroo. We will also show that the above provides a weaker definition of ovoidal subplanes. We proceed with a series of lemmas.

We will refer to the symps that contain a Y-ovoid as a *host symp*. As a first step, we show that any set Y satisfying (VV1) and (VV2) defines an ovoidal subplane of $\mathsf{E}_{6,1}(\mathbb{K})$ (independent of whether there is a collineation fixing it pointwise or not).

Lemma 4.8. *Two host symps intersect in a unique point, which automatically belongs to Y.*

Proof. Suppose first for a contradiction that two host symps ξ_1, ξ_2 have a 4-space U in common. By (VV1), U contains a unique point $y \in Y$. Select $y_2 \in Y \cap \xi_2 \setminus \{y\}$. By (VV1), y_2 is not collinear to y, so $y_2^\perp \cap \xi_1$ is a $4'$-space which does not contain y, but which, by the definition of ovoid, contains another point $y_1 \in Y$. Now any symp containing the collinear points $y_1, y_2 \in Y$ violates (VV1). Hence $\xi_1 \cap \xi_2 = \{p\}$, for some point $p \in X$. Suppose for a contradiction that $p \notin Y$. Then there exists $x_1 \in \xi_1 \cap Y$ collinear to p, and $x_2 \in \xi_2 \cap x_1^\perp \cap Y$ (by (VV1)). Again any symp containing the collinear points $x_1, x_2 \in Y$ violates (VV1). □

Let Ψ be the set of all host spaces.

Lemma 4.9. *The incidence structure (Y, Ψ), with natural incidence, is a projective plane. Hence Y defines an ovoidal subplane of $\mathsf{E}_{6,1}(\mathbb{K})$.*

Proof. This follows from (VV1) and Lemma 4.8. □

We now also have:

Lemma 4.10. *The set Ψ satisfies* (VV1) *in the dual $\mathsf{E}_{6,6}(\mathbb{K})$ of $\mathsf{E}_{6,1}(\mathbb{K})$.*

Proof. Let $y \in Y$ and select $\xi \in \Psi$ with $y \notin \xi$ (this exists by Lemma 4.9). The members of Ψ through y are given by the symps $\xi(y, x)$, with $x \in \xi \cap Y$. Since $\xi \cap Y$ is an ovoid of ξ, the assertion follows. □

Now comes the crux of the argument.

Lemma 4.11. *Every point x of $\mathsf{E}_{6,1}(\mathbb{K})$ belongs to some host symp, unique if $x \notin Y$.*

Proof. We apply approximately the same technique as in the proof of Lemma 4.4. So suppose first that $x \perp y \in Y$. By Lemma 4.10, there is a unique member $\xi \in \Psi$ containing the line xy (in the residue of y, the line xy is a 4-space of $\mathsf{E}_{6,6}(\mathbb{K})$). Hence we may assume x is not collinear to any member of Y. Let $y \in Y$ be arbitrary and select $z \in x^\perp \cap y^\perp$. Then z is contained in a host symp ξ by the foregoing. But $x^\perp \cap \xi$ is a $4'$-space, and so contains a member of Y, contradicting our hypothesis. The lemma is proved, taking into account Lemma 4.8. □

We can now show:

Proposition 4.12. *The collineation θ is a kangaroo.*

Proof. Let $x \in X$ be arbitrary. By Lemma 4.11, x belongs to a symp $\xi \in \Psi$. Since ξ contains at least two non-collinear fixed points, it is fixed by θ. Then Proposition 3.8 implies that θ restricted to ξ is a kangaroo. So if $x^\theta \neq x$ then $x^\theta \not\perp x$. Thus since θ is nontrivial, θ is a kangaroo. □

We have now everything in place to prove Theorem 4.1.

4.3. **Proof of Theorem 4.1.** Recall once again that, since $\mathscr{E}_6(\mathbb{K})$ is, by [16], the so-called *universal embedding* of $\mathsf{E}_{6,1}(\mathbb{K})$, the collineation θ extends to a collineation of $\mathsf{PG}(26,\mathbb{K})$. We distinguish between two cases.

Case 1: *Suppose θ is a linear collineation.* By Lemmas 3.12, 4.3 and 4.6, each member of Φ is a quadric of Witt index 1 in some 5-space of $\mathsf{PG}(26,\mathbb{K})$. Since different such 5-spaces are contained in distinct symps, they pairwise intersect in unique points; also, each such 5-space intersects X in that quadric of Witt index 1. Hence Main Result 4.3 of [7] implies that (F,Φ) is a quaternion Veronese variety in a 14-dimensional subspace U of $\mathsf{PG}(26,\mathbb{K})$. Since θ fixes U pointwise, no other point of X is contained in U and so $F = U \cap \mathscr{E}_6(\mathbb{K})$.

Now, conversely, suppose some collineation θ pointwise fixes a quaternion Veronese variety on $\mathscr{E}_6(\mathbb{K})$ arising as intersection with a subspace U. Then Proposition 4.12 implies that θ is a kangaroo.

Case 2: *Now let θ be a semilinear kangaroo collineation.* By Lemmas 3.13, 4.3 and 4.6, the restriction of θ to each subspace $\langle\xi\rangle$ of $\mathsf{PG}(26,\mathbb{K})$, with $\xi \in \Phi$, is a Baer involution whose fixed point set intersects ξ in an ovoid spanning the 9-space $\langle\xi\rangle$. Let x be any point of X. Lemma 4.4 (for $x^\theta \neq x$) and the dual of Lemma 4.6 (for $x^\theta = x$) imply that x is contained in a fixed symp ξ. In the latter, θ induces an involution, hence $x^{\theta^2} = x$, implying that θ is a Baer involution in $\mathsf{PG}(26,\mathbb{K})$, say fixing the projective space $\mathsf{PG}(26,\mathbb{F})$, with $\mathbb{K}/\mathbb{F}$ quadratic. It follows that F is the intersection of X with $\mathsf{PG}(26,\mathbb{F})$, that each intersection $\xi \cap \mathsf{PG}(26,\mathbb{F})$ is a quadric Q_ξ of Witt index 1 in some 9-space, that, for each pair of such distinct quadrics Q_ξ and Q_ζ, the intersection $\langle Q_\xi\rangle \cap \langle Q_\zeta\rangle$ is a singleton, and that each quadric Q_ξ has trivial nucleus. Main Result 4.3 of [7] again implies that (F,Φ) is an octonion Veronese variety in $\mathsf{PG}(26,\mathbb{F})$. The converse again follows from Proposition 4.12.

5. Existence and uniqueness of quaternion Veroneseans in $\mathsf{E}_{6,1}(\mathbb{K})$

In this section we complete the proof of Theorem A. Roughly, our aim is to prove that every equivalence class of norm forms of a quadratic division algebra of degree 4 over the field $\mathbb{K}$ gives rise to a projectively unique quaternion Veronese variety contained in $\mathscr{E}_6(\mathbb{K})$ (via $\mathsf{E}_{6,1}(\mathbb{K})$) in a subspace of dimension 14 of $\mathsf{PG}(26,\mathbb{K})$. Also, we determine the pointwise stabiliser, both as a subgroup of the automorphism group of $\mathsf{E}_{6,1}(\mathbb{K})$ and as an abstract group.

From here on, it is convenient to use the letter X for elements of some quadratic algebra over $\mathbb{K}$, and not anymore for the point set of $\mathsf{E}_{6,1}(\mathbb{K})$. This will cause no confusion.

We start with the case of a separable quaternion Veronese variety.

