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Axisymmetric, antidynamo theory for no generation of azimuthal electromotive force from an azimuthal magnetic field: The axisymmetric, $\alpha_{\phi\phi} = 0$, antidynamo theorem

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For the mean field induction equation $\partial_t\overline{\boldsymbol{B}} + \eta\nabla^2\overline{\boldsymbol{B}} = \boldsymbol{\nabla}\times\boldsymbol{F}$ in a conducting volume V, where $\overline{\boldsymbol{B}}$ is the mean magnetic field, ∂_t is rate of change, η is magnetic diffusivity, using the second order correlation approximation (SOCA) the electromotive force $\boldsymbol{F}$ is $\boldsymbol{F} = \boldsymbol{\alpha}\cdot\overline{\boldsymbol{B}}$. The following antidynamo theorem (ADT) is derived: if there is no generation of azimuthal $\boldsymbol{F}$ from azimuthal $\overline{\boldsymbol{B}}$, that is when $\mathbf{1}_\phi\cdot\boldsymbol{\alpha}\cdot\mathbf{1}_\phi = \alpha_{\phi\phi} = 0$, where $\mathbf{1}_\phi$ is the unit vector in the ϕ direction, (s, ϕ, z) cylindrical polar coordinates, then an axisymmetric magnetic field will decay. This $\alpha_{\phi\phi} = 0$ ADT is derived in two parts. Firstly, the magnetic field contained in meridional planes (containing the axis of symmetry) is shown to decay to zero. Once the meridional field has decayed, the azimuthal component of the magnetic field is shown to decay.

As a gauge of the magnetic energy, $\|b\|^2 = \int_V b^2\mathrm{d}V$, where V is a finite conductor, $b = \overline{\boldsymbol{B}}\cdot\mathbf{1}_\phi/s$, is considered. The resulting $\|b\|^2$ magnetic energy analysis demonstrates that; for $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s, z)$, and $\alpha_{\phi\phi} = 0$, once the meridional field has decayed, induction can contribute energy by increasing the Magnetic Reynolds number, however, diffusion detracts energy to more-than account for the inductive contributions and, consistent with the ADT, the field decays. Numerical results and field plots using the model $\boldsymbol{\alpha} = s\mathbf{1}_z\mathbf{1}_\phi$, illustrate the interaction mechanisms responsible for the diffusive dominance as induction is increased.

Using the SOCA and Green's-tensor analysis an explicit formulation for this critical $\alpha_{\phi\phi}$ is derived. It is shown for a conductor filling all space, for zero mean flow using the SOCA, if ever member of the ensemble of turbulent flows and the mean magnetic field are co-axisymmetric then $\alpha_{\phi\phi} = 0$.

The analysis of Braginskii (1964), where the fields are analysed as perturbations from axisymmetry, is extended to compressible velocity fields appropriate for the solar and stellar dynamos. This new analysis, as well as the original incompressible treatment in Braginskii (1964), also produce an $\alpha_{\phi\phi}$ component for a reformulation of the problem into 'effective' mean, magnetic and velocity fields. The work of Soward (1972) which generalises that of Braginskii (1964) to higher orders and more general field decompositions for incompressible flows, is analysed to provide a concise expression, and generation mechanism for $\alpha_{\phi\phi}$. Each of these disparate approaches provide insight into mechanisms for generating this critical $\alpha_{\phi\phi}$ regenerative component and produce remarkably similar generation mechanisms dependent on the helicity of the meridional perturbation velocity field.

Conclusions for non-magnetic stars are proposed and implications for hidden dynamos are drawn.
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1972 which generalises that of Braginskii (1964) to higher orders and more general field decompositions for incompressible flows, is analysed to provide a concise expression, and generation mechanism for $\alpha_{\phi\phi}$. Each of these disparate approaches provide insight into mechanisms for generating this critical $\alpha_{\phi\phi}$ regenerative component and produce remarkably similar generation mechanisms dependent on the helicity of the meridional perturbation velocity field.

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Keywords: $\alpha_{\phi\phi} = 0$ antidynamo theory; Mean field electrodynamics; $\alpha_{\phi\phi}$ generation; alpha phi phi antidynamo theorem; non-magnetic stars

1. Introduction

This work derives the mean field counterpart of the axisymmetric antidynamo theorem (ADT) originally established in Cowling (1934). A velocity can be said to act as a dynamo if a magnetic field satisfying (4) does not decay to zero as $t \to \infty$. For a self-excited dynamo, $\boldsymbol{v}$ and $\boldsymbol{B}$ must satisfy further conditions. An ADT may establish conditions under which such an interaction cannot perpetuate a dynamo. Thus an ADT is a collection of results that establish necessary conditions for dynamo action (Ivers 1984).

Ivers and Phillips (2014) prove the separate mean-field, two-dimensional (2D) and planar ADTs for homogeneous turbulence with zero mean flow. It is, however, possible to relax these conditions. Ivers and Phillips (2014) show that for a turbulent ensemble of 2D flows of the form $\boldsymbol{v}_{2\mathrm{D}} = v_x(x,y,t)\mathbf{1}_x + v_y(x,y,t)\mathbf{1}_y + v_z(x,y,t)\mathbf{1}_z$, where $\mathbf{1}_x$ is a Cartesian unit vector in the direction of x etc., then $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{2\mathrm{D}}$ given by (1a). For a derivation and definition of $\boldsymbol{\alpha}$, see subsection 2.1. The mean-field 2D ADT proves that for $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{2\mathrm{D}}$, then a 2D mean magnetic field will decay. Ivers and Phillips (2014) also show that for a turbulent ensemble of planar flows, given by $\boldsymbol{v}_{\mathrm{P}} = v_x(x,y,z,t)\mathbf{1}_x + v_y(x,y,z,t)\mathbf{1}_y$, then $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\mathrm{P}}$, (1b). The mean-field planar ADT proves that for $\boldsymbol{\alpha}_{\mathrm{P}}$, then a general magnetic field will decay.

$$\boldsymbol{\alpha}_{2\mathrm{D}} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & 0 \end{pmatrix}, \quad \boldsymbol{\alpha}_{\mathrm{P}} = \begin{pmatrix} 0 & 0 & \alpha_{xz} \\ 0 & 0 & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{pmatrix}, \quad \boldsymbol{\alpha}_{2\mathrm{DP}} = \begin{pmatrix} 0 & 0 & \alpha_{xz} \\ 0 & 0 & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & 0 \end{pmatrix} \tag{1}$$

Krause and Rüdiger (1974) conflate these concepts and define turbulence to be '*two-dimensional*', if any velocity in the turbulent ensemble is 2D and planar, $\boldsymbol{v}_{\mathrm{TP}} = v_x(x,y,t)\mathbf{1}_x + v_y(x,y,t)\mathbf{1}_y$. This more restricted form of velocity, is a consequence of the Proudman-Taylor theorem (Proudman 1916) for rapidly rotating fluids. Krause (1976) showed that for two-scale isotropic '*two-dimensional*' homogeneous turbulence then $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{2\mathrm{DP}}$, (1c). Krause argued that $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{2\mathrm{DP}}$ cannot produce dynamo action, even for three-dimensional mean magnetic fields with space and time dependence $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}+\gamma t}$, if $0 < |\boldsymbol{k}| \ll 1$. However, the approximate argument (Krause 1973, 1976) fails if $\boldsymbol{\alpha}$ does not make a positive contribution to $\mathrm{Re}\,\gamma$ and thus does not constitute a rigorous proof unless the alpha-effects, for which it is true, are characterized independently of the argument (see Ivers and Phillips (2014)). The individual mean field 2D and planar ADT are rigorously proven for all wave-vectors $\boldsymbol{k} \neq \mathbf{0}$ in Ivers and Phillips (2014).

Rüdiger (1978) invoked the arguments of Krause to explain the observation that magnetic [nonmagnetic] A-stars are mostly slow [fast] rotators. However, some care must be taken in generalising the results for $\boldsymbol{\alpha}$ given by (1) that are established for a infinite conducting region E^3, to finite conductors. The ADT established herein which proves that an axisymmetric magnetic field cannot be sustained if $\alpha_{\phi\phi} = 0$ ((r,θ,ϕ) spherical polar coordinates), can however, be invoked in a finite conductor and may be used to preclude dynamo action under such conditions.

In Moffatt (1970), an $\boldsymbol{\alpha}$ of the form $\boldsymbol{\alpha}_M = a_1(\boldsymbol{I} - A\mathbf{1}_z\mathbf{1}_z)$, where $\boldsymbol{I}$ is the identity tensor, is derived for a reflectionally-asymmetric random superposition of inertial waves, where $A \to 1$ in the rapid rotation limit. Again, here care needs to be taken in applying ADTs. As shown in Ivers and Phillips (2014), an ensemble of turbulent flows $\boldsymbol{v}_{\mathrm{TP}}$ in E^3 will produce $\boldsymbol{\alpha}_{\mathrm{2DP}}$. Thus, the Proudman-Taylor theorem and the combined 2D and Planar mean field ADT may together preclude dynamo action in the rapid rotation limit. However, $\boldsymbol{\alpha}_{\mathrm{M}}$ and $\boldsymbol{\alpha}_{\mathrm{2DP}}$ are consistent if and only if $\boldsymbol{\alpha} = \mathbf{0}$. Because $\boldsymbol{\alpha}_{\mathrm{M}}$, is derived in Moffatt (1970) and Busse (1970) from higher order, ageostrophic terms the dynamo escapes these ADTs.

It is also tempting to use the mean field 2D ADT as, for $A = 1$, $\alpha_{zz} = 0$ as in $\boldsymbol{\alpha}_{\mathrm{2D}}$. However, the arguments of Krause (1976) and the mean-field planar ADT of Ivers and Phillips (2014) do not apply to models in a finite volumes, as they assume a uniform conductor filling all space and cannot simply be applied to finite conductors. Thus this argument cannot be used to explain the lack of steady axisymmetric solutions at $A = 1$ of Busse and Miin (1979). Nor can they be used to explain the apparently asymptotic ADT behaviour near $A = 1$ for the axisymmetric solutions of Rüdiger (1980). The results of Phillips (1993), Phillips (2013), Phillips and Ivers (2014) provide counter examples for these apparent paradoxes and aim to correct Kono and Roberts (1994).

As an extension to the model above for rapid rotation, Phillips and Ivers (2014) also explored $\boldsymbol{\alpha} = a_1((1 - C)\boldsymbol{I} + C\mathbf{1}_z\mathbf{1}_z)$. As $C \to 1$, $\boldsymbol{\alpha} \to \boldsymbol{\alpha}_{zz} = \alpha_{zz}\mathbf{1}_z\mathbf{1}_z$. To explain the observed antidynamo behaviour as $C \to 1$, Phillips and Ivers (2014) prove that the axisymmetric interaction equations are independent of $\boldsymbol{\alpha}_{zz}$ at $C = 1$ and thus the field decays. This $\boldsymbol{\alpha}_{zz}$-ADT is proven for a sphere and thus is distinct from the $\boldsymbol{\alpha}_{\mathrm{2D}}$ and $\boldsymbol{\alpha}_{\mathrm{2DP}}$ ADT results of Ivers and Phillips (2014) proven in E^3 and in turn can be applied to finite conductors. For this $\boldsymbol{\alpha}_{zz}$-ADT all components are zero except for α_{zz}, and as such, is a special case of the ADT proven herein where just one, $\alpha_{\phi\phi}$, component is zero. The ADTs reviewed above are the mean field counterparts to the laminar; axisymmetric ADT of Cowling, and the Planar ADT, and many extensions thereof. Table 1 gives a selection of the most restricted laminar and mean field ADTs and shows how they are related; noting the many generalisations through many works.

Each of the ADTs in table 1 are closely related and share both analogous results and common methods. In section 10 it is proven that, if every member of the ensemble of fluctuating velocities is co-axisymmetric with the same axis of symmetry as the mean magnetic field in E^3, then $\alpha_{\phi\phi} = 0$. The ADT derived herein uses an extension of the ADT of Cowling (1934). In subsection 3.1, for $\alpha_{\phi\phi} = 0$ the contribution from $\boldsymbol{\alpha}$ is compared to a laminar flow ($\boldsymbol{v}_{\alpha 1}$). The methods of Ivers and James (1984), used to prove an extension of Cowling's theorem of Ivers (1984) and for compressible flows that are not necessarily zero on the boundary, are then adapted and extended to prove that the axisymmetric meridional field will decay to zero.

Once the meridional field has decayed, the azimuthal field is shown to decay in subsection 3.2 by using extensions and modification of Ivers (1984) and Ivers and James (1984). For other variations of Cowling's theorem see also Backus and Chandrasekhar (1956), Backus (1957), Braginskii (1964), Lortz (1968), Hide (1979)

The laminar, Cartesian analogue of the Cowling (1934) axisymmetric ADT is the 2D ADT. The 2D ADT proves that a 2D velocity, chosen to preserve 2D symmetry of the field, will result in the decay of a 2D field; see table 1. Results of varying generality have been established by Cowling (1957), Zel'dovich (1957), Lortz (1968), Vainshtein and Zel'dovich (1972), Lortz and Meyer-Spasche (1982a), Lortz and Meyer-Spasche (1982b), Lortz and Meyer-Spasche (1984), Lortz *et al.* (1984), Stredulinsky *et al.* (1986).

Just as Phillips and Ivers (2014) provide counter examples for the misuse of the mean field 2D and planar turbulence ADT for finite conductors, the work of Bachtiar *et al.* (2006) suggests a counter example for the misuse of the laminar planar velocity ADT, established in E^3, by modelling the interaction of a laminar planar velocity ($\boldsymbol{v}_{\mathrm{TP}}$) in a finite conductor.

There are other examples of inappropriate uses of ADT to explain physical systems. The ADT proven herein does provide an explanation of why a dynamo may fail in a finite conducting fluid, such as non magnetic stars or planets. The work herein also has implications

Table 1. Summary of a selection of antidynamo results showing the most restricted forms of the ADTs. For the laminar ADTs, the 'Velocity' rows give the form that will preserve the symmetry of the decaying field. For the mean field ADTs; the $\boldsymbol{\alpha}$ is proven to result from the accompanying turbulent velocities in E^3. The 'Field Decays' rows give the form of the magnetic field that will decay from the accompanying; $\boldsymbol{v}$ for laminar ADTs; and the $\boldsymbol{\alpha}$ for the mean field ADTs. Each of; Cowling's axisymmetric, the toroidal velocity and the $\alpha_{\phi\phi}=0$, ADTs, are proven for finite conducting fluid volumes.

Antidynamo theorems	Independent of one direction	No component in one direction
	Laminar antidynamo theorems	
	Two dimensional	Planar velocity
Velocity	$\boldsymbol{v}=v_x(x,y)\mathbf{1}_x+v_y(x,y)\mathbf{1}_y+v_z(x,y)\mathbf{1}_z$	$\boldsymbol{v}=v_x(x,y,z)\mathbf{1}_x+v_y(x,y,z)\mathbf{1}_y$
Field decays	$\boldsymbol{B}=B_x(x,y)\mathbf{1}_x+B_y(x,y)\mathbf{1}_y+B_z(x,y)\mathbf{1}_z$	
	Cowling's	Toroidal velocity
Velocity	$\boldsymbol{v}=v_r(r,\theta)\mathbf{1}_r+v_\theta(r,\theta)\mathbf{1}_\theta+v_\phi(r,\theta)\mathbf{1}_\phi$	$\boldsymbol{v}=v_\theta(r,\theta,\phi)\mathbf{1}_\theta+v_\phi(r,\theta,\phi)\mathbf{1}_\phi$
Field decays	$\boldsymbol{B}=B_r(r,\theta)\mathbf{1}_r+B_\theta(r,\theta)\mathbf{1}_\theta+B_\phi(r,\theta)\mathbf{1}_\phi$	
	Mean field antidynamo theorems	
	Two dimensional turbulence	Planar turbulence
Velocity	$\boldsymbol{v}'=v'_x(x,y)\mathbf{1}_x+v'_y(x,y)\mathbf{1}_y+v'_z(x,y)\mathbf{1}_z$	$\boldsymbol{v}'=v'_x(x,y,z)\mathbf{1}_x+v'_y(x,y,z)\mathbf{1}_y$
$\boldsymbol{\alpha}$	$\boldsymbol{v}' \overset{\text{in } E^3}{\longrightarrow} \begin{pmatrix}\alpha_{xx} & \alpha_{xy} & \alpha_{xz}\\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz}\\ \alpha_{zx} & \alpha_{zy} & 0\end{pmatrix}$	$\boldsymbol{v}' \overset{\text{in } E^3}{\longrightarrow} \begin{pmatrix}0 & 0 & \alpha_{xz}\\ 0 & 0 & \alpha_{yz}\\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz}\end{pmatrix}$
Field decays	$\overline{\boldsymbol{B}}=\overline{B}_x(x,y)\mathbf{1}_x+\overline{B}_y(x,y)\mathbf{1}_y+\overline{B}_z(x,y)\mathbf{1}_z$	
	The $\alpha_{\phi\phi}=0$ ADT	
Velocity	$\boldsymbol{v}'=v'_r(r,\theta)\mathbf{1}_r+v'_\theta(r,\theta)\mathbf{1}_\theta+v'_\phi(r,\theta)\mathbf{1}_\phi$	
$\boldsymbol{\alpha}$	$\boldsymbol{v}' \overset{\text{in } E^3}{\longrightarrow} \begin{pmatrix}\alpha_{rr} & \alpha_{r\theta} & \alpha_{r\phi}\\ \alpha_{\theta r} & \alpha_{\theta\theta} & \alpha_{\theta\phi}\\ \alpha_{\phi r} & \alpha_{\phi\theta} & 0\end{pmatrix}$	
Field decays	$\overline{\boldsymbol{B}}=\overline{B}_r(r,\theta)\mathbf{1}_r+\overline{B}_\theta(r,\theta)\mathbf{1}_\theta+\overline{B}_\phi(r,\theta)\mathbf{1}_\phi$	

for proposed hidden dynamos where the field is proposed to be wholly contained within the conductor; see for example Ivers and James (1984), Kaiser *et al.* (1994).

In section 2 the governing equations are derived. In section 3 the axisymmetric, $\alpha_{\phi\phi}=0$ ADT is proven. In section 4 the weighted energy analysis for $\int_V(\overline{B}_\phi/s)^2\mathrm{d}V$ is conducted. In section 5 the energy analysis is applied to a hidden dynamo model. In section 6 the spectral methods for the investigation of the hidden dynamo model are given. In section 7 the numerical methods for solving the hidden dynamo model are outlined. In section 8 the numerical methods for solving the energy analysis are derived. In section 9 the numerical results of the linear stability analysis and the magnetic energy analysis are presented. An explicit expression for $\alpha_{\phi\phi}$ using the second order correlation approximation and the Green's tensor is derived in section 10. This analysis is also used to show that axisymmetric turbulence in E^3 leads to $\alpha_{\phi\phi}=0$. In section 11 an alternate expression for $\alpha_{\phi\phi}$ is produced using the methods of Braginskii (1964) extended to compressible flows as appropriate in stellar interiors. In section 12 the work of Soward (1972) is examined as it produces a concise and informative insight for generating $\alpha_{\phi\phi}$. Discussion and conclusions are presented in section 13.

2. Governing equations

For an electrically conducting fluid, with prescribed velocity $\boldsymbol{v}$, using Ampere's Law $\boldsymbol{\nabla}\times\boldsymbol{B} = \mu\boldsymbol{J}$ in Ohm's Law $\boldsymbol{J}/\sigma = \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}$ yields

$$\boldsymbol{E} = \eta\boldsymbol{\nabla}\times\boldsymbol{B} - \boldsymbol{v}\times\boldsymbol{B}, \tag{2}$$

where $\boldsymbol{B}$ is the magnetic induction field, $\boldsymbol{E}$ is the electric field, and $\boldsymbol{J}$ is the electric current density. The permeability of free space μ, The electrical conductivity σ and the magnetic diffusivity $\eta = 1/(\mu\sigma)$, are uniform.

The magnetic vector potential $\boldsymbol{A}$, where $\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{B}$ and $\boldsymbol{E}$ are related by $\boldsymbol{E} = -\partial_t\boldsymbol{A} - \boldsymbol{\nabla}\Phi$, where Φ is the electric scalar potential and ∂_t is the derivative with respect to t etc. Eliminating $\boldsymbol{E}$ yields,

$$\partial_t\boldsymbol{A} = -\eta\boldsymbol{\nabla}\times\boldsymbol{B} + \boldsymbol{v}\times\boldsymbol{B} - \boldsymbol{\nabla}\Phi. \tag{3}$$

Using Faraday's Law of induction $\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\overline{\boldsymbol{B}}$ and Gauss' Law $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{B} = 0$ in the curl of (2) gives the induction equation

$$\partial_t\boldsymbol{B} - \eta\nabla^2\boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}). \tag{4}$$

2.1. *The second-order correlation approximation and the Green's tensor solution*

To produce mean field counterparts of (3) and (4), used herein, the velocity and magnetic fields are decomposed into mean and fluctuating parts $\boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}'$, $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{B}'$, where the overline denotes an ensemble average. Taking the mean of (4), using $\overline{\boldsymbol{F}'} = \boldsymbol{0}$, $\overline{\boldsymbol{F}'\times\overline{\boldsymbol{G}}} = \boldsymbol{0}$, $\overline{\boldsymbol{F}+\boldsymbol{G}} = \overline{\boldsymbol{F}} + \overline{\boldsymbol{G}}$, etc. gives

$$\boldsymbol{\mathcal{D}}\overline{\boldsymbol{B}} := \left(\partial_t - \eta\nabla^2\right)\overline{\boldsymbol{B}} - \boldsymbol{\nabla}\times(\overline{\boldsymbol{v}}\times\overline{\boldsymbol{B}}) = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}}. \tag{5}$$

where the mean turbulent emf $\boldsymbol{\mathcal{E}} := \overline{\boldsymbol{v}'\times\boldsymbol{B}'}$. Subtracting (5) from (4) yields

$$\boldsymbol{\mathcal{D}}\boldsymbol{B}' = \boldsymbol{\nabla}\times\left(\boldsymbol{v}'\times\overline{\boldsymbol{B}} + \boldsymbol{G}\right). \tag{6}$$

where $\boldsymbol{G} := \boldsymbol{v}'\times\boldsymbol{B}' - \overline{\boldsymbol{v}'\times\boldsymbol{B}'}$. If the second order correlation approximation (SOCA) is used, in which $\boldsymbol{\nabla}\times\boldsymbol{G}$ is neglected compared to the other terms, then

$$\boldsymbol{\mathcal{D}}\boldsymbol{B}' = \boldsymbol{\nabla}\times\left(\boldsymbol{v}'\times\overline{\boldsymbol{B}}\right) = \boldsymbol{\nabla}\times\boldsymbol{F}. \tag{7}$$

The solution of (7) is expressed in terms of the Green's tensor $\mathbf{G}(\boldsymbol{r},t;\boldsymbol{\xi},\tau)^\dagger$ solution to

$$\boldsymbol{\mathcal{D}}\mathbf{G}(\boldsymbol{r},t;\boldsymbol{\xi},\tau) = \left(\partial_t - \eta\nabla^2\right)\mathbf{G} - \boldsymbol{\nabla}\times(\overline{\boldsymbol{v}}\times\mathbf{G}) = \delta^3(\boldsymbol{\xi}-\boldsymbol{r})\delta(\tau - t)\boldsymbol{I}, \tag{8}$$

where δ^3 and δ are Dirac delta distributions and $\boldsymbol{I}$ is the identity tensor, subject to the conditions $\mathbf{G}(\boldsymbol{r},t;\boldsymbol{\xi},\tau) = \boldsymbol{0}$ for $t < \tau$ and $\mathbf{G}(\boldsymbol{r},t;\boldsymbol{\xi},\tau) \to \boldsymbol{0}$ as $|\boldsymbol{\xi}-\boldsymbol{r}| \to \infty$ (Bräuer 1973).

