PRO-p COMPLETIONS OF PD_n-GROUPS

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Abstract. We sharpen earlier work on the pro-p completions of orientable PD_3-groups. There are four cases, and we give examples of aspherical 3-manifolds representing each case. In three of the four cases the new results are best possible. We also consider the pro-p completion of some orientable PD_n groups for $n \leqslant 5$, including surface-by-surface groups.

1. Introduction

There are two definitions of a profinite Poincaré duality group G of dimension n at a prime p [21, 3.4.6], [30]. Both definitions differ on whether the profinite group G should be of type FP_∞ over $\mathbb{Z}_p$ i.e. whether the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ has a projective resolution with all projectives finitely generated $\mathbb{Z}_p[[G]]$-modules. The groups that satisfy the definition of [30] we call strong profinite PD_n groups at p and the groups that satisfy the original definition of Tate from [21], [27] we call profinite PD_n groups at p. By [30] every strong profinite PD_n group at p is a profinite PD_n group at p. By definition a group G which is a strong PD_n group at p has cohomological p-dimension $cd_p(G) = n$, has type FP_∞ over $\mathbb{Z}_p$ and $Ext^i_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, \mathbb{Z}_p[[G]]) = 0$ for $i \neq n$ and $Ext^n_{\mathbb{Z}_p[[G]]}(G, \mathbb{Z}_p[[G]]) \simeq \mathbb{Z}_p$. If the action of G on $Ext^n_{\mathbb{Z}_p[[G]]}(G, \mathbb{Z}_p[[G]]) \simeq \mathbb{Z}_p$ is trivial G is called orientable. For pro-p groups the notions of strong profinite PD_n group at p and profinite PD_n group at p coincide. We call such groups pro-p PD_n groups.

We are interested in pro-p and profinite completions of orientable PD_n groups. The cases $n = 1$ or 2 are well understood. The pro-p completions of orientable PD_2-groups are pro-p PD_2-groups. These are also known as Demuškin groups, and were completely classified in terms of pro-p generators and relations in [5], [6], [19], [26]. (Not all such groups are pro-p completions of PD_2-groups.) Profinite and pro-p completions of PD_3 groups were studied by Kochloukova and Zalesski in [18] and by Weigel in [32]. Some results on pro-p completions for arbitrary n were obtained by Hillman, Kochloukova and Lima in [12]. The notion of orientable profinite Poincaré duality pairs (over $\mathbb{F}_p$) was first suggested by Kochloukova in [16] and a more general notion of (in general non-orientable) profinite Poincaré duality pairs was developed by Wilkes in [32].

Sections 2 and 3 contain some basic definitions, lemmas and results from earlier work. In Section 4 we build upon the results of [18], and prove the following theorem.

Theorem A. *Let G be an orientable Poincaré duality group of dimension 3 and let $\widehat{G}_p$ be the pro-p completion of G. Then exactly one of the following conditions holds:*

a) $\widehat{G}_p$ is cyclic or quaternionic;

b) $\widehat{G}_p$ is an orientable pro-p PD_3-group;

c) there is no upper bound on the deficiency of the subgroups of finite index in $\widehat{G}_p$;

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d) $\widehat{G}_p$ *is* $\mathbb{Z}_p$ *or* $\widehat{D_{\infty 2}}$.

The statements of cases (a) and (d) sharpen the corresponding statements in [18, Thm B]. We also give simple criteria for when they arise. Here our results are essentially complete (except for $p = 2$). Several equivalent criteria for case (b) were given in [18, Thm A]. We augment these criteria, as a corollary of Theorem 4.4. This theorem also implies that $\widehat{G}_p$ cannot have cohomological p-dimension 2. However it is not yet clear what else might occur, and we do not have simple criteria for recognizing case (c).

In §5 we give examples of geometric flavour for each of the four cases listed above. We give an example of case (c) in which $\widehat{G}_p$ is a free pro-p group of rank 2, for all primes p. (We do not know whether there are examples of case (c) in which $\widehat{G}_p$ has cohomological p-dimension at least 3.)

In [27] Serre called an abstract group G good if for every finite G-module M the map $H^i(\widehat{G}, M) \to H^i(G, M)$, induced by the canonical map $G \to \widehat{G}$, is an isomorphism, where $\widehat{G}$ denotes the profinite completion of G. The group B is called p-good if for every finite pro-p $\mathbb{Z}_p[[\widehat{B}_p]]$-module M we have that the canonical map $B \to \widehat{B}_p$ induces an isomorphism $H^i(\widehat{B}_p, M) \to H^i(B, M)$, where $\widehat{B}_p$ is the pro-p completion of B. Groups that are p-good were previously studied in [14], [16], [18]. In [8] the term p-good group was used with a different (but related) meaning.

For a set $\mathcal{T}$ of normal subgroups of p-power index in a discrete group B we say that $\mathcal{T}$ is directed if for every $U_1, U_2 \in \mathcal{T}$ we have that there is $U \subseteq U_1 \cap U_2$ such that $U \in \mathcal{T}$. In the next section we give criteria for groups of type FP_∞ and with additional structure to be good, or p-good, and we prove the following theorem.

Theorem B. *Let* $1 \to A \to B \to C \to 1$ *be a short exact sequence of groups such that* A *is an orientable surface group and* C *is an orientable* PD_s *group, where* $s = 2$ *or* $s = 3$. *Let* $\mathcal{T}$ *be a directed set of normal subgroups of* p*-power index in* B *that defines the pro-*p *topology of* B. *Suppose that* $\varprojlim_{U \in \mathcal{T}} H_1(U \cap A, \mathbb{F}_p) = 0$ *and furthermore if* $s = 3$, *there is an upper bound on the deficiency of the subgroups of finite index in* $\widehat{C}_p$. *Then*

a) if $s = 2$, *then* $\widehat{B}_p$ *is a pro-*p PD_4*-group and* B *is* p*-good.*

b) if $s = 3$, *then* $\widehat{B}_p$ *is virtually a pro-*p PD_k*-group for some* $k \in \{2, 3, 5\}$. *If* $k = 5$ *then* B *and* C *are* p*-good.*

If additionally B *is orientable and* p*-good then* $\widehat{B}_p$ *is an orientable pro-*p PD_{2+s}*-group.*

The case of profinite completions of orientable PD_4-groups is easier and is considered in Proposition 6.6.

It is an open problem whether there is an orientable PD_n-group G such that $\widehat{G}_p$ is an orientable pro-p PD_n-group and G is not p-good.

