# ON IDENTITIES OF REES QUOTIENTS OF FREE INVERSE SEMIGROUPS DEFINED BY POSITIVE RELATORS

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Abstract. We give a complete description of Rees quotients of free inverse semigroups given by positive relators that satisfy nontrivial identities, including identities in signature with involution. They are finitely presented in the class of all inverse semigroups. Those that satisfy a nontrivial semigroup identity have polynomial growth and can be given by an irredundant presentation with at most four relators. Those that satisfy a nontrivial identity in signature with involution, but which do not satisfy a nontrivial semigroup identity, have exponential growth and fall within two infinite families of finite presentations with two generators. The first family involves an unbounded number of relators, and the other involves presentations with at most four relators of unbounded length. We give a new sufficient condition for which a finite set $X$ of reduced words over an alphabet $A \cup A^{-1}$ freely generates a free inverse subsemigroup of $FI_A$ and use it in our proofs.

## 1. Introduction

Presentations involving generators and relations may be viewed from different standpoints, depending on the algebraic context, with sometimes surprising outcomes and connections. A example of this would be Garside's early discovery [13] that the (positive) braid monoid embeds in the braid group, using monoid and group presentations, respectively, involving identical (Coxeter type A) relations that avoid inverses of letters from the underlying alphabet. Quite spectacularly, Paris [27], relying on results of Crisp [6], generalised Garside's result to show that all Artin monoids embed into Artin groups, now using Coxeter relations of any type, where again the relations in the respective monoid and group presentations are identical and only involve positive words (with alternating letters) that avoid formal inverses. In different directions, G. Baumslag [3] studied groups defined by one positive relator and showed that they are residually solvable, and, recently, Isherwood and Williams [18] investigate the Tits alternative for cyclically presented groups with positive relators of length four (and see also [12, 25] for cyclically presented groups with positive relators of length three). The presentation

$$\langle a, b \mid ab = 1 \rangle \ , \tag{1}$$

involves a positive relator of length two and yields an infinite cyclic group, as a group presentation, and clearly remains unchanged, as a group, by considering its cyclically presented counterpart $\langle a, b \mid ab = ba = 1 \rangle$. However, as a monoid presentation, (1) produces the bicyclic monoid, which does not embed in a group, and is in fact a well studied bisimple

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inverse semigroup that satisfies the celebrated semigroup identity

$$xy^2xxyxy^2x \;=\; xy^2xyxxy^2x \tag{2}$$

due to Adjan [1]. Note also in [1] (see monograph [2] with detailed proofs), Adjan introduced the class of finitely generated special monoids given by presentations whose right-hand sides are empty words, and established that if a special monoid satisfies a nontrivial identity, then it is a group or isomorphic to a bicyclic monoid or free monogenic monoid $\langle a, b \mid b = 1\rangle$. It follows from a result of Scheiblich [31] that the bicyclic monoid and the free monogenic inverse semigroup satisfy the same identities (see [33, Proposition 2.1]). This fact opens up possible connections with presentations of inverse semigroups with zero, and is exploited in [33, Sections 3 and 5], where the authors demonstrate that all finitely presented Rees quotients of free inverse semigroups have polynomial growth if and only if they satisfy a single, but complicated, semigroup identity closely related to Adjan's identity (2). (In fact, as demonstrated in Theorem 6.2 below, if, in addition to having polynomial growth, the relators are all positive words, everything simplifies and the identity (2) alone suffices.) By contrast, Izhakian and Margolis [19] prove that the monoid of $2 \times 2$ tropical matrices satisfies an identity closely related (2), but twice as long, and its submonoid of upper triangular matrices satisfies (2). Moreover, Daviaud, Johnson and Kambites [8] prove that the identities which hold in the monoid of $2 \times 2$ upper triangular tropical matrices are exactly the same as those which hold in the bicyclic monoid. Another variation is given by Kubat and Okninski [20], who prove that the plactic monoid of rank three, associated in a natural way with generalisations of the bicyclic monoid, satisfies a very long identity closely related to (2).

In 1996, the authors [33] initiated the study of asymptotic behaviour of the class of finitely presented Rees quotients of free inverse semigroups. They showed that all semigroups from that class have polynomial or exponential growth and proved that the type of growth is algorithmically recognisable, giving a precise anologue of a classical result due to Ufnarovsky [36] and Gilman [14] in the study of finitely presented associative monomial algebras, using the de Bruijn graph of a presentation [7]. From the point of view of inverse semigroups with zero, it is natural to replace 1 by 0 in (1) and then consider

$$\mathrm{Inv}\langle a, b \mid ab = 0\rangle \tag{3}$$

regarded as a presentation as an inverse semigroup with zero (and denoted by $S_3$ in the sequel below). By contrast with the bicyclic monoid, the inverse semigroup defined by (3) satisfies no nontrivial semigroup identity, because it is easily seen to contain a nonmonogenic free subsemigroup (see for example [10, Section 1]). However, the authors prove ( [10, Theorem 3.8], and see Theorem 2.1 below) that this inverse semigroup is the unique principal Rees quotient of a free inverse semigroup that is not trivial or monogenic with zero satisfying a nontrivial identity in signature with involution.

Our main goal in this paper is to give a complete description of Rees quotients of free inverse semigroups given by positive relators that satisfy nontrivial identities, including identities in signature with involution. It turns out, though this is not obvious, that all such inverse semigroups are finitely presented in the class of all inverse semigroups, so that we can use our methods to investigate connections with growth and look for complete syntactic descriptions of their presentations. Theorem 6.2 below is the main result concerning inverse semigroups from our class that satisfy a nontrivial semigroup identity. Theorem 7.10 below is the main result concerning inverse semigroups from our class that satisfy a nontrivial identity in signature with involution, but which do not satisfy a nontrivial semigroup identity. This final result relies upon and may be regarded as a development of the authors' work in [10].

This paper is organised as follows. Section 2 includes preliminaries, setting up the notation for presentations of Rees quotients of free inverse semigroups, which defines a general class $\mathfrak{C}$ of inverse semigroups with zero. The subclass $\mathfrak{P}$, consisting of semigroups given by presentations where all relators are positive words over the underlying alphabet, is the focus of attention for Sections 4 to 7. In Section 3, which may be of independent interest, we develop a sufficient condition that guarantees that a subset of a free inverse subsemigroup, consisting of reduced words, freely generates an inverse subsemigroup, relying on Reilly's criterion [28, 29]. In Section 4, we provide details, including illustrations of Ufnarovsky graphs, of several inverse semigroups with zero that play important roles in the theorems of later sections. In Section 5, which is short, we make a major reduction, by showing that inverse semigroups with zero from the class $\mathfrak{P}$ that satisfy nontrivial identities in signature with involution must be generated by at most two elements. In Section 6, we give several characterisations of inverse semigroups with zero from the class $\mathfrak{P}$ that satisfy a nontrivial semigroup identity. This includes a syntactic description involving only two finitely presented semigroups, using three and four relators respectively, and both of which have polynomial growth. In Section 7, the final section, we give characterisations and syntactic descriptions of all inverse semigroups with zero from the class $\mathfrak{P}$ that satisfy a nontrivial identity in signature with involution, but which do not satisfy any nontrivial semigroup identity. All semigroups that arise are finitely presented, have exponential growth and fall within two main classes, the first of which can involve presentations with arbitrarily many relators, and another infinite class with at most four relators, but of arbitrary length. In the latter class, pairs of relators appear, reminiscent of Coxeter relations, which involve words and their reversals that alternate between two letters.

## 2. Preliminaries

We assume familiarity with definitions and elementary results from the theory of semigroups, which may be found in any of [4], [15], [17] or [21]. Throughout let $A$ be a nonempty alphabet and put

$$B \;=\; A \cup A^{-1}$$

where the elements of $A^{-1}$ are formal inverses of corresponding elements of $A$ and vice-versa (so $A$ and $A^{-1}$ are disjoint and any $a$ in $A$ may also be denoted by $(a^{-1})^{-1}$). Let $I$ be an indexing set and suppose that $c_i \in B^+$ for each $i \in I$. Consider the inverse semigroup $S$ with zero given by the following presentation:

$$S \;=\; \mathrm{Inv}\langle\, A \mid c_i = 0 \;\text{ for }\; i \in I \,\rangle\,.$$

In this paper we only consider presentations within the class of inverse semigroups with zero and abbreviate the notation slightly to write

$$S \;=\; \langle\, A \mid c_i = 0 \;\text{ for }\; i \in I \,\rangle\,. \tag{4}$$

The words $c_i$ are called (*zero*) *relators*. Observe that $S$ may be regarded as (isomorphic to) the Rees quotient of the free inverse semigroup $FI_A$ generated by $A$ with respect to the ideal generated by the relators. Denote by $\mathfrak{C}$ the class of all inverse semigroups with zero given by presentations of type (4).

In the case that both $A$ and $I$ are finite and nonempty, say $I = \{1, \ldots, L\}$ where $L$ is a positive integer, then (4) may be written as a finite presentation:

$$S \;=\; \langle\, A \mid c_i = 0 \;\text{ for }\; i = 1, \ldots, L \,\rangle\,. \tag{5}$$

It could be noted that the simplest nontrivial example of (5) would be

$$\langle\, a \mid a^2 = 0 \,\rangle\,,$$

which defines the five-element Brandt inverse semigroup. The class of finitely presented inverse semigroups with zero defined by presentations (5) may now be formally referred to as $\mathfrak{M}_{FI}$, and has been the subject of investigations, from the point of view of growth and identities, in previous articles by the authors [10, 11, 33–35]. In particular, the reader is referred to Section 2 of [35] for a comprehensive summary of terminology, notation and past results relied upon in this paper.

We say that the inverse semigroup $S$ with zero is given by a *positive* presentation if the relators in (4) are all words over $A$ (that is avoiding letters from $A^{-1}$). The class of all such semigroups, given by positive presentations, is denoted by $\mathfrak{P}$. We say that a presentation of a semigroup from class $\mathfrak{P}$ is *irredundant* if none of the generators is a relator and no relator is a factor of any of the others. In Section 4, we focus on some particular examples from $\mathfrak{P}$ that play critical roles in the results in Sections 6 and 7 respectively. One of them is the inverse semigroup with zero given by the presentation

$$S_3 \;=\; \langle a, b \mid ab = 0 \rangle$$

mentioned in the introduction. The following theorem then becomes a slight adjustment of [10, Theorem 3.7], noting that in presentations of classical algebras, such as groups, associative rings, semigroups or monoids, if a generating set is infinite but the number of relations is finite, then there is always a free subalgebra on two generators (so that, in the case of an inverse semigroup presentation, no nontrivial identity in signature with involution can be satisfied).