5.1. **The separable case.**

5.1.1. *Linear ovoids of polar spaces of type* D_5. We have seen in Section 4.3 that a linear kangaroo collineation of $\mathsf{E}_{6,1}(\mathbb{K})$ pointwise fixes a quaternion Veronesean subvariety $\mathscr{V}(\mathbb{K},\mathbb{H})$ of $\mathscr{E}_6(\mathbb{K})$, with $\mathbb{H}$ a quaternion division algebra over $\mathbb{K}$. Recall the notion of *host spaces* of $\mathscr{V}(\mathbb{K},\mathbb{H})$ from Section 2.4, which here are 5-dimensional subspaces of $\mathsf{PG}(26,\mathbb{K})$. Let $\mathsf{PG}(5,\mathbb{K})$ be an arbitrary host space of $\mathscr{V}(\mathbb{K},\mathbb{H})$ with corresponding quadric O. Then we can choose coordinates in such a way that points are labeled with $(x_1,x_2,X) \in \mathbb{K}\times\mathbb{K}\times\mathbb{H}$ and O has equation $x_1x_2 = X\overline{X}$. Let $\mathbb{O}'$ be the split octonion

algebra obtained by the Cayley-Dickson doubling process applied to $\mathbb{H}$ with primitive element 1. Let ξ be the host symp of O. Then coordinatising the ambient projective space $\mathsf{PG}(9,\mathbb{K})$ as $(x_1,x_2,X)\in\mathbb{K}\times\mathbb{K}\times\mathbb{O}'$, the equation $x_1x_2=X\overline{X}$ describes a hyperbolic polar space Q isomorphic to $\mathsf{D}_{5,1}(\mathbb{K})$. Clearly O is contained in Q (as a linear ovoid), and we say it is contained in Q *in a standard way*, or it is the *standard inclusion*. We now show that this inclusion is projectively unique.

Lemma 5.1. *Let Q be a hyperbolic quadric in $\mathsf{PG}(9,\mathbb{K})$ and let $\mathbb{H}$ be a quaternion division algebra over $\mathbb{K}$. Then there exists a subspace U of $\mathsf{PG}(9,\mathbb{K})$ of dimension 5 such that $U\cap Q$ is a quadric of Witt index 1 isomorphic to the one with equation $x_1x_2=X\overline{X}$, where we label coordinates (x_1,x_2,X) in $\mathbb{K}\times\mathbb{K}\times\mathbb{H}$. Moreover, this inclusion is projectively equivalent to the standard inclusion.*

Proof. The existence of U follows straight from the considerations preceding the lemma, taking into account that Q is projectively unique as (nondegenerate) hyperbolic quadric in $\mathsf{PG}(9,\mathbb{K})$. It remains to show uniqueness.

The quaternion division algebra $\mathbb{H}$ is obtained by the Cayley-Dickson doubling process from a separable quadratic extension $\mathbb{L}$ of $\mathbb{K}$, with some primitive element ℓ. If $X\in\mathbb{H}$, then we can write $X=(z_1,z_2)$, with $z_1,z_2\in\mathbb{L}$, and $X\overline{X}=z_1\overline{z}_1+\ell z_2\overline{z}_2$, where, for $z\in\mathbb{L}$, $\overline{z}$ means the image of z under the Galois involution defining the extension $\mathbb{L}/\mathbb{K}$. It coincides with the standard involution of $\mathbb{H}$ restricted to $\mathbb{L}$. Let $\perp$ be the orthogonality relation defined by Q (if $\mathsf{char}\,\mathbb{K}=2$, this comes from an alternating form; if $\mathsf{char}\,\mathbb{K}\neq 2$, it comes from a symmetric bilinear form. In both cases the form is non-degenerate). With self-explaining notation, we set $W_1=(*,*,*,0)^\perp$ and $W_2=(*,*,0,*)^\perp$. From the equation of O we derive that $(*,*,0,0)\subseteq(0,0,*,*)^\perp$. With that, and using [10, Lemma 4.2] (which is stated there as folklore, but still proved for $\mathsf{char}\,\mathbb{K}\neq 2$, but the proof goes through for $\mathsf{char}\,\mathbb{K}=2$), we see that $W_i\cap Q$ are (non-degenerate) quadrics of Witt index 2. Let L_i and M_i be two opposite lines in $W_i\cap Q$, $i=1,2$. Then $Q_i=L_i^\perp\cap M_i^\perp$ is a hyperbolic quadric of Witt index 3 containing the quadric O_i of Witt index 1 obtained by intersecting Q_i with $(*,*,*,0)$ and $(*,*,0,*)$, for $i=1,2$, respectively. These quadrics are obtained from the field extension $\mathbb{L}/\mathbb{K}$ and hence, by the classification in [25], they are projectively unique. Hence, re-introducing coordinates for Q_1 so that its equation in $\langle Q_1\rangle$ becomes $x_1x_2+y_3y_4+y_5y_6=0$, we may take for $\langle O_1\rangle$ the subspace with equations $y_4=y_3-ty_5$ and $y_6=ny_5$, $n,t\in\mathbb{K}$, where x^2-tx+n is irreducible over $\mathbb{K}$, but reducible over $\mathbb{L}$. Then O_1 is described by the equation $x_1x_2+y_3^2-ty_3y_5+ny_5^2=0$. Let K_1 be the line with equations

$$\begin{cases} x_1=x_2=0,\\ y_4=y_3-ty_5,\\ y_6=ny_5. \end{cases}$$

Then one calculates that $K_1^\perp$ (now $\perp$ is with respect to Q_1) is disjoint from K_1, has equations $ny_5-ty_3+y_6=y_3+y_4=0$, and intersects Q_1 in a quadric O_1' of Witt index 1 isomorphic to O_1 and described by the equation $x_1yx_2-(y_3^2-ty_3y_5+ny_5^2)=0$. It follows that we can re-choose coordinates in $\langle Q_1\rangle$, labelled by $(x_1,x_2,z_3,z_4)\in\mathbb{K}\times\mathbb{K}\times\mathbb{L}\times\mathbb{L}$ such that Q_1 has equation $x_1x_2+z_3\overline{z}_3-z_4\overline{z}_4=0$. Performing similar coordinate changes in $\langle Q_2\rangle$ (keeping the original coordinates x_1 and x_2 at all times), we conclude that we can re-coordinatise the space using 4-tuples $(x_1,x_2,X_1,X_2$ in $\mathbb{K}\times\mathbb{K}\times\mathbb{H}\times\mathbb{H}$ such that Q has

equation $x_1x_2 + X_1\overline{X}_1 - X_2\overline{X}_2$. Noting that $X_1\overline{X}_1 - X_2\overline{X}_2$ is a norm in $\mathbb{O}'$, the lemma follows. □

5.1.2. *A construction of* $\mathscr{E}_6(\mathbb{K})$. In order to prove Theorem A we will need an explicit construction of $\mathscr{E}_6(\mathbb{K})$, the projectively unique embedding of $\mathsf{E}_{6,1}(\mathbb{K})$ in a 26-dimensional projective space over $\mathbb{K}$. Let V be a 27-dimensional vector space over the field $\mathbb{K}$, and write V as $V = \mathbb{K}\times\mathbb{K}\times\mathbb{K}\times\mathbb{O}'\times\mathbb{O}'\times\mathbb{O}'$. Write a generic vector in V as $(x_1, x_2, x_3; X_1, X_2, X_3)$. Then, by the last section of [6], the following equations determine 27 (degenerate) quadrics in the corresponding projective space $\mathsf{PG}(V) = \mathsf{PG}(26, \mathbb{K})$, whose intersection is precisely $\mathscr{E}_6(\mathbb{K})$:

$$\begin{aligned} x_2x_3 &= X_1\overline{X}_1, & X_2X_3 &= x_1\overline{X}_1,\\ x_3x_1 &= X_2\overline{X}_2, & X_3X_1 &= x_2\overline{X}_2,\\ x_1x_2 &= X_3\overline{X}_3, & X_1X_2 &= x_3\overline{X}_3. \end{aligned} \tag{$*$}$$