The main result of Section 7 is the following theorem.

Theorem C. *Let* $1 \to A \to B \to C \to 1$ *be a short exact sequence of groups such that* $A \simeq \mathbb{Z}^2$, B *is an orientable* PD_4 *group and* C *is an orientable surface group. Then one of the following holds:*

a) $\widehat{B}_p$ *is an orientable pro-*p PD_4*-group and* B *is* p*-good;*

b) $\widehat{B}_p$ is an orientable pro-p PD_2 group and the image of A in $\widehat{B}_p$ is trivial.

Remark. If B is not orientable, $p \neq 2$, then there is a third option for the closure $\overline{A}$ of the image of A in $\widehat{B}_p$ to be virtually $\mathbb{Z}_p$.

We show also that if B is an orientable PD_4 group and $\chi(B) = 0$ then $\widehat{B}_p$ cannot be a pro-p PD_3-group, and we give examples of orientable PD_4-groups which are fundamental groups of bundles with base and fibre aspherical closed surfaces, and for which the projection to the base induces an isomorphism on pro-p completions, for all primes p.

In [12] it was shown that under some conditions the pro-p completion of an orientable PD_n group is virtually a pro-p PD_r-group, for $r \leqslant n, r \neq n-1$. In the final Section 8 we give an example of an aspherical 5-manifold with perfect fundamental group, which completes the discussion of examples with "dimension drop" $n - r \neq 1$ in [12]. We do not know of any orientable PD_n-group G whose pro-p completion $\widehat{G}_p$ is virtually a pro-p PD_{n-1}-group. Note that if $\widehat{G}_p$ is virtually a pro-p PD_{n-1}-group then G has a subgroup of p-power index H such that $\widehat{H}_p$ is a pro-p PD_{n-1}-group and by Theorem 4.4 this is impossible for $n = 3$.

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2. Preliminaries

Let G be a group, and let $\{\gamma_i G\}$ be its lower central series, with $\gamma_1 G = G$ and $\gamma_{i+1}G = [G, \gamma_i G]$ for all $i \geqslant 1$. If p is a prime, let $X^p(G)$ be the subgroup generated by all pth powers of members of G. Let $D_\infty = \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$ be the infinite dihedral group.

Let G be a profinite group. By definition $\mathbb{Z}_p[[G]] = \varprojlim \frac{\mathbb{Z}}{p^i\mathbb{Z}}[[G/U]]$, where the inverse limit is over all $i \geqslant 1$ and U open subgroups of G. And $\mathbb{F}_p[[G]] = \mathbb{Z}_p[[G]]/p\mathbb{Z}_p[[G]] = \varprojlim \mathbb{F}_p[[G/U]]$ where the inverse limit is over all open subgroups U of G.

When G is a group, $H_i(G, V)$ denotes the ith homology of G in the respective category. Thus if G is an abstract group V is a $\mathbb{Z}G$-module, if G is a pro-p group V is a pro-p $\mathbb{Z}_p[[G]]$-module and if G is a profinite group V is a profinite $\widehat{\mathbb{Z}}[[G]]$-module. Furthermore $H^i(G, W)$ denotes the ith cohomology of G in the respective category. If G is an abstract group W is a $\mathbb{Z}G$-module, if G is a pro-p group or more generally a profinite group W is a discrete G-module and so $W = \cup W^U$ where the union is over all open subgroups U of G. In our applications V and W will be finite.

Since the pro-P completions of $\mathbb{Z}$ and of surface groups (PD_2-groups) are well understood, the first interesting case is in dimension 3.

Theorem 2.1. [18, Thm B] *Let G be an orientable Poincaré duality group of dimension 3 and let $\widehat{G}_p$ be the pro-p completion of G. Then exactly one of the following conditions holds*

a) $\widehat{G}_p$ is finite;

b) $\widehat{G}_p$ is an orientable pro-p PD_3-group;

c) there is no upper bound on the deficiency of the subgroups of finite index in $\widehat{G}_p$;

d) $\widehat{G}_p$ is virtually $\mathbb{Z}_p$.

By the proof of Theorem 2.1 if $\varprojlim_{U\in\mathcal{T}} H_2(U,\mathbb{F}_p)=0$ then case b) from Theorem 2.1 holds. Furthermore if a), b), c) do not hold (and so d) holds) then $\varprojlim_{U\in\mathcal{T}} H_2(U,\mathbb{F}_p)\simeq\mathbb{F}_p$.

Theorem 2.2. [14, Thm. 4] *Let G be an abstract Poincaré duality group of dimension m and let C be a directed set of normal subgroups of finite index in G. Suppose further that there is a subgroup G_0 of finite index in G such that G_0 is orientable, that there is some $U_0\in C$ such that $U_0\subseteq G_0$ and that, for all $i\geqslant 1$,*

$$\varprojlim_{U\in C} H_i(U,\mathbb{F}_p)=0.$$

Then $\widehat{G}_C$ is a strong profinite Poincaré duality group of dimension m at p, $\widehat{(G_0)}_C$ is a strong profinite Poincaré duality group of dimension m at p and $\chi_p(\widehat{G}_C)=\chi(G)$.

3. Some auxiliary results

We will need the following simple lemmas.

Lemma 3.1. *Let S be an orientable PD_2-group with a subnormal subgroup D of index p^k, where p is prime, and let $j:D\to S$ be the inclusion. Then $H_2(j;\mathbb{F}_p)=0$ and $H^2(j;\mathbb{F}_p)=0$.*

Proof. Since D is subnormal there is a chain $D=D_1<\cdots<D_m=S$, where D_i is normal in D_{i+1} and $[D_{i+1}:D_i]=p$, for all $i<m$. It shall suffice to show that $H_2(D_i;\mathbb{F}_p)\to H_2(D_{i+1};\mathbb{F}_p)$ is the zero map. Thus we can assume that D is a normal subgroup of S of index p.