**Theorem 2.1.** *The inverse semigroup with zero*

$$S_3 \;=\; \langle a, b \mid ab = 0 \rangle$$

*is the unique semigroup, up to isomorphism, from the class $\mathfrak{C}$ (so, in particular also from the class $\mathfrak{P}$), that has exactly one relator, is not trivial and not monogenic with zero and satisfies a nontrivial identity in signature with involution, in fact, the following identity:*

$$PQ \;=\; QP$$

*where*

$$P = P(x,y) = [yx^2y, x^{-1}y^{-2}x^{-1}] \qquad \text{and} \qquad Q = Q(x,y) = [y^{-1}x^2y^{-1}, x^{-1}y^2x^{-1}]\,.$$

## 3. Free generation of inverse subsemigroups

In this section (Proposition 3.2 below), we provide a sufficient condition for a subset of a free inverse semigroup to become a free generating set, namely that all elements of the subset be reduced words such that the length of each word exceeds the sum of the lengths of a largest prefix and a largest suffix shared by this word with other words in the set. This condition is applied in two cases below (Lemma 5.1 and 7.9), as stepping stones towards developing our main theorems in Sections 6 and 7.

We first recall Reilly's criterion [28, Theorem 2.2] [29] for identifying free generating sets in inverse semigroups:

> *If $K$ is a subset of an inverse semigroup $S$ then $K$ is a set of free generators for the inverse subsemigroup $K$ generates if and only if $K \cap K^{-1}$ is empty and if $y \in K \cup K^{-1}$ and $yy^{-1}$ is annihilated by a product of idempotents of the form $w_1 w_1^{-1} \dots w_n w_n^{-1}$ where $w_1, \dots, w_n$ are reduced words with respect to the alphabet $K$, then $y$ is the first letter of $w_j$ for some $j$.*

In the previous statement, consider the case that $S = FI_A$ happens to a free inverse semigroup with respect to some alphabet $A$. Suppose $y$ is a reduced word over $A$ such that $yy^{-1}$ is annihilated by a product of idempotents. Then $y$ labels some geodesic path from the initial vertex of the Munn tree of that product of idempotents to some vertex, which lies in the Munn tree of one of the idempotents. But this unique geodesic path must appear also in that particular Munn tree, since it contains the initial vertex of the Munn tree of the product of idempotents. Hence $yy^{-1}$ is in fact annihilated by just one of the idempotents used to form the product. Note also that if $K$ is a subset of $FI_A$ consisting of reduced words only, then $K \cap K^{-1}$ is empty automatically. Hence we have the following simplification of Reilly's criterion in this special case:

> *If $K$ is a subset of a free inverse semigroup $FI_A$ with respect to some alphabet $A$, such that all elements of $K$ are reduced, then $K$ is a set of free generators for the inverse subsemigroup $K$ generates if and only if, for all $y \in K \cup K^{-1}$, if $yy^{-1}$ is annihilated by an idempotent $ww^{-1}$ where $w$ is some reduced word with respect to the alphabet $K$, then $y$ is the first letter of $w$.*

The following terminology adapted from [11, Section 3] will be used in the proof of Proposition 3.2 below. If $w$ is a word over $B$ then we say $w$ *has the whisker property* if the word tree $T(w)$ contains an underlying chain $\mathcal{C}$ of vertices such that (i) each leaf vertex of $\mathcal{C}$ is a leaf of $T(w)$, (ii) each leaf of $T(w)$ that is not a leaf of $\mathcal{C}$ is connected to $\mathcal{C}$ by a chain of edges, (iii) degrees of vertices of $T(w)$ are only allowed to be 1, 2 or 3, and (iv) vertices of degree 3 only occur on $\mathcal{C}$. These connecting chains are called *whiskers* and are themselves word trees of reduced words.

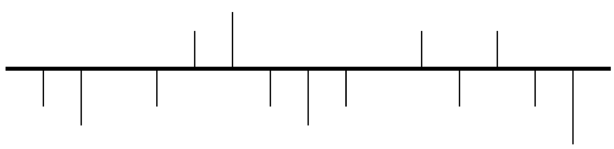

Figure 1. Word tree with whisker property

In the diagram above the thick line is intended to represent the underlying chain $\mathcal{C}$ and the thin lines represent the whiskers. Whiskers may be of varying length. The definition includes the possibility that $T(w)$ coincides with the chain $\mathcal{C}$, in which case there are no whiskers. This would occur, for example, if $w$ is a reduced word.

*Example* 3.1. Let $A = \{a, b, c\}$ and consider the following words over $B$:

$$u = a^2b^3 , \quad u_1 = b^{-2}c^4 , \quad u_2 = c^{-2}a^2 , \quad u_3 = a^{-1}b^3 .$$

Put

$$w = uu_1u_2u_3 = a^2b(b^2b^{-2})c^2(c^2c^{-2})a(aa^{-1})b^3 .$$

Then the word tree $T(w)$ has the whisker property (see Figure 2), with an underlying chain $\mathcal{C}$ traversed by the reduced part of $w$, which is the word $a^2bc^2ab^3$, and three whiskers traversed by $b^2$, $c^2$ and $a$ respectively. It is straightforward to check that condition (6) in the statement of Proposition 3.2 below is satisfied by $X = \{u, u_1, u_2, u_3\}$, so, in fact, $X$ freely generates a free inverse subsemigroup of $FI_A$.

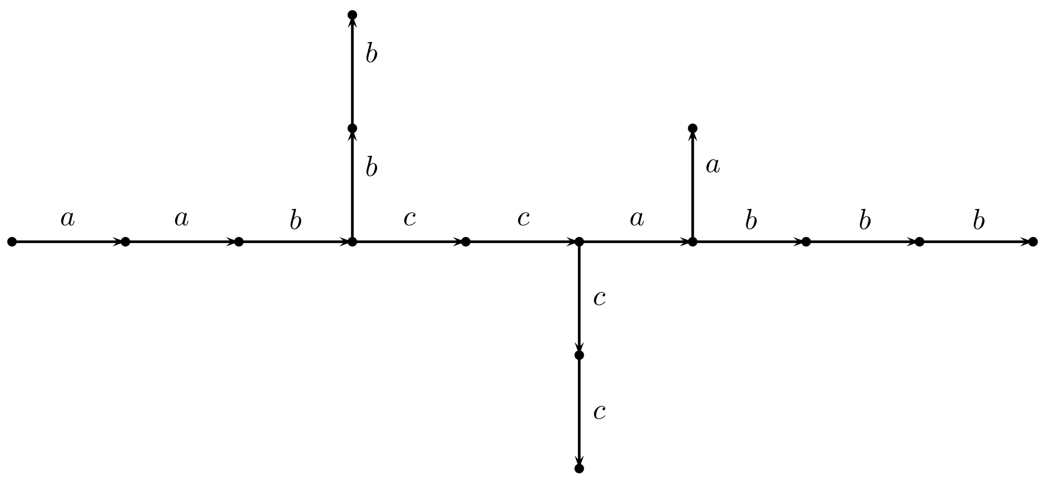


FIGURE 2. Word tree of $w = uu_1u_2u_3$

We now provide the following sufficient condition for free generation of subsets of free inverse semigroups, which becomes routine to check in our applications below (Lemmas 5.1 and 7.9 below):

**Proposition 3.2.** *Let $FI_A$ be the free inverse semigroup over an alphabet $A$. Let $X$ be a nonempty subset of $FI_A$ consisting of reduced words such that $X \cap X^{-1}$ is empty. For each $x \in X \cup X^{-1}$ let $m_x$ be the maximum length of any prefix of $x$ which $x$ has in common with any other element of $X \cup X^{-1}$, and let $M_x$ be the maximum length of any suffix of $x$ which $x$ has in common with any other element of $X \cup X^{-1}$. Suppose that the following condition holds:*

$$m_x + M_x < \ell(x) \text{ for all } x \in X, \text{ where } \ell(x) \text{ is the number of letters in } x. \tag{6}$$

*Then $X$ is a free generating set for the inverse subsemigroup generated by $X$.*

*Proof.* Put $Y = X \cup X^{-1}$, which is a disjoint union by assumption. For each $x \in Y$, denote by $x\varphi$ the prefix of $x$ obtained by deleting from $x$ the suffix of length $M_x$. Observe, by (6), firstly, that

$$x \text{ is not a prefix of } y \text{ for all distinct } x, y \in Y , \tag{7}$$

and, secondly, that

$$x\varphi \text{ is not a prefix of } y\varphi \text{ for all distinct } x, y \in Y . \tag{8}$$

For any $x, y \in Y$, such that $y \neq x^{-1}$, denote by $P(x, y)$ the length of the largest prefix of $y$ in common with $x^{-1}$. Hence, by (6),

$$P(x, y) \;\leq\; m_y \;<\; \ell(y) - M_y .$$

Let $u, u_1, \ldots, u_n \in Y$ and suppose that $w = uu_1 \ldots u_n$ is a reduced word with respect to $Y$. We will show that the word tree of $w$ has the *whisker property* (as explained above, before Figure 1), such that an initial segment of the underlying chain $\mathcal{C}$ is the word tree of $u\varphi$. This is certainly true if $n = 0$, because the word tree of a reduced word is a chain, so we may assume $n \geq 1$. Note that $u^{-1}$ and $u_1$ are different words, because $w$ is reduced with respect to $Y$. Hence, by (7), we have that $u^{-1}$ is not a prefix of $u_1$. Put $e_1 = \ell(u)$ and

$$d_1 \;=\; \ell(u) - P(u_1, u) .$$

Then $d_1$ is positive and the initial vertex of the Munn tree of $uu_1$ must be a leaf. Note that $d_1 = \ell(u)$ if and only if $uu_1$ is reduced. Now put

$$e_2 \;=\; \ell(u_1) - P(u_1, u) ,$$

which is positive by condition (6). Observe that the word tree $T(u)$ is a chain of length $e_1$ and $T(uu_1)$ is a word tree with the whisker property, having an underlying chain of length $d_1 + e_2$, with at most one whisker, of length $P(u_1, u)$, when $P(u_1, u)$ is positive, attached to the chain at the node located $d_1$ edges from the initial vertex (see Figure 3).