Contracting (8) with $\boldsymbol{\nabla}\times\boldsymbol{F}(\boldsymbol{\xi},t)$ and integrating over $\boldsymbol{\xi}$ and τ gives

$$\boldsymbol{\mathcal{D}}\iint_{\mathbb{R}^3\times\mathbb{R}}\mathbf{G}(\boldsymbol{r},t;\boldsymbol{\xi},\tau)\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{\xi}}\times\boldsymbol{F}(\boldsymbol{\xi},\tau)\mathrm{d}^3\boldsymbol{\xi}\mathrm{d}\tau = \boldsymbol{\nabla}\times\boldsymbol{F}(\boldsymbol{\xi},t). \tag{9}$$

Thus from (9) a particular solution of (7) is

$$\boldsymbol{B}'_{\mathrm{p}}(\boldsymbol{r},t) = \iint_{\mathbb{R}^3\times\mathbb{R}}\mathbf{G}(\boldsymbol{r},t;\boldsymbol{\xi},\tau)\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{\xi}}\times\boldsymbol{F}(\boldsymbol{\xi},\tau)\mathrm{d}^3\boldsymbol{\xi}\mathrm{d}\tau. \tag{10}$$

The solution of (7) is $\boldsymbol{B}'(\boldsymbol{r},t) = \boldsymbol{B}'_{\mathrm{h}}(\boldsymbol{r},t) + \boldsymbol{B}'_{\mathrm{p}}(\boldsymbol{r},t)$, where $\boldsymbol{B}'_{\mathrm{h}}$ is the solution to (7) for $\boldsymbol{\nabla}\times\boldsymbol{F} = \boldsymbol{0}$.

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†The upright Green's tensor $\mathbf{G}$ is distinguished from the vector $\boldsymbol{G}$.

For $\boldsymbol{B}'_{\rm h} = \boldsymbol{0}$, (10) yields

$$\boldsymbol{B}'(\boldsymbol{r},t) = \iint_{\mathbb{R}^3\times\mathbb{R}} \mathbf{G}(\boldsymbol{r},t;\boldsymbol{\xi},\tau)\cdot\boldsymbol{\nabla}_{\boldsymbol{\xi}}\times\left[\boldsymbol{v}'(\boldsymbol{\xi},\tau)\times\overline{\boldsymbol{B}}(\boldsymbol{\xi},\tau)\right]\,\mathrm{d}^3\boldsymbol{\xi}\mathrm{d}\tau. \tag{11}$$

Integration by parts in (11) shifts the derivative to $\mathbf{G}$

$$\boldsymbol{B}'(\boldsymbol{r},t) = \iint_{\mathbb{R}^3\times\mathbb{R}} \overline{\boldsymbol{B}}(\boldsymbol{\xi},\tau)\cdot\left[\boldsymbol{v}'(\boldsymbol{\xi},\tau)\times\boldsymbol{\nabla}_{\boldsymbol{\xi}}\times\mathbf{G}^{\rm T}(\boldsymbol{r},t;\boldsymbol{\xi},\tau)\right]\,\mathrm{d}^3\boldsymbol{\xi}\mathrm{d}\tau. \tag{12}$$

The mean electromotive force (emf) $\boldsymbol{\mathcal{E}}$ is the ensemble mean, of the vector product of $\boldsymbol{v}'(\boldsymbol{r},t)$ with (12), thus

$$\boldsymbol{\mathcal{E}} = -\iint_{\mathbb{R}^3\times\mathbb{R}} \overline{\boldsymbol{v}'(\boldsymbol{r},t)\times[\boldsymbol{\nabla}_{\boldsymbol{\xi}}\times\mathbf{G}^{\rm T}(\boldsymbol{r},t;\boldsymbol{\xi},\tau)]^T\times\boldsymbol{v}'(\boldsymbol{\xi},\tau)}\cdot\overline{\boldsymbol{B}}(\boldsymbol{\xi},\tau)\,\mathrm{d}^3\boldsymbol{\xi}\mathrm{d}\tau. \tag{13}$$

Expanding $\overline{\boldsymbol{B}}(\boldsymbol{\xi},\tau)$ in a Taylor series about $\boldsymbol{r}$ and t,

$$\overline{\boldsymbol{B}}(\boldsymbol{\xi},\tau) = \overline{\boldsymbol{B}}(\boldsymbol{r},t) + (\boldsymbol{\xi}-\boldsymbol{r})\cdot\boldsymbol{\nabla}_{\boldsymbol{r}}\overline{\boldsymbol{B}}(\boldsymbol{r},t) + \boldsymbol{\mathcal{O}}(|\boldsymbol{\xi}-\boldsymbol{r}|^2) + (\tau-t)\partial_t\overline{\boldsymbol{B}}(\boldsymbol{r},t) + \mathcal{O}(|\tau-t|^2). \tag{14}$$

Using (14) in (13) and neglecting terms $\boldsymbol{\mathcal{O}}(|\boldsymbol{\xi}-\boldsymbol{r}|^2)$, $\mathcal{O}(|t-\tau|)$ then the mean field induction equation (5) becomes

$$\partial_t\overline{\boldsymbol{B}} = \eta\nabla^2\overline{\boldsymbol{B}} + \boldsymbol{\nabla}\times\left(\boldsymbol{\alpha}\cdot\overline{\boldsymbol{B}} + \boldsymbol{\beta}\cdot\cdot\boldsymbol{\nabla}\overline{\boldsymbol{B}} + \overline{\boldsymbol{v}}\times\overline{\boldsymbol{B}}\right), \tag{15}$$

where, in coordinate-independent form

$$\boldsymbol{\alpha} = -\iint_{\mathbb{R}^3\times\mathbb{R}} \overline{\boldsymbol{v}'(\boldsymbol{r},t)\times[\boldsymbol{\nabla}_{\boldsymbol{\xi}}\times\mathbf{G}^{\rm T}(\boldsymbol{r},t;\boldsymbol{\xi},\tau)]^T\times\boldsymbol{v}'(\boldsymbol{\xi},\tau)}\,\mathrm{d}^3\boldsymbol{\xi}\mathrm{d}\tau, \tag{16}$$

$$\boldsymbol{\beta} = -\iint_{\mathbb{R}^3\times\mathbb{R}} \overline{\boldsymbol{v}'(\boldsymbol{r},t)\times[\boldsymbol{\nabla}_{\boldsymbol{\xi}}\times\mathbf{G}^{\rm T}(\boldsymbol{r},t;\boldsymbol{\xi},\tau)]^T\times\boldsymbol{v}'(\boldsymbol{\xi},\tau)}\otimes(\boldsymbol{\xi}-\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{\xi}\,\mathrm{d}\tau. \tag{17}$$

For $\overline{\boldsymbol{v}} = 0$ and a conductor filling all space then the Green's tensor is isotropic and given by $\mathbf{G}(\boldsymbol{r},t;\boldsymbol{\xi},\tau) = G(\boldsymbol{r}-\boldsymbol{\xi},t-\tau)\boldsymbol{I}$, where

$$G(\boldsymbol{r},t) = \begin{cases} 0, & t \le 0; \\ \dfrac{\exp(-|\boldsymbol{r}|^2/4\eta t)}{(4\pi\eta t)^{3/2}}, & t > 0. \end{cases} \tag{18}$$

For this specialisation $[\boldsymbol{\nabla}_{\boldsymbol{\xi}}\times\mathbf{G}^{\rm T}(\boldsymbol{r},t;\boldsymbol{\xi},\tau)]^T = [\boldsymbol{I}\times\boldsymbol{\nabla}_{\boldsymbol{\xi}}G(\boldsymbol{r}-\boldsymbol{\xi},t-\tau)]$. Changing the variables of integration to $\boldsymbol{\xi}' = \boldsymbol{r}-\boldsymbol{\xi}$, $\tau' = t-\tau$ and then dropping the primes in (16) and (17) gives

$$\boldsymbol{\alpha} = \iint_{\mathbb{R}^3\times\mathbb{R}} \overline{\boldsymbol{v}'(\boldsymbol{r},t)\times[\boldsymbol{I}\times\boldsymbol{\nabla}_{\boldsymbol{\xi}}G(\boldsymbol{\xi},\tau)]\times\boldsymbol{v}'(\boldsymbol{r}-\boldsymbol{\xi},t-\tau)}\,\mathrm{d}^3\boldsymbol{\xi}\,\mathrm{d}\tau, \tag{19}$$

$$\boldsymbol{\beta} = -\iint_{\mathbb{R}^3\times\mathbb{R}} \overline{\boldsymbol{v}'(\boldsymbol{r},t)\times[\boldsymbol{I}\times\boldsymbol{\nabla}_{\boldsymbol{\xi}}G(\boldsymbol{\xi},\tau)]\times\boldsymbol{v}'(\boldsymbol{r}-\boldsymbol{\xi},t-\tau)}\otimes\boldsymbol{\xi}\,\mathrm{d}^3\boldsymbol{\xi}\,\mathrm{d}\tau.$$

The analysis above can be applied to magnetic vector potential equation (3). Alternatively, 'uncurling' the mean field induction equation yields

$$\partial_t\overline{\boldsymbol{A}} = -\eta\boldsymbol{\nabla}\times\overline{\boldsymbol{B}} + \boldsymbol{\alpha}\cdot\overline{\boldsymbol{B}} + \boldsymbol{\beta}\cdot\cdot\boldsymbol{\nabla}\overline{\boldsymbol{B}} + \overline{\boldsymbol{v}}\times\overline{\boldsymbol{B}} - \boldsymbol{\nabla}\overline{\Phi}. \tag{20}$$

Henceforth, the Taylor series (14) is truncated at the first term. Thus $\boldsymbol{\beta}$ is omitted.

The following scaling is adopted

$$r/r_0 \to r, \quad v/v_0 \to v, \quad t\eta/r_0^2 \to t, \quad r_0\alpha_0/\eta \to R_\alpha r_0 v_0/\eta \to R_v, \quad \overline{\Phi}/\eta \to \overline{\Phi}, \tag{21}$$

where the r_0 denotes a typical dimension of the conducting volume V and the subscript $_0$ denotes a typical value. Using (21), (15) and (20) become

$$\partial_t \overline{\boldsymbol{B}} = \nabla^2 \overline{\boldsymbol{B}} + \boldsymbol{\nabla} \times \left(R_\alpha \boldsymbol{\alpha} \cdot \overline{\boldsymbol{B}} + R_v \overline{\boldsymbol{v}} \times \overline{\boldsymbol{B}} \right), \tag{22}$$

$$\partial_t \overline{\boldsymbol{A}} = -\boldsymbol{\nabla} \times \overline{\boldsymbol{B}} + R_\alpha \boldsymbol{\alpha} \cdot \overline{\boldsymbol{B}} + R_v \overline{\boldsymbol{v}} \times \overline{\boldsymbol{B}} - \boldsymbol{\nabla} \overline{\Phi}\,. \tag{23}$$

3. The axisymmetric $\alpha_{\phi\phi} = 0$ antidynamo theorem

A field $\boldsymbol{F}$ is *axisymmetric* in E^3 if there is a cylindrical polar coordinate system (s, ϕ, z), such that the cylindrical polar components of $\boldsymbol{F}$ are independent of ϕ. Thus,

$$\boldsymbol{F} = \boldsymbol{F}(s, z, t) = F_s(s, z, t)\mathbf{1}_s + F_\phi(s, z, t)\mathbf{1}_\phi + F_z(s, z, t)\mathbf{1}_z.$$

A meridional field is given by $\boldsymbol{F}_m := \boldsymbol{F} - \boldsymbol{F} \cdot \mathbf{1}_\phi$. Here $\overline{\boldsymbol{B}}$ is axisymmetric. Using $\boldsymbol{\nabla} \cdot \overline{\boldsymbol{B}} = 0$, then $\boldsymbol{\nabla} \cdot \overline{\boldsymbol{B}}_m = 0$, thus the mean magnetic field $\overline{\boldsymbol{B}}$ can be represented in terms of the azimuthal magnetic field $\overline{B}_\phi \mathbf{1}_\phi$ and the meridional flux function χ;

$$\overline{\boldsymbol{B}} = \boldsymbol{B}_\phi + \boldsymbol{B}_m = \overline{B}_\phi \mathbf{1}_\phi + \boldsymbol{\nabla} \times \left(\frac{\chi}{s} \mathbf{1}_\phi \right). \tag{24}$$

There is no imposition on either $\boldsymbol{\alpha}$ or $\overline{\boldsymbol{v}}$ to be axisymmetric. However, without loss of generality $\boldsymbol{\alpha}$ and $\overline{\boldsymbol{v}}$ are taken to be axisymmetric because departures from axisymmetry will generate non-axisymmetric $\overline{\boldsymbol{B}}$. The axisymmetric $\alpha_{\phi\phi} = 0$ ADT is proven in two parts. Firstly in subsection 3.1 $|\chi|$ is shown to decay to zero. Then in subsection 3.2, once χ has decayed, $\int_V |B_\phi/s| \mathrm{d}V$ is shown to decay.

3.1. *Decay of* χ

Theorem 3.1 Axisymmetric $\alpha_{\phi\phi} = 0$, $|\chi| \to 0$ ADT**:**
For a conducting volume V with non-conducting exterior $\widehat{V}$, where V and $\widehat{V}$ are individually connected[†]*, for $\overline{\boldsymbol{B}} = \overline{\boldsymbol{B}}(s, z, t)$, $\alpha_{\phi\phi} = 0$ in V and $|\overline{\boldsymbol{B}}| = \mathcal{O}(r^{-3})$ as $r \to \infty$, then $|\chi|$, decays to zero*[‡]*.*

The ϕ component of (23) in V is

$$\frac{1}{s}\partial_t \chi = \left[-\boldsymbol{\nabla} \times \overline{\boldsymbol{B}} + R_\alpha \boldsymbol{\alpha} \cdot \left(\overline{B}_\phi \mathbf{1}_\phi + \boldsymbol{\nabla} \times \left(\frac{\chi}{s} \mathbf{1}_\phi \right) \right) + R_v \overline{\boldsymbol{v}} \times \left(\overline{B}_\phi \mathbf{1}_\phi + \boldsymbol{\nabla} \times \left(\frac{\chi}{s} \mathbf{1}_\phi \right) \right) \right] \cdot \mathbf{1}_\phi. \tag{25}$$

Using $\boldsymbol{\nabla} \times (\mathbf{1}_\phi / s) = \mathbf{0}$, then

$$\left[\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} \times \left(\frac{\chi}{s} \mathbf{1}_\phi \right) \right] \cdot \mathbf{1}_\phi = \frac{1}{s} \mathbf{1}_\phi \cdot (\boldsymbol{\alpha} \times \boldsymbol{\nabla} \chi) \cdot \mathbf{1}_\phi = -\frac{1}{s} (\mathbf{1}_\phi \cdot \boldsymbol{\alpha} \times \mathbf{1}_\phi) \cdot \boldsymbol{\nabla} \chi. \tag{26}$$

The last term in (25) produces

$$\left[\overline{\boldsymbol{v}} \times \boldsymbol{\nabla} \times \left(\frac{\chi}{s} \mathbf{1}_\phi \right) \right] \cdot \mathbf{1}_\phi = \frac{1}{s} \left[\overline{\boldsymbol{v}} \times (\boldsymbol{\nabla} \chi \times \mathbf{1}_\phi) \right] \cdot \mathbf{1}_\phi = -\frac{1}{s} \overline{\boldsymbol{v}} \cdot \boldsymbol{\nabla} \chi. \tag{27}$$

Using (26) and (27) in (25) yields

$$\partial_t \chi - s^2 \boldsymbol{\nabla} \cdot (s^{-2} \boldsymbol{\nabla} \chi) = s R_\alpha \alpha_{\phi\phi} B_\phi - R_\alpha (\mathbf{1}_\phi \cdot \boldsymbol{\alpha} \times \mathbf{1}_\phi) \cdot \boldsymbol{\nabla} \chi - R_v \overline{\boldsymbol{v}} \cdot \boldsymbol{\nabla} \chi. \tag{28}$$

[†] V connected, means that any two points in V are associated with a curve that lies wholly in V; likewise for $\widehat{V}$.

[‡] It is not necessary to state the conditions $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s, z)$, $\overline{\boldsymbol{v}} = \overline{\boldsymbol{v}}(s, z)$, $V = V(s, z)$ as part of theorem 3.1 as, if they were not satisfied, then $\overline{\boldsymbol{B}}$ would not remain axisymmetric

By comparing the last two terms of (28) the action of $\boldsymbol{\alpha}$ through

$$\mathbf{1}_\phi\boldsymbol{\cdot}\boldsymbol{\alpha}\times\mathbf{1}_\phi = \alpha_{\phi z}\mathbf{1}_s - \alpha_{\phi s}\mathbf{1}_z := \overline{\boldsymbol{v}}_{\alpha 1} \tag{29}$$

is equivalent to the analogue laminar velocity $\overline{\boldsymbol{v}}_{\alpha 1}$. Thus, for $\alpha_{\phi\phi}=0$, (28) is transformed into the ϕ component of the scaled (3) for $R\boldsymbol{v} := R_v\overline{\boldsymbol{v}} + R_\alpha\overline{\boldsymbol{v}}_{\alpha 1}$, where $\boldsymbol{v}=\boldsymbol{v}(s,z)$ is an equivalent 'laminar' velocity. Because $\overline{\boldsymbol{v}}_{\alpha 1}$ and hence $\boldsymbol{v}$ are not necessarily incompressible and may flow across conductor boundaries then extensions to the Cowling (1934) ADT are required. To this end, the proof that $|\chi|$ decays to zero of Ivers (1984) and Ivers and James (1984) are is extended to include $\boldsymbol{\alpha}$ and adapted below.

The ϕ component of the scaled (3) gives[§]

$$\mathcal{P}\{\chi\} := \nabla^2\chi - \frac{2}{s}\mathbf{1}_s\boldsymbol{\cdot}\boldsymbol{\nabla}\chi - R\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\chi - \partial_t\chi = 0 \qquad \text{in } V. \tag{30}$$

The ϕ component of $\boldsymbol{\nabla}\times\boldsymbol{B}=0$ is

$$\mathcal{E}\{\chi\} = \nabla^2\chi - \frac{2}{s}\partial_s\chi = 0 \qquad \text{in } \widehat{V}. \tag{31}$$

The $\mathcal{P}$ and $\mathcal{E}$ operators are parabolic and elliptic, respectively.

Ivers and James (1984) (section 4.1) use theorem 5 and 7 of Protter and Weinberger (1967) to prove the following theorem, where $V_\infty = V\cup\widehat{V}$, stated here for uniform η.

Theorem 3.2 Comparison theorem for χ**:**
For $\mathcal{P}$ and $\mathcal{E}$ as in (30), (31), *if*

$$\mathcal{P}\{u\}\le 0 \quad \textit{in } V; \qquad \mathcal{E}\{u\}\le 0 \quad \textit{in } \widehat{V}; \tag{32}$$

$$|\chi|\le u \quad \textit{at} \quad t=0; \qquad u\ge 0 \quad \textit{as} \quad s\to 0 \quad \textit{or} \quad r\to\infty; \tag{33}$$

then $|\chi|\le u$ in V_∞ for $t\ge 0$.

Thus, if a comparison function u can be found that decays and satisfies conditions (32), (33), then, by theorem 3.2, $|\chi|$ will decay.

To establish a decaying u, Ivers and James (1984) section 4.2 considers the trial function

$$u(s,z,t) := \begin{cases} u_0F(s)\mathrm{e}^{-pt}, & s\le 1;\\ u_0F(1)\mathrm{e}^{-pt}, & s\ge 1;\end{cases} \tag{34}$$

where $u_0>0$, $F(s)\ge 0$, $p>0$ are determined from the conditions of theorem 3.2.

Using (34) in (30) and introducing κ and λ to aid in the solution for F then

$$\mathcal{P}\{u\} = \mathrm{e}^{-pt}u_0\left\{\left(\partial_s^2F - \frac{1}{s}\partial_sF + \kappa\partial_sF + \lambda\right) - (\kappa + Rv_s\partial_s)F + pF - \lambda\right\}.$$

Equation (32) is satisfied if, for $s\le 0$, $\kappa=\sup_V\{-Rv_s,0\}$, $p=\inf_V\{\lambda/F\}$, $\partial_sF\ge 0$, $\lambda>0$,

$$\partial_s^2F - \frac{1}{s}\partial_sF + \kappa\partial_sF = -\lambda. \tag{35}$$

The solution of (35) is $F(s)=F_0+\lambda F_1(s)$, where

$$F_1(s) = \int_0^s \rho\mathrm{e}^{-\kappa\rho}\int_\rho^1 \frac{1}{\zeta}\mathrm{e}^{\kappa\zeta}\,\mathrm{d}\zeta\,\mathrm{d}\rho. \tag{36}$$

Also (33) is satisfied if $u_0=\max\{|\chi|/F(s)\}$ at $t=0$.

The maximum of $F_1(s)$ occurs at $s=1$, thus $\max\{p\}=\lambda/F(1)>0$. From theorem 3.2

$$|\chi|\le u_0F(s)\mathrm{e}^{-\lambda t/F(1)} \le u_0F(1)\mathrm{e}^{-\lambda t/F(1)} = u_0F_0\left[1+a\tau\right]\mathrm{e}^{-t/(a^{-1}+\tau)} = u_0F_0H(t,\tau,a), \tag{37}$$

[§]The discussion of Ivers and James (1984) and Ivers (1984) is simplified here because η is uniform herein.

where $\tau = F_1(1)$, $a = \lambda/F_0$.

For each value of t and τ, $H(t, \tau, a)$ will have an optimum value of a. Using $\partial_a H(t, \tau, a) = 0$ gives $a = (t - \tau)/\tau^2$. Because $a = \lambda/F_0 \geq 0$ then this analysis is valid for $a \geq 0$, i.e. $t \geq \tau$. For $t \leq \tau$ the optimum for H is $a \to 0$ giving $H(t, \tau, 0) = 1$. Thus from (37)

$$|\chi| \leq u_0 F_0 E(t; \tau), \tag{38}$$

where the optimum envelope is

$$E(t;\tau) = \begin{cases} 1, & t \leq \tau; \\ \frac{t}{\tau} e^{1-t/\tau}, & t \leq \tau. \end{cases} \tag{39}$$

From (38), (39) the bounding function $u_0 F_0 E(t; \tau)$ decays monotonically to 0. For $t \gg \tau$ the decay time is approximately τ. From (38), the flux function will decay to zero, however, not necessarily monotonically. For details see section 4.2.1 of Ivers and James (1984).