Let $j_*=H_*(j;\mathbb{F}_p)$ and $j^*=H^*(j;\mathbb{F}_p)$, for simplicity of notation. Let $x\in H^1(S;\mathbb{F}_p)=Hom(S,\mathbb{Z}/p\mathbb{Z})$ be an epimorphism with kernel D. Since S is an orientable PD_2-group and $x\neq 0$ there is a $y\in H^1(S;\mathbb{F}_p)$ such that $x\cup y$ generates $H^2(S;\mathbb{F}_p)$. If we evaluate $x\cup y$ on the image of a class $\delta\in H_2(D;\mathbb{F}_p)$ we get $(x\cup y)(j_2\delta)=j^*(x\cup y)(\delta)=(j^*x\cup j^*y)(\delta)=0$, since $j^*x=0$ is the restriction of x to D. Hence $j_2\delta=0$, for all δ, and so $H_2(j;\mathbb{F}_p)=0$. The dual result $H^2(j;\mathbb{F}_p)=0$ follows immediately. □

For an abstract group U denote by $\widehat{U}_p$ the pro-p completion of U.

Lemma 3.2. *Let G be an abstract group, $\mathcal{M}$ be a directed set of normal subgroups of p-power index in G that define the pro-p topology on G . Then $\varprojlim_{M\in\mathcal{M}} H_1(M,\mathbb{F}_p)=0$.*

Proof. Let $\overline{M}$ be the closure of $M\in\mathcal{M}$ in $\widehat{G}_p$. Then since $\overline{M}\simeq\widehat{M}_p$ and $\bigcap_{M\in\mathcal{M}}\overline{M}=1$ we have

$$\varprojlim_{M\in\mathcal{M}} H_1(M,\mathbb{F}_p)\simeq\varprojlim_{M\in\mathcal{M}} H_1(\overline{M},\mathbb{F}_p)\simeq H_1(\varprojlim_{M\in\mathcal{M}}\overline{M},\mathbb{F}_p)=H_1(\bigcap_{M\in\mathcal{M}}\overline{M},\mathbb{F}_p)=0.$$

□

Proposition 3.3. *Let $1\to A\to B\to C\to 1$ be a short exact sequence of abstract groups. Let $\mathcal{T}$ be a directed set of subgroups in B. Suppose that each $H_j(U\cap A,\mathbb{F}_p)$ is finite and $\varprojlim_{U\in\mathcal{T}} H_j(U\cap A,\mathbb{F}_p)=0$. Then $\varprojlim_{U\in\mathcal{T}} H_i(U/(U\cap A),H_j(U\cap A,\mathbb{F}_p))=0$ for $i\geqslant 0$.*

Proof. Set $M_U = H_j(U \cap A, \mathbb{F}_p)$ and $V_U = U/(U \cap A)$ a subgroup of C. Let

$$\mathcal{R} : \ldots \to R_i \to R_{i-1} \to \ldots \to R_0 \to \mathbb{Z} \to 0$$

be a free resolution of the trivial $\mathbb{Z}C$-module $\mathbb{Z}$. Then $H_i(V_U, M_U) = H_i(\mathcal{R}_U)$, where $\mathcal{R}_U = \mathcal{R} \otimes_{V_U} M_U$. The maps of the inverse system $\{H_i(V_U, M_U) \mid U \in \mathcal{T}\}$ can be described as follows: if $U_1, U_2 \in \mathcal{T}$, where $U_1 \subseteq U_2$ the map $\varphi_{U_1,U_2} : H_i(V_{U_1}, M_{U_1}) \to H_i(V_{U_2}, M_{U_2})$ is induced by the map $id_{\mathcal{R}} \otimes d_{U_1,U_2}$: $\mathcal{R} \otimes_{V_{U_1}} M_{U_1} \to \mathcal{R} \otimes_{V_{U_2}} M_{U_2}$ that sends $r_i \otimes m$ to $r_i \otimes d_{U_1,U_2}(m)$ for $r_i \in R_i$ and $d_{U_1,U_2} : M_{U_1} \to M_{U_2}$ is induced by the inclusion map $U_1 \cap A \to U_2 \cap A$.

Since $\varprojlim_{U \in \mathcal{T}} M_U = 0$ and each M_U is finite, then for every $U_2 \in \mathcal{T}$ there is U_1 as above such that d_{U_1,U_2} is the zero map. Then φ_{U_1,U_2} is the zero map and hence $\varprojlim_{U \in \mathcal{T}} H_i(V_U, M_U) = 0$. □

Lemma 3.4. *Let A be an orientable surface group. Let $\mathcal{S}$ be a directed set of normal subgroups of p-power index in A. Suppose that $\varprojlim_{U \in \mathcal{S}} H_1(U, \mathbb{F}_p) = 0$. Then the completion $\overline{A} = \varprojlim_{U \in \mathcal{S}} A/U$ of A with respect to $\mathcal{S}$ is isomorphic to the pro-p completion $\widehat{A}_p$.*

Proof. Consider the cellular chain complex associated to the standard Cayley complex of A, i.e.

$$\mathcal{R} : 0 \to \mathbb{Z}A \to (\mathbb{Z}A)^d \to \mathbb{Z}A \to \mathbb{Z} \to 0$$

Consider the complexes $\overline{\mathcal{R}} = \mathbb{F}_p[[\overline{A}]] \otimes_{\mathbb{Z}A} \mathcal{R}$ and $\widehat{\mathcal{R}} = \mathbb{F}_p[[\widehat{A}_p]] \otimes_{\mathbb{Z}A} \mathcal{R}$. By [18, Lemma 2.1] $H_i(\overline{\mathcal{R}}) = \varprojlim_{U \in \mathcal{S}} H_i(U, \mathbb{F}_p)$. Thus $H_1(\overline{\mathcal{R}}) = 0$.

Note that $\widehat{\mathcal{R}}$ is a free resolution of the trivial $\mathbb{F}_p[[\widehat{A}_p]]$-module $\mathbb{F}_p$. Let $\mathcal{T}$ be the directed set of all normal subgroups of p-power index in A. Let $K = Ker(\widehat{A}_p \to \overline{A})$ and $\widehat{A}_p = \varprojlim_{U \in \mathcal{T}} A/U \to \overline{A} = \varprojlim_{U \in \mathcal{S}} A/U$ is the epimorphism induced by the identity maps $id_{A/U}$ for $U \in \mathcal{S} \subseteq \mathcal{T}$. Thus $\overline{\mathcal{R}} \simeq \mathbb{F}_p \otimes_{\mathbb{F}_p[[K]]} \widehat{\mathcal{R}}$. Then

$$H_1(K, \mathbb{F}_p) = H_1(\mathbb{F}_p \otimes_{\mathbb{F}_p[[K]]} \widehat{\mathcal{R}}) \simeq H_1(\overline{\mathcal{R}}) = 0$$