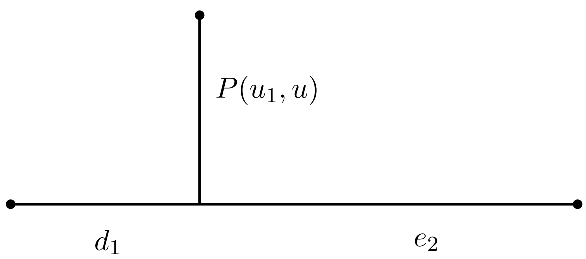


FIGURE 3. Word tree of $uu_1$

This is the start of an inductive process. If $n = 1$ then we have finished, so suppose that $n \geq 2$. Put $u_0 = u$. For $2 \leq i \leq n$, put

$$d_i \;=\; \ell(u_{i-1}) - P(u_{i-1}, u_{i-2}) - P(u_i, u_{i-1})$$

and, for $2 \leq i \leq n+1$, put

$$e_i \;=\; \ell(u_{i-1}) - P(u_{i-1}, u_{i-2}) ,$$

noting that this extends the definition of $e_2$, and both $d_i$ and $e_i$ are positive, by condition (6). Note also that, for $1 \leq i \leq n$,

$$e_i \;=\; d_i + P(u_i, u_{i-1})\ ,$$

and $e_i = d_i$ if and only if the word $u_{i-1}u_i$ is reduced with respect to $A$. Suppose, as inductive hypothesis, that $1 \leq k < n$ and the word tree $T(uu_1 \ldots u_k)$ has the whisker property with an underlying chain $\mathcal{C}_k$ of length

$$d_1 + \ldots + d_k + e_{k+1}$$

with at most $k$ whiskers, being the number of elements $i$ such that $1 \leq i \leq k$ and $u_{i-1}u_i$ is reducible with respect to $A$. All whiskers of lengths amongst the positive values of

$$P(u_1, u_0)\ ,\ P(u_2, u_1)\ ,\ \ldots,\ P(u_k, u_{k-1})$$

are attached to the nodes located

$$d_1\ ,\ d_1 + d_2\ ,\ \ldots,\ d_1 + \ldots + d_k$$

edges respectively from the initial vertex of $\mathcal{C}_k$. Observe that $u_k^{-1}$ cannot be a prefix of $u_{k+1}$, by (7), since $u_k^{-1}$ and $u_{k+1}$ are different words, as $w$ is reduced with respect to $Y$. We modify the chain $\mathcal{C}_k$ by removing edges corresponding to the suffix of $u_k$ of length $P(u_{k+1}, u_k)$ and then adding edges corresponding to the suffix of $u_{k+1}$ obtained by deleting its prefix (in common with $u_k^{-1}$) of length $P(u_{k+1}, u_k)$. This produces a new chain $\mathcal{C}_{k+1}$ of length

$$d_1 + \ldots + d_k + d_{k+1} + e_{k+2}\ .$$

We can now reinstate the edges corresponding to the common prefix of $u_k^{-1}$ and $u_{k+1}$ to add a whisker to the tree of length $P(u_{k+1}, u_k)$, in the case that this length is positive. This completes the construction of the word tree of $T(uu_1 \ldots u_k u_{k+1})$, which again has the whisker property, and establishes the inductive step. Figure 4 illustrates the final outcome of this construction. In the illustration, the whiskers are drawn with varying lengths, though of course whiskers could have the same lengths, and whiskers will not exist at nodes where the concatenation of consecutive words yields a reduced word.

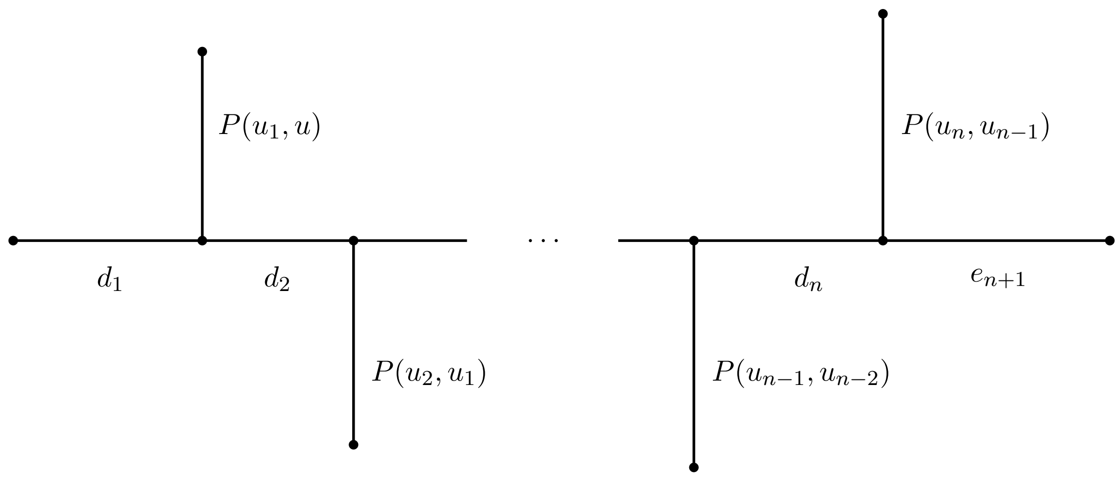


FIGURE 4. Word tree of $w = uu_1 \ldots u_n$

This shows that the word tree of $w = uu_1 \dots u_n$ has the whisker property. The initial segment of the underlying chain, up to and including the node to which the first whisker is attached, or the entire chain if there are no whiskers, becomes the Munn tree of a reduced word for which $u\varphi$ is a prefix, with the initial vertex being the initial vertex of the underlying chain.

Suppose now that $y \in Y$ and $yy^{-1}$ is annihilated by an idempotent $ww^{-1}$, where $w = uu_1 \dots u_n$ is a reduced word with respect to the alphabet $Y$. We want to show that $y = u$. Suppose to the contrary that $y \neq u$ and denote by

$$(T, \alpha, \alpha) \;=\; \big(T(w), \alpha(w), \alpha(w)\big)$$

the Munn tree of $w$ (where the initial and terminal vertices coincide). By what we proved, $T$ has the whisker property with respect to an underlying chain, where $\alpha$ is the initial vertex of this chain, and the initial segment of this chain, up to and including the vertex where the first whisker is attached, or the entire chain if there are no whiskers, extends (and possibly equals) the Munn tree of $u\varphi$. Denote by $(T', \alpha, \alpha)$ the Munn tree of the idempotent $(u\varphi)(u\varphi)^{-1}$, which typically would use just a fragment of the underlying chain of $T$. Since $yy^{-1}$ is annihilated by $w$, we have that $y\varphi$ labels some unique path in $T$ emanating from $\alpha$. Hence either (i) this path exhausts $T'$, so that $u\varphi$ is a prefix for $y\varphi$, or (ii) $y\varphi$ labels a path that terminates inside $T'$, so that $y\varphi$ is a prefix of $u\varphi$. Both cases (i) and (ii) contradict (8), since $y$ and $u$ are different words. Hence, in fact, $y = u$, so that Reilly's criterion is satisfied, and $X$ becomes a free generating set for the inverse subsemigroup of $FI_A$ that it generates. This completes the proof of the proposition. □

Condition (6) is equivalent to condition (N2) in the definition of a Nielsen basis [23, Chapter 1], so we have immediately the following:

**Corollary 3.3.** *Let $F(A)$ be the free group over a finite alphabet $A$. Let $X$ be a finite subset of $F(A)$ consisting of reduced words. If $X$ is a Nielsen basis then $X$ freely generates an inverse subsemigroup of the free inverse semigroup $FI_A$.*

Another related sufficient condition for a finite subset $X$ of reduced words over a finite alphabet $A \cup A^{-1}$ to be a basis for a free inverse subsemigroup $\mathrm{Inv}\langle X\rangle$ of $FI_A$ was found by Margolis and Meakin [24, remark following Theorem 3.6], who considered closed finitely generated inverse subsemigroups. We also mention their example [24, p. 88], the set $X = \{ab, ac, bc\}$, which freely generates a free inverse subsemigroup over the alphabet $\{a, b, c\}$. Since $X$ does not satisfy condition (6), this condition is not necessary in general for free generation of an inverse subsemigroup.

## 4. Examples

In this section, we provide details, including the graphs, of particular inverse semigroups with zero, namely $S_1$, $S_2$, $S_3$, $S_4$ and $S_8(1)$, which play key roles in the two main results of this paper (Theorems 6.2 and 7.10 below).

*Example* 4.1. The following inverse semigroups play a critical role in describing and characterising semigroups in the class $\mathfrak{P}$ that satisfy a nontrivial semigroup identity (Theorem

6.2 below):

$$S_1 = \langle a, b \mid a^2 = b^2 = ab = 0 \rangle \quad \text{and} \quad S_2 = \langle a, b \mid a^2 = b^2 = ab = ba = 0 \rangle \ .$$

They are both homomorphic images of the following semigroups:

$$S_3 = \langle a, b \mid ab = 0 \rangle \quad \text{and} \quad S_4 = \langle a, b \mid a^2 = b^2 = 0 \rangle \ ,$$

which, in turn, play a critical role in describing semigroups from $\mathfrak{P}$ that do not satisfy a nontrivial semigroup identity, but satisfy a nontrivial identity in signature with involution (Theorem 7.10 below). It could be noted that the presentation for $S_2$ would reduce to the three element null semigroup, if the presentation were interpreted within the class of all semigroups with zero.

