~~PREDICTION~~ Fitted Values

Data: $x_1, x_2, \dots, x_N$

Fitted Model: $x_t - \hat{\mu} = \hat{\alpha}_1(x_{t-1} - \hat{\mu}) + \dots + \hat{\alpha}_p(x_{t-p} - \hat{\mu}) + \hat{z}_t + \hat{\beta}_1 \hat{z}_{t-1} + \dots + \hat{\beta}_q \hat{z}_{t-q}$

Fitted (or predicted values): $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N$

Fitted values at time t, $\hat{x}_t$ $(t = 1, 2, \dots, N)$ are defined by

$$\boxed{\hat{x}_t = x_t - \hat{z}_t}$$

Therefore the fitted values are calculated by

$$\boxed{\hat{x}_t = \hat{\mu} + \hat{\alpha}_1(x_{t-1} - \hat{\mu}) + \dots + \hat{\alpha}_p(x_{t-p} - \hat{\mu}) + \hat{\beta}_1 \hat{z}_{t-1} + \dots + \hat{\beta}_q \hat{z}_{t-q}}$$

$t = 1, 2, \dots, N,$

Note: 1) The rhs of $\hat{x}_t$ involves information up to time $t-1$.

2) To calculate $\hat{x}_t$ we need pre-period values $x_0, x_{-1}, \dots, x_{1-p}$ & $\hat{z}_0, \dots, \hat{z}_{1-q}$.

3) Calculation of $\hat{x}_t$ can be thought of as $E(X_t | F_{t-1})$, where $F_{t-1} = \{X_{t-1}, X_{t-2}, \dots, Z_{t-1}, \dots\}$ is the history of the process since

$$E(X_t - \mu | F_{t-1}) = E\big(\alpha_1(X_{t-1} - \mu) + \dots + \alpha_p(X_{t-p} - \mu) + Z_t + \beta_1 Z_{t-1} + \dots + \beta_q Z_{t-q} \,|\, F_{t-1}\big)$$

$$= \alpha_1(X_{t-1} - \mu) + \dots + \alpha_p(X_{t-p} - \mu) + \beta_1 Z_{t-1} + \dots + \beta_q Z_{t-q}$$

4) $Z_t = X_t - E(X_t | F_{t-1})$.

Since we estimate x_t based on information from the past $F_{t-1} = \{x_{t-1}, x_{t-2}, \dots, \hat{z}_{t-1}, \dots, \hat{z}_{t-q}\}$, ~~this~~ $\hat{x}_t$ is called a one step-ahead prediction from the ~~time~~ origin $t-1$.

Examples:

1) AR(1): $x_t - \hat{\mu} = \hat{\alpha}_1(x_{t-1} - \hat{\mu}) + \hat{z}_t$

$\hat{x}_t = \hat{\mu} + \hat{\alpha}_1(x_{t-1} - \hat{\mu})$; $t = 1, 2, \dots, N$

2) AR(2): $\hat{x}_t = \hat{\mu} + \hat{\alpha}_1(x_{t-1} - \hat{\mu}) + \hat{\alpha}_2(x_{t-2} - \hat{\mu})$

3) MA(1): $\hat{x}_t = \hat{\mu} + \hat{\beta}_1 \hat{z}_{t-1}$

4) MA(2): $\hat{x}_t = \hat{\mu} + \hat{\beta}_1 \hat{z}_{t-1} + \hat{\beta}_2 \hat{z}_{t-2}$

~~NOTE: Prediction errors are $\hat{z}_t$; $t = 1, 2, \dots, N$~~

Note: predictions errors at $t=1,2,\dots,N$ are given by

$$\boxed{\hat{z}_t = x_t - \hat{x}_t}$$

Example: The price of a certain commodity can be modeled by an AR(1) process $x_t - 10 = 0.4(x_{t-1} - 10) + z_t$, $\{z_t\} \sim N(0, 2)$

Suppose the first ~~three~~ five values are:

t	1	2	3	4	5
x_t	9.5	10.75	9.0	9.75	11.25

Calculate the fitted values at times $t = 1, 2, 3, 4$ & 5.

Assuming $x_0 = 10$.