Here it is noted from (36) the 'eventual' decay time $\tau = F(1)$ is dependent on $\kappa = \sup\{-Rv_s, 0\}$ and thus dependent on Rv_s. So even though the envelope $u_0 F_0 E(t; \tau)$ will decay, the decay rate is dependent on R and v_s. This may have the effect of extending the decay of $|\chi|$. For example, Ivers and James (1984) (pp:180) observe that: $\tau \geq 10^{17}$ diffusion time units when $R \approx 10^2$.

This proof that $|\chi|$ decays to zero is independent of compressibility.† It is also noted that the behaviour of $\boldsymbol{v}$ at the boundary does not affect the conclusions. Thus the proof holds for $R\boldsymbol{v} = R_v \overline{\boldsymbol{v}} + R_\alpha \overline{\boldsymbol{v}}_{\alpha 1}$ given by (29). This concludes the proof of theorem 3.1.

3.2. *Decay of* $\int_V |\overline{B}_\phi/s| \mathrm{d}V$

Here the meridional field is assume to have decayed to zero. Thus $\chi \equiv 0$.

Theorem 3.3 Axisymmetric $\alpha_{\phi\phi} = 0$, decay of $\int_V |\overline{B}_\phi/s| \mathrm{d}V$ ADT**:**
For a conducting volume V with non-conducting exterior $\widehat{V}$, where V and $\widehat{V}$ are individually connected, for for $\overline{\boldsymbol{B}} = \overline{\boldsymbol{B}}(s, z, t)$, $\chi = 0$, $\alpha_{\phi\phi} = 0$ in V and $|\overline{\boldsymbol{B}}| = \mathcal{O}(r^{-3})$ as $r \to \infty$, then $\int_V |\overline{B}_\phi/s| \mathrm{d}V$, decays.‡

Using $\mathbf{1}_\phi \cdot \boldsymbol{\nabla} \times \overline{\boldsymbol{E}} = s \boldsymbol{\nabla} \cdot \left(\overline{\boldsymbol{E}} \times \mathbf{1}_\phi / s \right)$ on the ϕ component of the right hand side (22) yields

$$\partial_t \overline{B}_\phi = s \boldsymbol{\nabla} \cdot \left\{ \left(-\boldsymbol{\nabla} \times (\overline{B}_\phi \mathbf{1}_\phi) + R_\alpha \boldsymbol{\alpha} \cdot (\overline{B}_\phi \mathbf{1}_\phi) + R_v \overline{\boldsymbol{v}} \times (\overline{B}_\phi \mathbf{1}_\phi) \right) \times \mathbf{1}_\phi / s \right\}. \tag{40}$$

The diffusion term is simplified using $\boldsymbol{\nabla} \times (\mathbf{1}_\phi / s) = \mathbf{0}$,

$$\boldsymbol{\nabla} \times \left(\overline{B}_\phi \mathbf{1}_\phi \right) \times \mathbf{1}_\phi = \left(\boldsymbol{\nabla} (\overline{B}_\phi s) \times \mathbf{1}_\phi \right) \times \frac{1}{s} \mathbf{1}_\phi = -s \boldsymbol{\nabla} \left(\overline{B}_\phi / s \right) - 2 \overline{B}_\phi \mathbf{1}_s / s. \tag{41}$$

Using (41) in (40) and $b := \overline{B}_\phi / s$ gives

$$\partial_t b = \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} b + 2b \mathbf{1}_s / s - R_\alpha \mathbf{1}_\phi \times \boldsymbol{\alpha} \cdot \mathbf{1}_\phi b - R_v \overline{\boldsymbol{v}} b \right). \tag{42}$$

From here the working of Ivers and James (1984) section 5.4 is extended and modified to accomodate the mean field $\boldsymbol{\alpha}$ effect. The volume $V(t)$ is partitioned into component volumes V_i ($i = 1, 2, \ldots$), where for each V_i, $\overline{B}_\phi$ does not change sign. Because of the behaviour of $\overline{B}_\phi / s^2$ as $s \to 0$, each V_i is considered to exclude the region inside a cylinder radius $s = \epsilon$. The limit as $\epsilon \to 0$ is subsequently considered. Thus the surface of V_i is decomposed into the

†The proof of Ivers and James (1984) is also independent of variable conductivity.

‡As for theorem 3.1, it is not necessary to state the conditions $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s, z)$, $\overline{\boldsymbol{v}} = \overline{\boldsymbol{v}}(s, z)$, $V = V(s, z)$ as part of theorem 3.3 as, if they were not satisfied, then $\overline{\boldsymbol{B}}$ would not remain axisymmetric.

surface on the cylinder $S_{i\epsilon} = V \cap \{s = \epsilon\}$ and S_i. Integrating (42) over V_i, and using the divergence theorem gives

$$\int_{V_i} \partial_t |b| \mathrm{d}V_i = \lim_{\epsilon \to 0} \int_{S_i + S_{i\epsilon}} \{\boldsymbol{\nabla}|b| + 2|b|\mathbf{1}_s/s - R_\alpha \mathbf{1}_\phi \times \boldsymbol{\alpha}\cdot\mathbf{1}_\phi b - R_v \boldsymbol{v}|b|\} \cdot \mathrm{d}\boldsymbol{S}_i. \tag{43}$$

By comparing the last two terms of (43) the action of

$$\mathbf{1}_\phi \times \boldsymbol{\alpha}\cdot\mathbf{1}_\phi = \alpha_{z\phi}\mathbf{1}_s - \alpha_{s\phi}\mathbf{1}_z := \overline{\boldsymbol{v}}_{\alpha 2} \tag{44}$$

is compared to an analogue lamina velocity $\overline{\boldsymbol{v}}_{\alpha 2}$.

Using $|b| = 0$ on S_i, and $\boldsymbol{\nabla}|b|\cdot\mathrm{d}\boldsymbol{S}_i = \partial_n |b| \boldsymbol{n}\cdot\mathrm{d}\boldsymbol{S}_i$, then (43) becomes

$$\begin{aligned} \int_{V_i} \partial_t |b| \mathrm{d}V_i &= \int_{S_i} \partial_n |b| \mathrm{d}S - \lim_{s \to 0} \int_{I_i} \{\boldsymbol{\nabla}|b| + 2|b|\mathbf{1}_s/s - R_\alpha \overline{\boldsymbol{v}}_{\alpha 2} b - R_v \overline{\boldsymbol{v}}|b|\} \cdot 2\pi s \mathbf{1}_s \mathrm{d}z. \\ &= \int_{S_i} \partial_n |b| \mathrm{d}S - 4\pi \int_{I_i} |b| \mathrm{d}z + 2\pi \lim_{s \to 0} \int_{I_i} s\left\{-\partial_s |b| + R_v \overline{v}_s |b| + R_\alpha \alpha_{z\phi} |b|\right\} \mathrm{d}z, \end{aligned} \tag{45}$$

where $I_i = \lim_{\epsilon \to 0} S_{i\epsilon}$. As $\overline{\boldsymbol{B}}$ is differentiable and axisymmetric in V then $B_\phi = \mathcal{O}(s)$ as $s \to 0$. Thus the first term in the last integral of (45) is zero. Because $\overline{\boldsymbol{v}}$ is axisymmetric then $\overline{v}_s \to 0$ as $s \to 0$ and the second term in the last integral of (45) is zero. Similarly, $\boldsymbol{\alpha}$ is axisymmetric i.e. all the polar components have $\alpha_{z\phi} = \alpha_{z\phi}(s, z)$ etc. Expressing $\alpha_{z\phi}$ in terms of the Cartesian components gives $\alpha_{z\phi} = \alpha_{zy}\cos\phi - \alpha_{zx}\sin\phi$. Thus $\alpha_{z\phi} \to 0$ as $s \to 0$ and the last term in the last integral of (45) is zero. The s factor in the last two terms of (45) mean that they are zero for relatively well behaved $\overline{\boldsymbol{v}}$ and $\boldsymbol{\alpha}$. Likewise $b = 0$ on S_i results in (46) being independent of $\overline{\boldsymbol{v}}$ and $\boldsymbol{\alpha}$ on S.

Summing (45) over all V_i and using $\partial_t \int_V |b| \mathrm{d}V = \int_V \partial_t |b| \mathrm{d}V + \int_S |b| \boldsymbol{v}_S \cdot \mathrm{d}\boldsymbol{S}$, where the velocity of S, $\boldsymbol{v}_S = 0$ here, then (45) yields

$$\partial_t \int_V |b| \mathrm{d}V_i = \int_S \partial_n |b| \mathrm{d}S - 4\pi \int_I |b| \mathrm{d}z, \tag{46}$$

where $I = V \cap \{s = 0\}$ and S is the surface of V.

The decay of $\int_V |b| \mathrm{d}V$ is guaranteed unless there exists a speculative non-trivial solution $b = b_0(s, z, t)$ to the ϕ component of (22), $(\mathcal{L} + \mathcal{C})\{b_0\} = 0$, where

$$\begin{aligned} \mathcal{L} &:= \nabla^2 + \frac{2}{s}\mathbf{1}_s\cdot\boldsymbol{\nabla} - R_v \overline{\boldsymbol{v}}\cdot\boldsymbol{\nabla} - \partial_t \\ \mathcal{C} &:= -R_v \boldsymbol{\nabla}\cdot\overline{\boldsymbol{v}} - R_\alpha \boldsymbol{\nabla}\cdot\overline{\boldsymbol{v}}_{\alpha 2} = -R_v \boldsymbol{\nabla}\cdot\overline{\boldsymbol{v}} - R_\alpha(\partial_s(s\alpha_{z\phi})/s - \partial_z \alpha_{s\phi}), \end{aligned} \tag{47}$$

such that the RHS of (46) is zero. That is, at any time $t_0 > 0$ (i) $\partial_n b_0 \equiv 0$ on S_i, $\forall i$ and (ii) $b_0 \equiv 0$ on a I, given I is not empty.

To dismiss such a b_0 Ivers and James (1984) (pp. 205, 206) is adapted by considering the pre-constructed function $Z(s, z, t) = -|b_0|\mathrm{e}^{-c_\alpha t}$, where c_α is an upper bound on $\mathcal{C}$ of (47)†. Thus $(\mathcal{L} + \mathcal{C} - c_\alpha)\{Z\} = 0$.‡ Because Z takes its maximum of zero on the exterior $S_i \cup I_i$, then Protter and Weinberger (1967) theorem 7 implies that either (iii) $\partial_n Z > 0$ on $S_i \cup I_i$, or (iv) Z is constant for $t \leq t_0$. Because (i) and (iii) are inconsistent, as observed by Ivers and James (1984) (p. 206), then the only solution is $b_0 \equiv 0$ $\forall t$. Thus no such non-trivial b_0 exists.

Noting that, as $|b| \geq 0$ in V then $\partial_n |b| \leq 0$ on S, where n increases from in V to S, then for $b \not\equiv 0$ the RHS of (46) is negative. Thus $\int_V |b| \mathrm{d}V$ must be strictly monotonically decreasing for all t. This concludes the proof of theorem 3.3.

†It is reasonable to assert that $\alpha_{z\phi}/s + \partial_s \alpha_{z\phi} - \partial_z \alpha_{s\phi}$ is finite for well behaved, differentiable, axisymmetric $\boldsymbol{\alpha}$.

‡Because c_α is an upper bound on $\mathcal{C}$, then the undifferentiated term $\mathcal{C} - c_\alpha \leq 0$ and theorem 7 of Protter and Weinberger (1967) can be utilised.

Ivers and James (1984) section 5.2 and 5.3 provides a proof for the decay of $b = B_\phi/s$ under restrictive conditions. This proof uses analogous methods to those outlined in section 3.1 above for χ, but for b. The Comparison theorem for b is proven using 'fortuitous' extensions and observation of theorems 5, 6, 7, and remarks of Protter and Weinberger (1967). A comparison function b is then established using similar separation of variable methods as in section 3.1 above. This comparison function can be shown to decay monotonically to zero under the condition $c_s\tau_1(\kappa_b) < 1$, where c_s is an upper bound on $\mathcal{C}_v = -R\boldsymbol{\nabla\cdot v}$,

$$\tau_1(\kappa_b) = \int_0^1 \frac{1}{\zeta^3}\mathrm{e}^{\kappa_b\zeta} \int_0^\zeta \rho^3\mathrm{e}^{-\kappa_b\rho}\,\mathrm{d}\rho\,\mathrm{d}\zeta \tag{48}$$

is the counterpart to $\tau = F_1(1)$ (36), and $\kappa_b = \sup\{Rv_s, 0\}$ in V for $t \geq 0$.

However, to use this comparison theorem/function method for theorem 3.3 the analogous lamina velocity $\overline{\boldsymbol{v}}_{\alpha 2}$ (44) is identified, and $\overline{\boldsymbol{v}}_{\alpha 2}$ has no predetermined bound on the dilatation rate $\boldsymbol{\nabla\cdot}\overline{\boldsymbol{v}}_{\alpha 2}$. Thus this method proves ineffective for the purposes of establishing theorem 3.3.

4. Magnetic energy analysis

It is instructive to consider the magnetic energy budget for the magnetic field $\overline{B}_\phi\mathbf{1}_\phi$ once χ has decayed to examine the interaction of inductive and diffusive contributions. An alternate form of (42) with $\overline{\boldsymbol{v}} = \mathbf{0}$, obtained by using (44) and $\boldsymbol{\nabla\cdot}(b\,\mathbf{1}_s/s) = \boldsymbol{\nabla}(b)\cdot\mathbf{1}_s/s$, is

$$\partial_t b = \nabla^2 b + 2\boldsymbol{\nabla} b\cdot\mathbf{1}_s/s - R_\alpha\boldsymbol{\nabla\cdot}\left(b\,\overline{\boldsymbol{v}}_{\alpha 2}\right). \tag{49}$$

The integral of b^2 over V is considered here as a weighted gauge of the magnetic energy $\int_V \overline{B}_\phi^2\mathrm{d}V/2$. Multiplying (49) by b and using $\boldsymbol{\nabla}(b^2)\cdot(\mathbf{1}_s/s) = \boldsymbol{\nabla\cdot}(b^2\mathbf{1}_s/s)$, gives

$$\partial_t\frac{1}{2}b^2 = b\nabla^2 b + \boldsymbol{\nabla\cdot}\left(\frac{b^2}{s}\mathbf{1}_s\right) - R_\alpha b\boldsymbol{\nabla\cdot}\left(b\,\overline{\boldsymbol{v}}_{\alpha 2}\right). \tag{50}$$

Because of the behaviour of the first term on the RHS of (50) as $s \to 0$, V here, as in section 3.2, is considered to exclude the region inside a cylinder radius $s = \epsilon$. The surface of V is partitioned into the surface on the cylinder $S_\epsilon = V \cap \{s = \epsilon\}$ and S. The limit as $\epsilon \to 0$ is then considered. Thus, using $V = V(s,z)$ the integral over V of (50) becomes

$$\partial_t\int_V \frac{1}{2}b^2\mathrm{dV} = \int_V b\nabla^2 b\,\mathrm{dV} + \int_V \boldsymbol{\nabla\cdot}\left(\frac{b^2}{s}\mathbf{1}_s\right)\mathrm{dV} - R_\alpha\int_V b\boldsymbol{\nabla\cdot}\left(b\,\overline{\boldsymbol{v}}_{\alpha 2}\right)\mathrm{dV}. \tag{51}$$

Using $\boldsymbol{\nabla\cdot}(b\boldsymbol{\nabla} b) = \boldsymbol{\nabla} b\cdot\boldsymbol{\nabla} b + b\nabla^2 b$, and $b = 0$ on S, the first term on the RHS of (51) is

$$\int_V b\nabla^2 b\,\mathrm{d}V = \lim_{\epsilon\to 0}\int_{S+S_\epsilon} b\boldsymbol{\nabla} b\cdot\mathbf{d}\boldsymbol{S} - \int_V \boldsymbol{\nabla} b\cdot\boldsymbol{\nabla} b\,\mathrm{d}V = -\int_V (\boldsymbol{\nabla} b)^2\,\mathrm{d}V. \tag{52}$$

Using the divergence theorem on the second term on the RHS of (51), gives

$$\int_V \boldsymbol{\nabla\cdot}\left(\frac{b^2}{s}\mathbf{1}_s\right)\mathrm{d}V = \lim_{\epsilon\to 0}\left\{\int_S \frac{b^2}{s}\mathbf{1}_s\cdot\mathbf{d}\boldsymbol{S} + \int_{S_\epsilon}\frac{b^2}{s}\mathbf{1}_s\cdot\mathbf{d}\boldsymbol{S}\right\}. \tag{53}$$

For $b = 0$ on S the first term on the RHS of (53) vanishes. The last term in (53) becomes

$$\lim_{\epsilon\to 0}\int_{S_\epsilon}\frac{b^2}{s}\mathbf{1}_s\cdot\mathbf{d}\boldsymbol{S} = -2\pi\lim_{s\to 0} s\int_I \frac{b^2(s,z)}{s}\,\mathrm{d}z = -2\pi\int_I b^2(0,z)\mathrm{d}z, \tag{54}$$

where $b(0,z) := \lim_{s\to 0} b(s,z)$ and $I = V\cap\{s = 0\}$. The last term in (51) is transformed using

$$\boldsymbol{\nabla\cdot}(b^2\;\overline{\boldsymbol{v}}_{\alpha 2}) = b\boldsymbol{\nabla\cdot}(b\,\overline{\boldsymbol{v}}_{\alpha 2}) + \boldsymbol{\nabla}(b)\cdot(b\,\overline{\boldsymbol{v}}_{\alpha 2}) = b^2\boldsymbol{\nabla\cdot}\overline{\boldsymbol{v}}_{\alpha 2} + \boldsymbol{\nabla}(b^2)\cdot\overline{\boldsymbol{v}}_{\alpha 2},$$

the divergence theorem, and $b = 0$ on S, to give

$$\int_V b\boldsymbol{\nabla}\boldsymbol{\cdot}(b\,\overline{\boldsymbol{v}}_{\alpha 2})\,\mathrm{d}V = \lim_{\epsilon\to 0}\int_{S+S_\epsilon} b^2\,\overline{\boldsymbol{v}}_{\alpha 2}\cdot\mathbf{d}\boldsymbol{S} - \int_V b\,\boldsymbol{\nabla}(b)\cdot\overline{\boldsymbol{v}}_{\alpha 2}\,\mathrm{d}V = \frac{1}{2}\int_V b^2\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\boldsymbol{v}}_{\alpha 2}\,\mathrm{d}V. \tag{55}$$

Thus using (53)–(55) in (51) produces the b^2 'energy' equation for $\boldsymbol{\alpha}$,

$$\partial_t\int_V \frac{1}{2}b^2\mathrm{d}V = -\int_V(\boldsymbol{\nabla}b)^2\,\mathrm{d}V - 2\pi\int_I b^2(0,z)\mathrm{d}z - \frac{R_\alpha}{2}\int_V b^2\boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{1}_\phi\times\boldsymbol{\alpha}\boldsymbol{\cdot}\mathbf{1}_\phi)\,\mathrm{d}V. \tag{56}$$

It is possible to increase the magnitude of the last induction term of (56) by choosing, for instance, $\boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{1}_\phi\times\boldsymbol{\alpha}\boldsymbol{\cdot}\mathbf{1}_\phi)$ to be negative throughout the sphere and increase R_α. The subsequent contribution of induction to the rate of change of $\int_V b^2/2\,\mathrm{d}V$ would increase. Because the first two terms on the right hand side of (56) do not include R_α explicitly, it is tempting to imagine that simply increasing R_α would sustain a working dynamo. Indeed, if the contribution from the diffusion terms in (56) where not to counteract a positive increase from induction then the resulting energy would grow. Such a dynamo would satisfy the conditions for a so-called *hidden dynamo* because the resulting magnetic field $\overline{B}_\phi\mathbf{1}_\phi$ would be invisible from outside V. However, such a dynamo would contradict the antidynamo theorem proved herein. Thus the energy analysis indicates that the diffusion terms in (56) must detract more than the induction term contributes for all such models.

5. Decay of axisymmetric $\overline{\boldsymbol{B}}$ for $\alpha_{\phi\phi} = 0$: No hidden dynamo

To demonstrate the decay and energy budget of an $\alpha_{\phi\phi} = 0$ model, consider:

Model 1
$$\boldsymbol{\alpha} = s\mathbf{1}_z\mathbf{1}_\phi, \tag{57}$$

in an electrically conducting sphere $r \le 1$. The magnetic field is axisymmetric and the meridional component has decayed ($\chi = 0$). The exterior is electrically insulating and there are no sources at infinity, thus $\overline{B}_\phi = 0$ at $r \ge 1$.

Model 1 has $\alpha_{\phi\phi} = 0$, and thus satisfies the ADT established herein. The contribution of (57) to (56) is given by $\boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{1}_\phi\times\boldsymbol{\alpha}\boldsymbol{\cdot}\mathbf{1}_\phi) = \partial_s(s\alpha_{z\phi})/s - \partial_z\alpha_{s\phi} = 2$. Also, from $s\mathbf{1}_z\mathbf{1}_\phi = -x\mathbf{1}_z\mathbf{1}_x + y\mathbf{1}_z\mathbf{1}_y$, the model is analytic at the origin. Thus (57) avoids unbounded (Cartesian) derivative at the origin that could produce numerical convergence problems and unphysical contributions. The differentiability of scalar and vector spherical harmonics is given in subsection 6.2.

For a single uncoupled eigenmode where $b(s,z,t)\to \mathrm{e}^{\gamma t}b(s,z)$, then (56), (57) reduces to

$$\gamma\int_V b^2\mathrm{d}V = -\int_V(\boldsymbol{\nabla}b)^2\,\mathrm{d}V - 2\pi\int_I b^2(0,z)\mathrm{d}z - R_\alpha\int_V b^2\mathrm{d}V. \tag{58}$$

Using the p-norm notation with $p = 2$,

$$\|\boldsymbol{\nabla}b\|^2 := \int_V(\boldsymbol{\nabla}b)^2\,\mathrm{d}V, \quad |b|^2 := 2\pi\int_I b^2(0,z)\mathrm{d}z, \quad \|b\|^2 := \int_V b^2\,\mathrm{d}V, \tag{59}$$

then (58) becomes

$$\gamma\|b\|^2 = -\|\boldsymbol{\nabla}b\|^2 - |b|^2 - R_\alpha\|b\|^2. \tag{60}$$

It will be demonstrated that for model 1, as the contribution to induction is increased, through increasing $-R_\alpha$, the contribution to diffusion through $-|b|^2$ and $-\|\boldsymbol{\nabla}b\|^2$ will decrease to counteract induction. Thus, even though the contribution from $-R_\alpha\|b\|^2$ in (60) may be regenerative, γ remains negative through the counteractive, depleting effects of diffusion and the dynamo eventually fails.