The inverse semigroup $S_1$ has the following Ufnarovsky graph, leading to a detailed analysis in [11, Example 2.2]:

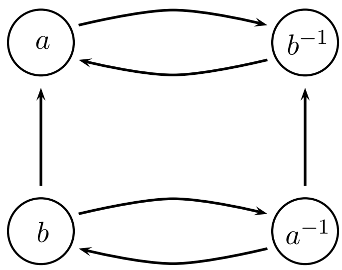


FIGURE 5. Ufnarovsky graph of $S_1$

By adding the relator $ba$, we get the inverse semigroup $S_2$, which has the following simplified graph:

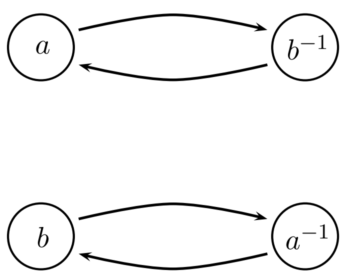


FIGURE 6. Ufnarovsky graph of $S_2$

By contrast, the graph of $S_3$ is as follows:

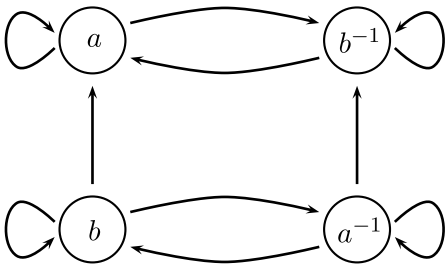


FIGURE 7. Ufnarovsky graph of $S_3$

and the graph of $S_4$ is as follows:

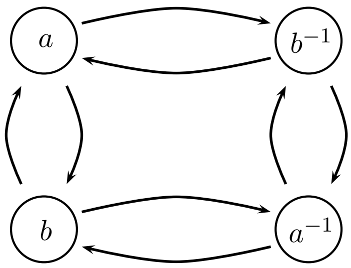


FIGURE 8. Ufnarovsky graph of $S_4$

The semigroups $S_3$ and $S_4$ clearly have exponential growth, as their graphs contain vertices in different cycles. There are many ways of seeing that $S_2$ has polynomial growth. It follows from the fact that $S_1$ has polynomial growth, explained from several different points of view in [11], and the fact that $S_2$ is a homomorphic image of $S_1$. The language of nonzero reduced words in $S_2$ can be read from the graph and is

$$(1 \cup a)(b^{-1}a)^*(1 \cup b^{-1}) \cup (1 \cup a)(b^{-1}a)^*(1 \cup b^{-1}) \ .$$

The Gelfand-Kirillov dimension of $S_1$ was calculated in [11, Example 2.3] to be four. In the case of $S_2$, the language of nonzero reduced words has Gelfand-Kirillov dimension one (because of the isolated cycles), and the calculation (which is an adaptation of the technique used in [32]) involving word trees that are chains is the same, associating with every reduced word of length $m$ some subset that consists of a quadratic function in $m$ distinct nonzero elements of the semigroup $S_2$. It follows that the Gelfand-Kirillov dimension of $S_2$ is three. Observe further that $S_2$ is a terminal object in the subclass of $\mathfrak{P}$, in the sense that $S_2$ has no proper homomorphic images involving positive relators other than inverse semigroups that are trivial or the five element Brandt inverse semigroup $\langle a \mid a^2 = 0\rangle$.

*Example* 4.2. By contrast with $S_1$ and $S_2$, the following inverse semigroup has exponential growth and plays a central role in describing and characterising semigroups in the class $\mathfrak{P}$ that satisfy an identity in signature with involution:

$$S_\Pi = \langle a, b \mid a^2 = b^2 = aba = bab = 0 \rangle$$

(see Proposition 7.8 and Theorem 7.10 below, where $S_\Pi = S_8(1)$). It has the following Ufnarovsky graph:

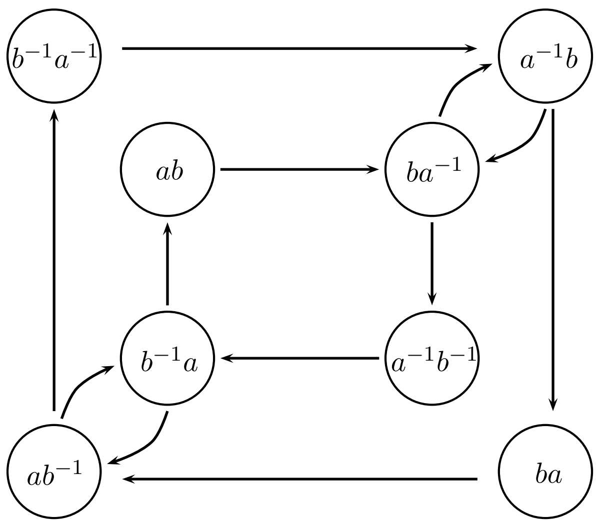


FIGURE 9. Ufnarovsky graph of $S_\Pi$

It is clear that $S_\Pi$ has exponential growth because there are vertices in the graph contained in different cycles. Observe that $S_\Pi$ also lies on a boundary, in the sense that it has no proper homomorphic images having exponential growth within the class of semigroups in $\mathfrak{P}$. Any proper homomorphic image of $S_\Pi$ obtained by adding positive relators is either trivial or isomorphic to the inverse monogenic semigroup $\langle a \mid a^2 = 0 \rangle$, $S_1$ or $S_2$, described in the previous example.

## 5. Reduction to at most two generators

In this short section, we show that inverse semigroups with zero from the class $\mathfrak{P}$ that satisfy some nontrivial identity in signature with involution (which, of course, includes those members of the class satisfying some nontrivial semigroup identity) have a presentation involving at most two generators. This major reduction relies on the following simple lemma, which is an application of Proposition 3.2 and which in turn relies on Reilly's criterion [28].

**Lemma 5.1.** *Let $A$ be an alphabet containing three distinct letters $a$, $b$ and $c$ and let $T$ be the inverse subsemigroup of $FI_A$ generated by*

$$u = a^{-1}bc^{-1}a \quad \text{and} \quad v = c^{-1}ba^{-1}c\,.$$

*Then*

(a) *$T$ is freely generated by $u$ and $v$.*
(b) *The only reduced words over the alphabet $\{a, b, c\}$ that divide elements of $T$ in $FI_A$, and which are positive with respect to $\{a, b, c\}$, are the letters $a$, $b$ and $c$.*

*Proof.* Consider the alphabet $Y = \{u, v\}$. Observe, by inspection, that, for all $y \in Y \cup Y^{-1}$, we have $\ell(y) = 4$ and $m_y = M_y = 1$, so that

$$m_y + M_y \;=\; 2 \;<\; \ell(y) \;=\; 4\,,$$

so that (6) is satisfied. By Proposition 3.2, (a) holds. By inspection of word trees of elements of $T$, (b) holds, completing the proof of the lemma. □

**Theorem 5.2.** *Consider an inverse semigroup with zero*

$$S \;=\; \langle\, A \mid c_i = 0 \textit{ for } i \in I\,\rangle$$

*from the class $\mathfrak{P}$ such that $c_i \notin A$ for all $i \in I$. If $S$ does not contain a nonmonogenic free inverse subsemigroup then $|A| \leq 2$. In particular, if $S$ satisfies a nontrivial identity in signature with involution then $|A|$ cannot be larger than two.*

*Proof.* Suppose that $S$ does not contain a nonmonogenic free inverse subsemigroup and that $|A| \geq 3$, so that $A$ contains distinct letters $a$, $b$ and $c$. let $T$ be the inverse subsemigroup of $FI_A$ generated by $a^{-1}bc^{-1}a$ and $c^{-1}ba^{-1}c$. By Lemma 5.1, $T$ is a free nonmonogenic inverse semigroup. If no element of $T$ is zero in $S$ then $S$ contains a free nonmonogenic inverse subsemigroup, which is a contradiction. Hence at least one element of $T$ is divided by a positive relator $c_i$ for some $i \in I$. By Lemma 5.1, $c_i$ equals one of $a$, $b$ or $c$, so that $c_i \in A$, producing another contradiction, completing the proof. □

## 6. Semigroup identities and polynomial growth

We begin this section with the following theorem, which tells us that $S_1$ (and hence also its homomorphic image $S_2$) satisfies Adjan's identity for the bicyclic monoid [1]. This result is a stepping stone to providing a cascade of characterisations, and a complete syntactic description, given later in this section (Theorem 6.2), of semigroups from the class $\mathfrak{P}$ that satisfy a nontrivial semigroup identity. The syntactic descriptions, in the case where such semigroups are nontrivial and nonmonogenic with zero, involve only $S_1$, using three relators, and $S_2$, using four relators.

**Theorem 6.1.** *The inverse semigroup with zero*

$$S_1 \;=\; \langle a, b \mid a^2 = b^2 = ab = 0\rangle$$

*satisfies Adjan's identity:*

$$xy^2xxyxy^2x \;=\; xy^2xyxxy^2x\,. \tag{9}$$

*Proof.* Let $A = \{a, b\}$. Consider a reduced word $w$ over $A \cup A^{-1}$ that is nonzero in $S_1$. By inspection (see [11, Example 2.3]), $w$ or $w^{-1}$ may be written in one of the following forms:

$$b^\alpha (a^{-1}b)^\beta\,,\;\; b^\alpha (a^{-1}b)^\beta a (b^{-1}a)^\gamma b^{-\delta} \tag{10}$$

for some nonnegative integers $\beta, \gamma$ and for some $\alpha, \delta \in \{0, 1\}$. If follows from (10) and the relators defining $S_1$ that if $w^2$ is nonzero in $S_1$, then

$$w = (a^{-1}b)^\gamma \quad \text{or} \quad w = (ab^{-1})^\gamma \tag{11}$$

for some nonzero integer $\gamma$.

Let $u, v \in S_1$. If $u^2$, $v^2$, $uv$ or $vu$ evaluate to zero in $S_1$ then (9) holds trivially when $u$ and $v$ are substituted for $x$ and $y$ respectively. Hence we may suppose that all of $u^2$, $v^2$ $uv$ and $vu$ and their reduced parts

$$(\overline{u})^2\,,\; (\overline{v})^2\,,\; \overline{u}\,\overline{v}\,,\; \overline{v}\,\overline{u} \quad \text{are nonzero in } S_1\,. \tag{12}$$

If follows from (11) and (12) that $\overline{u}$ and $\overline{v}$ are both nontrivial integer powers of $a^{-1}b$ or $ab^{-1}$. There is no loss in generality in assuming that

$$\overline{u} \;=\; (ab^{-1})^\alpha \quad \text{and} \quad \overline{v} = (ab^{-1})^\beta \tag{13}$$

for some nonzero integers $\alpha$ and $\beta$. We can write both $u$ and $v$ in their respective Schein canonical forms:

$$u \;=\; u_1 u_1^{-1} \ldots u_k u_k^{-1} \overline{u} \quad \text{and} \quad v = v_1 v_1^{-1} \ldots v_\ell v_\ell^{-1} \overline{v} \tag{14}$$

where $k$ and $\ell$ are nonnegative integers and $u_1, \ldots, u_k, v_1, \ldots, v_\ell$ are reduced. From (14), it follows that, being divisors of $u^2$ and $v^2$ respectively,

$$u_i^{-1} \overline{u} u_i \quad \text{and} \quad v_j^{-1} \overline{v} v_j \quad \text{are nonzero in } S_1 \tag{15}$$

for $1 \le i \le k$ and $1 \le j \le \ell$. From (10), (13) and (15) it follows that each $u_i$ and $v_j$ has the form

$$(ab^{-1})^\gamma a^\varepsilon \quad \text{or} \quad (ba^{-1})^\gamma b^\varepsilon \tag{16}$$

for some nonnegative integer $\gamma$ and for some $\varepsilon \in \{0, 1\}$ such that $\gamma + \varepsilon > 0$. Put

$$L \;=\; \big\{ (ab^{-1})^{-\alpha} b^\varepsilon b^{-\varepsilon} (ab^{-1})^{\alpha+\beta+\gamma} a^\delta a^{-\delta} (ab^{-1})^{-\gamma} \mid \alpha, \beta, \gamma \ge 0\,, \qquad \varepsilon, \delta \in \{0, 1\}\,,\; \alpha + \beta + \gamma > 0 \big\}$$

and

$$I \;=\; L \cup L^{-1}\,.$$

If follows, from (14) and (16), and a simple induction, that $u, v \in I$. But $I$ embeds in the free monogenic semigroup (see [33, Lemma 5.1]), which satisfies Adjan's identity (by [31] and [33, Proposition 2.1]). Hence (9) holds when $u$ and $v$ are substituted for $x$ and $y$ respectively. This completes the proof of the theorem. $\square$

In the main result of this section, we give equivalent conditions for inverse semigroups from $\mathfrak{P}$ to satisfy some nontrivial semigroup identity.