Hence $K = 1$ and $\widehat{A}_p \simeq \overline{A}$. □

Lemma 3.5. *Let G be a group with pro-p completion $\widehat{G}_p$. Denote $\mu^i : H^i(\widehat{G}_p, \mathbb{F}_p) \to H^i(G, \mathbb{F}_p)$ the map induced by the canonical map $G \to \widehat{G}_p$. Then we have a commutative diagram*

$$\begin{tikzcd}
H^i(\widehat{G}_p, \mathbb{F}_p) \times H^j(\widehat{G}_p, \mathbb{F}_p) \arrow[r, "\cup"] \arrow[d, "\mu^i \times \mu^j"'] & H^{i+j}(\widehat{G}_p, \mathbb{F}_p) \arrow[d, "\mu^{i+j}"] \\
H^i(G, \mathbb{F}_p) \times H^j(G, \mathbb{F}_p) \arrow[r, "\cup"] & H^{i+j}(G, \mathbb{F}_p)
\end{tikzcd}$$

where the horizontal maps are the cup products in the categories of pro-p and abstract groups.

Proof. Following [27] consider the set $C^n(\widehat{G}_p, \mathbb{F}_p)$ of all continuous maps $\widehat{G}_p^n \to \mathbb{F}_p$. Then there is a map $\cup : C^i(\widehat{G}_p, \mathbb{F}_p) \times C^j(\widehat{G}_p, \mathbb{F}_p) \to C^{i+j}(\widehat{G}_p, \mathbb{F}_p)$ defined by $(f \cup h)(g_1, \ldots, g_{i+j}) = f(g_1, \ldots, g_i)h(g_{i+1}, \ldots, g_{i+j})$ that induces the cup product $\cup : H^i(\widehat{G}_p, \mathbb{F}_p) \times H^j(\widehat{G}_p, \mathbb{F}_p) \to H^{i+j}(\widehat{G}_p, \mathbb{F}_p)$.

Similarly we can consider the set $C_0^n(G,\mathbb{F}_p)$ of all maps $G^n \to \mathbb{F}_p$. Then there is a map $\cup$: $C_0^i(G,\mathbb{F}_p) \times C_0^j(G,\mathbb{F}_p) \to C_0^{i+j}(G,\mathbb{F}_p)$ defined by $(f \cup h)(g_1,\dots,g_{i+j}) = f(g_1,\dots,g_i)h(g_{i+1},\dots,g_{i+j})$ that induces the cup product $\cup : H^i(G,\mathbb{F}_p) \times H^j(G,\mathbb{F}_p) \to H^{i+j}(G,\mathbb{F}_p)$.

The canonical map $G \to \widehat{G}_p$ induces maps $\nu^i : C^i(\widehat{G}_p,\mathbb{F}_p) \to C_0^i(G,\mathbb{F}_p)$, that induce the maps μ^i. Then by the definition of the cup product we have a commutative diagram

$$\begin{array}{ccc} C^i(\widehat{G}_p,\mathbb{F}_p) \times C^j(\widehat{G}_p,\mathbb{F}_p) & \xrightarrow{\ \cup\ } & C^{i+j}(\widehat{G}_p,\mathbb{F}_p) \\ {\scriptstyle \nu^i\times\nu^j}\big\downarrow & & \big\downarrow{\scriptstyle \nu^{i+j}} \\ C_0^i(G,\mathbb{F}_p) \times C_0^j(G,\mathbb{F}_p) & \xrightarrow{\ \cup\ } & C_0^{i+j}(G,\mathbb{F}_p) \end{array}$$

and this commutative diagram induces the commutative diagram from the statement of the lemma. $\square$

4. PRO-p COMPLETIONS OF PD_3-GROUPS

In this section we shall sharpen some of the results of [18]. Cases (a), (b), (c) and (d) shall refer to the four possibilities in the statement of Theorem 2.1.

We begin by refining the statements of cases (a) and (d). Theorem A of the introduction is then an immediate consequence.

Lemma 4.1. *Let G be a finitely generated group and p be a prime, and let K be the kernel of the natural homomorphism from G to $\widehat{G}_p$. Suppose that $\widehat{G}_p$ is virtually $\widehat{\mathbb{Z}}_p$. Then G/K has a finite normal subgroup F which is a p-group, and such that $G/K \cong F \rtimes \mathbb{Z}$ if p is odd, while the quotient of G/K by F is $\mathbb{Z}$ or D_∞ if $p = 2$.*

Proof. Since $\widehat{G}_p$ is virtually $\widehat{\mathbb{Z}}_p$, there is a short exact sequence

$$0 \to \widehat{\mathbb{Z}}_p \to \widehat{G}_p \to T \to 0,$$

where T is a finite p-group. Hence there is a short exact sequence $1 \to A \to G/K \to T \to 1$, where $A \cong \mathbb{Z}$. Therefore G/K has two ends, and so it has a maximal finite normal subgroup F with quotient $\mathbb{Z}$ or D_∞. The subgroup F maps injectively to T, and so is a p-group. If p is odd then A is central, since $Aut(A) = \{\pm 1\}$ has order 2. Since $[G/K : A]$ is finite, $(G/K)'$ is finite, by a lemma of Schur. Hence $G/K \cong F \rtimes \mathbb{Z}$, with F finite. $\square$

Theorem 4.2. *Let G be an orientable PD_3-group and p be a prime. Then*

(1) if $\widehat{G}_p$ is finite then it is cyclic or quaternionic;

(2) if $\widehat{G}_p$ is virtually $\widehat{\mathbb{Z}}_p$ then either $\widehat{G}_p \cong \widehat{\mathbb{Z}}_p$ or $p = 2$ and $\widehat{G}_2 \cong \widehat{D_\infty}_2$.

Proof. Let K be the kernel of the natural homomorphism from G to $\widehat{G}_p$. Suppose first that $\widehat{G}_p$ is finite. Then $P = G/K \cong \widehat{G}_p$ is a finite p-group, and K has no quotient which is a finite p-group. Hence $H_1(K;\mathbb{Z})$ is finite, of order prime to p, and $H^1(K;\mathbb{Z}) = 0$. Consider the LHS spectral sequence

$$E_2^{p,q} = H^p(P; H^q(K;\mathbb{Z})) \Rightarrow H^{p+q}(G;\mathbb{Z})$$

for the cohomology of G. The group P acts trivially on $H^0(K;\mathbb{Z})$ and $H^3(K;\mathbb{Z})$, since G is orientable, while $H^1(K;\mathbb{Z}) = 0$ and $H^2(K;\mathbb{Z}) \cong H_1(K;\mathbb{Z})$. Therefore $E_2^{p,q} = 0$ if $p > 0$ and $q \neq 0$ or 3, or if $p = 0$ and $q \neq 0$, 2 or 3.