6. Spectral representation of the linear stability eigenvalue problem

6.1. *Vector spectral equations*

Solutions to (22) are sought in the form

$$\partial_t \overline{\boldsymbol{B}} - \nabla^2 \overline{\boldsymbol{B}} = R_\alpha \boldsymbol{\nabla} \times \boldsymbol{F}. \tag{61}$$

Here, $\overline{\boldsymbol{B}}$ and $\boldsymbol{F}$ are represented as vector spherical harmonics, see Jones (1970), James (1976),

$$\overline{\boldsymbol{B}} = \sum_{m=-\infty}^{\infty} \sum_{n=\max(|m|,1)}^{\infty} \sum_{n_1=n-1}^{n+1} B^m_{n,n_1} \boldsymbol{Y}^m_{n,n_1}, \tag{62}$$

where $$\boldsymbol{Y}^m_{n,n_1}(\theta,\phi) = (-1)^{n-m} \mathit{\Lambda}(n) \sum_{\xi=-1,0,1} \begin{pmatrix} n & n_1 & 1 \\ m & \xi - m & -\xi \end{pmatrix} Y^{m-\mu}_{n_1} \boldsymbol{e}_\xi, \tag{63}$$

$\boldsymbol{e}_{-1} = (\mathbf{1}_x - \mathrm{i}\mathbf{1}_y)/2^{1/2}$, $\boldsymbol{e}_0 = \mathbf{1}_z$, $\boldsymbol{e}_1 = -(\mathbf{1}_x + \mathrm{i}\mathbf{1}_y)/2^{1/2}$, $\mathit{\Lambda}(n) = (2n+1)^{1/2}$, the 2×3 array is a Wigner 3J coefficient (see, e.g. Brink and Satchler 1968),

$$Y^m_n(\theta,\phi) = (-1)^m \mathit{\Lambda}(n)(1-\mu^2)^{m/2} \left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} \partial^m_\mu P_n(\mu)\, \mathrm{e}^{\mathrm{i}m\phi}, \text{ for } m \geq 0;$$

$Y^m_n = (-1)^m Y^{-m}_n$ for, $m < 0$; and $P_n(\mu) = \partial^n_\mu(\mu^2-1)^n/(2^n n!)$, $\mu = \cos\theta$.

From $1/4\pi \oint \boldsymbol{Y}^m_{n,n_1} \boldsymbol{\cdot} \overline{\boldsymbol{Y}}^M_{N,N_1} \mathrm{d}\Omega = \delta_{nN}\delta_{n_1N_1}\delta_{mM}$; where $\delta_{ij} = 1$, if $i = j$; $\delta_{ij} = 0$, if $i \neq j$; and $\overline{\boldsymbol{Y}}$ (or $\overline{Y}$) indicates complex conjugation, the $\boldsymbol{Y}^M_{N,N_1}$ components of $\overline{\boldsymbol{B}}$ and $\boldsymbol{F}$ are given by,

$$F^M_{N,N_1} = \frac{1}{4\pi} \oint \boldsymbol{F} \boldsymbol{\cdot} \overline{\boldsymbol{Y}}^M_{N,N_1}\, \mathrm{d}\Omega. \tag{64}$$

To incorporate $\boldsymbol{\nabla \cdot} \overline{\boldsymbol{B}} = 0$ the toroidal, poloidal representation is used

$$\overline{\boldsymbol{B}} = \boldsymbol{T} + \boldsymbol{S} = \boldsymbol{\nabla} \times (T\boldsymbol{r}) + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{r}S),$$

where $\boldsymbol{r} = r\mathbf{1}_r$. The potential functions T, S are represented as scalar spherical harmonics as, $[T,S] = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} [T^m_n(r,t), S^m_n(r,t)] Y^m_n(\theta,\phi) \mathrm{e}^{\gamma t}$. The B^m_{n,n_1} are related to T^m_n and S^m_n by

$$B^m_{n,n-1} = \frac{(n+1)n^{1/2}}{\mathit{\Lambda}(n)} \partial^{n-1}_n S^m_n,\ B^m_{n,n} = -\mathrm{i}[n(n+1)]^{1/2} T^m_n,\ B^m_{n,n+1} = \frac{n(n+1)^{1/2}}{\mathit{\Lambda}(n)} \partial^{n+1}_n S^m_n, \tag{65}$$

where $\partial^{n-1}_n := \partial_r + (n+1)/r$ and $\partial^{n+1}_n := \partial_r - n/r$.

By inverting (65) the toroidal, poloidal spectral forms of the induction equation are

$$(\gamma - D_N)\, T^M_N = -\frac{R_\alpha}{\mathit{\Lambda}(N)} \left[\frac{\partial^N_{N-1}}{N^{1/2}} - \frac{\partial^N_{N+1}}{(N+1)^{1/2}}\right] F^M_{N,N-1}, \quad (\gamma - D_N)\, S^M_N = \frac{\mathrm{i}RF^M_{N,N}}{[N(N+1)]^{1/2}}, \tag{66}$$

where $D_n := \partial^n_{n-1}\partial^{n-1}_n$.

Here, the general spectral equations are derived for $\boldsymbol{F} = a \sin\theta \mathbf{1}_z \mathbf{1}_\phi \boldsymbol{\cdot} \overline{\boldsymbol{B}}$, where $a = a(r)$. For model 1, (57) $a(r) = r$. From Phillips (1993) or the polar forms of $\boldsymbol{Y}^m_{n,n_1}$ (see, James 1976),

$$\mathbf{1}_z = \boldsymbol{Y}^0_{1,0} \qquad \text{and} \qquad \mathbf{1}_\phi = -\frac{1}{\sin\theta}\sqrt{\frac{2}{3}}\, \mathrm{i}\boldsymbol{Y}^0_{1,1}. \tag{67}$$

Thus by representing a as, $a = \sum_{m_a=-\infty}^{\infty} \sum_{n_a=|m_a|}^{\infty} a_{n_a}^{m_a} Y_{n_a}^{m_a}$, (64) and (67) yield

$$\begin{aligned} F_{N,N_1}^M &= \frac{1}{4\pi} \oint a \sin\theta\, \mathbf{1}_z \cdot \overline{\boldsymbol{B}}\ \mathbf{1}_\phi \cdot \overline{\boldsymbol{Y}}_{N,N_1}^M \, \mathrm{d}\Omega \\ &= -\sqrt{\frac{2}{3}}\, \mathrm{i} \sum_{n_a,m_a} \sum_{n,n_1,m} a_{n_a}^{m_a} B_{n,n_1}^m \frac{1}{4\pi} \oint Y_{n_a}^{m_a}\, \boldsymbol{Y}_{1,0}^0 \cdot \boldsymbol{Y}_{n,n_1}^m\ \boldsymbol{Y}_{1,1}^0 \cdot \overline{\boldsymbol{Y}}_{N,N_1}^M \, \mathrm{d}\Omega. \end{aligned} \tag{68}$$

The integral of five scalar and vector spherical harmonics in (68) may be reduced by expressing the product of two harmonics as one. For example, $\boldsymbol{Y}_{1,1}^0 \cdot \boldsymbol{Y}_{N,N_1}^m = \sum_{N',N_1',M'} 1/4\pi \oint \boldsymbol{Y}_{1,1}^0 \cdot \boldsymbol{Y}_{N,N_1}^M \overline{Y}_{N',N_1'}^{M'} \mathrm{d}\Omega\ Y_{N',N_1'}^{M'}$. To evaluate such integrals the following result of Jones (1970) is used

$$\begin{aligned} &\frac{1}{4\pi} \oint \boldsymbol{Y}_{n_a,n_{1a}}^{m_a} \cdot \boldsymbol{Y}_{n_b,n_{1b}}^{m_b} Y_{n_c}^{m_c} \mathrm{d}\Omega \\ &\qquad = (-1)^{n_a+n_{1a}} \Lambda(n_a, n_{1a}, n_b, n_{1b}, n_c) \begin{Bmatrix} n_a & n_{1a} & 1 \\ n_{1b} & n_b & n_c \end{Bmatrix} \begin{pmatrix} n_{1a} & n_{1b} & n_c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_a & n_b & n_c \\ m_a & m_b & m_c \end{pmatrix}, \end{aligned} \tag{69}$$

where the 2×3 array in brasses is a Wigner 6J coefficient and $\Lambda(n, n_1) = \Lambda(n)\Lambda(n_1)$ etc. From Brink and Satchler (1968) useful special cases of 3J and 6J coefficients are

$$\begin{pmatrix} n_a & n_b & 0 \\ m_a & m_b & 0 \end{pmatrix} = \frac{(-1)^{n_a - m_a} \delta_{n_a n_b} \delta_{m_a\, -m_b}}{\Lambda(n_a)} \quad \text{and} \quad \begin{Bmatrix} n_a & n_b & 0 \\ n_d & n_c & n_f \end{Bmatrix} = \frac{(-1)^{n_b+n_d-n_f} \delta_{n_a n_b} \delta_{n_c n_d}}{\Lambda(n_a, n_c)}, \tag{70}$$

where orders and degrees take only integer values for the formulæ herein.

Using (67a), (69), (70), the symmetry of 3J and 6J coefficients, and the complex conjugate relations $\overline{Y}_n^m = (-1)^m Y_n^{-m}$, $\overline{\boldsymbol{Y}}_{N,N_1}^M = (-1)^{N+N_1+1+M}\, \boldsymbol{Y}_{N,N_1}^{-M}$ then

$$\mathbf{1}_z \cdot \boldsymbol{Y}_{n,n_1}^m = (-1)^{n+m} \Lambda(n) \begin{pmatrix} n & n_1 & 1 \\ m & -m & 0 \end{pmatrix} Y_{n_1}^m. \tag{71}$$

Similarly, using (67b), (69) to reduce $\boldsymbol{Y}_{1,1}^0 \cdot \boldsymbol{Y}_{N,N_1}^M$ to a sum of scalar Y's and (71) in (64) results in an integral of three scalar Y's. The remaining integral has been evaluated by Adams (1900) as

$$\frac{1}{4\pi} \oint Y_{n_a}^{m_a} Y_{n_b}^{m_b} Y_{n_c}^{m_c} \mathrm{d}\Omega = \Lambda(n_a, n_b, n_c) \begin{pmatrix} n_a & n_b & n_c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_a & n_b & n_c \\ m_a & m_b & m_c \end{pmatrix}.$$

Thus the vector spherical harmonic components of $\boldsymbol{F} = a \sin\theta \mathbf{1}_\phi \cdot \overline{\boldsymbol{B}}\, \mathbf{1}_z$ are given by

$$\begin{aligned} F_{N,N_1}^M = &\sum_{n_a,m_a} \sum_{n,n_1,m} \sum_{N',M'} a_{n_a}^{m_a} B_{n,n_1}^m \sqrt{6}\, \mathrm{i}\, (-1)^{N+1+m} \Lambda(n_a, n, n_1, N, N_1, N', N') \times \\ &\begin{Bmatrix} n_1 & n & 1 \\ 1 & 1 & N' \end{Bmatrix} \begin{pmatrix} n_1 & N' & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} N & N_1 & 1 \\ M & -M & 0 \end{pmatrix} \begin{pmatrix} n & N' & 1 \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} N' & N_1 & n_a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} N' & N_1 & n_a \\ m & -M & m_a \end{pmatrix}. \end{aligned} \tag{72}$$

For model 1, from the result of subsection 3.1 and $\chi = 0$, only the toroidal equation (66a) is required. The F_{N,N_1}^M contributions to (66a) are calculating using (72) directly. The B_{n,n_1}^m contributions are converted to T_n^m contributions using (65b). Here, for $\boldsymbol{\alpha} = s \mathbf{1}_z \mathbf{1}_\phi$, $a = r$, $a_0^0 = r$, and all other a_n^m are zero. This method of using the $\boldsymbol{Y}_{N,N_1}$ formalism has the advantage that the spectral forms are summarised in (72) and any errors at this level are usually catastrophic and immediately evident. This same method has been utilised for a wide spectrum of problems from laminar velocity models to models of anisotropic diffusion and other mean field $\boldsymbol{\alpha}$ models. Thus many, common components for these problems in the numerical program have been benchmarked. The method also avoids errors in converting back to interaction-type spectral equations.

6.2. *Behaviour of α at the origin*

From the Herglotzian definition of the spherical harmonics, for any integer $k \geq 0$, $r^{n+2k} Y_n^m(\theta,\phi)$ is a homogeneous polynomial in x, y, z, and thus, an analytic function of x, y, z at the origin. However, $r^{n+2k+1} Y_n^m(\theta,\phi)$ has singular Cartesian derivatives in x, y, z of order $n+2k+1$ at the origin. Thus, the scalar function f, is an analytic function of x, y, z at the origin, if the coefficients of its spherical harmonic expansion $f = \sum_{n,m} f_n^m(r) Y_n^m(\theta,\phi)$ are of the form, $f_n^m(r) = r^n \sum_{k=0}^{\infty} f_n^{m,k} r^{2k}$, where the $f_n^{m,k}$ are independent of r. Likewise, from (63), $\boldsymbol{Y}_{n,n_1}^m$ is a linear combination of $Y_{n_1}(\theta,\phi)$ of order n_1, and complex cartesian unit vectors $\boldsymbol{e}_\xi$. Thus $\boldsymbol{F} = \sum_{n,n_1,m} F_{n,n_1}^m(r) \boldsymbol{Y}_{n,n_1}^m(\theta,\phi)$ is an analytic function of x, y, z at the origin, if $F_{n,n_1}^m(r) = r^{n_1} \sum_{k=0}^{\infty} F_{n,n_1}^{m,k} r^{2k}$, where the $F_{n,n_1}^{m,k}$ are independent of r. As an example for model 1 $\boldsymbol{\alpha} = s\mathbf{1}_z\mathbf{1}_\phi = \mathrm{i}\sqrt{2/3}\, r^0 \boldsymbol{Y}_{10}^0\, r^1 \boldsymbol{Y}_{11}^0$ will be analytic at the origin.

7. Formulation of the eigen-problem for model 1

For model 1, from the result of subsection 3.1 (or $3J$ and $6J$ selection rules), $\chi = 0$, (66a), (72) and (65b), then the radial functions $T_n(r)$ always occur in one of the forms $T_n(r)$, $\partial_{n_1}^n T_n(r)$, $D_n T_n(r)$, where $T_n(r) = T_n^0(r)$ etc. These specific forms are discretised using second-order finite-differences on a uniform grid, $r_j = jh$, $j = 1, 2, \ldots, J_{\max}$, $h = 1/J_{\max}$. This programming technique adds extra levels of redundancy. An analytic extension of the radial functions across the origin is used to fold-back the differencing scheme at $r = 0$ to $r > 0$ (see Ivers and Phillips 2003). From subsection 6.2, if

$$T_n = \mathcal{O}(r^n + r^{n+2} + r^{n+4} + \ldots), \tag{73}$$

then T is a polynomial in (x, y, z) and thus analytic at the origin. To satisfy $\overline{B}_\phi = 0$ at $r = 1$ and (73), $T_n(0) = T_n(1) = 0$ and the T_n equations are omitted at $r = 0$ and $r = 1$.

For a single mode $B_{n,n_1}^m(r,t) = B_{n,n_1}^{m\lambda}(r)\mathrm{e}^{\lambda t}$ of (62), the original problem is represented by the eigenvalue problem $[\mathbf{A} + R_\alpha \mathbf{B}]\, \boldsymbol{e} = \gamma\, \boldsymbol{e}$, for the eigenvalue–eigenvector pair $(\gamma, \boldsymbol{e})$ where the magnetic Reynolds number R_α is prescribed, and $\mathbf{A}$, $\mathbf{B}$, are matrices discretising; the diffusion, and the induction term with $\boldsymbol{F} = a\sin\theta \mathbf{1}_z \boldsymbol{\cdot} \overline{\boldsymbol{B}} \mathbf{1}_\phi$ of (61).

The eigensolutions $(\gamma, \boldsymbol{e})$ were sought using inverse iteration with partial pivoting, for a single eigenvalue or with quadratic root-finding to determine the two eigenvalues closest to a shift, and the implicitly restarted Arnoldi method (Sorensen 1992).

7.1. *Decoupling the magnetic field interactions*

Herein a vector field $\boldsymbol{F}(\boldsymbol{r})$ at the point $\boldsymbol{r}$ is denoted as symmetric [anti-symmetric] in the equatorial plane $(\theta = \pi/2)$ if its planar reflection $\boldsymbol{F}^{\mathrm{ref}}(\boldsymbol{r})$ is equal to positive [negative] the un-reflected quantity at the reflected point $\boldsymbol{F}(\boldsymbol{r}^{\mathrm{ref}})$, that is $\boldsymbol{F}^{\mathrm{ref}}(\boldsymbol{r}) = \boldsymbol{F}(\boldsymbol{r}^{\mathrm{ref}})$ $[\boldsymbol{F}^{\mathrm{ref}}(\boldsymbol{r}) = -\boldsymbol{F}(\boldsymbol{r}^{\mathrm{ref}})]$. Likewise, a scalar function f is symmetric [anti-symmetric], if $f(r, \pi-\theta, \phi) = f(r,\theta,\phi)$ $[f(r,\pi-\theta,\phi) = -f(r,\theta,\phi)]$.

For $\boldsymbol{\alpha} = s\mathbf{1}_z\mathbf{1}_\phi$ the scalar function s is symmetric, the vector fields $\mathbf{1}_z$ is anti-symmetric and $\mathbf{1}_\phi$ is symmetric. Hence, the tensor $s\mathbf{1}_z\mathbf{1}_\phi$ is antisymmetric. The symmetric [antisymmetric] part of (61) with $\boldsymbol{F} = \boldsymbol{\alpha}\boldsymbol{\cdot}\overline{\boldsymbol{B}}$ and $\boldsymbol{\alpha} = s\mathbf{1}_z\mathbf{1}_\phi$ (noting $\times$ is antisymmetric) reveals that symmetric [antisymmetric] $\overline{\boldsymbol{B}}$ will decouple and evolve independently. By the same argument symmetric $\boldsymbol{\alpha}$ will destroy this decoupling. Thus for $\chi = 0$, $m = 0$, $\boldsymbol{B}$ decouples into the symmetric $(T_1, T_3, T_5, T_7, \ldots)$ and antisymmetric $(T_2, T_4, T_6, T_8, \ldots)$ classes.

8. Energy budget analysis: spectral methods for evaluating $\|b\|^2$, $\|\nabla b\|^2$ and $|b|^2$

For $\overline{\boldsymbol{B}}$ axisymmetric and $\chi = 0$, then $\overline{\boldsymbol{B}} = -\partial_\theta T\,\mathbf{1}_\phi$. Here $T = \sum_{n=1}^{\infty} T_n \Lambda(n) P_n(\cos\theta)$, thus

$$\overline{\boldsymbol{B}} = -\sum_{n=1}^{\infty} T_n \Lambda(n) \partial_\theta P_n(\cos\theta) \mathbf{1}_\phi. \tag{74}$$

The T_n from the eigen solutions $\boldsymbol{e}$ and γ could be used directly in (74) to calculate $\overline{\boldsymbol{B}}$, which in turn can be used to calculate $\|b\|^2$, $\|\nabla b\|^2$ and $|b|^2$ for the energy analysis of (60). However, it proves simpler and less error prone, to use a spectral representation of b directly as

$$b(r,\theta) = \sum_{n=0}^{\infty} b_n(r) P_n(\cos\theta), \tag{75}$$

calculate the b_n from the T_n obtained from the eigensolution $(\lambda, \boldsymbol{e})$ (using (79)) below, and then use b_n directly for $\|b\|^2$, $\|\nabla b\|^2$ and $|b|^2$. Thus an expression for b_n in terms of the T_n is required.

Using the identity, $\partial_\mu \left(P_{n+1}(\mu) - P_{n-1}(\mu)\right) = \Lambda^2(n) P_n(\mu)$ recursively in the form $\partial_\mu P_n(\mu) = \Lambda^2(n-1) P_{\mu-1}(\mu) + \partial_\mu P_{\mu-2}(\mu)$ yields

$$\partial_\mu P_n(\mu) = \Lambda^2(n-1) P_{n-1}(\mu) + \Lambda^2(n-3) P_{n-3}(\mu) + \ldots = \sum_{i=n-1,-2}^{i\geq 0} \Lambda^2(i) P_i(\mu), \tag{76}$$

where $\Lambda^2(n) = (\Lambda(n))^2$ etc. and $\sum_{i=a,c}^{b}$ denotes a sum from a to b in steps of c. For (76) the upper limit for the sum is to the least non-negative integer. Thus, from (74) with $\mu = \cos\theta$

$$b = \frac{\overline{B}_\phi}{s} = -\frac{1}{r\sin\theta}\partial_\theta T = -\frac{1}{r}\sum_{n=1}^{\infty} T_n \Lambda(n) \frac{1}{\sin\theta}\partial_\theta P_n(\cos\theta) = \frac{1}{r}\sum_{n=1}^{\infty} T_n \Lambda(n) \partial_\mu P_n(\mu). \tag{77}$$

Using (76) in (77) then $b = r^{-1} \sum_{n=0}^{\infty} \Lambda^2(n) P_n(\mu) \sum_{i=n+1,2}^{\infty} \Lambda(i) T_i$. The orthogonality of Legendre polynomials,

$$\int_{-1}^{1} P_i(\mu) P_j(\mu) \mathrm{d}\mu = 2\delta_{ij}/\Lambda^2(n), \tag{78}$$

is then used with (75) to give

$$b_n = \frac{\Lambda^2(n)}{r} \sum_{i=n+1,2}^{\infty} \Lambda(i) T_i. \tag{79}$$

8.1. *Evaluating* $\int_V b^2 dV$

Thus using (78),

$$\int_V b^2 \mathrm{d}V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sum_{i=0}^{\infty} b_i P_i(\mu) \sum_{j=0}^{\infty} b_j P_j(\mu)\; r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = 4\pi \int_0^1 \sum_{n=0}^{\infty} \frac{b_n^2}{\Lambda^2(n)}\, r^2 \mathrm{d}r.$$

8.2. *Evaluating* $\int_V (\nabla b)^2 \mathrm{d}V$

From $\nabla f = \partial_r f \mathbf{1}_r + \partial_\theta(f)\mathbf{1}_\theta/r$, $\nabla b = \sum_{n=0}^{\infty} \partial_r b_n P_n(\mu) \mathbf{1}_r - \frac{\sin\theta}{r} \sum_{n=0}^{\infty} b_n \partial_\mu P_n(\mu) \mathbf{1}_\theta$. Thus $\int_V (\nabla b)^2 \mathrm{d}V = I_1 + I_2$, where, using (78),

$$I_1 = \int_V \sum_{l=0}^{\infty} \partial_r b_l P_l(\mu) \sum_{n=0}^{\infty} \partial_r b_n P_n(\mu) \mathrm{d}V = 4\pi \int_0^1 r^2 \sum_{n=0}^{\infty} \frac{(\partial_r b_n)^2}{\Lambda^2(n)} \mathrm{d}r, \tag{80}$$

$$I_2 = \int_V \frac{\sin^2\theta}{r^2} \sum_{l=0}^{\infty} b_l \partial_\mu P_l(\mu) \sum_{n=0}^{\infty} b_n \partial_\mu P_n(\mu) \mathrm{d}V. \tag{81}$$

To evaluate I_2 the orthogonality of the θ derivatives of the Legendre Polynomials is established using integration by parts, $\partial_\mu \left((1-\mu^2)\partial_\mu P_l(\mu)\right) = -l(l+1)P_l(\mu)$, and (78). Thus

$$\begin{aligned}\int_0^\pi \partial_\theta(P_n(\mu))\partial_\theta(P_l(\mu)) \sin\theta \, \mathrm{d}\theta &= -\int_0^\pi P_n(\mu) \frac{1}{\sin\theta} \partial_\theta(\sin\theta \partial_\theta P_l(\mu)) \sin\theta \, \mathrm{d}\theta \\ &= \int_0^\pi P_n(\mu) l(l+1) P_l(\mu) \sin\theta \, \mathrm{d}\theta = \frac{2l(l+1)}{\Lambda^2(l)} \delta_{nl}. \end{aligned} \tag{82}$$

Using (82) in (81)

$$I_2 = 2\pi \int_0^1 \int_{-1}^1 \sum_{l=0}^{\infty} b_l \partial_\theta P_l(\mu) \sum_{n=0}^{\infty} b_n \partial_\theta P_n(\mu) \mathrm{d}\mu \, \mathrm{d}r = 4\pi \int_0^1 \sum_{n=0}^{\infty} b_n^2 \frac{n(n+1)}{\Lambda^2(n)} \mathrm{d}r. \tag{83}$$

Combining (80) and (83) gives

$$\int_V (\boldsymbol{\nabla} b)^2 \mathrm{d}V = 4\pi \int_0^1 r^2 \sum_{n=0}^{\infty} \frac{(\partial_r b_n)^2}{\Lambda^2(n)} \mathrm{d}r + 4\pi \int_0^1 \sum_{l=0}^{\infty} b_l^2 \frac{l(l+1)}{\Lambda^2(l)} \mathrm{d}r.$$

8.3. *Evaluating* $\int_I b^2(0,z)\mathrm{d}z$.