**Theorem 6.2.** *Consider an inverse semigroup with zero*

$$S \;=\; \langle\, A \mid c_i = 0 \text{ for } i \in I \,\rangle$$

*from the class* $\mathfrak{P}$. *Then the following conditions are equivalent:*

(a) $S$ *satisfies a nontrivial semigroup identity.*
(b) $S$ *satisfies Adjan's identity for the bicyclic monoid.*
(c) $S$ *does not contain a nonmonogenic free subsemigroup.*
(d) $S$ *is finitely presented and has polynomial growth.*
(e) $S$ *is trivial, a monogenic inverse semigroup with zero or isomorphic to*

$$S_1 = \langle a, b \mid a^2 = b^2 = ab = 0\rangle \quad \text{or} \quad S_2 = \langle a, b \mid a^2 = b^2 = ab = ba = 0\rangle\ .$$

*Proof.* If $S$ is trivial or a monogenic inverse semigroup with zero then $S$ is a homomorphic image of the free monogenic inverse semigroup, which satisfies Adjan's identity (9). If $S = S_2$ then $S$ is a homomorphic image of $S_1$. By Theorem 6.1, $S_1$ satisfies (9). Thus, in all cases, if $S$ is trivial, a monogenic inverse semigroup with zero, or isomorphic to $S_1$ or $S_2$, then $S$ satisfies (9). Thus (e) implies (b). Clearly (b) implies (a), and (a) implies (c).

We now prove that (c) implies (e). Suppose that $S$ does not contain a nonmonogenic free subsemigroup and that $S$ is not trivial and not a monogenic inverse semigroup with zero. We show that $S$ is isomorphic to $S_1$ or $S_2$. In particular, $|A| \geq 2$. If $c_i \in A$ for some $i \in I$ then we may remove $c_i$ from $A$ and all relators containing $c_i$, without changing $S$ up to isomorphism. Hence, there is no loss of generality in assuming that $c_i \notin A$ for all $i \in I$. Certainly, $S$ does not contain a nonmonogenic free inverse subsemigroup, so, by Theorem 5.2, $|A| \leq 2$, whence $|A| = 2$. We may suppose that $A = \{a, b\}$.

Let $U$ be the subsemigroup of $FI_A$ generated by $a^2b^{-1}$ and $ab^{-1}$, which is clearly free. Hence, some element of $U$ must be zero in $S$. But the only positive words dividing elements of $U$ in $FI_A$ are $a$, $b$ and $a^2$. Since $a$ and $b$ are not relators, we conclude that $a^2$ must be a relator. By a similar argument, $b^2$ must also be a relator.

Let $V$ be the subsemigroup of $FI_A$ generated by $ab^{-1}aba^{-1}b$ and $ab^{-1}$, which is easily shown to be free. Hence, some element of $V$ must be zero in $S$. But the only positive words dividing elements of $V$ are $a$, $b$, $ab$ and $ba$. Since $a$ and $b$ are not relators, we conclude that $ab$ or $ba$ must be a relator. In either case, $S$ must be a homomorphic image of $S_1$. But all positive words over $A$ that are not $a$ or $b$ must contain $a^2$, $b^2$, $ab$ or $ba$ as a subword. Hence all homomorphic images of $S_1$ defined by positive presentations are isomorphic to $S_1$ or $S_2$. This completes the proof that (c) implies (e).

If (e) holds then clearly $S$ is finitely presented, and, in each case, $S$ has polynomial growth, in particular, because $S_1$ has polynomial growth, by [11, Example 2.3], so that (d) holds. That (d) implies (c) follows from [33, Theorem 1], and the proof of the theorem is complete. □

## 7. Identities in signature with involution and exponential growth

In this final section, we provide a sequence of lemmas and propositions leading to two main theorems. Theorem 7.3 exhibits a nontrivial identity in signature with involution for the inverse semigroup $S_4$. (A nontrivial identity in signature with involution for the inverse semigroup $S_3$ had already been found in [10], documented above in Theorem 2.1.)

Propositions 7.4 and 7.8 give complete syntactic descriptions of proper homomorphic images of $S_3$ and $S_4$ respectively. The final result, Theorem 7.10 below, gives characterisations of inverse semigroups from $\mathfrak{P}$ that satisfy a nontrivial identity in signature with involution but satisfy no nontrivial semigroup identity.

The next two lemmas refer to the inverse semigroup with zero

$$S_4 \ = \ \langle a, b \mid a^2 = b^2 = 0 \rangle \ .$$

and are used to find a nontrivial identity in signature with involution satisfied by $S_4$ (see Theorem 7.3 below). Put $A = \{a, b\}$ and $B = A \cup A^{-1}$. Because of the relations $a^2 = b^2 = 0$, nonempty reduced words $w$ over $A \cup A^{-1}$ that are nonzero in $S_4$ must have the form

$$w \ = \ a^{\varepsilon_1} b^{\delta_1} a^{\varepsilon_2} b^{\delta_2} \ldots a^{\varepsilon_k} b^{\delta_k} \qquad (17)$$

for some positive integer $k$ and integers $\varepsilon_1, \delta_1, \ldots, \varepsilon_k, \delta_k$, which are all $\pm 1$, except for $\varepsilon_1$ and $\delta_k$, which may be 0 or $\pm 1$.

**Lemma 7.1.** *Let $w \in S_4$ be a nonempty reduced word over $B$ that is nonzero in $S_4$. Then each of the following conditions implies that $w^2 = 0$ in $S_4$:*

(a) *$w = c$ or $cw'c$ for some letter $c \in B$ and reduced word $w'$ (possibly empty) over $B$.*
(b) *$w$ is reduced but not cyclically reduced.*
(c) *$w$ has odd length.*

*Proof.* Sufficiency of (a) is clear because $a^2 = b^2 = 0$ in $S_4$. Suppose that $w$ is a reduced but not cyclically reduced word that is nonzero in $S_4$ and let $\widehat{w}$ denote its reduced part. By (17), the length of $w$ must be odd, which implies also that the length of $\widehat{w}$ is odd. By the form of (17) applied to $\widehat{w}$, it follows that $\widehat{w}$ starts and finishes with the same letter, so that $(\widehat{w})^2 = 0$ in $S_4$, by the sufficiency of (a). Since $(\widehat{w})^2$ divides $w^2$ in $FI_A$, it follows that $w^2 = 0$ in $S_4$. This proves sufficiency of (b) and simultaneously sufficiency of (c). ☐

**Lemma 7.2.** *Let $u$ and $v$ be nonempty reduced words over $B$ such that $u$, $v$ and $uv$ are nonzero in $S_4$. Suppose that the word $uv$ is not reduced. Then either $v^{-1}$ is a suffix for $u$, or $u^{-1}$ is a prefix for $v$. In particular, if $u$ and $v$ have the same length then $u = v^{-1}$.*

*Proof.* We have that $u = z_1 t$ and $v = t^{-1} z_2$ for some reduced words $z_1$, $z_2$, $t$, where $t$ is nonempty and $z_1 z_2$ is reduced. The lemma will be proved by showing that either $z_1$ or $z_2$ is empty. Suppose to the contrary that both $z_1$ and $z_2$ are nonempty. Applying the form (17) to each of $u$ and $v$, using the fact that $t$ is nonempty, we deduce that the terminal letter of $z_1$ coincides with the initial letter of $z_2$. Then $a^2$ or $b^2$ divides $z_1 z_2$, which in turn divides $uv$ in $FI_A$. But then $uv = 0$ in $S_4$, which contradicts our hypothesis. Hence at least one of the words $z_1$, $z_2$ must be empty, and the lemma is proved. ☐

We can now prove the following theorem, which becomes one of the main steps in finding a complete description (Theorem 7.10 below) of semigroups from $\mathfrak{P}$ that satisfy a nontrivial identity in signature with involution, but which do not have polynomial growth.

**Theorem 7.3.** *The inverse semigroup with zero given by the presentation*

$$S_4 \;=\; \langle a, b \mid a^2 = b^2 = 0\rangle$$

*satisfies the following identity in signature with involution:*

$$x\,[x,y]^2\,y^{-1} \;=\; x\,[x,y]\,y^{-1}\;,$$

*or equivalently,*

$$x^2yx^{-1}y^{-1}xyx^{-1}y^{-2} \;=\; x^2yx^{-1}y^{-2}\;. \tag{18}$$

*Proof.* Consider any $x, y \in S_4$. We may suppose that all of $x^2$, $y^2$, $xy$ and $x^{-1}y^{-1}$ are nonzero in $S_4$, for otherwise (18) holds trivially. By Lemma 7.1, both $\overline{x}$ and $\overline{y}$ have even length greater than or equal to two, and, by (17), consecutive letters in each word alternate between the alphabets $\{a^{\pm1}\}$ and $\{b^{\pm1}\}$. Put

$$u \;=\; \overline{xy} \;=\; \overline{\overline{x}\,\overline{y}} \qquad\text{and}\qquad v \;=\; \overline{x^{-1}y^{-1}} \;=\; \overline{\overline{x^{-1}}\,\overline{y^{-1}}}\;.$$

Because of the alternation of the alphabets and even lengths, either $\overline{x}\,\overline{y}$ is reduced, in which case $\overline{y}\,\overline{x}$ is also reduced, and

$$\ell(u) \;=\; \ell(\overline{x}) + \ell(\overline{y}) \;=\; \ell(v)\;,$$

or $\overline{x}\,\overline{y}$ is not reduced, in which case $\overline{y}\,\overline{x}$ is also not reduced, and, by Lemma 7.2,

$$\ell(u) \;=\; \big|\ell(\overline{x}) - \ell(\overline{y})\big| \;=\; \ell(v)\;.$$