Solution: Fitted values are given by

$$\hat{x}_t - 10 = 0.4(x_{t-1} - 10) \quad \text{[Current error has been ignored]}$$

$\therefore t=1$: $\hat{x}_1 - 10 = 0 \Rightarrow \hat{x}_1 = 10$

$\therefore \hat{z}_1 = x_1 - \hat{x}_1 = 9.5 - 10 = -0.5$

$t = 2$:
$$\hat{x}_2 - 10 = 0.4(x_1 - 10)$$
$$= 0.4 \times (9.5 - 10)$$
$$= 0.4 \times (-0.5)$$
$$\therefore \hat{x}_2 = 9.80$$
$$\therefore \hat{z}_2 = x_2 - \hat{x}_2 = 10.75 - 9.80 = 0.95$$

$t = 3$:
$$\hat{x}_3 - 10 = 0.4(x_2 - 10)$$
$$= 0.4(0.75)$$
$$\therefore \hat{x}_3 = 10.3$$
$$\therefore \hat{z}_3 = 9.0 - 10.3 = -1.3$$

$t = 4$:
$$\hat{x}_4 = 10 + 0.4(9.00 - 10) = 9.6$$
$$\therefore \hat{z}_4 = 9.75 - 9.6 = +0.15$$

$t = 5$:
$$\hat{x}_5 = 10 + 0.4(9.75 - 10) = 10.1$$
$$\hat{z}_5 = 11.25 - 10.10 = 1.15$$

Example: MA(1) process $X_t = 3 + Z_t + 0.5 Z_{t-1}$ has been fitted to a time series. The first four values are

t	1	2	3	4
x_t	3.25	4.75	2.25	1.75

Calculate the fitted values at $t = 1, 2, 3, 4$ given $\hat{z}_0 = 0$.

Solution: Fitted values are given by

$\hat{x}_t = 3 + \hat{z}_{t-1}$ (current error has been ignored)

$\therefore \hat{x}_1 = 3 + \hat{z}_0 = 3$

$\therefore \hat{z}_1 = x_1 - \hat{x}_1 = 0.25$

$\hat{x}_2 = 3 + \hat{z}_1 = 3.25$

$\therefore \hat{z}_2 = 4.75 - 3.25 = 1.50$

$\hat{x}_3 = 3 + \hat{z}_2 = 4.50$

$\therefore \hat{z}_3 = 2.25 - 4.50 = -2.25$

$\hat{x}_4 = 3 + \hat{z}_3 = 0.75$

$\therefore \hat{z}_4 = 1.75 - 0.75 = 1.00$

Forecasting Using ARIMA Models

W13, L)

Suppose that $x_1, \dots, x_n$ follows a stationary and invertible ARMA(p,q) process. Now we are interested of forecasting future values using this ARMA model.

~~It can be shown~~ $\underbrace{x_1, \dots, x_n}_{\text{ob}}, \underbrace{x_{n+1}, \dots, x_{n+l}, \hat{x}_{n+l+1}, \dots}_{\text{future}}$

Let $x_n(l)$ be the l-step-ahead forecast value of future x_{n+l}. Clearly, this estimate $\hat{x}_n(l)$ need to obtain based on the available data.

The l-step-ahead forecast error, $\varepsilon_n(l) = x_{n+l} - \hat{x}_n(l)$

In practice we expect to minimize the error and therefore find $x_n(l)$ based on minimizing $E[\{\varepsilon_n(l)\}^2]$. The corresponding forecast value is called the minimum mean square error (mmse) forecast of x_{n+l}.

Theorem: $x_n(l) = E[x_{n+l} \mid x_n, x_{n-1}, x_{n-2}, \dots]$

Note: This best mmse forecast is the conditional mean of x_{n+l} conditional on the history or observed values.

Let $F_n = \{x_n, x_{n-1}, \dots\}$ be the history.

$$\therefore \boxed{x_n(l) = E[x_{n+l} \mid F_n]}$$

Examples: ① one-step-ahead

$$\hat{x}_n(1) = E[x_{n+1} \mid x_n, x_{n-1}, \dots]$$

forecast error: $\varepsilon_n(1) = x_{n+1} - \hat{x}_n(1)$

② two-step-ahead

$$\hat{x}_n(2) = E[x_{n+2} \mid x_n, x_{n-1}, \dots]$$

forecast error: $\varepsilon_n(2) = x_{n+2} - \hat{x}_n(2)$

etc.