If the volume V is the unit sphere, then $I = [-1, 1]$ thus

$$\int_I b^2(0,z)\mathrm{d}z = \int_{z=-1}^{0} b^2(0,z)\mathrm{d}z + \int_{z=0}^{1} b^2(0,z)\mathrm{d}z = \int_{z=0}^{1} b^2(0,-z)\mathrm{d}z + \int_{z=0}^{1} b^2(0,z)\mathrm{d}z. \tag{84}$$

To calculate (84), the limit as $\delta \to 0$ is considered for

$$\begin{aligned} I_3 &:= \int_{r=0}^{1} b^2(r, \pi-\delta)\mathrm{d}r + \int_{r=0}^{1} b^2(r,\delta)\mathrm{d}r. \\ &= \int_{r=0}^{1} \left[\sum_{n=0}^{\infty} b_n(r) P_n(\cos(\pi-\delta))\right]^2 \mathrm{d}r + \int_{r=0}^{1} \left[\sum_{l=0}^{\infty} b_l(r) P_l(\cos\delta)\right]^2 \mathrm{d}r. \end{aligned}$$

Now $P_n(\cos 0) = 1$ and $P_n(\cos\pi) = (-1)^n$ thus

$$\lim_{\delta\to 0} I_3 = \int_{r=0}^{1} \left[\sum_{n=0}^{\infty} b_n(r)(-1)^n\right]^2 \mathrm{d}r + \int_{r=0}^{1} \left[\sum_{l=0}^{\infty} b_l(r)\right]^2 \mathrm{d}r. \tag{85}$$

It is noted for n and l either odd or even, the integrals in (85) will sum to twice the second.

8.4. *Boundary conditions for* $b_{n,j}$

Using (79), and (73), then $b_n = \mathcal{O}(r^n + r^{n+2} + r^{n+4} + \ldots)$ at the origin. Thus, at $r = 0$, $b_n = 0$ for all $n \geq 1$. However, using (79), and L'Hopital's rule

$$b_{0,0} = \Lambda(1) \lim_{r\to 0}(T_1/r) = \Lambda(1) \lim_{r\to 0} \partial_r T_1,$$

where $b_{n,j}$ is b_n at the jth gridpoint; likewise for $T_{n,j}$. From $T_1 = \mathcal{O}(r^1 + r^3 \ldots)$ an odd extension of T_1 is used across the origin. Thus by using central differencing across $r = 0$, $\partial_r T_1 \approx (T_{1,1} - T_{1,-1})/2h = T_{1,1}/h$. Thus $b_{0,0} \approx \Lambda(1) T_{1,1}/h$.

From (79), (73) then $\partial_r b_{n,0} = 0$, for $n = 0, n \geq 2$. However, for $n = 1$ using L'Hopital's rule

$$\partial_r b_{1,0} = \Lambda(1,1,2) \lim_{r \to 0} \partial_r \left(T_2/r\right) = \Lambda(1,1,2)\partial_r^2 T_{2,0}/2.$$

By using an even extension of T_2 across the origin and central differencing, $\partial_r^2 T_{2,0} \approx (T_{2,-1} - 2T_{2,0} + T_{2,1})/h^2 = T_{2,1}/h^2$. Thus $\partial_r b_{1,0} \approx \Lambda(1,1,2)T_{2,1}/h^2$.

At $r = 1$, left sided differencing is used, thus $\partial_r b_{n,\mathrm{Max}j} \approx (b_{n,\mathrm{Max}j-2} - 4b_{n,\mathrm{Max}j-1} + 3b_{n,\mathrm{Max}j})/2h$.

9. Numerical results

To simplify the discussion of the results it is useful to normalise (60) by $\|b\|^2$, thus

$$\gamma = -\|\boldsymbol{\nabla} b\|^2/\|b\|^2 - |b|^2/\|b\|^2 - R_\alpha = -D_V - D_Z - R_\alpha. \tag{86}$$

Here the contributions to γ from diffusion are quantified by; $D_Z = |b|^2/\|b\|^2$, along the z axis; and $D_V = \|\boldsymbol{\nabla} b\|^2/\|b\|^2$, from the gradients of b throughout the volume V. The contributions to γ from induction are quantified by $-R_\alpha$.

From (86) a benchmark of the convergence of the numerical code is the relative error

$$\epsilon_{\mathrm{rel}} := \left(\gamma + D_V + D_Z + R_\alpha\right)/|R_\alpha|. \tag{87}$$

Table 2 shows the least negative growth rate γ, the effects of D_V, D_Z and the relative error ϵ_{rel} given by (87), for $R_\alpha = -100, 100$ for model 1.

Table 2. Numerical results for model 1, $\boldsymbol{\alpha} = s\mathbf{1}_z\mathbf{1}_\phi$, magnetic Reynolds numbers R_α, decay rates (γ), D_V, D_Z and the relative error ϵ_{rel}. All solutions are steady ($\mathrm{Im}\{\gamma\} = 0$). The truncation is $n_{\max} = 40$, $J_{\max} = 800$.

	Symmetric $T_1, T_3, T_5 \dots$				Antisymmetric $T_2, T_4, T_6 \dots$			
R_α	γ	D_V	D_Z	ϵ_{rel}	γ	D_V	D_Z	ϵ_{rel}
-100	-202.595	102.535	200.057	-7.682 E-06	-210.356	110.129	200.225	-1.626 E-05
-80	-162.629	82.554	160.075	-2.206 E-06	-170.498	90.206	160.292	-3.339 E-06
-60	-122.693	62.589	120.104	-1.610 E-06	-130.755	70.343	120.412	-2.640 E-06
-40	-82.849	42.672	80.177	-1.546 E-06	-91.350	50.655	80.695	-3.409 E-06
-20	-43.732	23.094	40.638	-1.748 E-06	-53.830	31.854	41.976	-5.633 E-06
0	-20.191	10.043	10.147	∞	-33.217	20.416	12.801	∞
20	-43.732	20.570	3.166	2.112 E-04	-53.830	29.329	4.503	1.339 E-04
40	-82.848	40.296	2.597	1.119 E-03	-91.350	47.980	3.401	7.825 E-04
60	-122.691	60.376	2.476	2.682 E-03	-130.754	67.733	3.137	1.940 E-03
80	-162.625	80.588	2.424	4.835 E-03	-170.497	87.753	3.028	3.553 E-03
100	-202.589	100.945	2.396	7.523 E-03	-210.35	107.943	2.970	5.583 E-03

To discuss these results it is useful to observe the corresponding field configuration and geometries. Figure 1 shows the level contours of $\overline{B}_\phi$ in the meridional half plane, of the slowest decaying mode for; (a) $R_\alpha = -100$, (b) $R_\alpha = 100$, $\overline{\boldsymbol{B}}$ equatorially symmetric (T_n, n odd); (c) $R_\alpha = -100$, (d) $R_\alpha = 100$, $\overline{\boldsymbol{B}}$ equatorially antisymmetric (T_n, n even). For the symmetric mode at $R_\alpha = -100$ the resulting field rolls are distributed principally within the North and South hemispheres. However, for $R_\alpha = 100$ the field rolls are expelled outward toward the boundary of the sphere and the resulting toroidal field gradients are principally concentrated in narrow regions closer to $r = 1$. For the corresponding antisymmetric modes, the field gradients are likewise concentrated toward $r = 1$ for $R_\alpha = 100$. Additionally, this antisymmetric mode must have $\overline{B}_\phi = 0$ on the equatorial plane.

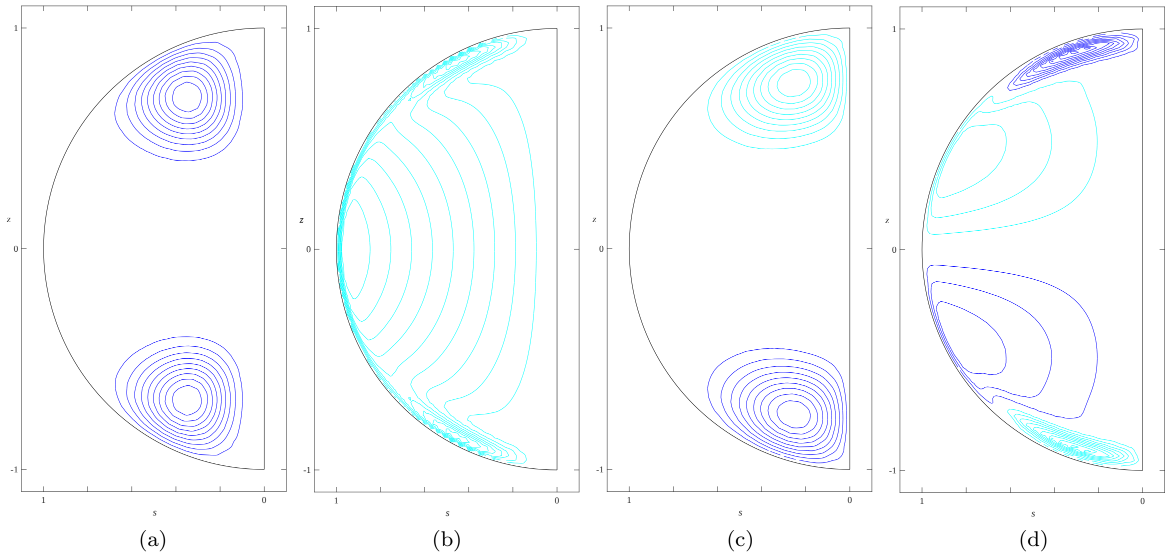

Figure 1. Level contours of $\overline{B}_\phi$ ($\overline{\boldsymbol{B}} = \overline{B}_\phi \mathbf{1}_\phi$) of the slowest decaying mode for, from left to right; (a) $R_\alpha = -100$, (b) $R_\alpha = 100$, $\overline{\boldsymbol{B}}$ equatorially symmetric (T_n, n odd); (c) $R_\alpha = -100$, (d) $R_\alpha = 100$, $\overline{\boldsymbol{B}}$ equatorially antisymmetric (T_n, n even). Because (61) with $\boldsymbol{F} = r\theta \mathbf{1}_z \mathbf{1}_\phi \boldsymbol{\cdot} \overline{\boldsymbol{B}}$ is invariant for $\overline{B}_\phi \to -\overline{B}_\phi$ ($\chi = 0$), the light contours levels of $\overline{B}_\phi$ are negative the dark contour levels in (c) and (d).

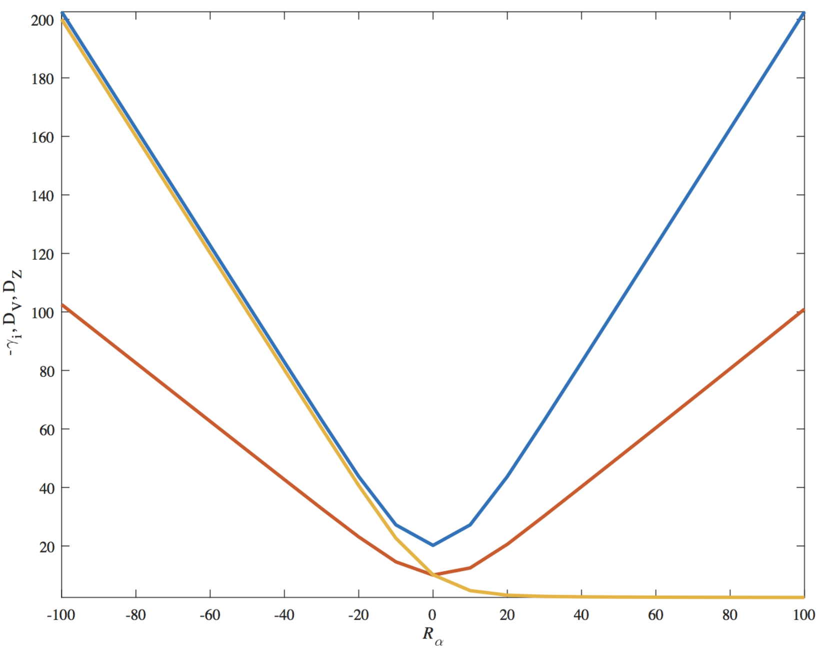

Figure 2. Graphs of numerical solutions for the equatorially symmetric mean magnetic field $\overline{\boldsymbol{B}}$ (T_n, n odd) for model 1; $\boldsymbol{\alpha} = s\mathbf{1}_z\mathbf{1}_\phi$ in a sphere with insulating exterior. The results are for the modes with greatest γ; all solutions are steady ($\mathrm{Im}\{\gamma\} = 0$). The graphs show $-\gamma$ blue (top at $R_\alpha = 100$), $D_V = \|\boldsymbol{\nabla} b\|^2/\|b\|^2$ red (middle at $R_\alpha = 100$) and $D_Z = |b|^2/\|b\|^2$ yellow (bottom at $R_\alpha = 100$) vs R_α. The results demonstrate the inductive and diffusive contributions to equation (60). As the induction contribution to (56) is increased through increasing $-R_\alpha$, diffusion counteracts this and the growth rate γ remains negative.

Figure 2 demonstrates how γ, D_V and D_Z respond to differing inductive effects characterised by $-R_\alpha$, for the symmetric mode with greatest γ of model 1. All solutions are steady. For increasing $|R_\alpha|$ the dynamo decays more rapidly.

For R_α increasing from zero, induction detracts more energy from the system. The diffusion due to the field along the z axis ($-D_Z$) reduces in magnitude. The D_Z contribution originates from the $\boldsymbol{\nabla}s$ component of the diffusive part of $\overline{\boldsymbol{E}} \times \mathbf{1}_\phi/s$ (see (40) and (41)). The resulting $\boldsymbol{\nabla}\boldsymbol{\cdot}(b^2\mathbf{1}_s/s)$ term in (50) results in the integral along z through the divergence theorem. To evaluate $\int_I b^2(0,z)\partial z$ the limit as $\epsilon \to 0$ is taken for $b^2(\epsilon, z)$ as in (54). Using L'Hopital's rule $\lim_{\epsilon\to 0} b^2(\epsilon, z) = \partial_s \overline{B}_\phi(s,z)_{s=0}$. Thus, D_Z is a measure of the s gradients of $\overline{B}_\phi$ as $s \to 0$. From figure 1 (b), the expulsion of the fields towards $r = 1$ results in lower field gradients along $s = 0$ over the interval $z = [-1, 1]$. This, in turn, progressively reduces $|D_Z|$ as $R_\alpha \to 100$. In this R_α interval the diffusive effects of the gradients in V and induction, through $-D_V - R_\alpha$, almost completely account for the decay rate γ.

In contrast, for R_α decreasing from zero, induction will contribute energy to the system. As indicated above, it is tempting to infer that the dynamo may be driven by this induction. However, as $-R_\alpha$ increases, both D_V and D_Z increase to counteract the inductive contributions and each dynamo remains decaying. It is interesting that, as induction increases ($R_\alpha \to -100$), D_V alone closely counteracts the energy contributions from induction as $-R_\alpha - D_V = \mathcal{O}(-3)$. As a result, γ closely follows $-D_Z$ as $R_\alpha \to -100$. Thus, as $R_\alpha \to -100$, the increase in D_Z almost completely accounts for the increasing decay rate[†].

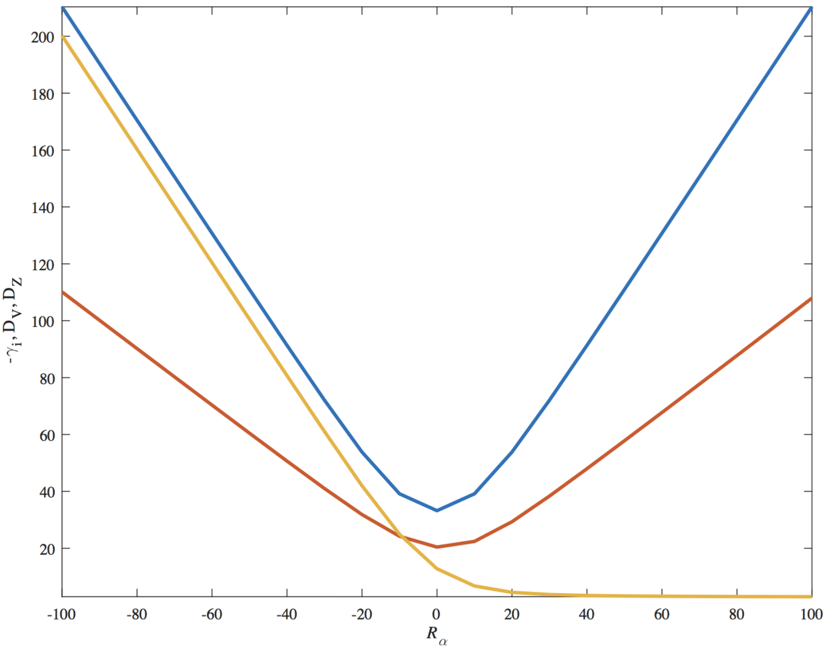

Figure 3. Graphs of $-\gamma$, $\|\boldsymbol{\nabla}b\|^2/\|b\|^2$ and $|b|^2/\|b\|^2$ vs R_α as in figure 3, except the mean magnetic field $\overline{\boldsymbol{B}}$ is equatorially antisymmetric (T_n, n even).

Figure 3 shows γ, D_V and D_Z vs R_α, for the symmetric mode with greatest γ of model 1. Again, for this antisymmetric mode the field is expelled toward $r = 1$, at $R_\alpha = 100$, as indicated in figure 1 (d). Thus, the s field gradient is reduced along $s = 0$ in $z = [-1, 1]$, and again $D_Z \to 0$, for $R_\alpha \to 100$. For $R_\alpha < 0$, the D_Z component of diffusion is close to that of the symmetric mode. However, the D_V diffusive component for the antisymmetric mode is greater than for the symmetric mode over the full range of R_α. This is due to greater field gradients throughout the sphere for the antisymmetric model as is apparent in figure 1 (c), (d). The fact that this antisymmetric mode must be zero on the equatorial plane also contributes to greater D_V for this model. As a result of the greater contribution from D_V, the decay rate is significantly different than D_Z ($\gamma + D_Z = \mathcal{O}(-10)$).

[†]This observation exemplifies how care must be taken in analysing the behaviour of diffusion as $s \to 0$, as D_Z may play a critical part in the dynamo behaviour.

The values of ϵ_{rel}, (table 2), generally increase for $|R_\alpha|$ increasing from zero. It is proposed that this is due to the fields exhibiting greater structure as $|R_\alpha|$ increases, thus requiring higher truncation levels to resolve. This hypothesis is reinforced by the greater field structure and localised field gradients for $R_\alpha = 100$ than for $R_\alpha = -100$ (see figure 1) which results in a larger $|\epsilon_{\mathrm{rel}}|$. The values of ϵ_{rel}, however, lie within satisfactory limits.

For both symmetry modes of model 1, the inductive contributions, through increasing $-R_\alpha$, are more than counteracted by the energy depleting effects of diffusion, through decreasing $-D_V - D_Z$, and the dynamos continue to fail.