In either case, $\ell(u) = \ell(v)$. If $\overline{xyx^{-1}y^{-1}}$ is zero in $S$ then (18) holds trivially. Hence we may suppose that

$$\overline{uv} \;=\; \overline{xyx^{-1}y^{-1}}$$

is nonzero in $S_4$. But then $uv$ is nonzero and not reduced, so that $u = v^{-1}$, by Lemma 7.2. Hence

$$\overline{[x,y]} \;=\; \overline{xyx^{-1}y^{-1}} \;=\; \overline{uv} \;=\; 1$$

in the free group. This shows that $[x, y]$ is idempotent in $S_4$, and (18) holds, completing the proof of the theorem. ☐

We now develop some notation and apparatus that enables us to describe, in a compact way (see Proposition 7.4 below), all homomorphic images of the semigroup $S_3$. Let $n$ be a positive integer and suppose that

$$\mathbf{x} \;=\; (x_1, x_2, \ldots, x_n) \quad\text{and}\quad \mathbf{y} \;=\; (y_1, y_2, \ldots, y_n)$$

are finite sequences of nonnegative integers such that $\mathbf{x}$ is decreasing, $\mathbf{y}$ is increasing and at least one entry is nonzero. Define

$$S_{\mathbf{x},\mathbf{y}} \;=\; \langle a, b \mid ab = b^{x_1}a^{y_1} = \ldots = b^{x_n}a^{y_n} = 0\rangle\;, \tag{19}$$

which is a member of our class $\mathfrak{P}$ and a homomorphic image of the inverse semigroup $\langle a, b \mid ab = 0\rangle$. For example, if $\mathbf{x} = (4, 3, 2, 1, 0)$ and $\mathbf{y} = (0, 1, 2, 3, 4)$ then

$$S_{\mathbf{x},\mathbf{y}} \;=\; \langle a, b \mid ab = b^4 = b^3a = b^2a^2 = ba^3 = a^4 = 0\rangle\;.$$

The notion of *irredundancy* of an arbitrary inverse semigroup presentation of the form (4) is given in [35, Section 2], and, in our context (as defined above in the Preliminaries), a presentation is *irredundant* if none of the generators are relators and no relator is a factor of any of the others. Observe, by inspection (see (20), (21) and (22) below), that the presentation (19) violates irredundancy if and only if $a$ or $b$ is a relator, which occurs if and only if $x_1 = 1$ and $y_1 = 0$, or $x_n = 0$ and $y_n = 1$, that is,

$$(\mathbf{x}, \mathbf{y}) \;\in\; \left\{ \big((1),(0)\big)\,,\; \big((0),(1)\big)\,,\; \big((k,0),(0,1)\big)\,,\; \big((1,0),(0,k)\big) \;\big|\; k \geq 1 \right\} .$$

In particular, the presentation (19) is irredundant if $n \geq 3$. To decongest the notation slightly, in the case that $n = 1$, we can write

$$S_{i.j} \;=\; S_{(i),(j)} \;.$$

Note that $S_{0,0}$ is not defined and

$$S_{1,1} \;=\; \langle a, b \mid ab = ba = 0 \rangle \,.$$

Observe that $S_{1,1}$ is a homomorphic image of $S_{\mathbf{x},\mathbf{y}}$ for any positive integer $n$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are sequences of positive integers. Further, $S_{1,1}$ has exponential growth, by [11, Theorem 5.2], and is a terminal object in the subclass of $\mathfrak{P}$ consisting of semigroups where none of the generators are nilpotent, in the sense that $S_{1,1}$ is unchanged upon adding further positive relators that avoid $a^2$ and $b^2$. Observe further that, in general, if $\overline{\mathbf{s}}$ is the sequence obtained by reversing a finite sequence $\mathbf{s}$, then

$$S_{\overline{\mathbf{y}},\overline{\mathbf{x}}} \;\cong\; S_{\mathbf{x},\mathbf{y}} \;,$$

under the isomorphism induced by mapping $a$ to $b^{-1}$ and $b$ to $a^{-1}$.

Most of the time, the finitely presented inverse semigroup $S_{\mathbf{x},\mathbf{y}}$ has exponential growth (see Lemma 7.4 below), but there are some exceptions:

$$S_{1,0} \;=\; \langle a, b \mid ab = b = 0 \rangle \;=\; \langle a \rangle \cup \{0\} \;\cong\; S_{0,1} \tag{20}$$

is free monogenic with zero,

$$S_{(1,0),(0,1)} \;=\; \langle a, b \mid ab = b = a = 0 \rangle \;=\; \{0\} \tag{21}$$

is trivial,

$$S_{(1,0),(0,k)} \;=\; \langle a, b \mid ab = b = a^k = 0 \rangle \;=\; \langle a \mid a^k = 0 \rangle \;\cong\; S_{(k,0),(0,1)} \tag{22}$$

is monogenic with zero, for any integer $k \geq 2$,

$$S_{(2,0),(0,2)} \;=\; \langle a, b \mid ab = b^2 = a^2 = 0 \rangle \tag{23}$$

has polynomial growth (see [11, Example 2.3]), and

$$S_{(2,1,0),(0,1,2)} \;=\; \langle a, b \mid ab = b^2 = ba = a^2 = 0 \rangle \tag{24}$$

also has polynomial growth, being a homomorphic image of $S_{(2,0),(0,2)}$. By contrast,

$$S_{(3,0),(0,2)} \;=\; \langle a, b \mid ab = b^3 = a^2 = 0 \rangle \;\cong\; S_{(2,0),(0,3)} \tag{25}$$

and

$$S_{(1,0),(1,2)} \;=\; \langle a, b \mid ab = ba = a^2 = 0 \rangle \;\cong\; S_{(2,1),(0,1)} \tag{26}$$

have exponential growth, by [11, Theorem 6.1]. Consider, also,

$$S_{(3,1,0),(0,1,2)} \;=\; \langle a, b \mid ab = b^3 = ba = a^2 = 0 \rangle \;\cong\; S_{(2,1,0),(0,1,3)} \tag{27}$$

and

$$S_{(3,2,1,0),(0,1,2,3)} \;=\; \langle a,b \mid ab = b^3 = b^2a = ba^2 = a^3 = 0\rangle \;. \tag{28}$$

Both of these have exponential growth. One can see this, for example, by observing that the subsemigroup $T$ of $FI_{\{a,b\}}$ generated by $a^{-1}b$ and $a^{-1}b^2$ is free (see [11, Lemma 3.1]), no element of which is divided by any of the relators in (27) or (28), so that $T$ embeds in both semigroups. Put

$$\begin{aligned}\mathcal{L} = \{ \; &\big((0),(0)\big)\,,\ \big((1),(0)\big)\,,\ \big((0),(1)\big)\,,\ \big((k,0)\,,(0,1)\big)\,,\ \big((1,0),(0,k)\big)\,,\\ &\big((2,0),(0,2)\big)\,,\ \big((2,1,0),(0,1,2)\big) \;\big|\; k\geq 1 \;\}\;.\end{aligned} \tag{29}$$

Now we can fully describe elements of $\mathfrak{P}$ that are homomorphic images of $\langle a,b \mid ab = 0\rangle$, and identify those that have exponential growth:

**Proposition 7.4.** *Let $S$ be a semigroup from the class $\mathfrak{P}$ given by an inverse semigroup presentation*

$$\langle a,b \mid c_i = 0 \;\text{ for }\; i\in I\rangle$$

*such that $ab$ is zero in $S$. Then $S$ is finitely presented in the class of inverse semigroups and is isomorphic to one of the following finitely presented Rees quotients of free inverse semigroups over the alphabet $\{a,b\}$:*

$$S_3 \;=\; \langle a,b \mid ab = 0\rangle \quad \textit{or} \quad S_{\mathbf{x},\mathbf{y}} \quad \textit{given by } (19)\;,$$

*for some finite decreasing sequence $\mathbf{x}$ and some increasing sequence $\mathbf{y}$ of nonnegative integers of the same length such that at least one entry is nonzero. Further, $S$ has exponential growth if and only if $S$ is isomorphic to $S_3$ or $S_{\mathbf{x},\mathbf{y}}$ for some $(\mathbf{x},\mathbf{y})\notin\mathcal{L}$, where $\mathcal{L}$ is given by (29).*

*Remark* 7.5. It should be noted that it is well-known that the semigroup over the alphabet $A=\{a,b\}$ given by positive relators, one of which is $ab$, must be finitely presented. Here, the language of all words over $A$ that avoid $ab$ is $b^*a^*$, and every ascending chain of ideals in the free semigroup $A^+$, starting from the principal ideal generated by $ab$, stabilises in a finite number of steps (see König's Lemma, [30, Corollary 2.4]). Our goal here, however, is to give a syntactic description of all possible ascending chains, distinguishing the cases of polynomial and exponential growth that arise when we consider inverse semigroup presentations.

*Proof of Proposition 7.4.* The first part of this proof is direct and elementary, and may be regarded as a proof of the two-dimensional case of the well-known Dickson's lemma [9] (see Remark 7.6 below). An alternative proof, using Higman's lemma [16], is also given below in Remark 7.7. Suppose that $S$ is a proper homomorphic image of $S_3$, so must contain at least one positive word $w$ that avoids $ab$ as a subword and evaluates to zero. We may choose $w$ to be of the form

$$w \;=\; b^{x_1}a^{y_1} \;=\; 0$$

for some nonnegative integers $x_1$ and $y_1$ such that $x_1+y_1>0$ and $y_1$ is as small as possible. But then, having fixed $y_1$, we may suppose that $x_1$ is as small as possible (since any word with larger exponent of $b$ would evaluate to zero as a consequence). Hence $S$ is a homomorphic image of

$$S_{x_1,y_1} \;=\; \langle a,b \mid ab = b^{x_1}a^{y_1} = 0\rangle \;.$$

If the image is not proper then $S = S_{\mathbf{x},\mathbf{y}}$, where $\mathbf{x} = (x_1)$ and $\mathbf{y} = (y_1)$ so that $S$ is given by (19), in an instance where $n = 1$. Note that at least one of $x_1$ or $y_1$ is positive.