We also use the notation $\mathscr{E}_6(\mathbb{K})$ for the set of vectors of V whose coordinates satisfy the 27 quadratic equations. It is shown in [6] for $|\mathbb{K}| > 2$ that $\mathscr{E}_6(\mathbb{K})$ is the *projective closure* of the image of the (split octonion) Veronese map $\rho' : \mathbb{O}'\times\mathbb{O}' \to V : (X, Y) \mapsto (X\overline{X}, Y\overline{Y}, 1, Y, \overline{X}, X\overline{Y})$, that is, it is the smallest point set with the property that, whenever all points, except possibly one, of a line are contained in the point set, then all points of that line are contained in it. (This can also easily be seen directly: the given vectors satisfy the above equation, as can be checked by simple calculations, and since the $\mathscr{E}_6(\mathbb{K})$ is projectively closed, the image of ρ' is contained in $\mathscr{E}_6(\mathbb{K})$; a vector with coordinates $(*, *, 1, Y, \overline{X}, *)$ satisfying the above equations $(*)$ is readily seen to have coordinates $(X\overline{X}, Y\overline{Y}, 1, Y, \overline{X}, X\overline{Y})$.)

Consider the cubic form C on V given by

$$C(x_1, x_2, x_3, X_1, X_2, X_3) = x_1x_2x_3 - x_1X_1\overline{X}_1 - x_2X_2\overline{X}_2 - x_3X_3\overline{X}_3 + (X_1X_2)X_3 + \overline{X}_3(\overline{X}_2\overline{X}_1).$$

A tedious calculation shows that the linear combination of any pair of images under ρ' vanishes under C. Taking projective closure, this implies that C is the cubic form associated to $\mathscr{E}_6(\mathbb{K})$ (see [1] and [26]), as $\mathscr{E}_6(\mathbb{K})$ is determined by its host spaces. The cubic form C has the characteristic property that $C(v) = 0$ if and only if the projective point $\langle v\rangle$ defined by v is contained in a host space. We have the following lemma.

Lemma 5.2. *For each triple of pairwise non-collinear points r_1, r_2, r_3 of $\mathscr{E}_6(\mathbb{K})$, the symps $\xi(r_1, r_2), \xi(r_2, r_3)$ and $\xi(r_1, r_3)$ have a nontrivial common intersection if, and only if, $C(r_1 + r_2 + r_3) = 0$. In other words, r_i is far from $\xi(r_j, r_k)$, $\{1, 2, 3\} = \{i, j, k\}$, if, and only if, $C(r_1 + r_2 + r_3) \neq 0$.*

Proof. We first claim and show that, with r_1, r_2, r_3 pairwise non-collinear, the point r_3 is far from $\xi(r_1, r_2)$ if, and only if, $\xi(r_1, r_2)\cap\xi(r_2, r_3)\cap\xi(r_3, r_1) = \varnothing$. Indeed, the "only if" part follows from the observation that, if r_3 is far from $\xi(r_1, r_2)$, then $\xi(r_1, r_2)\cap\xi(r_3, r_1) = \{r_1\}$ (a line in the intersection would yield a point of $\xi(r_1, r_2)$ collinear to r_3). Now assume r_3 is close to $\xi(r_1, r_2)$. Then clearly $\xi(r_1, r_2)$ and $\xi(r_3, r_1)$ share a hyperplane of $\xi(r_1, r_2)\cap r_3^\perp$. Hence, repeating this argument with $\xi(r_2, r_3)$, we deduce that $\xi(r_1, r_2)\cap\xi(r_2, r_3)\cap\xi(r_3, r_1)$ contains a plane of $\xi(r_1, r_2)\cap r_3^\perp$. The claim is proved.

Since the automorphism group of $\mathscr{E}_6(\mathbb{K})$ acts transitively on the set of pairs of non-collinear points (as can easily be deduced from the construction of buildings of type E_6

in [21] and the existence of a natural Tits system (or BN-pair) in the automorphism group), we may assume $r_1 = (1,0,0,0,0,0)$ and $r_2 = (0,1,0,0,0,0)$. Suppose first that r_3 is close to $\xi(r_1,r_2)$. Set $U = r_3^\perp \cap \xi(r_1,r_2)$. Then $\langle U, r_1+r_2\rangle$ intersects $\xi(r_1,r_2)$ in a degenerate quadric consisting of U and a second 4-dimensional subspace U'. It follows that $\langle r_3, r_1+r_2, U\rangle$ is 6-dimensional, contains a singular 5-space $\langle r_3, U\rangle$ and a singular 4-space U', and hence contains a line through $r_1+r_2+r_3$ intersecting $\langle r_3, U\rangle \cup U'$ in two distinct points a and b. Then $r_1+r_2+r_3 \in \langle \xi(a,b)\rangle$ and so $C(r_1+r_2+r_3) = 0$.

Now let r_3 be any point in the image of ρ'. Then an elementary calculation shows that $C(r_1+r_2+r_3) = 1$. Since the points close to $\xi(r_1,r_2)$ span a hyperplane H_1 of $\mathsf{PG}(V)$ (see [5, §3.2], where it is called a white hyperplane), and since the image of ρ' is contained in the complement of a unique hyperplane H_2, we conclude that $H_1 = H_2$ and the lemma follows. □

5.1.3. *Embedding the separable quaternion Veronesean into $\mathscr{E}_6(\mathbb{K})$.* Restricting $\mathbb{O}'$ to $\mathbb{H}$, we now see that the image $\mathscr{V}(\mathbb{K},\mathbb{H})$ of the quaternion Veronesean map $\rho : \mathbb{H}\times\mathbb{H}\times\mathbb{H} \to V : (x,y,z) \mapsto (x\overline{x}, y\overline{y}, z\overline{z}, y\overline{z}, z\overline{x}, x\overline{y})$ is fully contained in $\mathscr{E}_6(\mathbb{K})$. Our goal is to show that this containment is projectively unique and that it is pointwise fixed by a non-trivial group of automorphisms isomorphic to the multiplicative subgroup of $\mathbb{H}^\times$ consisting of the elements with norm 1. We refer to this containment as a *standard inclusion of the quaternion Veronese variety in* $\mathsf{E}_{6,1}(\mathbb{K})$. Since it depends on the given representation of $\mathsf{E}_{6,1}(\mathbb{K})$, the projective uniqueness is not obvious. As a first step we prove Proposition 5.3 below. Define the ovoid O_1 to be the image $\{\rho(0,y,z) \mid y,z \in \mathbb{H}\} \subseteq \mathscr{V}(\mathbb{K},\mathbb{H})$. This coincides with the image $\{\rho'((0,0),(y,0)) \mid y \in \mathbb{H}\} \cup \{\rho(0,1,0)\}$. Likewise define $O_2 = \{\rho(x,0,z) \mid x,z \in \mathbb{H}\} = \{\rho'((x,0),(0,0)) \mid x \in \mathbb{H}\} \cup \{\rho(1,0,0)\}$ and $O_2' = \{\rho'((0,x),(0,0)) \mid x \in \mathbb{H}\} \cup \{\rho(1,0,0)\}$. Let us call a collineation of $\mathscr{E}_6(\mathbb{K})$ *linear* if its extension to $\mathsf{PG}(V)$ belongs to $\mathsf{PGL}_{27}(\mathbb{K})$.