10. Derivation of $\alpha_{\phi\phi}$ using the second-order correlation approximation and the Green's tensor solution

Here the analysis of section 2.1 is used to find $\alpha_{\phi\phi}$, where $\boldsymbol{\alpha}$ is derived using the emf $\boldsymbol{\mathcal{E}} := \overline{\boldsymbol{v}' \times \boldsymbol{B}'}$ with $\boldsymbol{B}'$ given by (11). To find $\alpha_{\phi\phi}$ the $\overline{\boldsymbol{B}} = \overline{B}_\phi \mathbf{1}_\phi$ contribution to $\boldsymbol{B}'$ is used to calculate $\mathbf{1}_\phi \boldsymbol{\cdot} \boldsymbol{\mathcal{E}}$. Here the $\overline{\boldsymbol{B}} = \overline{B}_\phi \mathbf{1}_\phi$ contribution to $\boldsymbol{B}'$, is denoted by $\boldsymbol{B}'^{\phi}(\boldsymbol{r}, t)$. Using (11)

$$\boldsymbol{B}'^{\phi}(\boldsymbol{r}, t) = \iint_{\mathbb{R}^3 \times \mathbb{R}} \mathbf{G}(\boldsymbol{r}, t; \boldsymbol{\xi}, \tau) \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{\xi}} \times \left[\boldsymbol{v}'(\boldsymbol{\xi}, \tau) \times \overline{B}_\phi(\boldsymbol{\xi}, \tau) \mathbf{1}_\phi\right] \mathrm{d}^3\boldsymbol{\xi} \mathrm{d}\tau. \tag{88}$$

From

$$\mathbf{1}_\phi \boldsymbol{\cdot} \boldsymbol{\mathcal{E}}(\boldsymbol{r}, t) = \mathbf{1}_\phi \boldsymbol{\cdot} \overline{\boldsymbol{v}'(\boldsymbol{r}, t) \times \boldsymbol{B}'(\boldsymbol{r}, t)} = \overline{v'_z B'_s - v'_s B'_z}, \tag{89}$$

to calculate $\alpha_{\phi\phi}$ the components $B_s'^{\phi} := \boldsymbol{B}'^{\phi} \boldsymbol{\cdot} \mathbf{1}_s$ and $B_z'^{\phi} := \boldsymbol{B}'^{\phi} \boldsymbol{\cdot} \mathbf{1}_z$ are required. For $\overline{\boldsymbol{v}} = 0$ then, the Green's tensor is isotropic $\mathbf{G}(\boldsymbol{r}, t, \boldsymbol{\xi}, \tau) = G(\boldsymbol{r} - \boldsymbol{\xi}, t - \tau)\boldsymbol{I}$, where G is given by (18). Thus (88) gives

$$B_s'^{\phi}(\boldsymbol{r}, t) = \iint_{\mathbb{R}^3 \times \mathbb{R}} G(\boldsymbol{r} - \boldsymbol{\xi}, t - \tau) s_{\boldsymbol{\xi}}^{-1} \partial_{\phi_{\boldsymbol{\xi}}} (v'_s(\boldsymbol{\xi}, \tau) \overline{B}_\phi(\boldsymbol{\xi}, \tau)) \, \mathrm{d}^3\boldsymbol{\xi} \mathrm{d}\tau, \tag{90}$$

$$B_z'^{\phi}(\boldsymbol{r}, t) = \iint_{\mathbb{R}^3 \times \mathbb{R}} G(\boldsymbol{r} - \boldsymbol{\xi}, t - \tau) s_{\boldsymbol{\xi}}^{-1} \partial_{\phi_{\boldsymbol{\xi}}} (v'_z(\boldsymbol{\xi}, \tau) \overline{B}_\phi(\boldsymbol{\xi}, \tau)) \, \mathrm{d}^3\boldsymbol{\xi} \mathrm{d}\tau. \tag{91}$$

Using $\partial_\phi \overline{B}_\phi = 0$, changing the variables of integration to $\boldsymbol{\xi}' = \boldsymbol{r} - \boldsymbol{\xi}$, $\tau' = t - \tau$, then dropping the primes, (90), (91) become

$$B_s'^{\phi}(\boldsymbol{r}, t) = \iint_{\mathbb{R}^3 \times \mathbb{R}} G(\boldsymbol{\xi}, \tau) s_{\boldsymbol{r}-\boldsymbol{\xi}}^{-1} \partial_{\phi_{\boldsymbol{r}-\boldsymbol{\xi}}} (v'_s(\boldsymbol{r} - \boldsymbol{\xi}, t - \tau)) \overline{B}_\phi(\boldsymbol{r} - \boldsymbol{\xi}, t - \tau) \, \mathrm{d}^3\boldsymbol{\xi} \mathrm{d}\tau, \tag{92}$$

$$B_z'^{\phi}(\boldsymbol{r}, t) = \iint_{\mathbb{R}^3 \times \mathbb{R}} G(\boldsymbol{\xi}, \tau) s_{\boldsymbol{r}-\boldsymbol{\xi}}^{-1} \partial_{\phi_{\boldsymbol{r}-\boldsymbol{\xi}}} (v'_z(\boldsymbol{r} - \boldsymbol{\xi}, t - \tau)) \overline{B}_\phi(\boldsymbol{r} - \boldsymbol{\xi}, t - \tau) \, \mathrm{d}^3\boldsymbol{\xi} \mathrm{d}\tau. \tag{93}$$

Expanding $\overline{B}_\phi(\boldsymbol{r} - \boldsymbol{\xi}, t - \tau)$ in a Taylor series, $\overline{B}_\phi(\boldsymbol{r} - \boldsymbol{\xi}, t - \tau) = \overline{B}_\phi(\boldsymbol{r}, t) - \boldsymbol{\xi} \boldsymbol{\cdot} \boldsymbol{\nabla} \overline{B}_\phi(\boldsymbol{r}, t) + \mathcal{O}(|\boldsymbol{\xi}|^2)$, neglecting terms of order $\boldsymbol{\xi}$, in (92), (93) then using these in (89) yields

$$\begin{aligned} \alpha_{\phi\phi}(\boldsymbol{r}, t) = & \iint_{\mathbb{R}^3 \times \mathbb{R}} G(\boldsymbol{\xi}, \tau) s_{\boldsymbol{r}-\boldsymbol{\xi}}^{-1} \overline{v'_z(\boldsymbol{r}, t) \partial_{\phi_{\boldsymbol{r}-\boldsymbol{\xi}}} v'_s(\boldsymbol{r} - \boldsymbol{\xi}, t - \tau)} \, \mathrm{d}^3\boldsymbol{\xi} \, \mathrm{d}\tau \\ & - \iint_{\mathbb{R}^3 \times \mathbb{R}} G(\boldsymbol{\xi}, \tau) s_{\boldsymbol{r}-\boldsymbol{\xi}}^{-1} \overline{v'_s(\boldsymbol{r}, t) \partial_{\phi_{\boldsymbol{r}-\boldsymbol{\xi}}} v'_z(\boldsymbol{r} - \boldsymbol{\xi}, t - \tau)} \, \mathrm{d}^3\boldsymbol{\xi} \, \mathrm{d}\tau. \end{aligned} \tag{94}$$

If every member of the ensemble of fluctuating velocities is co-axisymmetric with the same axis of symmetry as the mean magnetic field, then $\partial_\phi v'_z(\boldsymbol{r}, t) = \partial_\phi v'_s(\boldsymbol{r}, t) = 0$. Using $\partial_\phi v'_z(\boldsymbol{r}, t) = \partial_\phi v'_s(\boldsymbol{r}, t) = 0$ in (94) yields $\alpha_{\phi\phi} = 0$. This concludes the proof of theorem 10.1 below.

Theorem 10.1 $\alpha_{\phi\phi} = 0$ for co-axisymmetric $\boldsymbol{v}'$ and $\overline{\boldsymbol{B}}$**:**
If every member of the ensemble of fluctuating velocity fields $\boldsymbol{v}'$ is co-axisymmetric with the

mean magnetic field $\overline{\boldsymbol{B}}$ *and* $\overline{\boldsymbol{v}} = \mathbf{0}$ *in* E^3*, then* $\alpha_{\phi\phi} = 0$.

An alternative approach of producing an expression for $\alpha_{\phi\phi}$ is to expand (7) as

$$\partial_t \boldsymbol{B}' - \eta\nabla^2\boldsymbol{B}' = \overline{\boldsymbol{B}}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{v}' - \boldsymbol{v}'\boldsymbol{\cdot}\boldsymbol{\nabla}\overline{\boldsymbol{B}} + \boldsymbol{v}'\,\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\boldsymbol{B}} - \overline{\boldsymbol{B}}\,\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}'. \tag{95}$$

By using $\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\boldsymbol{B}} = 0$, assuming $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}' = 0$, and imposing $\boldsymbol{v}'\boldsymbol{\cdot}\boldsymbol{\nabla}\overline{\boldsymbol{B}} = 0$ then only the first term on the RHS of (95) is retained, as used in Jones (2011). The result of imposing these assumptions is that an extra term is produced in $B_s'^{\phi}$, where $\partial_\phi v_s'$ is replaced by $\partial_\phi v_s' - v_\phi'$ in (92). To proceed it would be necessary to impose assumptions about the comparative magnitude of v_ϕ'. However, this extra $-v_\phi'$ term is a result of the derivative of the coordinate vector, $\mathbf{1}_\phi$ and as such, is an artefact of the coordinate system. Indeed, if the $-\boldsymbol{v}'\boldsymbol{\cdot}\boldsymbol{\nabla}\overline{\boldsymbol{B}}$ term in (95) is retained, another term, which again results from the derivative of $\mathbf{1}_\phi$, will cancel with the extra $-\boldsymbol{v}_\phi'$ in (92). Thus, care must be taken here because the term $-\boldsymbol{v}'\boldsymbol{\cdot}\boldsymbol{\nabla}\overline{\boldsymbol{B}}$ includes terms proportional to $\overline{B}_\phi$ from $\partial_\phi\mathbf{1}_\phi$ and as such, cannot be discarded as only including derivatives of $\overline{\boldsymbol{B}}$. Hence, the term $-\boldsymbol{v}'\boldsymbol{\cdot}\boldsymbol{\nabla}\overline{\boldsymbol{B}}$ contributes to $\boldsymbol{\alpha}$. The result of not discarding terms in (95) and thus producing (94) means that the result applies to a greater range of problems because it uses fewer assumptions in the derivation†.

The proof of theorem 10.1, applies to a conductor filling all space. To extend this analysis to a finite conducting region, the Green's function for this problem would have to be investigated in greater detail and is beyond the scope of the present study.

To obtain a simplified physical interpretation of this $\alpha_{\phi\phi}$ interaction, consider the asymptotic limit $G(\boldsymbol{\xi},\tau) = \delta^3(\boldsymbol{\xi})\delta(\tau)$. For this zero-distance-of-influence regime, (94) reduces to

$$\alpha_{\phi\phi}(\boldsymbol{r},t) = s^{-1}\overline{[v_z'(\boldsymbol{r},t)\partial_\phi v_s'(\boldsymbol{r},t) - v_s'(\boldsymbol{r},t)\partial_\phi v_z'(\boldsymbol{r},t)]}. \tag{96}$$

Thus (96) demonstrates that a departure from axisymmetric $\boldsymbol{v}_m'$ can generate $\alpha_{\phi\phi}$. Section 3 proves that this $\alpha_{\phi\phi}$ mechanism for generating meridional $\overline{\boldsymbol{B}}$ from azimuthal $\overline{\boldsymbol{B}}$, is critical for axisymmetric dynamo maintenance. In this sense, the departure from axisymmetry in $\boldsymbol{v}_m'$ mitigates Cowling's axisymmetric antidynamo theorem through this $\alpha_{\phi\phi}$ interaction. The work of Jones (2011) which follows Olson *et al.* (1999) also provides and informative insight into $\alpha_{\phi\phi}$ under different assumptions.

11. Derivation of $\boldsymbol{\alpha}_{\phi\phi}$ with compressible velocity using the methods of Braginskii

An alternative derivation an $\alpha_{\phi\phi}$ component is produced in Braginskii (1964) using the analysis of nearly axisymmetric dynamos for highly conducting fluids. Here the analysis of Braginskii (1964) is reviewed and extended to compressible fluids for applications in stellar interiors.

The analysis of Braginskii (1964) uses an azimuthal average given by

$$\langle F\rangle = \overline{F} := \frac{1}{2\pi}\int_0^{2\pi} F(s,\phi,z)\,\mathrm{d}\phi.$$

Herein, the overline or angled brackets are used alternatively, depending on context and convenience. The mean of a vector is defined as the mean of the cylindrical polar coordinates,

$$\overline{\boldsymbol{v}} := \overline{v}_s\mathbf{1}_s + \overline{v}_\phi\mathbf{1}_\phi + \overline{v}_z\mathbf{1}_z \qquad \text{etc.}$$

The magnetic and velocity fields are progressively decomposed as mean, $\overline{\boldsymbol{B}}$, $\overline{\boldsymbol{v}}$; and perturbation $\boldsymbol{B}'$, $\boldsymbol{v}'$' components, where the mean components are axisymmetric and the perturbation

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†Indeed this alternative approach does not lead to the proof of theorem 10.1 without the imposition of further assumptions.

components have no axisymmetric part ($\overline{\boldsymbol{B}'} = 0$, etc). These components are then decomposed into, mean and perturbation; azimuthal and meridional parts;

$$\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{B}' = \overline{B}_\phi \mathbf{1}_\phi + \overline{\boldsymbol{B}}_m + \boldsymbol{B}' = \overline{B}_\phi \mathbf{1}_\phi + \boldsymbol{\nabla}\times(A\mathbf{1}_\phi) + B'_\phi \mathbf{1}_\phi + \boldsymbol{B}'_m, \tag{97}$$

$$\boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}' = \overline{v}_\phi \mathbf{1}_\phi + \overline{\boldsymbol{v}}_m + \boldsymbol{v}' = \overline{v}_\phi \mathbf{1}_\phi + \overline{\boldsymbol{v}}_m + v'_\phi \mathbf{1}_\phi + \boldsymbol{v}'_m, \tag{98}$$

where meridional (m subscript) indicates no azimuthal component. The analysis of Braginskii (1964) assumes

$$\boldsymbol{\nabla\cdot v} = \boldsymbol{\nabla\cdot}\overline{\boldsymbol{v}} = \boldsymbol{\nabla\cdot}(\overline{v}_\phi \mathbf{1}_\phi) = \boldsymbol{\nabla\cdot}\overline{\boldsymbol{v}}_m = \boldsymbol{\nabla\cdot v}' = \boldsymbol{\nabla\cdot}(v'_\phi \mathbf{1}_\phi) = \boldsymbol{\nabla\cdot v}'_m = 0. \tag{99}$$

Herein all of these conditions are relaxed and all of the velocity components (98) are compressible. The effect of relaxing the incompressible conditions extend the analysis considerably.

The mean of, the ϕ components of, (3) and (4), noting $\partial_\phi \Phi = 0$ gives

$$\partial_t A + \frac{1}{s}\overline{\boldsymbol{v}}_m \cdot \boldsymbol{\nabla}(sA) = \eta\Delta_1 A + \boldsymbol{\mathcal{E}} \cdot \mathbf{1}_\phi, \tag{100}$$

$$\partial_t \overline{B}_\phi + s\overline{\boldsymbol{v}}_m \cdot \boldsymbol{\nabla}\left(\frac{\overline{B}_\phi}{s}\right) = \eta\Delta_1 \overline{B}_\phi + \boldsymbol{\nabla}\left(\frac{\overline{v}_\phi}{s}\right) \times \boldsymbol{\nabla}(sA)\cdot \mathbf{1}_\phi + \boldsymbol{\nabla}\times(\boldsymbol{\mathcal{E}})\cdot\mathbf{1}_\phi - (\boldsymbol{\nabla\cdot}\overline{\boldsymbol{v}}_m)\,\overline{B}_\phi, \tag{101}$$

where $\boldsymbol{\mathcal{E}} = \langle \boldsymbol{v}' \times \boldsymbol{B}' \rangle$ and

$$\Delta_1 f(s,z) = -\mathbf{1}_\phi \cdot \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(f\mathbf{1}_\phi) = \nabla^2 f - \frac{f}{s^2} = s\boldsymbol{\nabla\cdot}\left(\frac{1}{s^2}\boldsymbol{\nabla}(sf)\right) = \frac{1}{s}\boldsymbol{\nabla\cdot}\left(s^2\boldsymbol{\nabla}\left(\frac{f}{s}\right)\right).$$

To find an expression for $\boldsymbol{B}'$ to use in (101), the mean of (4) is subtracted from (4), yielding

$$\partial_t \boldsymbol{B}' - \eta\nabla^2\boldsymbol{B}' = \boldsymbol{\nabla}\times\left(\overline{\boldsymbol{v}} \times \boldsymbol{B}' + \boldsymbol{v}' \times \overline{\boldsymbol{B}} + \boldsymbol{G}\right), \tag{102}$$

where again $\boldsymbol{G} = \boldsymbol{v}' \times \boldsymbol{B}' - \overline{\boldsymbol{v}' \times \boldsymbol{B}'}$. Equation (102) compares with (6) for different averaging operations. However, Braginskii (1964) refrains from setting $\boldsymbol{G} = \mathbf{0}$ (as used in the SOCA). A useful way to proceed is to find an expression for $\boldsymbol{B}'_m$ and then assume $\boldsymbol{\nabla\cdot B}' = 0$ to determine B'_ϕ. To this end, using the full decomposition of (97), (98), subtracting the ϕ projection of (102) from (102) gives

$$\begin{aligned}\partial_t \boldsymbol{B}'_m - \eta\left[\nabla^2\boldsymbol{B}'\right]_m = -\frac{\overline{v}_\phi}{s}\partial_{1\phi}\boldsymbol{B}'_m + \boldsymbol{B}'_m \cdot \boldsymbol{\nabla}\overline{\boldsymbol{v}}_m - \overline{\boldsymbol{v}}_m \cdot \boldsymbol{\nabla}\boldsymbol{B}'_m + \frac{\overline{B}_\phi}{s}\partial_{1\phi}\boldsymbol{v}'_m \\ + \left[\boldsymbol{\nabla}\times\left(\boldsymbol{v}' \times \overline{\boldsymbol{B}}_m + \boldsymbol{G}\right)\right]_m - (\boldsymbol{\nabla\cdot}\overline{\boldsymbol{v}}_m)\,\boldsymbol{B}'_m,\end{aligned} \tag{103}$$

where, for a vector $\boldsymbol{v}$, $\partial_{1\phi}\boldsymbol{v} = \partial_\phi(v_s)\mathbf{1}_s + \partial_\phi(v_\phi)\mathbf{1}_\phi + \partial_\phi(v_z)\mathbf{1}_z$.

To solve for $\boldsymbol{B}'_m$, an antiderivative of $\partial_{1\phi}$ is used. For f', where $\langle f' \rangle = 0$, then $\widehat{f'} := \int f' \mathrm{d}\phi$ such that $\langle \widehat{f'} \rangle = 0$. For a perturbation vector $\boldsymbol{v}'$, then

$$\widehat{\boldsymbol{v}'} := \int v'_s \mathrm{d}\phi\, \mathbf{1}_s + \int v'_\phi \mathrm{d}\phi\, \mathbf{1}_\phi + \int v'_z \mathrm{d}\phi\, \mathbf{1}_z \qquad \text{such that} \qquad \langle \widehat{\boldsymbol{v}'} \rangle = \mathbf{0}. \tag{104}$$

Thus integrating $\boldsymbol{\nabla\cdot B} = 0$, gives $B_\phi = -s\widehat{\boldsymbol{\nabla\cdot B}_m}$ and by assuming $\boldsymbol{\nabla\cdot B}' = 0$ $[\boldsymbol{\nabla\cdot}\overline{\boldsymbol{B}} = 0]$, then $B'_\phi = -s\widehat{\boldsymbol{\nabla\cdot B}'_m}$ $[\overline{B}_\phi = -s\widehat{\boldsymbol{\nabla\cdot}\overline{\boldsymbol{B}}_m}]$.

Using the operator (104) on (103) and the material derivative $D_t := \partial_t + \overline{\boldsymbol{v}}_m \cdot \boldsymbol{\nabla}$ gives

$$\begin{aligned}\frac{\overline{v}_\phi}{s}\boldsymbol{B}'_m = \frac{\overline{B}_\phi}{s}\boldsymbol{v}'_m + [\boldsymbol{\nabla}\times(\widehat{\boldsymbol{v}'} \times \overline{\boldsymbol{B}}_m + \widehat{\boldsymbol{G}})]_m + \eta[\nabla^2\widehat{\boldsymbol{B}'}]_m \\ + \widehat{\boldsymbol{B}'}_m \cdot \boldsymbol{\nabla}\overline{\boldsymbol{v}}_m - D_t\widehat{\boldsymbol{B}'}_m - (\boldsymbol{\nabla\cdot}\overline{\boldsymbol{v}}_m)\,\widehat{\boldsymbol{B}'}_m.\end{aligned} \tag{105}$$

11.1. *Expansion of perturbation magnetic field components as a series in* $R^{-1/2}$

The perturbation magnetic field is then expanded in a power series in $\epsilon = (\eta/L\mathcal{V})^{1/2} = R^{-1/2}$, where L is a characteristic length and $\mathcal{V} \sim \overline{v}_\phi$ thus, inferred from (B3.6) (equation (3.6) of Braginskii (1964) etc.) is

$$\boldsymbol{B}'_m = \sum_{n=1}^{\infty} \boldsymbol{B}'^{(n)}_m, \qquad \boldsymbol{B}'_\phi = \sum_{n=1}^{\infty} \boldsymbol{B}'^{(n)}_\phi, \qquad \boldsymbol{B}'^{(n)}_m, \boldsymbol{B}'^{(n)}_\phi \sim \overline{\boldsymbol{B}}_\phi \epsilon^n. \tag{106}$$

From Braginskii (1964) (p:729) we also infer that:

$$\boldsymbol{B}'_m, \boldsymbol{B}'_\phi \sim \overline{\boldsymbol{B}}_\phi \epsilon, \quad \boldsymbol{v}'_m, \boldsymbol{v}'_\phi \sim \overline{\boldsymbol{v}}_\phi \epsilon, \quad \overline{\boldsymbol{v}}_m \sim \overline{\boldsymbol{v}}_\phi \epsilon^2, \quad \overline{\boldsymbol{B}}_m \sim \overline{\boldsymbol{B}}_\phi \epsilon^2, \quad \partial_t \sim \eta/L^2. \tag{107}$$

Terms of order ϵ in (105) give

$$\boldsymbol{B}'^{(1)}_m = \frac{\overline{B}_\phi}{\overline{v}_\phi} \boldsymbol{v}'_m. \tag{108}$$

Additionally, assuming $\boldsymbol{\nabla\cdot}\left(\boldsymbol{B}'^{(1)}_m + \boldsymbol{B}'^{(1)}_\phi\right) = 0$ then $B'^{(1)}_\phi = -s\widehat{\boldsymbol{\nabla\cdot B}'^{(1)}_m}$. Thus from (108)

$$\boldsymbol{B}'^{(1)}_\phi = -s\boldsymbol{\nabla\cdot}\left(\frac{\overline{B}_\phi}{\overline{v}_\phi}\widehat{\boldsymbol{v}}'_m\right)\mathbf{1}_\phi = -s\left(\frac{\overline{B}_\phi}{\overline{v}_\phi}\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}'_m + \boldsymbol{\nabla}\left(\frac{\overline{B}_\phi}{\overline{v}_\phi}\right)\cdot\widehat{\boldsymbol{v}}'_m\right)\mathbf{1}_\phi. \tag{109}$$

For compressible flow the $\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}'_m$ in (109) does not vanish and the analysis diverges more significantly from Braginskii (1964). Here $\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}' = \boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}'_m + v'_\phi/s$, is used in (109), because this produces a term proportional to $\boldsymbol{v}'$ in $\boldsymbol{B}'^{(1)}$. This, in turn, simplifies $\boldsymbol{\mathcal{E}}^{(2)} = \overline{\boldsymbol{v}' \times \boldsymbol{B}'^{(1)}}$ as follows. Equation (109) then becomes

$$\boldsymbol{B}'^{(1)}_\phi = \frac{\overline{B}_\phi}{\overline{v}_\phi} v'_\phi \mathbf{1}_\phi - s\boldsymbol{\nabla}\left(\frac{\overline{B}_\phi}{\overline{v}_\phi}\right)\cdot\widehat{\boldsymbol{v}}'_m\mathbf{1}_\phi - s\frac{\overline{B}_\phi}{\overline{v}_\phi}(\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}')\mathbf{1}_\phi. \tag{110}$$

Combining (108) and (110)

$$\boldsymbol{B}'^{(1)} = \frac{\overline{B}_\phi}{\overline{v}_\phi}\boldsymbol{v}' - s\boldsymbol{\nabla}\left(\frac{\overline{B}_\phi}{\overline{v}_\phi}\right)\cdot\widehat{\boldsymbol{v}}'_m\mathbf{1}_\phi - s\frac{\overline{B}_\phi}{\overline{v}_\phi}(\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}')\mathbf{1}_\phi. \tag{111}$$

From (108), $\boldsymbol{\mathcal{E}}^{(2)}_\phi = \langle \boldsymbol{v}'_m \times \boldsymbol{B}'^{(1)}_m \rangle = \mathbf{0}$ because $\boldsymbol{v}'_m$ and $\boldsymbol{B}'^{(1)}_m$ are parallel. Using (111), the vector triple product and integration by parts

$$\begin{aligned}
\boldsymbol{\mathcal{E}}^{(2)}_m = \langle \boldsymbol{v}' \times \boldsymbol{B}'^{(1)} \rangle &= -s\left[\boldsymbol{\nabla}\left(\frac{\overline{B}_\phi}{\overline{v}_\phi}\right)\cdot\left\langle\widehat{\boldsymbol{v}}'_m\,\boldsymbol{v}'_m\right\rangle\right]\times\mathbf{1}_\phi \; - s\frac{\overline{B}_\phi}{\overline{v}_\phi}\left\langle(\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}')\,\boldsymbol{v}'_m\right\rangle\times\mathbf{1}_\phi \\
&= \frac{s}{2}\left[\left\langle\boldsymbol{v}'_m\times\widehat{\boldsymbol{v}}'_m\right\rangle\times\boldsymbol{\nabla}\left(\frac{\overline{B}_\phi}{\overline{v}_\phi}\right)\right]\times\mathbf{1}_\phi - s\frac{\overline{B}_\phi}{\overline{v}_\phi}\left\langle(\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}')\,\boldsymbol{v}'_m\right\rangle\times\mathbf{1}_\phi \\
&= \overline{v}_\phi^2 w\boldsymbol{\nabla}\left(\frac{\overline{B}_\phi}{\overline{v}_\phi}\right) - s\frac{\overline{B}_\phi}{\overline{v}_\phi}\left\langle(\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}')\,\boldsymbol{v}'_m\right\rangle\times\mathbf{1}_\phi,
\end{aligned} \tag{112}$$

where $w = \langle[\boldsymbol{u}'_m \times \widehat{\boldsymbol{u}}'_m]_\phi\rangle s/2$, $\boldsymbol{u}'_m = \boldsymbol{v}'_m/\overline{v}_\phi$. The last term in (112) extends (B3.13).