Suppose then that $S$ is a proper image of $S_{x_1,y_1}$, so $S$ must contain at least one positive word $w$ that evaluates to zero and avoids $ab$ and $b^{x_1}a^{y_1}$, so that

$$w' \;=\; b^{x_2}a^{y_2} \;=\; 0$$

for some nonnegative integers $x_2$ and $y_2$ such that $x_2 + y_2 > 0$ and $y_2$ is as small as possible. Then, having fixed $y_2$, we may suppose that $x_2$ is as small as possible. By minimality of $y_1$, however, $y_2 \geq y_1$. Since $b^{x_1}a^{y_1}$ does not divide $w'$ we have that $x_2 < x_1$. If $y_2 = y_1$ then we have a contradiction, by the minimality of $x_1$. Hence $y_2 > y_1$, and $S$ is a homomorphic image of $S_{\mathbf{x},\mathbf{y}}$ where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. If the image is not proper then $S = S_{\mathbf{x},\mathbf{y}}$ is given by (19), in an instance where $n = 2$.

We may suppose then that $S$ is a proper homomorphic image of this $S_{\mathbf{x},\mathbf{y}}$, and we move to the next step in an inductive process.

Suppose, more generally, that $n \geq 2$ and $S$ is a proper homomorphic image of $S_{\mathbf{x},\mathbf{y}}$ for some decreasing sequence $\mathbf{x} = (x_1, \ldots, x_n)$ and increasing sequence $\mathbf{y} = (y_1, \ldots, y_n)$ of nonnegative integers. In particular, $x_1$ and $y_n$ are positive. We may assume, as part of this inductive hypothesis, that, for $2 \leq i \leq n$, we have chosen $y_i$ to be the smallest nonnegative integer such that $b^{x_i}a^{y_i}$ evaluates to zero in $S$, for some $x_i < x_{i-1}$, and that for this $y_i$, we have chosen $x_i$ as small as possible. Because $S$ is a proper homomorphic image, $S$ must contain at least one positive word $w''$ that evaluates to zero but avoids each of $ab$, $b^{x_1}a^{y_1}, \ldots, b^{x_n}a^{y_n}$, so that

$$w'' \;=\; b^{x_{n+1}}a^{y_{n+1}} \;=\; 0$$

for some nonnegative integers $x_{n+1}$ and $y_{n+1}$ such that $x_{n+1} + y_{n+1} > 0$ and $y_{n+1}$ is as small as possible. Then, having fixed $y_{n+1}$, we may suppose that $x_{n+1}$ is as small as possible. Since $b^{x_n}a^{y_n}$ does not divide $w''$ we have that either $x_{n+1} < x_n$ or $y_{n+1} < y_n$. Suppose that $x_{n+1} \geq x_n$, so $y_{n+1} < y_n$. By minimality of $y_1$, however, $y_{n+1} \geq y_1$. Hence there is a least integer $i$ such that $2 \leq i \leq n$ and

$$y_{i-1} \;\leq\; y_{n+1} \;<\; y_i \;.$$

If, also, $x_{i-1} \leq x_{n+1}$, then the word $b^{x_{i-1}}a^{y_{i-1}}$ divides $w''$, which is a contradiction. Hence $x_{n+1} < x_{i-1}$. But this now contradicts the conditions for the minimality of $y_i$ (as part of the inductive hypothesis). This proves that $x_{n+1} < x_n$. If $y_{n+1} \leq y_n$ then we contradict either the minimality of $y_n$ (if $y_{n+1} < y_n$) or the minimality of $x_n$ (if $y_{n+1} = y_n$). Hence $y_{n+1} > y_n$. Now put

$$\mathbf{x}' = (x_1, \ldots, x_n, x_{n+1}) \quad \text{and} \quad \mathbf{y}' = (y_1, \ldots, y_n, y_{n+1}) \;.$$

We have that $S$ is a homomorphic image of $S_{\mathbf{x}',\mathbf{y}'}$, establishing the inductive step.

This inductive process however must terminate after a finite number of steps because the first sequence is decreasing with nonnegative entries. This shows that $S$ is finitely presented and $S$ is given by (19) for some positive integer $n$, some decreasing sequence $\mathbf{x}$ and some increasing sequence $\mathbf{y}$, such that at least one of the entries is positive.

Suppose that $S$ is isomorphic to $S_3$ or to $S_{\mathbf{x},\mathbf{y}}$ for $(\mathbf{x}, \mathbf{y}) \notin \mathcal{L}$. As noted in observations preceding this proposition, all of these are defined by irredundant presentations, in the sense of [35, Section 2]. Observe that $S_3$ is defined by one relator and $S_{\mathbf{x},\mathbf{y}}$ is defined by two relators, if $\mathbf{x}$ and $\mathbf{y}$ are sequences of length 1, so have exponential growth, by [11, Theorem 5.2]. If $\mathbf{x}$ and $\mathbf{y}$ are sequences of length 2 then, because $(\mathbf{x},\mathbf{y}) \notin \mathcal{L}$, it follows that $S_{\mathbf{x},\mathbf{y}}$ has $S_{(3,0),(0,2)}$ or $S_{(2,1),(0,1)}$ as a homomorphic image, so has exponential growth (explained above following (25) and (26)). If $\mathbf{x}$ and $\mathbf{y}$ are sequences of length 3 then, because $(\mathbf{x},\mathbf{y}) \notin \mathcal{L}$, it follows that $S_{\mathbf{x},\mathbf{y}}$ has $S_{(3,1,0),(0,1,2)}$ as a homomorphic image, so has exponential growth (explained above following (27)). Suppose finally that $\mathbf{x}$ and $\mathbf{y}$ are sequences of length $n \geq 4$. Then $S_{\mathbf{x},\mathbf{y}}$ has $S_{(3,2,1,0),(0,1,2,3)}$ as a homomorphic image, so has exponential growth (explained above following (28)). In all cases $S$ has exponential growth.

Finally, by observations above relating to (20), (21), (22), (23) and (24), if $(\mathbf{x},\mathbf{y}) \in \mathcal{L}$ then $S_{\mathbf{x},\mathbf{y}}$ has polynomial growth. This completes the proof of the proposition. $\square$

*Remark* 7.6. The proof of the first part of Proposition 7.4 regarding the finiteness of the presentation of $S$ follows quickly from a classical lemma of Dickson [9], which says that the partially ordered set $(\mathbb{N}^m, \leq)$ has no infinite antichains. Indeed, let

$$S \;=\; \langle a, b \mid ab = 0\,,\; b^{x_i}a^{y_i} = 0 \;\; (i = 1, 2, \ldots)\rangle \tag{30}$$

be any positive presentation of $S$. Then the relation $b^{x_j}a^{y_j} = 0$ can be deduced from the relation $b^{x_i}a^{y_i} = 0$ if and only if $(x_i, y_i) \leq (x_j, y_j)$ in $(\mathbb{N}^2, \leq)$. Thus the relation $b^{x_j}a^{y_j}$ occurs in an irredundant presentation of $S$ if and only if $(x_i, y_i)$ is a minimal element, with respect to the partial order, of the subset of ordered pairs $(x, y)$ such that $b^x a^y = 0$ in $S$. By Dickson's lemma, we immediately deduce that the irredundant presentation of $S$ equivalent to (30) must be finite. The first part of the proof of Proposition 7.4, in fact, becomes a proof of Dickson's lemma for $m = 2$.

*Remark* 7.7. An alternative proof of the first part of Proposition 7.4 is related to a result of Higman [16]. Consider the free monoid $A^*$ with respect to a finite nonempty set $A$. For $u, v \in A^*$, say that $u$ *embeds in* $v$, and write $u \leq_e v$, if $u$ can be obtained from $v$ by cancelling some letters. Then $u \leq_e v$ means that if $u = u_1u_2 \ldots u_n$ has a letter by letter factorisation over $A$, then the equation

$$v \;=\; v_1u_1v_2u_2 \ldots v_nu_nv_{n+1}$$

holds in $A^*$ for some words $v_1, \ldots, v_{n+1}$. In modern parlance, $u$ is called a *scattered subword* of $v$. If $(E, \leq)$ is a partially ordered set and $X \subseteq E$, then the set

$$\overline{X} \;=\; \{y \in E \mid (\exists x \in X)\; x \leq y\}$$

is called the *closure of* $X$ (or *ideal generated by* $X$). It is easy to see that $(A^*, \leq_e)$ is a partially ordered set that does not have infinite descending chains. This is called the *division ordering* on $A^*$. One of the corollaries of Higman's theorem [16], proved for abstract algebras, and applied to division orderings, is that for any infinite sequence of words $\{u_n\}_{n=1}^{\infty}$ over a finite alphabet $A$, one can find two indices $i$ and $j$ such that $i < j$ and $u_i \leq_e u_j$. A short proof of this result, which we refer to as Higman's lemma, was found by Conway [5, pp. 52-53] (and see [22] also for more details and comments). In particular, Higman's lemma shows that, for any nonempty subset $H$ of $A^*$, the partially ordered set $(H, \leq_e)$ has a finite number of minimal elements and every ideal is finitely generated. Denote the set of all minimal elements of $(H, \leq_e)$ by $\min(H)$. Thus $\min(H)$ is a finite nonempty set whose closure is $H$. We apply these observations in the context of the presentation (30) of $S$, using $A = \{a, b\}$ and

$$H \;=\; \{b^\alpha a^\beta \mid b^\alpha a^\beta = 0 \;\text{ in } S\} \;.$$

A relation $U \equiv (b^{x_j} a^{y_j} = 0)$ can be deduced from a relation $V \equiv (b^{x_i} a^{y_i} = 0)$, in the class of all semigroups with zero (as well as in the class of all inverse semigroups with zero), if and only if $x_i \leq x_j$ and $y_i \leq y_j$, in which case $V$ can be obtained from $U$ by deleting $x_j - x_i$ occurrences of the letter $b$ and $y_j - y_i$ occurrences of the letter $a$, that is, $b^{x_i} a^{y_i} \leq_e b^{x_j} a^{y_j}$. This means that $\overline{H} = \overline{\min H} \;=\; H$ and relators in the irredundant presentation of $S$ equivalent to (30) are precisely elements of the set $\min(H) \cup \{ab\}$. Now suppose that the irredundant presentation of $S$ consists of $n$ relators, so that we may suppose

$$S \;=\; \langle a, b \mid ab = 0\,,\; b^{x_i} a^{y_i} = 0 \;\; (i = 1, 2, \ldots, n)\rangle \tag{31}$$

where all relators are pairwise incomparable in $(A^*, \leq_e)$. Then $i \neq j$ implies that $x_i \neq x_j$ and $y_i \neq y_j$. Also $x_i > x_j$ implies $y_i < y_j$. Without loss of generality, then, we may suppose in (31) that $x_1 > x_2 > \ldots > x_n$ and $y_n < y_{n-1} < \ldots < y_1$, so that the finite sequence $\mathbf{x} = (x_1, \ldots, x_n)$ is decreasing whilst the sequence $\mathbf{y} = (y_1, \ldots, y_n)$ is increasing, and we have recovered the presentation (19), completing this alternative proof of the first part of Proposition 7.4.