Proposition 5.3. *Any linear collineation φ of $\mathscr{E}_6(\mathbb{K})$ fixing the four points $p_1 := \rho(1,0,0)$, $p_2 := \rho(0,1,0)$, $p_3 := \rho(0,0,1)$ and $p_{23} := \rho(0,1,1)$ and stabilising O_1 either stabilises both O_2 and O_2' or interchanges O_2 with O_2'. Also, the group of linear collineations pointwise fixing $\mathscr{V}(\mathbb{K},\mathbb{H})$ coincides with*

$$\{g_a : \mathsf{PG}(V) \to \mathsf{PG}(V) \colon (x,y,z,(x_1,x_2),(y_1,y_2),(z_1,z_2)) \mapsto (x,y,z,(x_1,x_2a),(y_1,y_2a),(z_1,z_2a)) \mid a \in \mathbb{H}, a\overline{a} = 1\}$$

and is hence abstractly isomorphic to the (multiplicative) subgroup of $\mathbb{H}^\times$ of elements of norm 1. It is a subgroup of index 2 in the pointwise stabiliser of $O_1 \cup \{p_1\}$.

Proof. Since φ is linear, and since the tangent hyperplane at p_{23} to O_1 in $\langle O_1\rangle$ is stabilised and intersects the line $\langle p_2,p_3\rangle$ in a point distinct from both p_2 and p_3, the line $\langle p_2,p_3\rangle$ is pointwise fixed by φ. Then, since φ stabilises the symps $\xi(p_1,p_2)$, $\xi(p_2,p_3)$, and $\xi(p_1,p_3)$, φ can be written as $(x,y,z,X,Y,Z) \mapsto (kx, y, z, X^{\theta_1}, Y^{\theta_2}, Z^{\theta_3})$, with $k \in \mathbb{K}$ and $\theta_1,\theta_2,\theta_3$ linear maps acting on $\mathbb{O}'$ (as a vector space over $\mathbb{K}$). In addition, since φ stabilises O_1 and fixes the point p_{23}, θ_1 stabilises $\mathbb{H} \subseteq \mathbb{O}'$ and fixes $1 \in \mathbb{H}$. Since the bilinear form β must be preserved, up to a scalar, we can write $(a,b)^{\theta_1} = (a^\theta, b^{\theta^*})$. Now $(*)$ yields

$$Y^{\theta_2}Z^{\theta_3} = k\overline{(\overline{Y}\overline{Z})^{\theta_1}}, \text{ for all } Y,Z \in \mathbb{O}'. \tag{5.1}$$

Setting subsequently $Y = 1$ and $Z = 1$ yields $Y^{\theta_2} = \overline{Y^{\theta_1}}C_2$ and $Z^{\theta_3} = C_3\overline{Z^{\theta_1}}$, for all $Y, Z \in \mathbb{O}'$, and some constants $C_2 = k(1^{\theta_3})^{-1}, C_3 = k(1^{\theta_2})^{-1} \in \mathbb{O}'$. Setting $Y = Z = 1$ we achieve $C_2C_3 = k$. We obtain, replacing Z with $\overline{Z}$ and Y with $\overline{Y}$ in Equation 5.1, and setting $\overline{C}_3 = C$,

$$(ZY)^{\theta_1} = (Z^{\theta_1}C)(C^{-1}Y^{\theta_1}), \text{ for all } Y, Z \in \mathbb{O}'.$$

Set $C = (c_1, c_2)$, $c_1, c_2 \in \mathbb{H}$. Suppose first $Z = (z, 0) \in \mathbb{H}$ and $Y = (y, 0) \in \mathbb{H}$. Then

$$(zy)^\theta(c_1\overline{c}_1 - c_2\overline{c}_2, 0) = (z^\theta c_1, \overline{z^\theta}c_2)(\overline{c}_1 y^\theta, -y^\theta c_2), \text{ for all } y, z \in \mathbb{H},$$

which yields

$$(zy)^\theta(c_1\overline{c}_1 - c_2\overline{c}_2) = z^\theta y^\theta c_1\overline{c}_1 - y^\theta z^\theta c_2\overline{c}_2, \tag{5.2}$$

$$\overline{c}_1(\overline{z^\theta}y^\theta - y^\theta\overline{z^\theta})c_2 = 0, \tag{5.3}$$

for all $y, z \in \mathbb{H}$. Clearly, since $\mathbb{H}^\times$ is not commutative, Equation 5.3 implies either $c_1 = 0$ or $c_2 = 0$. If $c_2 = 0$, then θ is an automorphism of $\mathbb{H}$. The expression for θ_3 becomes $Z^{\theta_3} = \overline{c}_1 Z^{\theta_1}$, hence θ_3, and similarly θ_2, preserve $\mathbb{H}$. In this case we see that φ preserves O_2. If $c_1 = 0$, then θ is an anti-automorphism. The expression for θ_3 becomes $(z_1, z_2)_3^\theta = (0, -c_2)(z_1^\theta, z_2^{\theta^*}) = (-z_2^{\theta^*}\overline{c}_2, -z_1^\theta c_2)$. Hence θ_3, and similarly θ_2, interchange $\{(x, 0) \mid x \in \mathbb{H}\}$ and $\{(0, x) \mid x \in \mathbb{H}\}$. In this case clearly φ interchanges the ovoids O_2 and O_2'. To see that such a collineation actually exists, we set $c_2 = k = 1$, $c_1 = 0$ and $X^{\theta_1} = \overline{X}$, for all $X \in \mathbb{O}'$. This shows the last assertion of the proposition.

Now assume that φ pointwise fixes $\mathscr{V}(\mathbb{K}, \mathbb{H})$. Then the associated linear map θ on $\mathbb{H}$ is the identity. Then Equation 5.2 yields $c_2 = 0$. We also have that both θ_2 and θ_3 pointwise fix $\mathbb{H}$. In particular, they both fix 1, and so $C_2 = C_3 = k = 1$. We easily deduce $\theta_1 = \theta_2 = \theta_3$. By linearity of θ_1, Equation 5.1 implies that θ_1 is an automorphism of $\mathbb{O}'$ which commutes with the standard involution. Hence we have, for all $(x_1, x_2), (y_1, y_2) \in \mathbb{O}'$,

$$x_1y_1 + y_2\overline{x}_2 = x_1y_1 + y_2^{\theta^*}\overline{x_2^{\theta^*}} \text{ and } (\overline{x}_1y_2 + y_1x_2)^{\theta^*} = \overline{x}_1y_2^{\theta^*} + y_1x_2^{\theta^*}.$$

Setting $a = 1^{\theta^*}$, we readily deduce that, for all $x \in \mathbb{H}$, $x^{\theta^*} = xa$, with $a\overline{a} = 1$. Conversely, every such automorphism clearly pointwise fixes $\mathscr{V}(\mathbb{K}, \mathbb{H})$. Hence the group that pointwise fixes $\mathscr{V}(\mathbb{K}, \mathbb{H})$ inside $\mathscr{E}_6(\mathbb{K})$ is isomorphic to the multiplicative group of norm 1 elements of $\mathbb{H}$.

The proposition is proved. □

5.1.4. *Quaternion Veroneseans on $\mathscr{E}_6(\mathbb{K})$ that contain O_1 and p_1.* In the next proof we use the following terminology. For a point p far from two symps ζ_1, ζ_2, we call the map $\zeta_1 \to \zeta_2 : q \mapsto \xi(p, q) \cap \zeta_2$ the *projection* (of ζ_1) from p onto ζ_2. Let $p_1 \in \zeta_1$. Noting that the points of ζ_2 close to $\xi(p, p_1)$ are precisely the points of ζ_2 collinear to the projection p_2 of p_1 from p onto ζ_2, that projection is, by Lemma 5.2, characterised by the property that it is the unique point p_2 such that $C(p + p_1 + p_2) = 0$ and collinear to all points $q_2 \in \zeta_2$ for which $C(p + p_1 + q_2) = 0$.