A slight extension of the argument of [1, Lemma IV.6.2] shows that P has periodic cohomology (with period dividing 4). Since P is a p-group, it must be either cyclic (of prime power order) or quaternionic (of order a power of 2). (Note that a finite group has periodic cohomology if and only if every abelian subgroup is cyclic.)

Suppose now that $\widehat{G}_p$ is virtually $\widehat{\mathbb{Z}}_p$. It follows from Lemma 4.1 that G has normal subgroups $K < L$ such that $G/L \cong \mathbb{Z}$ or D_∞ and $J = L/K$ is a finite p-group, while K has no non-trivial quotient which is a p-group. On passing to a subgroup of index 2 in G, if necessary, we may assume that $G/L \cong \mathbb{Z}$.

Let $\Lambda = \mathbb{F}_p[G/K]$. Then $H_i(K;\mathbb{F}_p) = H_i(G;\Lambda)$, for all i. Clearly $H_0(K;\mathbb{F}_p) = \mathbb{F}_p$ and $H_1(K;\mathbb{F}_p) = 0$, while $H_i(K;\mathbb{F}_p) = 0$ for $i > 2$, since K has infinite index in G and so $cd(K) < 3$ [29]. We also have $H_2(K;\mathbb{F}_p) = H_2(G;\Lambda) \cong H^1(G;\Lambda)$, by Shapiro's Lemma and Poincaré duality. Now $H_0(G;\Lambda) = \mathbb{F}_p$ and $H_1(G;\Lambda) = H_1(K;\mathbb{F}_p) = 0$. Hence $H^1(G;\Lambda) \cong Ext^1_\Lambda(\mathbb{F}_p,\Lambda)$, by the Universal Coefficient spectral sequence (or by an *ad hoc*low-degree argument). Since G/K has two ends, $Ext^1_\Lambda(\mathbb{F}_p,\Lambda) \cong \mathbb{F}_p$. Thus we conclude that $H_2(K;\mathbb{F}_p) \cong H^1(G;\Lambda) \cong \mathbb{F}_p$.

The LHS spectral sequence for the $\mathbb{F}_p$-cohomology of L associated to the extension $1 \to K \to L \to T \to 1$ may be identified with the Leray-Serre spectral sequence for the fibration of $K(L,1)$ over $K(T,1)$. The fibre $K(K,1)$ is a $\mathbb{F}_p$-homology 2-sphere. Since T is a p-group it acts trivially on $\mathbb{F}_p$, and therefore acts trivially on $H_*(K;\mathbb{F}_p)$. Hence this spectral sequence reduces to a Gysin sequence

$$\cdots \to H^{k+2}(L;\mathbb{F}_p) \to H^k(T;\mathbb{F}_p) \to H^{k+3}(T;\mathbb{F}_p) \to H^{k+3}(L;\mathbb{F}_p) \to \dots,$$

where the middle homomorphism is given by cup-product with a class $z \in H^3(T;\mathbb{F}_p)$, as in [20, Example 5.C]. Since $H^i(L;\mathbb{F}_p) = 0$ for $i \geqslant 3$ it follows that these cup products induce isomorphisms $H^i(T;\mathbb{F}_p) \to H^{i+3}(T;\mathbb{F}_p)$, for all $i \geqslant 0$. Hence T has cohomological period (dividing) 3. But a non-trivial finite group with periodic cohomology has even cohomological period [3, Exercise VI.9.1]. Hence T must be trivial. □

Theorem A now follows immediately from Theorems 2.1 and 4.2.

We may easily identify the orientable PD_3-groups with pro-p completion of type (a) or (d), when p is odd. (We do not yet have a comparably simple characterization when $p = 2$.)

Corollary 4.3. *If p is an odd prime then $\widehat{G}_p$ is finite if and only if G/G' is finite and has cyclic p-torsion, while $\widehat{G}_p \cong \widehat{\mathbb{Z}}_p$ if and only if $G/G' \cong \mathbb{Z} \oplus T$, where T is finite and $(p,|T|) = 1$.* □

The criteria for recognizing when case (b) or (c) occurs are less complete.

Theorem 4.4. *Let G be an orientable PD_3-group. If the restriction from $H^3(\widehat{G}_p;\mathbb{F}_p)$ to $H^3(G;\mathbb{F}_p)$ is trivial then $\widehat{G}_p$ is a free pro-p group. In particular, $cd_p(\widehat{G}_p) \neq 2$, and so $\widehat{G}_p$ cannot be a Demuškin group.*

Proof. Let $j : G \to \widehat{G}_p$ be the canonical homomorphism. Then $H_1(j;\mathbb{F}_p)$ and $H^1(j;\mathbb{F}_p)$ are isomorphisms, while $H_2(j;\mathbb{F}_p)$ is an epimorphism and $H^2(j;\mathbb{F}_p)$ is a monomorphism, for any group G If $\gamma \in H^2(\widehat{G}_p;\mathbb{F}_p)$ is non-zero then there is an $\alpha \in H^1(\widehat{G}_p;\mathbb{F}_p)$ such that $j^*(\alpha \cup \gamma) = j * \alpha \cup j^*\gamma \neq 0$, by the non-degeneracy of Poincaré duality for G. Hence if $H^3(j)=0$ then $H^2(\widehat{G}_p;\mathbb{F}_p)=0$, and so $\widehat{G}_p$ is a free pro-p group [27, Prop. 21]. ◻

Proofs of much of the following corollary can be found in [18], but the arguments here differ in some respects. Condition (3) is closely related to one of the hypotheses in [18, Theorem 3.1], while (2) and the implication (4) ⇒ (1) appear to be new. We recall that $H^i(\widehat{G}_p;\mathbb{F}_p)\cong \varinjlim H^i(G/U;\mathbb{F}_p)$, for all i, the limit being taken over the directed system of normal subgroups U of p-power index in G.