The next proposition gives a complete description of proper homomorphic images of the inverse semigroup $S_4$. It is notable that cascades of examples occur (semigroups $S_6(i)$ and $S_8(i)$, for $i \geq 1$, in the statement of the proposition below), where relators occur with equations resembling Coxeter relations.

**Proposition 7.8.** *Let $S$ be a semigroup from the class $\mathfrak{P}$ which is a proper homomorphic image of the inverse semigroup*

$$S_4 \;=\; \langle a, b \mid a^2 = b^2 = 0\rangle \;.$$

*Then $S$ is finitely presented and either $S$ is trivial or monogenic with zero, or $S$ is isomorphic to*

$$S_5(i) = \langle a, b \mid a^2 = b^2 = (ab)^i = 0\rangle \;,$$
$$S_6(i) = \langle a, b \mid a^2 = b^2 = (ab)^i = (ba)^i = 0\rangle \;,$$
$$S_7(i) = \langle a, b \mid a^2 = b^2 = (ab)^i a = 0\rangle \;\text{ or}$$
$$S_8(i) = \langle a, b \mid a^2 = b^2 = (ab)^i a = (ba)^i b = 0\rangle \;,$$

*for some integer $i \geq 1$. Then $S$ has exponential growth in all cases, except when $S$ is trivial or monogenic with zero, or when $S$ is isomorphic to $S_5(1)$ or $S_6(1)$, in which case $S$ has polynomial growth.*

*Proof.* First note that there must be some positive word $w$ over the alphabet $\{a, b\}$ that is zero in $S$ but is not divisible by $a^2$ or $b^2$. We may assume that $w$ is as short as possible. If $w = a$ or $w = b$ then $S$ must be trivial or monogenic with zero.

Suppose that $w$ is not equal to $a$ or $b$. Then $w$ must be one of

$$(ab)^i\,,\; (ba)^i\,,\; (ab)^i a\,,\; (ba)^i a\;,$$

for some positive integer $i$. Suppose first that $w = (ab)^i$. Then $S$ is a homomorphic image of $S_5(i)$. If this image is proper then there must be another positive word $w'$ that evaluates to zero in $S$, has length at least $2i$ but is not divisible by $a^2$, $b^2$ or $(ab)^i$. The only such word is

$$w' = (ba)^i$$

so that $S$ is a homomorphic image of $S_6(i)$. But all words of length greater than $2i$ that are not divisible by $a^2$ or $b^2$ are divisible by $w$ or $w'$, so that $S$ must be isomorphic to $S_6(i)$.

If $w = (ba)^i$ then, interchanging the roles of $a$ and $b$, we again conclude that $S$ is isomorphic to $S_5(i)$ or $S_6(i)$.

By a similar argument, if $w = (ab)^i a$ or $w = (ba)^i b$ then $S$ is isomorphic to $S_7(i)$ or $S_8(i)$.

Observe that $S_5(1)$ has polynomial growth, by [11, Theorem 6.1], Hence $S_6(1)$, being a homomorphic image, also has polynomial growth.

Note that

$$S_8(1) = \langle a, b \mid a^2 = b^2 = aba = bab = 0 \rangle$$

is a homomorphic image of $S_5(i)$, $S_6(i)$, $S_7(j)$ and $S_8(j)$ for all $i \geq 2$ and $j \geq 1$. But it is easy to see that the words $ba^{-1}$ and $b^{-1}aba^{-1}$ generate a free subsemigroup of $FI_{\{a,b\}}$, no element of which is divisible by $a^2$, $b^2$, $aba$ or $bab$, so that $S_8(1)$ contains a nonmonogenic free subsemigroup. Hence $S_8(1)$ has exponential growth. Hence $S_5(i)$, $S_6(i)$, $S_7(j)$ and $S_8(j)$ have exponential growth for all $i \geq 2$ and $j \geq 1$, completing the proof of the proposition. □

The following lemma is our second application of Proposition 3, and is used in Theorem 7.10 below, to severely limit the possibilities of inverse semigroups with zero from $\mathfrak{P}$ that can satisfy a nontrivial identity in signature with involution.

**Lemma 7.9.** *Let $A$ be an alphabet containing two distinct letters $a$ and $b$ and let $T$ be the inverse subsemigroup of $FI_A$ generated by*

$$u = ab^{-1}a^2b^{-1}a \quad \text{and} \quad v = ba^{-1}ba^{-1} .$$

*Then*

(a) *$T$ is freely generated by $u$ and $v$.*
(b) *The only positive reduced words over $A$ that divide elements of $T$ in $FI_A$ are $a$, $b$, $a^2$, $ab$ and $ba$.*

*Proof.* Consider the alphabet $Y = \{u, v\}$. Observe, by inspection, that,

$$m_u = M_v = m_{v^{-1}} = M_{v^{-1}} = 3$$

and

$$M_u = m_v = m_{u^{-1}} = M_{v^{-1}} = 0 ,$$

whilst

$$\ell(u) = \ell(u^{-1}) = 6 \qquad \ell(v) = \ell(v^{-1}) = 4 ,$$

from which it is immediate that the set $Y$ satisfies condition (6). Then, by Proposition 3.2, condition (a) of the lemma holds. By inspection of word trees of elements of $T$, (b) holds, completing the proof of the lemma. □

In the final result, we give a complete description of inverse semigroups from $\mathfrak{P}$ that satisfy some nontrivial identity in signature with involution, but which do not satisfy a semigroup identity. Such semigroups are all finitely presented and have exponential growth. The descriptions rely crucially on the inverse semigroup $S_3$, which has a central role in the results of [10], and whose homomorphic images are listed in Proposition 7.4, and on the inverse semigroup $S_4$, whose homomorphic images are listed in Proposition 7.8. Some details regarding both $S_3$ and $S_4$, including the graphs associated with their presentations, are given above in Example 4.1.

**Theorem 7.10.** *Consider an inverse semigroup with zero*

$$S \;=\; \langle\, A \mid c_i = 0 \textit{ for } i \in I \,\rangle$$

*from the class $\mathfrak{P}$. Suppose that $S$ does not satisfy a nontrivial semigroup identity. Then the following conditions are equivalent:*

(a) *$S$ satisfies a nontrivial identity in signature with involution.*
(b) *$S$ does not contain a nonmonogenic free inverse subsemigroup.*
(c) *$S$ is finitely presented and is isomorphic to*

$$S_3 = \langle a, b \mid ab = 0 \rangle \,,$$

$$S_{\mathbf{x},\mathbf{y}} = \langle a, b \mid ab = b^{x_1}a^{y_1} = \ldots = b^{x_n}a^{y_n} = 0 \rangle \,,$$

*for some integer $n \geq 1$, some finite decreasing sequence $\mathbf{x}$ and some increasing sequence $\mathbf{y}$ of nonnegative integers of the same length, such that $(\mathbf{x}, \mathbf{y})$ is not in the list $\mathcal{L}$ of exceptions* (29), *or $S$ is isomorphic to*

$$S_4 = \langle a, b \mid a^2 = b^2 = 0 \rangle \,,$$

$$S_5(i) = \langle a, b \mid a^2 = b^2 = (ab)^i = 0 \rangle \quad \textit{or}$$

$$S_6(i) = \langle a, b \mid a^2 = b^2 = (ab)^i = (ba)^i = 0 \rangle$$

*for some integer $i \geq 2$, or*

$$S_7(j) = \langle a, b \mid a^2 = b^2 = (ab)^j a = 0 \rangle \quad \textit{or}$$

$$S_8(j) = \langle a, b \mid a^2 = b^2 = (ab)^j a = (ba)^j b = 0 \rangle \,,$$

*for some integer $j \geq 1$.*

*If $S$ satisfies any (and hence all) of these conditions then $S$ is finitely presented and has exponential growth.*

*Proof.* We show that (c) implies (a). If $S$ is isomorphic to $S_3$ or $S_4$ then $S$ satisfies a nontrivial identity in signature with involution, by Theorems 2.1 and 7.3 respectively. Observe that each $S_{\mathbf{x},\mathbf{y}}$ is a homomorphic image of $S_3$, and each $S_5(i)$, $S_6(i)$, $S_7(j)$ and $S_8(j)$ is a homomorphic image of $S_4$. Hence, if $S$ is isomorphic to any of the semigroups listed in (c) then $S$ satisfies a nontrivial identity in signature with involution. Thus (c) implies (a). Clearly (a) implies (b).

It remains to prove that (b) implies (c). Suppose that $S$ does not contain any nonmonogenic free inverse subsemigroup. Certainly $|A| \geq 2$. As before, there is no loss of generality in assuming that no relator belongs to the alphabet $A$. By Theorem 5.2, $|A| = 2$, so that we may assume $A = \{a, b\}$.

Let $T$ be the inverse subsemigroup of $FI_A$ generated by $ab^{-1}a^2b^{-1}a$ and $ba^{-1}ba^{-1}$. By Lemma 7.9, $T$ is a free nonmonogenic subsemigroup, and only $a$, $b$, $a^2$, $ab$ and $ba$ can be words over the positive alphabet $A$ that are divisors of elements of $T$ in $FI_A$. Hence, some element of $T$ must be zero in $S$. Since $a$ and $b$ are not relators, we conclude that $a^2$, $ab$ or $ba$ must be a relator.

If $a^2$ is not a relator then $ab$ or $ba$ is a relator, so that $S$ is a homomorphic image of $S_3$. In this case, by Proposition 7.4, $S$ is isomorphic to $S_3$ or $S_{\mathbf{x},\mathbf{y}}$ for some $\mathbf{x}$ and $\mathbf{y}$ such that $(\mathbf{x},\mathbf{y}) \notin \mathcal{L}$.

By symmetry, the same conclusion follows if $b^2$ is not a relator.

The remaining case is that $a^2$ and $b^2$ are both relators, and then $S$ must be a homomorphic image of $S_4$, so that, by Proposition 7.8, $S$ is isomorphic to $S_4$, $S_5(i)$, $S_6(i)$, $S_7(j)$ or $S_8(j)$ for some $i \geq 2$ or $j \geq 1$. This completes the proof that (b) implies (c).

If $S$ satisfies any (and hence all) of these conditions then, in particular, $S$ is finitely presented, and by hypothesis does not satisfy a nontrivial semigroup identity, so that $S$ cannot have polynomial growth, by Theorem 6.2, whence $S$ has exponential growth, by [33, Corollary 1]. This completes the proof of the theorem. □

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