Proposition 5.4. *There are precisely two projectively equivalent quaternion Veronesean varieties contained in $\mathscr{E}_6(\mathbb{K})$ that contain O_1 and p_1. One contains O_2 and the other one O_2'.*

Proof. With the previous notation, let O_3 be the ovoid of $\mathscr{V}(\mathbb{K},\mathbb{H})$ containing the points p_1 and p_2 (hence $O_3 = \{\rho((x,0),(y,0),0) \mid x,y \in \mathbb{H}\}$). Denote the symps of $\mathscr{E}_6(\mathbb{K})$ determined by O_1, O_2, O_3 by ξ_1, ξ_2, ξ_3, respectively. Also, for a point p far from two symps ζ_1, ζ_2, we call the map $\zeta_1 \to \zeta_2 : q \mapsto \xi(p,q) \cap \zeta_2$ the *projection* (of ζ_1) from p onto ζ_2.

We first claim that, if a quaternion Veronesean variety $\mathscr{H}$ contained in $\mathscr{E}_6(\mathbb{K})$ and containing $O_1 \cup \{p_1\}$, contains any point p of $O_2 \setminus \{p_1, p_3\}$ (with $p_3 = \rho(0,0,1)$), then $\mathscr{H} = \mathscr{V}(\mathbb{K},\mathbb{H})$ as defined earlier. Likewise, if $\mathscr{H}$ contains a point of $O_2' \setminus \{p_1, p_3\}$, then $\mathscr{H} = \mathscr{V}(\mathbb{K},\mathbb{H})'$. Indeed, for each point $q \in O_1$, the symp $\xi(p,q)$ intersects ξ_3 in a unique point of $\mathscr{H}$ that belongs to O_3, and varying q, all points of O_3 are obtained. Likewise, all points of O_2 are also contained in $\mathscr{H}$. An arbitrary point $r \in \mathscr{H} \setminus (O_1 \cup O_2 \cup O_3)$ is the intersection of $\xi(p_1,q_1)$ and $\xi(p_2,q_2)$, with $q_i \in O_i$, $i = 1,2$, such that $r \in \xi(p_i,q_i)$. But then $r \in \mathscr{V}(\mathbb{K},\mathbb{H})$. The claim follows.

By the previous claim, it suffices to show that no point t_2 of $\xi_2 \setminus (O_2 \cup O_2' \cup p_1^\perp \cup p_3^\perp)$ lies in a quaternion Veronesean of $\mathscr{E}_6(\mathbb{K})$ together with O_1 and p_1. Such a point t_2 can be written as $(Y\overline{Y}, 0, 1, 0, \overline{Y}, 0)$, with $Y \in \mathbb{O}'$ invertible. On the other hand, a generic point t_1 of $\xi_1 \setminus (p_2^\perp \cup p_3^\perp)$ can be written as $(0, 1, X\overline{X}, \overline{X}, 0, 0)$, with $X \in \mathbb{O}'$ invertible. We now seek $Z_0 \in \mathbb{O}'$ such that t_1, t_2 and $t_3 = (1, Z_0\overline{Z}_0, 0, 0, 0, \overline{Z}_0)$ are contained in the same symp.

After an elementary calculation, and setting $t_Z = (1, Z\overline{Z}, 0, 0, 0, \overline{Z})$, one obtains

$$C(t_1 + t_2 + t_Z) = 0 \iff 1 + X\overline{X}.Y\overline{Y}.Z\overline{Z} + (\overline{XY})\overline{Z} + Z(YX) = 0,$$

which is equivalent to $(1 + Z(YX))(1 + \overline{Z(YX)}) = 0$. It follows that the set of points of ξ_3 collinear to t_3, but not to p_2, is given by

$$\{(1, Z\overline{Z}, 0, 0, 0, \overline{Z}) \mid \exists N \in \mathbb{O}', N\overline{N} = 0, Z = (N-1)X^{-1}Y^{-1}\}.$$

One sees that the point with $N = 0$, namely $(1, Z\overline{Z}, 0, 0, 0, \overline{Z})$ with $Z = -X^{-1}Y^{-1}$, is collinear to all others of that set, and so, by our little argument preceding this proposition, $t_3 = (1, Z\overline{Z}, 0, 0, 0, \overline{Z})$, with $Z = -X^{-1}Y^{-1}$. Note that also the "cyclically obtained" conditions hold: $Y = -Z^{-1}X^{-1}$ and $X = -Y^{-1}Z^{-1}$ (each one is equivalent to $ZYX = -1$).

It follows that the projection O_3^* of O_1 from t_2 onto ξ_3 consists of the set of points

$$\{(1, (Y\overline{Y}x\overline{x})^{-1}, 0, 0, 0, -\overline{Y}^{-1}\overline{x}^{-1}) \mid x \in \mathbb{H}^\times\} \cup \{(p_1, p_2\}.$$

Note that we just proved that three points $(0, 1, X\overline{X}, \overline{X}, 0, 0)$, $(Y\overline{Y}, 0, 1, 0, \overline{Y}, 0)$ and $(1, Z\overline{Z}, 0, 0, 0, \overline{Z})$, with X, Y, Z invertible, are contained in a common symp if, and only if, $ZYX = -1$.

With that we easily compute the projection of O_3^* from each point $(0, 1, x'\overline{x}', \overline{x}', 0, 0)$, $x' \in \mathbb{H}$, of O_1 back on ξ_2. We obtain the set of points

$$O_2^* = \{Y\overline{Y}x\overline{x}(x'\overline{x}')^{-1}, 0, 1, 0, \overline{x}'^{-1}(\overline{x}\overline{Y}), 0) \mid x, x' \in \mathbb{H}^\times\} \cup \{p_1, p_3\}.$$

A necessary condition that O_2^* is an ovoid of a quaternion Veronesean is clearly that the sets

$$\mathscr{Y} = \{\overline{x}\overline{Y} \mid x \in \mathbb{H}^\times\} \text{ and } \mathscr{Y}' = \{\overline{x}'^{-1}(\overline{x}\overline{Y}) \mid x, x' \in \mathbb{H}^\times\}$$

coincide. Setting $Y = (y_1, y_2) \in \mathbb{H} \times \mathbb{H}$ we easily compute $\overline{x}'^{-1}(\overline{x}\overline{Y}) = (\overline{x}'^{-1}\overline{x}\,\overline{y}_1, -x'^{-1}xy_2)$.

Suppose first that $y_2 \neq 0$. Then each element of $\mathscr{Y}$ is determined by its second coordinate. Hence the same is true for $\mathscr{Y}'$. However, setting $x' = 1$ again and replacing x by $x'^{-1}x$, we obtain the element $(\overline{x}\,\overline{x}'^{-1}\overline{y}_1, -x'^{-1}xy_2) \in \mathscr{Y}'$. That element must hence coincide with $(\overline{x}'^{-1}\overline{x}\,\overline{y}_1, -x'^{-1}xy_2)$. Choosing x and x' non-commuting, we see that this is only possible if $y_1 = 0$. But then $O_2^* = O_2'$. This completes the case $y_2 \neq 0$. If $y_2 = 0$, then we immediately see that $O_2^* = O_2$.

Hence we have shown that there are exactly two quaternion Veroneseans in $\mathscr{E}_6(\mathbb{K})$ containing O_1 and p_1, and they are mutually projectively equivalent by Proposition 5.3. □

Now we turn to the case where the fixed point structure is a Veronesean over an inseparable extension of degree 4.

5.2. **The inseparable case.**

5.2.1. *Linear* inseparable *ovoids of polar spaces of type* $\mathsf{D_5}$. In this paragraph, we prove the analogous result to Lemma 5.1 for the inseparable case.