Corollary 4.5. *Let G be an orientable PD_3-group and p be a prime. Then the following are equivalent.*

(1) $\widehat{G}_p$ is a pro-p PD_3-group;

(2) G has a normal subgroup U of p-power index such that inflation from $H^2(G/U;\mathbb{F}_p)$ to $H^2(G;\mathbb{F}_p)$ is an epimorphism, and each such U has a proper subgroup $V<U$ which is normal and of p-power index in G and such that inflation from $H^3(G/U;\mathbb{F}_p)$ to $H^3(G/V;\mathbb{F}_p)$ has rank 1;

(3) every subgroup $U<G$ of p-power index has a proper subgroup $V<U$ of p-power index which is normal in U and such that inflation from $H^2(U/V;\mathbb{F}_p)$ to $H^2(U;\mathbb{F}_p)$ is an epimorphism;

(4) $\widehat{G}_p$ has cohomological p-dimension 3 and $\chi(\widehat{G}_p)=0$.

Proof. Let $j:G\to\widehat{G}_p$ be the canonical homomorphism. If $\widehat{G}_p$ is a pro-p PD_3-group then

$$\beta_2(\widehat{G}_p;\mathbb{F}_p)=\beta_1(\widehat{G}_p;\mathbb{F}_p)=\beta_1(G;\mathbb{F}_p)=\beta_2(G;\mathbb{F}_p),$$

and so $H^2(j;\mathbb{F}_p)$ is an isomorphism. Therefore G has a normal subgroup U such that $[G:U]$ is a power of p, $H^1(G/U;\mathbb{F}_p)\cong H^1(G;\mathbb{F}_p)$ and inflation from $H^2(G/U;\mathbb{F}_p)$ to $H^2(G;\mathbb{F}_p)$ is onto. Since $H^1(\widehat{G}_p;\mathbb{F}_p)\neq 0$ it follows that inflation from $H^3(G/U;\mathbb{F}_p)$ to $H^3(G;\mathbb{F}_p)$ is an epimorphism, by the non-degeneracy of Poincaré duality for G. Since $\varinjlim H^3(G/U;\mathbb{F}_p)=H^3(\widehat{G}_p;\mathbb{F}_p)\cong\mathbb{F}_p$, there is in turn a subgroup $V<U$ which is normal and of p-power index in G and such that inflation from $H^3(G/U;\mathbb{F}_p)$ to $H^3(G/V;\mathbb{F}_p)$ has rank 1, i.e, the image is $\mathbb{F}_p$. Hence (1) ⇒ (2).

Conversely, if these conditions hold then $\widehat{G}_p$ is infinite, since G has subgroups of unbounded p-power index, and $H^2(j;\mathbb{F}_p)$ is an isomorphism. (In particular, $\widehat{G}_p$ is not a free pro-p group.) Moreover, there is a sequence $U_{i+1}<U_i$ of normal subgroups of p-power index such that the inflation from $H^2(G/U_i;\mathbb{F}_p)$ to $H^2G;\mathbb{F}_p)$ is an epimorphism and the inflation from $H^3(G/U_i;\mathbb{F}_p)$ to $H^3(G/U_{i+1};\mathbb{F}_p)$ has rank 1. This together with Theorem 4.4 implies that $H^2(j;\mathbb{F}_p)$ is an isomorphism and that $H^3(\widehat{G}_p;\mathbb{F}_p)\cong\mathbb{F}_p$. Hence $H^3(j;\mathbb{F}_p)$ is also an isomorphism, by the Theorem, and so (2) ⇒ (1), by [27, Sect. 4.5, Prop. 32], together with Lemma 3.5.

A similar argument applies for each subgroup of finite index in G, since such subgroups are also orientable PD_3-groups. Hence (1) ⇒ (3).

If (3) holds and $j_U:U\to\widehat{U}_p$ is the canonical map then $H^2(j_U;\mathbb{F}_p)$ is an isomorphism, so $\beta_2(\widehat{G}_p;\mathbb{F}_p)=\beta_2(G;\mathbb{F}_p)$. Hence $H_2(j_U;\mathbb{F}_p)$ is also an isomorphism, and so (3) ⇒ (4), by [18, Theorem 3.1].

If (4) holds then $H^3(j;\mathbb{F}_p)\neq 0$, since $\widehat{G}_p$ has cohomological p-dimension >1, by Theorem 4.4. Since $\beta_3(\widehat{G}_p;\mathbb{F}_p)\geqslant 1=\beta_3(G;\mathbb{F}_p)$ and $\chi(\widehat{G}_p)=\chi(G)=0$, we have

$$\beta_2(\widehat{G}_p;\mathbb{F}_p)\geqslant\beta_1(\widehat{G}_p;\mathbb{F}_p)=\beta_1(G;\mathbb{F}_p)=\beta_2(G;\mathbb{F}_p).$$

On the other hand $H_2(j;\mathbb{F}_p)$ is surjective, hence $\beta_2(\widehat{G}_p;\mathbb{F}_p) \leqslant \beta_2(G;\mathbb{F}_p)$. Hence $\beta_2(\widehat{G}_p;\mathbb{F}_p) = \beta_2(G;\mathbb{F}_p)$ and $\beta_3(\widehat{G}_p;\mathbb{F}_p) = \beta_3(G;\mathbb{F}_p) = 1$, and so $H_*(j;\mathbb{F}_p)$ and $H^*(j;\mathbb{F}_p)$ are isomorphisms in all degrees. It then follows from Lemma 3.5 that $\widehat{G}_p$ is a pro-p PD_3-group. (See also [31, Prop. 3.2].) Thus (4) $\Rightarrow$ (1). $\square$

We note that whether there is a PD_3-group G with $3 < cd_p(\widehat{G}_p) < \infty$ remains open.

We remark finally that if G is finitely generated then the order of the torsion subgroup of G/G' is divisible by only finitely many primes. Hence if G is an orientable PD_3-group then $\widehat{G}_p$ is finite for all primes p if and only if either G/G' is finite cyclic or G/G' is the direct sum of a finite cyclic group with a cyclic 2-group, and the 2-lower central series of G terminates after finitely many steps. If $\widehat{G}_p$ is of type (b) or (c) for all primes p then $\beta_1(G;\mathbb{Q}) \geqslant 2$. Every pro-$p$ completion of G is of type (d) if and only if $G/G' \cong \mathbb{Z}$.

5. EXAMPLES ILLUSTRATING THEOREM 2.1

In this section we shall gives examples of aspherical 3-manifolds whose fundamental groups represent each of the four cases of Theorem 2.1.