To find $\boldsymbol{B}'^{(2)}_m$, terms of order ϵ^2 in (105) yield

$$\boldsymbol{B}'^{(2)}_m = \frac{s}{\overline{v}_\phi}[\boldsymbol{\nabla}\times\widehat{\boldsymbol{G}}^{(2)}]_m. \tag{113}$$

Using $[\boldsymbol{\nabla}\times(f\boldsymbol{v}_m \times \mathbf{1}_\phi)]_m = \partial_{1\phi}(f\boldsymbol{v}_m)/s$ and $[\boldsymbol{\nabla}\times\widehat{\boldsymbol{\mathcal{E}}}^{(2)}_m]_m = \mathbf{1}_\phi \times \boldsymbol{\mathcal{E}}^{(2)}_m/s$ then from (113)

$$\begin{aligned}\boldsymbol{B}'^{(2)}_m &= \frac{s}{\overline{v}_\phi}[\boldsymbol{\nabla}\times(\widehat{\boldsymbol{v}'\times\boldsymbol{B}'^{(1)}})]_m - \frac{s}{\overline{v}_\phi}[\boldsymbol{\nabla}\times\widehat{\boldsymbol{\mathcal{E}}}^{(2)}_m]_m\\ &= -\frac{s}{\overline{v}_\phi}\boldsymbol{\nabla}\left(\frac{\overline{B}_\phi}{\overline{v}_\phi}\right)\cdot\widehat{\boldsymbol{v}}'_m\,\boldsymbol{v}'_m - \overline{v}_\phi w\mathbf{1}_\phi\times\boldsymbol{\nabla}\left(\frac{\overline{B}_\phi}{\overline{v}_\phi}\right) - \frac{s\overline{B}_\phi}{\overline{v}^2_\phi}\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}'\,\boldsymbol{v}'_m + \frac{s\overline{B}_\phi}{\overline{v}^2_\phi}\langle(\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}')\,\boldsymbol{v}'_m\rangle.\end{aligned} \tag{114}$$

The last two terms in (114) extend (B3.15). From (114) and $\langle\boldsymbol{v}'\rangle = \mathbf{0}$, despite the new terms, $\boldsymbol{\mathcal{E}}^{(3)}_\phi = \mathbf{0}$ and thus the next term to generate $\overline{\boldsymbol{B}}_m$ is $\boldsymbol{\mathcal{E}}^{(4)}_\phi$.

To find $\boldsymbol{\mathcal{E}}^{(4)}_\phi = \langle\boldsymbol{v}'_m\times\boldsymbol{B}'^{(3)}_m\rangle$, (105) is again used with the appropriate terms for $\boldsymbol{B}'^{(3)}_m$. The new compressible term in (105) now contributes together with terms arising from $\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}'$. Using vector calculus, integration by parts and some algebra I/we???? obtain

$$\begin{aligned}\boldsymbol{\mathcal{E}}^{(4)}_\phi &= \overline{v}_\phi^{-1}\langle\boldsymbol{\nabla\cdot}\left(s[\boldsymbol{v}'_m\times\boldsymbol{B}'_m]_\phi\widehat{\boldsymbol{v}}'_m)\right\rangle - \overline{v}_\phi^{-1}\langle s\left[\boldsymbol{v}'_m\times\boldsymbol{B}'_m\right]_\phi\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}'\rangle\\ &+\overline{v}_\phi^{-1}\langle\boldsymbol{\nabla\cdot}(s\left[\boldsymbol{v}'_m\times\overline{\boldsymbol{B}}_m\right]_\phi\widehat{\boldsymbol{v}}'_m)\rangle - \overline{v}_\phi^{-1}\langle s[\boldsymbol{v}'_m\times\overline{\boldsymbol{B}}_m]_\phi\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}'\rangle\\ &+\overline{v}_\phi^{-1}\langle s[\boldsymbol{v}'_m\times(\widehat{\boldsymbol{B}'}\cdot\boldsymbol{\nabla}\overline{\boldsymbol{v}}_m)]_\phi\rangle - \overline{v}_\phi^{-1}(\boldsymbol{\nabla\cdot}\overline{\boldsymbol{v}}_m)\langle s[\boldsymbol{v}'_m\times\widehat{\boldsymbol{B}'}_m]_\phi\rangle\\ &+\eta\,\overline{v}_\phi^{-1}\langle s[\boldsymbol{v}'_m\times[\nabla^2\widehat{\boldsymbol{B}'}]_m]_\phi\rangle - \overline{v}_\phi^{-1}\langle s[\boldsymbol{v}'_m\times D_t\widehat{\boldsymbol{B}'}_m]_\phi\rangle.\end{aligned} \tag{115}$$

Terms 2, 4, 6 in (115) extend (B3.17), the $\widehat{\boldsymbol{v}}'_m$ in term 3 differs by a possible typo.

The last term in (115) simplifies using (108), and $\langle D_t(\boldsymbol{v}\times\widehat{\boldsymbol{v}})\rangle = 2\langle\boldsymbol{v}\times D_t\widehat{\boldsymbol{v}}\rangle$, thus

$$-\overline{v}_\phi^{-1}\langle s[\boldsymbol{v}'_m\times D_t\widehat{\boldsymbol{B}'}_m]_\phi\rangle = -swD_t\left(\overline{B}_\phi s^{-1}\right) - s^{-1}D_t\left(sw\overline{B}_\phi\right) + s^{-1}\overline{v}_s\overline{B}_\phi w. \tag{116}$$

The $D_t(\overline{B}_\phi s^{-1})$ in (116) is eliminated using (101). Using (116), (101) in (115) yields

$$\begin{aligned}\boldsymbol{\mathcal{E}}^{(4)}_\phi &= \overline{v}_\phi^{-1}\langle\boldsymbol{\nabla\cdot}\left([s\boldsymbol{v}'_m\times\boldsymbol{B}'_m]_\phi\widehat{\boldsymbol{v}}'_m)\right\rangle - \overline{v}_\phi^{-1}\langle s\left[\boldsymbol{v}'_m\times\boldsymbol{B}'_m\right]_\phi\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}'\rangle\\ &+\overline{v}_\phi^{-1}\langle\boldsymbol{\nabla\cdot}(s\left[\boldsymbol{v}'_m\times\overline{\boldsymbol{B}}_m\right]_\phi\widehat{\boldsymbol{v}}'_m)\rangle - \overline{v}_\phi^{-1}\langle s[\boldsymbol{v}'_m\times\overline{\boldsymbol{B}}_m]_\phi\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}'\rangle\\ &+\overline{v}_\phi^{-1}\langle s[\boldsymbol{v}'_m\times(\widehat{\boldsymbol{B}'}\cdot\boldsymbol{\nabla}\overline{\boldsymbol{v}}_m)]_\phi\rangle - \overline{v}_\phi^{-1}(\boldsymbol{\nabla\cdot}\overline{\boldsymbol{v}}_m)\langle s[\boldsymbol{v}'_m\times\widehat{\boldsymbol{B}'}_m]_\phi\rangle\\ &+\eta\,\overline{v}_\phi^{-1}\langle s[\boldsymbol{v}'_m\times[\nabla^2\widehat{\boldsymbol{B}'}]_m]_\phi\rangle - w\eta\Delta_1\overline{B}_\phi - w[\boldsymbol{\nabla}(\overline{v}_\phi s^{-1})\times\boldsymbol{\nabla}(sA)]_\phi\\ &-w[\boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}}^{(2)}]_\phi - s^{-1}D_t(sw\overline{B}_\phi) + s^{-1}\overline{v}_s\overline{B}_\phi w + w(\boldsymbol{\nabla\cdot}\overline{\boldsymbol{v}}_m)\overline{B}_\phi.\end{aligned} \tag{117}$$

To complete the calculation for $\boldsymbol{\mathcal{E}}^{(4)}_\phi$, $\boldsymbol{B}'^{(2)}$ is used in terms 1 and 2, $\boldsymbol{\mathcal{E}}^{(2)}$ is used in term 10 and $\boldsymbol{B}'^{(1)}$ is used in the remaining terms. New, compressible terms will result from; term 1 using (114), term 7 using (111), term 10 using (112). The new terms 2, 4, 6 will also produce new contributions using $\boldsymbol{B}'^{(1)}$, $\boldsymbol{B}'^{(2)}$, together with the new term 13. After using extensive vector calculus and algebra, together with integration by parts I/we???? obtain

$$\begin{aligned}\boldsymbol{\mathcal{E}}^{(4)}_\phi &= -s^{-1}D_t(sw\overline{B}_\phi) - s^{-1}\boldsymbol{\nabla}\times(w\overline{v}_\phi\mathbf{1}_\phi)\cdot\boldsymbol{\nabla}(sA) - s^{-1}\boldsymbol{\nabla}\times(w\overline{v}_\phi\mathbf{1}_\phi)\cdot\boldsymbol{\nabla}(sw\overline{B}_\phi)\\ &-\eta w\Delta_1\overline{B}_\phi + \eta\overline{v}_\phi^{-1}\langle s[\boldsymbol{v}'_m\times[\nabla^2\widehat{\boldsymbol{B}'_0}]_m]_\phi\rangle + 2\eta\,\overline{v}_\phi^{-1}\boldsymbol{\rho}\cdot\mathbf{1}_z\overline{B}_\phi\\ &+\langle\overline{v}_\phi^{-1}\boldsymbol{\nabla\cdot}[s^2[\boldsymbol{u}'_m\times\boldsymbol{\rho}]_\phi\widehat{\boldsymbol{v}}'_m]\rangle\,\overline{B}_\phi - s^{-1}(-s\boldsymbol{\rho}\cdot\boldsymbol{\nabla}(sw\overline{B}_\phi)) - \boldsymbol{\rho}\cdot\boldsymbol{\nabla}(sw)\,\overline{B}_\phi\\ &-sw\overline{v}_\phi^{-1}\boldsymbol{\rho}\cdot\boldsymbol{\nabla}\overline{v}_\phi\,\overline{B}_\phi + \boldsymbol{\rho}\cdot\boldsymbol{\nabla}(sA) + w[\boldsymbol{\nabla}\times(s\boldsymbol{\rho}\times\mathbf{1}_\phi)]_\phi\,\overline{B}_\phi - w(\boldsymbol{\nabla\cdot}\overline{\boldsymbol{v}}_m)\overline{B}_\phi,\end{aligned} \tag{118}$$

where $\boldsymbol{\rho}(s,z) = \big\langle(\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}')\,\boldsymbol{u}'_m\big\rangle$. The notation $\boldsymbol{B}'_0$ corresponds to $\boldsymbol{B}'^{(1)}$ (111) with $\boldsymbol{\nabla\cdot}\widehat{\boldsymbol{v}}' = 0$, for comparison with Braginskii (1964), thus the last eight terms in (118) are the compressible extensions to (B3.18). The following mapping summarises the extensive algebra: $3+9\to 2$,

$1+10 \to 3+7+12$, $7 \to 5+6$, $2 \to 8+9+10$, $4 \to 11$, $6+13 \to 13$, $12+5=0$, where $i \to j$ indicates the ith term(s) of (117) produce the jth term(s) of (118).

Using (118) in (100), yields the *effective* azimuthal mean field vector potential equation,

$$\partial_t A_e + s^{-1}\overline{\boldsymbol{v}}_{e\boldsymbol{\rho}} \cdot \boldsymbol{\nabla}(sA_e) = \eta\Delta_1 A_e + \Gamma_e \overline{B}_\phi, \tag{119}$$

where the new *effective* vector potential and mean velocity are

$$A_e := A + w\overline{B}_\phi, \qquad \overline{\boldsymbol{v}}_{e\boldsymbol{\rho}} := \overline{\boldsymbol{v}}_m + \boldsymbol{\nabla}\times(w\overline{v}_\phi\,\mathbf{1}_\phi) - s\boldsymbol{\rho}, \tag{120}$$

$$\Gamma_e := \eta\Gamma_{\text{Brag}} + 2\eta\,\overline{v}_\phi^{-1}\boldsymbol{\rho}\cdot\mathbf{1}_z + \Gamma_{\boldsymbol{\rho}}, \tag{121}$$

$$\Gamma_{\text{Brag}} = s^{-1}\langle[\boldsymbol{u}'_m \times \widehat{\boldsymbol{u}}'_m]_\phi\rangle + s^{-1}\langle[\boldsymbol{u}'_m \times \partial^1_\phi \boldsymbol{u}'_m]_\phi\rangle + 2\langle\boldsymbol{\nabla}_m(zu'_z + su'_s)\cdot\boldsymbol{\nabla}_m\widehat{u}_z\rangle, \tag{122}$$

$\boldsymbol{\nabla}_m = \mathbf{1}_s\partial_s + \mathbf{1}_z\partial_z$, $\boldsymbol{u}'_m = u'_s\mathbf{1}_s + u'_z\mathbf{1}_z$, and

$$\Gamma_{\boldsymbol{\rho}} = \langle\boldsymbol{\nabla}(s^2[\boldsymbol{u}'_m \times \boldsymbol{\rho}]_\phi)\boldsymbol{\cdot}\widehat{\boldsymbol{u}}'_m\rangle - \boldsymbol{\rho}\cdot\boldsymbol{\nabla}\,(sw) - sw\overline{v}_\phi^{-1}\boldsymbol{\rho}\cdot\boldsymbol{\nabla}\overline{v}_\phi - w\boldsymbol{\nabla}_m\boldsymbol{\cdot}(s\boldsymbol{\rho}) - w\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\boldsymbol{v}}_m. \tag{123}$$

Using (112) in (101) yields the *effective* azimuthal mean field induction equation.

$$\partial_t\overline{B}_\phi + s\overline{\boldsymbol{v}}_{e\boldsymbol{\rho}}\cdot\boldsymbol{\nabla}(\overline{B}_\phi s^{-1}) = \eta\,\Delta_1\overline{B}_\phi + \boldsymbol{\nabla}(\overline{v}_\phi s^{-1})\times\boldsymbol{\nabla}(sA_e)\cdot\mathbf{1}_\phi + \Psi\,\overline{B}_\phi, \tag{124}$$

$$\text{where} \qquad \Psi = s^{-1}\boldsymbol{\nabla}_m\boldsymbol{\cdot}(s^2\boldsymbol{\rho}) - \boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\boldsymbol{v}}_m. \tag{125}$$

Equation (119) may be regarded as the azimuthal component of

$$\partial_t\boldsymbol{A}_e = -\eta\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}_e + \boldsymbol{\alpha}_e\boldsymbol{\cdot}\overline{\boldsymbol{B}}_e + \overline{\boldsymbol{v}}_{e\boldsymbol{\rho}}\times\overline{\boldsymbol{B}}_e - \boldsymbol{\nabla}\Phi_e, \tag{126}$$

where the effective axisymmetric magnetic field is given by

$$\overline{\boldsymbol{B}}_e := \overline{B}_\phi\,\mathbf{1}_\phi + \boldsymbol{\nabla}\times(\boldsymbol{A}_e) = \overline{B}_\phi\,\mathbf{1}_\phi + \boldsymbol{\nabla}\times(A_e\,\mathbf{1}_\phi). \tag{127}$$

Likewise, (124) may be regarded as the azimuthal component of

$$\partial_t\overline{\boldsymbol{B}}_e - \eta\nabla^2\overline{\boldsymbol{B}}_e = \boldsymbol{\nabla}\times(\boldsymbol{\alpha}_e\cdot\overline{\boldsymbol{B}}_e) + \boldsymbol{\nabla}\times(\overline{\boldsymbol{v}}_{e\boldsymbol{\rho}}\times\overline{\boldsymbol{B}}_e). \tag{128}$$

Equations (126)–(128), represent a reformulation of the physical problem into an effective model.

For $\Psi = 0$, the $\Gamma_e\overline{B}_\phi$ term in (119) generates an azimuthal component of the effective emf through $\partial_t A_e$ from $\overline{B}_\phi$. In this context, the Γ_e of (119) represents an alpha effect from $\alpha_{\phi\phi}\mathbf{1}_\phi\otimes\mathbf{1}_\phi$. Below, for simplicity, the tensor product will be dropped $\mathbf{1}_\phi\mathbf{1}_\phi := \mathbf{1}_\phi\otimes\mathbf{1}_\phi$ etc. It is also significant that the $\Gamma_e\overline{\boldsymbol{B}}$ term is the only 'mean-field'† term beyond the effective-laminar terms involving $\overline{\boldsymbol{v}}_{e\boldsymbol{\rho}}$ in (119) and (124)‡. Thus no other components of $\boldsymbol{\alpha}_e$ are generated. As a consequence, for the effective model (126)–(128), for $\Psi = 0$, then $\boldsymbol{\alpha}_e = \alpha_{\phi\phi}\mathbf{1}_\phi\mathbf{1}_\phi = \Gamma_e\mathbf{1}_\phi\mathbf{1}_\phi$.

For $\Psi \neq 0$, however, the $\Psi\overline{B}_\phi$ term of (124) generates $\overline{B}_\phi$ (through $\partial_t\overline{B}_\phi$) from $\overline{B}_\phi$. An alpha effect of the form $\boldsymbol{\alpha}_\Psi = \alpha_{s\phi}\mathbf{1}_s\mathbf{1}_\phi + \alpha_{z\phi}\mathbf{1}_z\mathbf{1}_\phi$ will generate $\overline{B}_\phi$ (and its derivatives) from $\overline{B}_\phi$. However, such an $\boldsymbol{\alpha}_\Psi$ will also result in Ψ-dependent terms in (119), which are not generated with the present analysis. Thus the compressible Ψ contribution to (124) is not readily expressed in terms of an analogous $\boldsymbol{\alpha}_e$–effect and as such constitutes an effect beyond that produced by conventional mean-field analysis.

Because the $\Psi\overline{B}_\phi$ term in (124) represents an interaction that generated $\overline{B}_\phi$ from $\overline{B}_\phi$, it is tempting to consider this Ψ effect as a possible mechanism to perpetuate a hidden dynamo. However, in general for $\boldsymbol{v}' \neq 0$, then $\Gamma_e \neq 0$ and A_e will also be generated through (119).

Alternatively, consider $\Gamma_e = 0$ and $\Psi \neq 0$, which could result from, (i) $\boldsymbol{v}'_m = \mathbf{0}$ but $v'_\phi \neq 0$ and $\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\boldsymbol{v}}_m \neq 0$, or (ii) $\boldsymbol{v}' \equiv 0$, but $\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\boldsymbol{v}}_m \neq 0$. For (i) or (ii), then (124) and (125) indicate

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†Acknowledging that $\overline{\boldsymbol{v}}_{e\boldsymbol{\rho}}$, B_e and A_e are generated through $\boldsymbol{v}'$.

‡To leading order in ϵ, for $\Psi = 0$.

that $\overline{B}_\phi$ is generated from $\overline{B}_\phi$. However, for (i), (119) and (124) are equivalent to equations governing compressible laminar flow. For (ii) the flow is compressible and laminar. Thus for (i) or (ii) the analysis of Ivers and James (1984) proves that an axisymmetric field will decay. As indicated above, it is noted that the decay time of such a compressible laminar model may be many free times. Interestingly the inclusion of $v'_\phi \neq 0$ in (i) does not affect the antidynamo result.

Of course, (124)–(128) represent A_e and $\overline{B}_\phi$ as independent, however, $A_e = A + w\overline{B}_\phi$ and hence the generation of A_e from $\Gamma_e \overline{B}_\phi$ in (119) may result in generation of both meridional $\boldsymbol{\nabla}\times(A\mathbf{1}_\phi)$ *and* azimuthal $\overline{B}_\phi \mathbf{1}_\phi$, components of the original field. Thus, this Γ_e contribution is not confined to an $\boldsymbol{\alpha} = \alpha_{\phi\phi}\mathbf{1}_\phi\mathbf{1}_\phi$ contribution in the original (un-effective), formulation of the model. However, even though this analysis does not lead to an $\alpha_{\phi\phi}$–effect of the original problem the analysis of section 3 equally applies to equations (124)–(128) for the effective model. Thus for $\Gamma_e = 0$, then $\mathbf{1}_\phi\boldsymbol{\cdot}\boldsymbol{\alpha}_e\boldsymbol{\cdot}\mathbf{1}_\phi = 0$ and section 3 proves that $\overline{\boldsymbol{B}}_e$ will decay

12. Derivation of $\boldsymbol{\alpha}_{\phi\phi}$ with reference to Soward 1972

Soward (1972) greatly extends the results of Braginskii (1964) to encompass different ordering of the field components and extending the analysis to higher orders of ϵ. This work also derives expressions for $\alpha_{\phi\phi}$ in more generality and provides a valuable insight into the generation of this component. In Soward (1972) the different axisymmetric, perturbation, azimuthal and meridional components of $\boldsymbol{B}$ and $\boldsymbol{v}$ are re-partitioned to explain and extend the analysis of Braginskii (1964). In Soward (1971) it is shown that given $\overline{v}_\phi\mathbf{1}_\phi$ and $\boldsymbol{v}'$, then an axisymmetric velocity $\overline{\boldsymbol{V}}$ could be found so that $\overline{v}_\phi\mathbf{1}_\phi + \boldsymbol{v}' + \overline{\boldsymbol{V}}$ has closed stream lines. Soward (1972) is restricted to $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} = 0$.

As an example of the analysis of Soward (1972) applied to the ordering of Braginskii (1964) the velocity is re-partitioned as

$$\boldsymbol{v} = \overline{v}_\phi\mathbf{1}_\phi + \boldsymbol{v}' + \overline{\boldsymbol{v}}_m \tag{129}$$

$$\boldsymbol{v} = \overline{v}_\phi\mathbf{1}_\phi + \boldsymbol{v}' + \overline{\boldsymbol{V}} + \overline{\boldsymbol{v}}_{em} = \boldsymbol{v}_0 + \overline{\boldsymbol{v}}_{em}, \tag{130}$$

where $\boldsymbol{v}' \sim \mathcal{O}(\epsilon)$, $\overline{\boldsymbol{v}}_m, \overline{\boldsymbol{v}}_{em}, \overline{\boldsymbol{V}} \sim \mathcal{O}(\epsilon^2)$ and $\boldsymbol{v}_0 = \boldsymbol{v}_0(s,\phi,z)$. By comparing (129) and (130), $\overline{\boldsymbol{V}} = \overline{\boldsymbol{v}}_m - \overline{\boldsymbol{v}}_{em} = -\boldsymbol{\nabla}\times(\overline{v}_\phi w\mathbf{1}_\phi)$, discovered in Braginskii (1964).