Suppose $\mathbb{K}$ is a non-perfect field of characteristic 2. Choose an element $\ell_1 \in \mathbb{K}$ that is not a square and suppose that $\mathbb{K}^2 + \ell_1\mathbb{K}^2 \neq \mathbb{K}$. Select an element $\ell_2 \in \mathbb{K} \setminus (\mathbb{K}^2 + \ell_1\mathbb{K}^2)$. Then the quadruplets $(x_0, x_1, x_2, x_3) \in \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K}$, with componentwise addition and multiplication as follows, form an inseparable 4-dimensional extension $\mathbb{H} := \mathbb{K}(\sqrt{\ell_1}, \sqrt{\ell_2})$ of $\mathbb{K}$:

$$\begin{aligned}&(x_0, x_1, x_2, x_3) \cdot (y_0, y_1, y_2, y_3) = \\ &(x_0y_0 + \ell_1x_1y_1 + \ell_2x_2y_2 + \ell_1\ell_2x_3y_3, x_0y_1 + x_1y_0 + \ell_2x_2y_3 + \ell_2x_3y_2, \\ &x_0y_2 + x_2y_0 + \ell_1x_1y_3 + \ell_1x_3y_1, x_0y_3 + x_1y_2 + x_2y_1 + x_3y_0).\end{aligned}$$

In this structure, conjugation (the "standard involution") is the identity, and the norm is given by $\mathsf{n}(x_0, x_1, x_2, x_3) = x_0^2 + \ell_1x_1^2 + \ell_2x_2^2 + \ell_1\ell_2x_3^2 = (x_0, x_1, x_2, x_3)^2$.

Lemma 5.5. *Let $\mathbb{K}$ be as above, and let Q be a hyperbolic quadric in $\mathsf{PG}(9, \mathbb{K})$. Then there exists a subspace U of $\mathsf{PG}(9, \mathbb{K})$ of dimension 5 such that $U \cap Q$ is a quadric of Witt index 1 isomorphic to the one with equation $x_1x_2 = \mathsf{n}(X)$, where we label coordinates (x_1, x_2, X) in $\mathbb{K} \times \mathbb{K} \times \mathbb{H}$. Moreover, this inclusion is projectively unique.*

Proof. The existence of U is proved via the standard equation of Q : $x_{-5}x_5 + x_{-4}x_4 + \cdots + x_{-1}x_1 = 0$, setting U equal to the subspace with equations $x_{-4} + x_4 = x_{-3} + \ell_1x_3 = x_{-2} + \ell_2x_2 = x_{-1} + \ell_1\ell_2x_1 = 0$.

Now we show uniqueness. Set $O = U \cap Q$. As in the proof of Lemma 5.1, we can choose coordinates $(x_1, \ldots, x_{10}) \in \mathbb{K}^{10}$ such that U has equations $x_7 = x_8 = x_9 = x_{10} = 0$, O is described by $x_1x_2 = \mathsf{n}(x_3, x_4, x_5, x_6)$, and there exist subspaces U_1 and U_2, which we may assume to have equations $x_5 = x_6 = x_9 = x_{10} = 0$ and $x_3 = x_4 = x_7 = x_8 = 0$, respectively, such that $Q_i := U_i \cap Q$ is a hyperbolic quadric of Witt index 3, $i = 1, 2$, with $Q_i \cap O$ an ovoid O_i of Q_i. By the projective uniqueness of O_i in Q_i (see [25]), we may assume that Q_1 has equation $x_1x_2 + x_3x_7 + x_4x_8$ (and $Q_1 \cap U$ has equations $x_7 + x_3 = x_8 + \ell_1x_4 = 0$), and Q_2 has equation $x_1x_2 + x_5x_9 + x_6x_{10} = 0$ (and $Q_2 \cap U$ has equations $x_9 + \ell_2x_5 = x_{10} + \ell_1\ell_2x_6 = 0$). We then obtain the same description as in the first paragraph and the lemma is proved. □

The ovoid O of Q in the above proof is called a *linear inseparable ovoid* of Q.

5.2.2. *Embedding an inseparable quaternion Veronesean into $\mathscr{E}_6(\mathbb{K})$.* The split Cayley algebra $\mathbb{O}'$ over $\mathbb{K}$ can be defined with Zorn's matrices as follows (see [27]). Write a generic element X of the 8-dimensional vector space $\mathbb{O}'$ and its conjugate $\overline{X}$ in matrix form as follows (with $x_i \in \mathbb{K}$, $i = 0, 1, \ldots, 7$).

$$X = \begin{pmatrix} x_0 & \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & x_7 \end{pmatrix}, \ \overline{X} = \begin{pmatrix} x_7 & \begin{pmatrix} -x_4 \\ -x_5 \\ -x_6 \end{pmatrix} \\ \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} & x_0 \end{pmatrix}.$$

Denoting, with self-explaining notation, $X = \begin{pmatrix} x_0 & x_{456} \\ x_{123} & x_7 \end{pmatrix}$ and $Y = \begin{pmatrix} y_0 & y_{456} \\ y_{123} & y_7 \end{pmatrix}$, the multiplication is given by

$$XY = \begin{pmatrix} x_0y_0 + x_{456}.y_{123} & x_0y_{456} + y_7x_{456} + x_{123} \times y_{123} \\ y_0x_{123} + x_7y_{123} - x_{456} \times y_{456} & x_7y_7 + x_{123}.y_{456} \end{pmatrix},$$

where $a.b$ and $a \times b$ are the ordinary dot and vector product, respectively, of vectors a and b in 3-space.

Let $\ell_1, \ell_2 \in \mathbb{K}$ and $\mathbb{H} = \mathbb{K}(\sqrt{\ell_1}, \sqrt{\ell_2})$ be as above. Using the matrix representation of $\mathbb{O}'$ and Lemma 5.5, we can embed $\mathbb{H}$ in $\mathbb{O}'$ as follows.

$$\mathbb{H} \hookrightarrow \mathbb{O}' : (x_0, x_1, x_2, x_3) \mapsto \begin{pmatrix} x_0 & \begin{pmatrix} x_1 \\ x_2 \\ \ell_1\ell_2x_3 \end{pmatrix} \\ \begin{pmatrix} \ell_1x_1 \\ \ell_2x_2 \\ x_3 \end{pmatrix} & x_0 \end{pmatrix}.$$

Note that the restriction of the standard involution of $\mathbb{O}'$ indeed acts trivially on $\mathbb{H}$. Now the inseparable Veronese map ρ is defined as follows:

$$\rho : \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{K}^{15} : (X, Y, Z) \mapsto (X^2, Y^2, Z^2, YZ, ZX, XY).$$

The image of ρ in $\mathsf{PG}(14, \mathbb{K})$ is an inseparable Veronese variety $\mathscr{V}(\mathbb{K}, \mathbb{H})$. It embeds in $\mathsf{E}_{6,1}(\mathbb{K})$ just like the (separable) quaternion Veronese variety above.

Again, we want to determine all the automorphisms θ of $\mathscr{E}_6(\mathbb{K})$ that fix the inseparable quaternion Veronese variety $\mathscr{V}(\mathbb{K}, \mathbb{H})$ pointwise.