Examples with $\widehat{G}_p$ cyclic are easily found. If M is an aspherical Seifert fibred homology 3-sphere then it admits a natural S^1-action with finitely many exceptional orbits with nontrivial finite isotropy subgroups. If n is prime to the orders of these isotropy subgroups then the subgroup of nth roots of unity in S^1 acts freely on M, with quotient $\overline{M}$, say. Hence $G = \pi_1(\overline{M})$ is an orientable PD_3-group with perfect commutator subgroup $G' = G''$ and $G/G' \cong \mathbb{Z}/n\mathbb{Z}$. In particular, if $n = p^k$ for some prime p and $k \geqslant 1$ then $\widehat{G}_p \cong \mathbb{Z}/p^k\mathbb{Z}$. (For example, if q, r, s are pairwise relative prime and $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} < 1$ then the Brieskorn manifold $M(q,r,s)$ is an aspherical $\mathbb{Z}$-homology 3-sphere, and we may take p relatively prime to qrs.)

Since the quaternionic groups $Q(2^n)$ act freely on S^3 (for all $n \geqslant 3$), the Dehn surgery argument of [4, Theorem 2.6] may be used to show that these groups act freely on hyperbolic $\mathbb{Q}$-homology 3-spheres. By taking the Dehn surgery slope to be a large enough odd number we may ensure that $H_1(M;\mathbb{Z})$ has odd order, where M is the resulting $\mathbb{Q}$-homology 3-sphere. The quotient $\overline{M} = M/(Q(2^n))$ is then an aspherical orientable 3-manifold, and $G = \pi_1(\overline{M})$ is an orientable PD_3-group with $\widehat{G}_2 \cong Q(2^n)$. However we do not have explicit examples of this type.

The simplest example of case (b) of Theorem 2.1 is $G = \mathbb{Z}^3$, the fundamental group of the 3-torus. More generally, every finitely generated, torsion free nilpotent group is residually a finite p-group for all p, by Theorem 4 of [25, Chapter 1]. Thus the pro-p completion of a nilpotent PD_3-group is a pro-p Poincaré duality group of dimension 3. (This does not extend to the virtually nilpotent case. The group $G = \pi_1(M(3_1))$ mentioned below is virtually $\mathbb{Z}^3$, but $\widehat{G}_p \cong \widehat{\mathbb{Z}}_p$, for all primes p.)

Examples of aspherical 3-manifolds whose fundamental groups illustrate cases (c) and (d) may be constructed by surgery on links. Let $M(L)$ be the closed orientable 3-manifold obtained by 0-framed surgery on the components of an m-component link L in S^3. The fundamental group $\pi_1(M(L))$ is the quotient of the link group πL by the normal subgroup generated by the longitudes of L. The inclusion of a set of meridians determines a homomorphism from the free group $F(m)$ to the link group πL which induces an isomorphism on abelianization. If L is a boundary link (in particular, if $m = 1$ and so L is a knot) this homomorphism is split by an epimorphism from πL to $F(m)$, and the longitudes of L are in the kernel of any such epimorphism. The induced homomorphisms between the quotients of the lower central series $\pi L/\gamma_k \pi L \to F(m)/\gamma_k F(m)$ are isomorphisms, for all $k \geqslant 1$ [28]. Hence $J = \pi_1(M(L))$ is an extension of $F(m)$ by $\gamma_\omega J = \cap_{k\in\mathbb{N}}\gamma_k J$, and $\widehat{J}_p \cong \widehat{F(m)}_p$, for all primes p. In our first such example we shall show that $M(L)$ is aspherical; in the second we show that $M(L)$ must have an aspherical summand with the requisite properties.

For the first such example we shall let L be the link obtained by replacing each component of the Hopf link 2^2_1 by an untwisted Whitehead double [10, Figure 1.6]. (There is a choice involved, but that is irrelevant for our purposes.) This is a boundary link, since each component of L bounds a punctured torus inside a tubular neighbourhood of the corresponding component of the Hopf link.

The components of L are separated by a torus $T \subset S^3$. Each component of $S^3\backslash T$ is homeomorphic to $X(Wh)$, the exterior of the Whitehead link $Wh = 5^2_1$. (The notation 5^2_1 refers to the tables of knots and links in [24].) Then $M(L) \cong N \cup_f N$, where f is a homeomorphism between the boundaries of the two copies of the 3-manifold N obtained by attaching a solid torus to the boundary of $X(Wh)$ so that ∂D^2 is a longitude. We shall show that N is aspherical and the inclusion of $\pi_1(\partial N$ into $\nu = \pi_1(N)$ is injective. Hence $M(L)$ is aspherical and so $G = \pi_1(M(L))$ is a PD_3-group.

The link group πWh has a presentation

$$\langle a, b, w, x, y \mid axa^{-1} = bwb^{-1} = y,\ waw^{-1} = xax^{-1} = b,\ yxy^{-1} = w\rangle,$$

and the longitudes for a and x are represented by $\lambda_a = x^{-1}w$ and $\lambda_x = a^{-1}byx^{-1}$, respectively. We may assume that N is obtained by attaching $D^2 \times S^1$ to the component with meridian x, so that the image of λ_x in $\nu = \pi_1(N)$ is trivial. We have $\lambda_x = a^{-1}bab^{-1}$, since $y = axa^{-1}$. Hence ν has the presentation

$$\langle a, b, \lambda, x \mid axa^{-1} = bx\lambda b^{-1},\ a\lambda = \lambda a,\ xax^{-1} = b,$$
$$axa^{-1}xax^{-1}a^{-1} = x\lambda,\ ab = ba\rangle.$$

(Here we have written λ for λ_a and replaced w by $x\lambda$ and y by axa^{-1}.) This presentation simplifies to

$$\langle a, b, \lambda, x \mid a\lambda = \lambda a,\ ab = ba,\ xax^{-1} = b,\ x\lambda b^{-1}ax^{-1} = b^{-1}a\rangle,$$

since the relation $\lambda = x^{-1}axa^{-1}xax^{-1}a^{-1}$ follows from the others. Thus ν is an HNN extension with base the group $\langle a, b, \lambda\rangle \cong \mathbb{Z} \times F(2)$, associated subgroups $\langle a, \lambda b^{-1}\rangle$ and $\langle a, b\rangle$, and stable letter x. Hence ν has one end. The image of $\pi_1(\partial N)$ is the subgroup $\langle a, \lambda\rangle \cong \mathbb{Z}^2$, and so ∂N is incompressible in N. It follows from the exact sequence of $(N, \partial N)$ with coefficients $\mathbb{Z}[\nu]$, Poincaré-Lefshetz duality and the facts that ν has one end and the components of ∂N are aspherical that N is aspherical. (See [11, Lemma 3.1].)