The process then is to take the leading term in the expansion of $\boldsymbol{v}$ to be the closed-streamline velocity $\boldsymbol{v}_0 = \overline{v}_\phi\mathbf{1}_\phi + \boldsymbol{v}' - \boldsymbol{\nabla}\times(\overline{v}_\phi w\mathbf{1}_\phi)$, rather than $\overline{v}_\phi\mathbf{1}_\phi$. This approach has the advantage that the perturbation field $\boldsymbol{v}'$ is absorbed into the leading $\overline{\boldsymbol{v}}_0$ term, however it produces the added complication of integrating along the closed-streamline paths of $\overline{\boldsymbol{v}}_0$. To address the analysis, a transformation is considered from axisymmetric circles with position vector $\boldsymbol{x}$ to the the closed streamline loops with position vector $\boldsymbol{X}$,

$$\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t). \tag{131}$$

Figure 4 illustrates the transformation (131). The process is to consider that the real magnetic and velocity fields on the closed loops $\boldsymbol{B}(\boldsymbol{X}, t)$ and $\boldsymbol{v}(\boldsymbol{X}, t)$ are transformed from imagined magnetic and velocity fields $\boldsymbol{b}(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{x}, t)$ on the axisymmetric circles.

Restricting $\boldsymbol{X}$ to preserve volume is equivalent to the Jacobian determinant condition, (132a)

$$|\boldsymbol{J}(\boldsymbol{X})| = |\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{X}| = 1, \qquad \boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{\cdot}(\partial_t\boldsymbol{X}) = 0, \tag{132}$$

where $\boldsymbol{\nabla}_{\boldsymbol{x}}$ is the gradient with respect to $\boldsymbol{x}$, etc.

Using (132a) and the imposition (132b), then the transformation given by

$$\boldsymbol{B}(\boldsymbol{X}, t) = \boldsymbol{b}(\boldsymbol{x}, t)\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{X}, \qquad \boldsymbol{v}(\boldsymbol{X}, t) = \partial_t\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{x}, t)\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{X}, \tag{133}$$

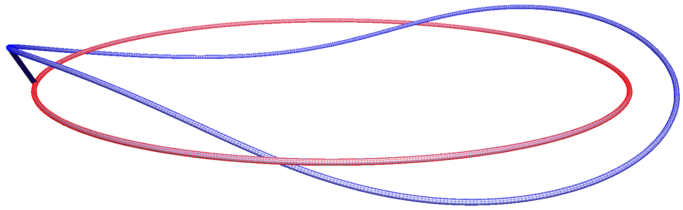

Figure 4. The (blue) 'loop' is helically wound around the (red) axisymmetric circle. The position vector to the loop [circle] is $\boldsymbol{X}$ [$\boldsymbol{x}$]. The transformation to the real magnetic [velocity] field on the loop $\boldsymbol{B}(\boldsymbol{X},t)$ [$\boldsymbol{v}(\boldsymbol{X},t)$] from the imagined magnetic [velocity] field on the circle $\boldsymbol{b}(\boldsymbol{X},t)$ [$\boldsymbol{u}(\boldsymbol{X},t)$] is given by (133a) [(133b)]. The (black) vector from the circle to the loop represents $\boldsymbol{\eta} = \boldsymbol{X} - \boldsymbol{x}$.

preserves the solenoidal magnetic field and the incompressible velocity conditions as,

$$\boldsymbol{\nabla}_{\boldsymbol{X}}\cdot\boldsymbol{B}(\boldsymbol{X},t) = \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{b}(\boldsymbol{x},t) \quad \text{and} \quad \boldsymbol{\nabla}_{\boldsymbol{X}}\cdot\boldsymbol{v}(\boldsymbol{X},t) = \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{u}(\boldsymbol{x},t).$$

Conditions (132) with the transformation (133) are also shown to preserve the advection part of the induction equation as

$$[\partial_t\boldsymbol{b} - \boldsymbol{\nabla}_{\boldsymbol{x}} \times (\boldsymbol{u}\times\boldsymbol{b})]\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{X} = \boldsymbol{\nabla}_{\boldsymbol{X}} \times [\partial_t\boldsymbol{B} - \boldsymbol{\nabla}_{\boldsymbol{X}} \times (\boldsymbol{v}\times\boldsymbol{B})]. \tag{134}$$

The result of tailoring the transformation (133) and using (132) is that the order $\boldsymbol{v}'$ components of order $\epsilon = R^{-1/2}$, in Braginskii (1964) will appear in the transformed diffusion term.

To investigate the resulting diffusion operating on $\boldsymbol{b}(\boldsymbol{x},t)$ the imagined fields are decomposed into axisymmetric and perturbation components as $\boldsymbol{b}(\boldsymbol{x},t) = \overline{\boldsymbol{b}} + \boldsymbol{b}'$, $\boldsymbol{u}(\boldsymbol{x},t) = \overline{\boldsymbol{u}} + \boldsymbol{u}'$ where $\boldsymbol{b}'$, $\boldsymbol{u}' \sim \mathcal{O}(\epsilon^2)$. With this less restrictive condition on the ordering of the component fields, the transformed diffusion operator produces a wide range of effects as given by $\boldsymbol{\Gamma}$ and $\boldsymbol{A}$ in (S2.46)† and (S2.47). In particular, using $\alpha_{\phi\phi} = R\Gamma_{22}$, (S2.44), (S2.45) and (S2.31b), produces new expressions for $\alpha_{\phi\phi}$ for more general, and higher (ϵ) order models.

In view of absorbing $\boldsymbol{v}' \sim \mathcal{O}(\epsilon)$ into $\boldsymbol{X}$, the transformation (SC1) is considered as

$$\boldsymbol{X}(\boldsymbol{x},t) = \boldsymbol{x} + \epsilon\boldsymbol{\eta}(R,\boldsymbol{x},t). \tag{135}$$

To considerably restrict the analysis and reproduce the results of Braginskii (1964) the real field representation of Braginskii (1964) on the loops $\boldsymbol{X}$ needs to be related to the imagined fields on the circles $\boldsymbol{x}$. To this end the real velocity field at $\boldsymbol{x}$ is Taylor expanded about $\boldsymbol{X}$ as

$$\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{v}(\boldsymbol{X},t) - \epsilon\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{v}(\boldsymbol{X},t) + \mathcal{O}(\epsilon^2). \tag{136}$$

The real field $\boldsymbol{v}(\boldsymbol{X},t)$ is then related to the imagined field at $\boldsymbol{x}$, $\boldsymbol{u}(\boldsymbol{x},t)$ by (133b) and (135)

$$\boldsymbol{v}(\boldsymbol{X},t) = \boldsymbol{u}(\boldsymbol{x},t) + \epsilon\{\partial_t\boldsymbol{\eta} + \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{\eta}\}. \tag{137}$$

Using (137) twice in (136) relates the real field to the imagined field and $\boldsymbol{\eta}$, thus

$$\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t) + \epsilon\{\partial_t\boldsymbol{\eta} + \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}(\boldsymbol{x},t)\} + \mathcal{O}(\epsilon^2). \tag{138}$$

Comparing the representation (138) with the ordering of Braginskii (1964), $\boldsymbol{v} = \overline{v}_\phi + \boldsymbol{v}' + \mathcal{O}(\epsilon^2)$ provides a relationship between $\boldsymbol{v}'$, $\overline{v}_\phi$ and $\boldsymbol{\eta}$ to $\mathcal{O}(\epsilon)$ as

$$s\boldsymbol{v}'_m = \left(s\partial_t + \overline{v}_\phi\partial^1_\phi\right)\boldsymbol{\eta}_m \qquad sv'_\phi = \left(s\partial_t + \overline{v}_\phi\partial_\phi\right)\eta_\phi + \eta_s\overline{v}_\phi - s\boldsymbol{\eta}\cdot\boldsymbol{\nabla}\overline{v}_\phi. \tag{139}$$

Using (139), $s\boldsymbol{\nabla}\cdot\boldsymbol{v}' = \overline{v}_\phi\partial_\phi\left(\boldsymbol{\nabla}\cdot\boldsymbol{\eta}\right)$. Thus, using $\boldsymbol{\nabla}\cdot\boldsymbol{\eta} = 0$ reproduces the $\boldsymbol{\nabla}\cdot\boldsymbol{v}' = 0$ condition of Braginkii. The condition (132a), used extensively throughout the analysis, produces $\boldsymbol{\nabla}\cdot\boldsymbol{\eta} = 0$, to leading order.

† Equation (2.46) in Soward (1972) etc.

To reproduce Γ_{Brag} (122), using (S C10) for $|\partial_t \boldsymbol{\eta}| \ll |\partial_{\boldsymbol{x}} \boldsymbol{\eta}|$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\eta} = 0$ then I/we obtain

$$R\Gamma_{22} = \frac{2}{s^3}\langle \eta_s \partial_\phi \eta_z + \partial_\phi(\eta_z)\partial_\phi^2 \eta_s \rangle \\ + \frac{2}{s}\langle \partial_s(\partial_\phi \eta_s)\partial_s(\eta_z) + \partial_z(\partial_\phi \eta_s)\partial_z \eta_z + \frac{1}{s}(\partial_z \eta_z + \partial_s \eta_s)\partial_\phi \eta_z \rangle + \mathcal{O}(\epsilon). \tag{140}$$

Only by including the extra $\partial_s \eta_s$ term in (140) and using (139a) with $|\partial_t \boldsymbol{\eta}| \ll |\partial_{\boldsymbol{x}} \boldsymbol{\eta}|$, do we recover the Γ_{Brag}, (122), of Braginkii.

Soward (1972) produces the most concise and possibly the most intuitive representation for general $\alpha_{\phi\phi} = R\Gamma_{22}$. For $\boldsymbol{X}(\boldsymbol{x}, t)$ given by (133) and (132), using (S2.45) gives $\gamma_{22}(\boldsymbol{X}) = \mu_{22}(\boldsymbol{X})$. Then using (S2.31b) for $\mu_{22}(\boldsymbol{X})$ and $\Gamma_{ij}(s, z, t) = \langle \gamma_{ij}(\boldsymbol{X}) \rangle$ yields

$$\alpha_{\phi\phi} = R\Gamma_{22}(\boldsymbol{X}) = -\frac{R}{2\pi}\int_0^{2\pi} \frac{1}{s}\partial_\phi(\boldsymbol{X})\cdot[\boldsymbol{\nabla}_{\boldsymbol{X}} \times (\frac{1}{s}\partial_\phi \boldsymbol{X})]\mathrm{d}\,\phi. \tag{141}$$

Equation (141) demonstrates that $\alpha_{\phi\phi}$ is an azimuthal average of the helicity of $\partial_\phi(\boldsymbol{X})/s$, and as such is a measure of the rate that the path of $\boldsymbol{X}$ wraps around the path of $\boldsymbol{x}$ as represented in figure 4. This helicity in $\boldsymbol{X}$ represents the departure of the magnetic field from axisymmetry and is responsible, through diffusion, for generating ε_ϕ and $\boldsymbol{B}_m$, and thus can provide a means of circumventing the $\alpha_{\phi\phi} = 0$ ADT proven herein and Cowlings theorem.

The extension of the Braginkii analysis to compressible flow, given by (119)-(125), produces a new *effective* velocity field $\overline{\boldsymbol{v}}_{\boldsymbol{e\rho}}$ (120b), which includes a compressible component due to $\boldsymbol{\rho}$. Thus an extension of the work of Soward (1972) to incorporate compressible flow would need to accomodate the new $\overline{\boldsymbol{v}}_{\boldsymbol{e\rho}}$. To do so would require a new, more general transformation that would need to encompass a new compressible $\boldsymbol{v}'$. As such the volume preserving properties of a new transformation would also require new analysis to ensure the solenoidal condition for the magnetic field is preserved. The extension of Soward (1972) to compressible flow, which would reproduce (119)-(125) as a byproduct, would provide generalised, compressible results to higher order and is the subject of current investigation.

13. Discussions and conclusions

The mean field counterpart to the theorem of Cowling (1934) is proven in section 3. This $\alpha_{\phi\phi} = 0$ antidynamo theorem proves that an axisymetric field will fail in a finite conducting volume V, if just one of the components of $\boldsymbol{\alpha}$, namely $\alpha_{\phi\phi}$, is zero†. By comparing the emf, in (23) then the emf arising from $\overline{\boldsymbol{v}}$ can be expressed as $\overline{\boldsymbol{v}} \times \boldsymbol{B} = \boldsymbol{\alpha_v}\cdot\boldsymbol{B}$, where

$$\boldsymbol{\alpha_v} = \begin{bmatrix} \alpha_{ss} & \alpha_{s\phi} & \alpha_{sz} \\ \alpha_{\phi s} & \alpha_{\phi\phi} & \alpha_{\phi z} \\ \alpha_{zs} & \alpha_{z\phi} & \alpha_{zz} \end{bmatrix} = \begin{bmatrix} 0 & -\overline{v}_z & \overline{v}_\phi \\ \overline{v}_z & 0 & -\overline{v}_s \\ -\overline{v}_\phi & \overline{v}_s & 0 \end{bmatrix}. \tag{142}$$

Thus, stated in terms of $\boldsymbol{\alpha_v}$, Cowling's theorem requires the six conditions,

$$\alpha_{\phi s} = -\alpha_{s\phi}, \quad \alpha_{sz} = -\alpha_{zs}, \quad \alpha_{z\phi} = -\alpha_{\phi z}, \quad \alpha_{ss} = 0, \quad \alpha_{zz} = 0, \tag{143}$$

and $\alpha_{\phi\phi} = 0$. The $\alpha_{\phi\phi} = 0$ ADT reduces these six conditions to just one; $\alpha_{\phi\phi} = 0$.

Indeed once the $\alpha_{\phi\phi} = 0$ ADT is established then Cowlings theorem can be proven by simply identifying that for laminar $\overline{v}$, the $\alpha_{\phi\phi}$ component of $\boldsymbol{\alpha_v}$ is zero (142) and Cowling's theorem is thus proven.

†Again, it is not necessary to specify that $\boldsymbol{\alpha}$ be axisymmetric as violation of this condition would destroy the axisymmetric magnetic field condition.

Of course, subsections 3.1 and 3.2 use theory that has been specifically developed, in the present work and Ivers and James (1984), to demonstrate the decay of χ for (30) in V and (31) in $\widehat{V}$, and for the decay of $b = \overline{B}_\phi/s$ in (42). However, this theory can be established independently of Cowling's theorem.

Because the theory developed herein demonstrates that all of the five conditions of (143) are redundant for the proof of the $\alpha_{\phi\phi} = 0$ ADT. And because Cowling's theorem can be proven using the $\alpha_{\phi\phi} = 0$ ADT, then the conditions (143) are superfluous, even irrelevant for the proof Cowling's Theory and many of its extensions.

Furthermore, if just this one $\alpha_{\phi\phi} = 0$ condition is satisfied then none of the remaining α_{ss}, $\alpha_{s\phi}$, α_{sz}, $\alpha_{\phi s}$, $\alpha_{\phi z}$, α_{zs}, $\alpha_{z\phi}$, α_{zz}, components can prevent the eventual decay of an axisymetric field. For instance, even though the $\alpha_{\phi s}$ and $\alpha_{\phi z}$ components can reproduce meridional field from meridional field (see (28) and (29)), this is insufficient to sustain the dynamo. Likewise, $\alpha_{s\phi}$ and $\alpha_{z\phi}$ can reproduce azimuthal field from azimuthal field (see (42) and (44)). However again, this alone will not indefinitely sustain an axisymmetric dynamo (see, for instance, section 9 for model 1). Also none of the α_{ss}, α_{sz}, α_{zs} α_{zz} components appear in (28) or (42) and hence do not provide regenerative maintenance in axisymmetry. The spectral interaction equations for the $\alpha_{\phi\phi}$ component and all others were first derived in Phillips (1993).

From the discussion above, the physical interactions that generate $\alpha_{\phi\phi}$ provide valuable insight into the critical mechanisms for axisymmetric dynamo maintenance. To examine these mechanisms a number of different analyses that produce $\alpha_{\phi\phi}$ are explored.

The derivation of $\alpha_{\phi\phi}$ using the second order correlation approximation and the Green's function analysis in section 10, shows that $\alpha_{\phi\phi}$ is generated through the interactions of the fluctuating meridional velocity $\boldsymbol{v}'_m$ and gradients of $\boldsymbol{v}'_m$, (see, for instance, (94)). Moreover, for the conditions of (94), it is the cross-correlation between the orthogonal, s and z components and gradients, which generate $\alpha_{\phi\phi}$. Of course $\alpha_{\phi\phi}$ can also be generated from other mechanisms such as from $\overline{\boldsymbol{v}}$ through the green's function for non-zero mean flow (see, (8) and (10)) or from $\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\boldsymbol{v}}_m$.

This analysis also demonstrates circumstances under which this $\alpha_{\phi\phi}$ vanishes. One such case is when all members of the ensemble of perturbation $\boldsymbol{v}'$ are co-axisymmetric with $\overline{\boldsymbol{B}}$ in E^3, then $\alpha_{\phi\phi} = 0$ (see table 1).

New $\alpha_{\phi\phi}$ expressions are derived in section 11 using the methods of Braginskii (1964). This independent method of using a decomposition of $\boldsymbol{B}$ and $\boldsymbol{v}$ into axisymmetric and 'purely variable in ϕ' components, and an expansion of order $\epsilon = (\eta/L\mathcal{V})^{1/2}$, produces different $\alpha_{\phi\phi}$ expression and regenerative mechanism. For the assumption that all components of the velocity are incompressible (99) then $\alpha_{\phi\phi} = \Gamma_{\mathrm{Brag}}{}^\dagger$, (122) for the *effective* fields (120) with $\boldsymbol{\rho} = \boldsymbol{0}$.

To extend this analysis to the compressible plasmas of stelar interiors the work of Braginskii (1964) is generalised by relaxing the conditions of incompressibility (99) for all components of $\boldsymbol{v}$. For the present analysis the stellar interactions would need to be nearly axisymmetric. However, the extension of this work to different ordering of field components using the methods of Soward (1972) could relax this condition. For this extension of the analysis, an $\alpha_{\phi\phi}$ is again generated where $\alpha_{\phi\phi} = \Gamma_e$ (121), together with a new regenerative mechanism for $\overline{B}_\phi$ given by $\Psi\overline{B}_\phi$; see (124), (125).

This independent method of producing $\alpha_{\phi\phi} = \Gamma_e$ demonstrates that this effect can be generated by contributions from many sources such as $\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\boldsymbol{v}}_m$, $\boldsymbol{\nabla}\boldsymbol{\cdot}\widehat{\boldsymbol{v}}'$, $\boldsymbol{\nabla}\overline{v}_\phi$. However, one dominating generation mechanism is the interaction of $\boldsymbol{u}'_m = \boldsymbol{v}'_m/\overline{v}_\phi$ with the primitives (antiderivatives), means and gradients of $\boldsymbol{u}'_m$ (see (122)).

These two different approaches demonstrate that a common generation mechanism for this

†To simplify discussion the Γ_e and Γ_{Brag} in section 13 are the scaled counterparts using (21).

critical $\alpha_{\phi\phi}$ component is the interaction of the meridional perturbation velocities[‡] with their gradients and antiderivatives.

The core mechanisms that generate these $\alpha_{\phi\phi}$ interactions for these different approaches can be more clearly demonstrated for specific physical conditions. For instance, using the SOCA, and the Green's function in the asymptotic limit $G(\boldsymbol{\xi}, \tau) = \delta^3(\boldsymbol{\xi})\delta(\tau)$, (96) is

$$\alpha_{\phi\phi} = \frac{1}{s}\overline{[v'_s \partial_\phi v'_z - v'_z \partial_\phi v'_s]} = \overline{\boldsymbol{v}'_m \boldsymbol{\cdot} \boldsymbol{\nabla} \times \boldsymbol{v}'_m}. \tag{144}$$

Using the alternative approach of an azimuthal average, expansion in order ϵ, assumptions (99), (106), (107), as in Braginskii (1964), then the first two terms of (122) give

$$\alpha_{\phi\phi} = \frac{1}{s}\langle [\boldsymbol{u}'_m \times \widehat{\boldsymbol{u}}'_m]_\phi \rangle + \frac{1}{s}\langle [\boldsymbol{u}'_m \times \partial^1_\phi \boldsymbol{u}'_m]_\phi \rangle = \langle \widehat{\boldsymbol{u}}'_m \boldsymbol{\cdot} \boldsymbol{\nabla} \times \widehat{\boldsymbol{u}}'_m \rangle - \langle \boldsymbol{u}'_m \boldsymbol{\cdot} \boldsymbol{\nabla} \times \boldsymbol{u}'_m \rangle. \tag{145}$$

The most concise insight into the generation mechanism for this $\alpha_{\phi\phi}$ is given using Soward (1972) as

$$\alpha_{\phi\phi} = -R\langle \frac{1}{s}\partial_\phi(\boldsymbol{X}) \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{X}} \times (\frac{1}{s}\partial_\phi \boldsymbol{X}) \rangle. \tag{146}$$

Equations (144), (145) and (146) demonstrate that, whether we use; an ensemble mean, the SOCA, and a Greens function analysis; an azimuthal average, expansion of order ϵ and adopt '*effective* fields; or we use a transformation $\boldsymbol{X}$ to incorporate the perturbation fields, this critical $\alpha_{\phi\phi}$ can be generated by the mean helicity of meridional perturbation velocities, that is either $\boldsymbol{v}'_m$ in (144), or $\widehat{\boldsymbol{u}}'_m$ or $\boldsymbol{u}'_m$ in (145) or the transformation $\boldsymbol{X}$ as a means of representing non-axisymmetric perturbation components of the field and flow. The different derivations of $\alpha_{\phi\phi}$ demonstrates the critical importance of the helicities of the meridional perturbation velocities for circumventing either Cowling's ADT or the $\alpha_{\phi\phi} = 0$ ADT derived herein.

The derivation of the $\alpha_{\phi\phi} = 0$ ADT is the first mean field ADT which derives conditions for $\boldsymbol{\alpha}$ under which an axisymmetric dynamo will fail within a finite conductor. All other mean field ADT's are proven for infinite conductors filling all space (see table 1). Thus this $\alpha_{\phi\phi} = 0$ ADT establishes the first conditions for a finite conductor where the magnetohydrodynamic interactions may produce no axisymmetric field. Thus this is the first mean field ADT that give mechanism where a turbulent conducting fluid may have no accompanying axisymmetric magnetic field. And as such may provide an insight into the physical mechanism within conducting fluids that produce no observable magnetic fields such as non-magnetic stars.

Conversely the analysis of the generating mechanisms for producing $\alpha_{\phi\phi}$ may provide an insight into the physical mechanisms that save a mean field axisymmetric dynamo from failing.

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[‡]Normalised as $\boldsymbol{u}'_m = \boldsymbol{v}'_m/\overline{v}_\phi$ in the case Γ_e.

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Appendix A:

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Appendix B:

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