The proof of the following lemma is straightforward but cumbersome, and therefore left to the reader. For each $a, b, c, d \in \mathbb{K}$, with $a + a^2 + \ell_1b^2 + \ell_2c^2 + \ell_1\ell_2d^2 = 0$, we define the following matrix:

$$A = \begin{pmatrix} 1+a & \ell_1 b & \ell_2 c & d & b & c & \ell_1\ell_2 d & a \\ b & 1+a & \ell_2 d & \ell_1^{-1}c & \ell_1^{-1}a & d & \ell_2 c & b \\ c & \ell_1 d & 1+a & \ell_2^{-1}b & d & \ell_2^{-1}a & \ell_1 b & c \\ \ell_1\ell_2 d & \ell_1\ell_2 c & \ell_1\ell_2 b & 1+a & \ell_2 c & \ell_1 b & \ell_1\ell_2 a & \ell_1\ell_2 d \\ \ell_1 b & \ell_1 a & \ell_1\ell_2 d & c & 1+a & \ell_1 d & \ell_1\ell_2 c & \ell_1 b \\ \ell_2 c & \ell_1\ell_2 d & \ell_2 a & b & d_2 d & 1+a & \ell_1\ell_2 b & \ell_2 c \\ d & c & b & \ell_1^{-1}\ell_2^{-1}a & \ell_1^{-1}c & \ell_2^{-1}b & 1+a & d \\ a & \ell_1 b & \ell_2 c & d & b & c & \ell_1\ell_2 d & 1+a \end{pmatrix}$$

We will use the notation $A(a,b,c,d)$ for this matrix.

Lemma 5.6. *Writing a generic element of $\mathbb{O}'$ as above as a vector $(a_0,a_1,a_2,\ldots,a_7)$, an invertible matrix M, acting as an automorphism of $\mathbb{O}'$ on the right, pointwise fixes $\mathbb{H}\subseteq\mathbb{O}'$ (embedded as above), if, and only if, $M=A(a,b,c,d)$, for some $a,b,c,d\in\mathbb{K}$ with $a+a^2+\ell_1b^2+\ell_2c^2+\ell_1\ell_2d^2=0$.*

Let us denote $\mathscr{A}=\{A(a,b,c,d)\mid a,b,c,d\in\mathbb{K}, a+a^2+\ell_1b^2+\ell_2c^2+\ell_1\ell_2d^2=0\}$. We then have the following result.

Proposition 5.7. *The group of linear collineations of $\mathsf{E}_{6,1}(\mathbb{K})$ pointwise fixing the inseparable Veronese variety $\mathscr{V}(\mathbb{K},\mathbb{H})$ embedded in $\mathsf{E}_{6,1}(\mathbb{K})$ as above, coincides with*

$$\{g_A:\mathsf{PG}(V)\to\mathsf{PG}(V)\colon (x,y,z,X,Y,Z)\mapsto(x,y,z,XA,YA,ZA)\mid A\in\mathscr{A}\}$$

and is isomorphic to $\mathscr{A}$ (for the matrix multiplication), which, on its turn, is abstractly isomorphic to the (additive) subgroup $\{0\}\cup\{(1,b,c,d)^{-1}\mid b,c,d\in\mathbb{K}\}$ of $\mathbb{H}$.

Proof. We copy the relevant parts of the proof of Proposition 5.3 up to the point where a generic linear collineation φ of $\mathsf{E}_{6,1}(\mathbb{K})$ pointwise fixing $\mathscr{V}(\mathbb{K},\mathbb{H})$ is of the form $(x,y,z,X,Y,Z)\mapsto(x,y,z,X^\theta,Y^\theta,Z^\theta)$, with θ a linear bijective transformation of $\mathbb{O}'$ pointwise fixing $\mathbb{H}\subseteq\mathbb{O}'$ (and then θ is an automorphism of $\mathbb{O}'$). By Lemma 5.6, $\varphi=g_A$, for some $A\in\mathscr{A}$. Hence $\mathscr{A}$ is isomorphic to the sought group.

Now, it is easily calculated that $A(a_1,b_1,c_1,d_1)A(a_2,b_2,c_2,d_2)=A((a_1,b_1,c_1,d_1)+(a_2,b_2,c_2,d_2))$. Hence $\mathscr{A}$ is isomorphic to an additive subgroup of $\mathbb{H}$. Let us call, in this proof, an element (a,b,c,d) of $\mathbb{H}$ *admissible* if $a+a^2+\ell_1b^2+\ell_2c^2+\ell_1\ell_2d^2=0$, that is, $a=(a,b,c,d)^2$, and note that the square is the norm, which we denote by $\mathsf{n}(a,b,c,d)$. The proposition will be proved if we show that the set of admissible elements of $\mathbb{H}$ equals $\{0\}\cup\{(1,b,c,d)^{-1}\mid b,c,d\in\mathbb{K}\}$.

Suppose first that (a,b,c,d) is admissible and distinct from 0. Then

$$0=a^{-2}(a+a^2+\ell_1b^2+\ell_2c^2+\ell_1\ell_2d^2)=a^{-1}+(1+\ell_1b'^2+\ell_2c'^2+\ell_1\ell_2d'^2),$$

with $b'=ba^{-1}$, $c'=ca^{-1}$ and $d'=da^{-1}$. Hence on the one hand $a=\mathsf{n}(a,b,c,d)$, and on the other hand $a=\mathsf{n}(1,b',c',d')^{-1}$. Since the norm is just squaring, we obtain

$$(a,b,c,d)=(1,b',c',d')^{-1}.$$

Conversely, for all $b,c,d\in\mathbb{K}$, one can write $(1,b,c,d)^{-1}=(n^{-1},bn^{-1},cn^{-1},dn^{-1})$, with $n=\mathsf{n}(1,b,c,d)$. Then clearly

$$\mathsf{n}(n^{-1},bn^{-1},cn^{-1},dn^{-1})=n^{-2}n=n^{-1},$$

which implies that $(n^{-1},bn^{-1},cn^{-1},dn^{-1})$, and hence $(1,b,c,d)^{-1}$, is admissible. □

Remark 5.8. Proposition 5.7 implies that the group of linear collineations of $\mathsf{E}_{6,1}(\mathbb{K})$ pointwise fixing the inseparable Veronese variety $\mathscr{V}(\mathbb{K},\mathbb{H})$ is abstractly a subgroup of $\mathbb{H}$ each element of which depends on three free variables over $\mathbb{K}$. We can compare this with the separable quaternion case, where that group was the subgroup of $\mathbb{H}^\times$ of elements of norm 1, and hence, in a certain sense, also has dimension 3.

5.2.3. *The projective uniqueness of $\mathscr{V}(\mathbb{K},\mathbb{H})$ in $\mathscr{E}_6(\mathbb{K})$ in the inseparable case.* It remains to show that all Veronesean varieties isomorphic to $\mathscr{V}(\mathbb{K},\mathbb{H})$ and contained in $\mathscr{E}_6(\mathbb{K})$ are mutually equivalent. In fact we will show that there is a unique $\mathscr{V}(\mathbb{K},\mathbb{H})$ containing a given ovoid O of a given symp ξ of $\mathscr{E}_6(\mathbb{K})$ arising from the inclusion $\mathbb{H} \subseteq \mathbb{O}'$, and a given point p of $\mathscr{E}_6(\mathbb{K})$ opposite ξ.

Proposition 5.9. *There is a unique quaternion Veronesean variety contained in $\mathscr{E}_6(\mathbb{K})$ that contains the point p_1 and the ovoid of $\mathscr{V}(\mathbb{K},\mathbb{H})$ in $\xi(p_2,p_3)$, and that is $\mathscr{V}(\mathbb{K},\mathbb{H})$ itself.*

Proof. The proof of Proposition 5.4 can be repeated here word for word, until we have to determine the invertible elements $Y \in \mathbb{O}'$ such that $\mathscr{Y} := \{xY \mid x \in \mathbb{H}\}$ coincides with $\mathscr{Y}' := \{x'(xY) \mid x', x \in \mathbb{H}\}$. This time we will show that Y itself has to belong to $\mathbb{H}$, which implies the proposition.

In order to do so, it is rather convenient to use the following form of $\mathbb{O}'$. Let $\mathbb{L}$ be the split (separable) extension $\mathbb{K}(t)$ of $\mathbb{K}$ defined by the element t with $t^2 = t$. Then the corresponding natural involution $x \mapsto \overline{x}$, $x \in \mathbb{L}$, interchanges t with $t+1$. Now we apply to $\mathbb{L}$ the Cayley-Dickson process twice: first using ℓ_1 as a primitive element, the second time using ℓ_2. Then we obtain $\mathbb{O}'$ as $\mathbb{L}\times\mathbb{L}\times\mathbb{L}\times\mathbb{L}$ with the following explicit form of the multiplication:

$$\begin{aligned}(x_1,x_2,x_3,x_4)(y_1,y_2,y_3,y_4) \;=\; & (x_1y_1 + \ell_1\overline{x}_2y_2 + \ell_2\overline{x}_3y_3 + \ell_1\ell_2x_4\overline{y}_4,\\ & \overline{x}_1y_2 + \;\; x_2y_1 + \ell_2\overline{x}_3y_4 + \;\; \ell_2x_4\overline{y}_3,\\ & \overline{x}_1y_3 + \ell_1\overline{x}_2y_4 + \;\; x_3y_1 + \;\; \ell_1x_4\overline{y}_2,\\ & x_1y_4 + \;\; x_2y_3 + \;\; x_3y_2 + \;\;\; x_4\overline{y}_1),\end{aligned}$$

for all $x_1,\dots,x_4,y_1,\dots,y_4 \in \mathbb{L}$. The embedding $\mathbb{H} \subseteq \mathbb{O}'$ is via $\mathbb{H} = \mathbb{K}\times\mathbb{K}\times\mathbb{K}\times\mathbb{K}$. Setting $Y = (y_1,y_2,y_3,y_4)$, it follows from $\mathscr{Y} = \mathscr{Y}'$ that there exists $x \in \mathbb{H}$ such that

$$(0,0,0,1)(\overline{y}_1,\overline{y}_2,\overline{y}_3,\overline{y}_4) = (0,1,0,0)((0,0,1,0)(y_1,y_2,y_3,y_4)) = uY.$$

Since $(0,0,0,1)$ is invertible, we deduce that $(\overline{y}_1,\overline{y}_2,\overline{y}_3,\overline{y}_4) \in \mathscr{Y}' = \mathscr{Y}$. Then the additivity of $\mathscr{Y}$ (by its very definition) yields

$$w := (y_1+\overline{y}_1, y_2+\overline{y}_2, y_3+\overline{y}_3, y_4+\overline{y}_4) \in \mathscr{Y},$$

implying the existence of $x \in \mathbb{H}^\times$ such that $xY = u$. Since x is invertible, this yields $Y = x^{-1}u \in \mathbb{H}$ and we are done. □

This completes the proof of Theorem A.

The following is now a consequence of Propositions 5.3, Proposition 5.4, 5.7 and 5.9.

Corollary 5.10. *Whenever a separable or inseparable quaternion Veronesean is embedded in $\mathscr{E}_6(\mathbb{K})$, and each point off a quadric is far from the corresponding symp of $\mathsf{E}_{6,1}(\mathbb{K})$, then it is a standard inclusion and hence it is the fix point set of a kangaroo of $\mathsf{E}_{6,1}(\mathbb{K})$.*

Proof. Since quaternion Veroneseans have 5-dimensional host spaces, the corresponding quadrics are easily seen to be linear ovoids of their host symps. Then the corollary follows from the previous results. □

Remark 5.11. Let P be an ovoidal subplane of $\mathsf{E}_{6,1}(\mathbb{K})$. Then arguments similar to those in Section 4.3 quickly show that P can be thought of a Veronesean variety contained in $\mathscr{E}_6(\mathbb{K})$, spanning either a 14-dimensional or a 26-dimensional subspace. If it spans a 14-dimensional subspace, then it is a quaternion Veronesean and hence, by Corollary 5.10, the fix structure of a linear collineation. If it spans the whole space, then it is shown in [15] that, if $\mathsf{char}\,\mathbb{K} \neq 2$, then it arises as the fix structure of a semilinear kangaroo collineation. If $\mathsf{char}\,\mathbb{K} = 2$, however, then the existence of a Veronesean in $\mathscr{E}_6(\mathbb{K})$ with trivial fix group is an open question. The analogous question (whether there exists a Veronesean with trivial fix group) in $\mathsf{A}_{5,2}(\mathbb{K})$ has an affirmative answer, see [25].

6. Restriction to an equator geometry

Let $\mathscr{A}_5(\mathbb{K})$ be the restriction of $\mathscr{E}_6(\mathbb{K})$ to $\mathbb{K}\times\mathbb{K}\times\mathbb{K}\times\mathbb{H}'\times\mathbb{H}'\times\mathbb{H}'$, with $\mathbb{H}'$ a split quaternion subalgebra of $\mathbb{O}'$. In the separable case, we consider a subfield $\mathbb{L}$ of $\mathbb{H}$ containing $\mathbb{K}$ such that $\mathbb{L}$ is 2-dimensional over $\mathbb{K}$ (but $\mathbb{L}$ may be an inseparable field extension of $\mathbb{K}$). We then choose $\mathbb{H}'$ to be the subalgebra $\mathbb{L}\times\mathbb{L}$ of $\mathbb{O}' = \mathbb{H}\times\mathbb{H}$. In the inseparable case, we restrict Zorn's matrices, with above notation, to the subspace $x_2 = x_3 = x_5 = x_6 = 0$. In that case we define $\mathbb{L}$ as the subfield of $\mathbb{H}$ defined by restricting to the first two coordinates of the elements. Then we see that we obtain a geometry isomorphic to $\mathsf{A}_{5,2}(\mathbb{K})$. Moreover, the quaternion Veronesean gets restricted to a Veronesean variety isomorphic to $\mathscr{V}(\mathbb{K},\mathbb{L})$. Translated to the projective space $\mathsf{PG}(5,\mathbb{K})$ corresponding to $\mathsf{A}_{5,2}(\mathbb{K})$, this corresponds to a line spread.

The inclusion $\mathsf{A}_{5,2}(\mathbb{K}) \subseteq \mathsf{E}_{6,1}(\mathbb{K})$ (and $\mathscr{A}_5(\mathbb{K}) \subseteq \mathscr{E}_6(\mathbb{K})$) can be considered as an equator geometry with respect to two opposite 5-spaces of $\mathsf{E}_{6,1}(\mathbb{K})$. This can be easily seen directly in the inseparable case where the 5-spaces are defined as follows. Let X_{26} and X_{35} be the subspaces of $\mathbb{O}'$ (in the Zorn's representation) defined by all coordinates zero, except for x_3, x_6 and x_3, x_5, respectively. Then $(0,0,0,X_{26},X_{26},X_{26})$ and $(0,0,0,X_{35},X_{35},X_{35})$ define opposite 5-space U and U', respectively, such that $\mathsf{A}_{5,2}(\mathbb{K})$ above consists of the set of points collinear to 3-spaces in each of U and U' (and hence, $\mathsf{A}_{5,2}(\mathbb{K})$ is the so-called *equator geometry with poles U and U'*). A similar construction holds in the separable case. Indirectly, this follows from [8, Proposition 4.4].

Since we will not really use this result, but merely state it as a curiosity to strengthen our motivation (see the introduction), we do not insist on precise and detailed proofs for it (but the reader might reconstruct the proofs from the information given above).

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Department of Mathematics, Computer Science & Statistics, Ghent University, Belgium

Email address: `Yannick.Neyt@UGent.be`

School of Mathematics and Statistics, University of Sydney, Australia

Email address: `james.parkinson@sydney.edu.au`

Department of Mathematics Computer Science & Statistics, Ghent University, Belgium

Email address: `Hendrik.VanMaldeghem@UGent.be`

Department of Mathematics, Computer Science & Statistics, Ghent University, Belgium

Email address: `MagaliVictoor@gmail.com`