In our second example the PD_3-group G does not map onto a nonabelian free group, although the pro-p completions of G are free pro-p groups. Let $L = L_1 \cup L_2$ be the 2-component link of [10, Figure 8.1]. The homomorphism from $F(2)$ to πL determined by a pair of meridians induces isomorphisms $F(2)/\gamma_n F(2) \cong \pi L/\gamma)n\pi L$, for all $n \geqslant 1$, and the longitudes of L lie in $\cap_{n\geqslant 1}\gamma_n \pi L$. Hence $\pi_1(M(L))/\gamma_n\pi_1(M(L)) \cong \pi L)/\gamma_n\pi L$, for all $n \geqslant 1$. The link L is not an homology boundary link: there is no epimorphism from πL to $F(2)$, and so $\pi_1(M(L))$ does not map onto $F(2)$. Thus if $M(L) = \sharp_{i=1}^r M_i$ is a factorization of $M(L)$ as a connected sum of indecomposables all but one of the summands must be homology 3-spheres. We may assume that M_1 is not an homology 3-sphere, and so $H_1(M_1; \mathbb{Z}) \cong \mathbb{Z}^2$. Hence M_1 is aspherical, since it is indecomposable, and $\pi_1(M_1)$ is infinite and not virtually $\mathbb{Z}$. (It is likely that $M(L)$ is itself aspherical, but we do not need to know this.) Thus $G = \pi_1(M_1)$ is an orientable PD_3-group. The natural epimorphism from $\pi_1(M)$ to G induces isomorphisms $\pi_1(M(L))/\gamma_n\pi_1(M(L)) \cong G/\gamma_n G$, for all $n \geqslant 1$, since the fundamental groups of the other summands of $M(L)$ are all perfect. Hence $F(2)/\gamma_n F(2) \cong G/\gamma_n G$, for all $n \geqslant 1$. On passing to the p-lower central series and pro-p completion, we conclude that $\widehat{F(2)}_p \cong \widehat{G}_p$, for all primes p.

The fundamental groups of orientable closed 3-manifolds which fibre over non-orientable aspherical surfaces give further examples of type (c). The simplest such are the semidirect products $G = \mathbb{Z} \rtimes_w C$, where C is a PD_2-group with orientation character $w : C \to \mathbb{Z}^\times$. Such groups G are orientable PD_3-groups. If C is orientable then $\widehat{G}_p$ is a pro-p PD_3-group, for all primes p. If C is non-orientable then $\widehat{G}_2$ is again a pro-2 PD_3-group, but if p is odd then $\widehat{G}_p \cong \widehat{C}_p$, by Lemma 6.9, and this is a finitely generated free pro-p group, by Lemma 6.5.

We have not yet found any examples of case (c) for which the pro-p completion is not a free pro-p group.

The simplest examples of case (d) of Theorem 2.1 are semidirect products $G = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, where $A \in SL(2,\mathbb{Z})$. All such groups are solvable PD_3-groups. If p is a prime such that $(\det(A-I), p) = 1$ then $\widehat{G}_p \cong \mathbb{Z}_p$, but if p divides $\det(A-I)$ then $\widehat{G}_p$ is a pro-p PD_3-group. (Note that if one eigenvalue of A is congruent to 1 *mod* (p) then so is the other, since they are mutually inverse.)

We may construct further examples of case (d) by 0-framed surgery on knots. If K is a nontrivial fibred knot then $M(K)$ fibres over S^1, with fibre F a closed orientable surface of genus $\geqslant 1$. Taking K to be the trefoil knot 3_1 or the figure-eight knot 4_1 gives examples with fibre the torus T, and $\pi_1(M(K)) \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, where $A = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, respectively. The knots $K = 6_2$ and 6_3 give examples with fibre of genus 2. Thus if K is a non-trivial fibred knot then $G = \pi_1(M(K))$ is a PD_3-group which is an extension of $\mathbb{Z}$ by the PD_2-group $\phi = \pi_1(F)$, and $G' = \phi$. Hence $G/G' \cong \mathbb{Z}$, and so the lower central series for G stabilizes at $\gamma_n G = \gamma_2 G = \phi$. Therefore $\widehat{G}_p \cong \widehat{\mathbb{Z}}_p$, for all primes p, and so G is in case (d).

Examples of dihedral type for case (d) may also be constructed in terms of knot theory, but require a little more work. Let K be a knot which is carried onto itself by an orientation-reversing involution h of S^3 which also reverses the orientation of K. (Such knots are said to be "strongly -amphicheiral".) We may assume that $h(X) = X$, where X is the exterior of K. Suppose also that h has just two fixed points. Then $Fix(h) \subset K$, and so h restricts to a fixed-point free involution of X which inverts the generator of $H_1(X;\mathbb{Z})$.

Let X_1 and X_2 be two copies of X, with a fixed homeomorphism $j : X_1 \to X_2$, and let $DX = X_1 \cup_{\partial X} X_2$ be the double of X along its boundary, obtained by setting $x = j(x)$ for all $x \in \partial X_1$. Then $H_3(DX;\mathbb{Z}) \cong \mathbb{Z}$ and so X is an orientable closed 3-manifold. We may define an involution ϕ by $\phi(x) = j(h(x))$ for $x \in X_1$ and $\phi(j(x)) = h(x)$ for $j(x) \in X_2$. This involution clearly acts freely on DX, and is orientation-preserving, so $M = DX/\langle\phi\rangle$ is a closed orientable 3-manifold. However ϕ inverts the generator of $H_1(DX;\mathbb{Z}) \cong \mathbb{Z}$, and so $G = \pi_1(M)$ maps onto D_∞, with kernel $\pi_1(DX)'$. The abelianization $\pi_1(DX)'/\pi_1(DX)''$ is annihilated by the Alexander polynomial $\Delta_K(t)$.

If K is the unknot then $X \cong S^1 \times D^2$, $DX \cong S^1 \times S^2$ and $M \cong \mathbb{RP}^3 \# \mathbb{RP}^3$, and so $G \cong D_\infty$. If K is non-trivial then X is aspherical and $\partial X \to X$ is π_1-injective, and so DX is aspherical. Hence M is aspherical . If $\Delta_K(t) \equiv 1$ *mod* (2) then $\pi_1(DX)'/\pi_1(DX)''$ is a torsion abelian group of odd exponent. It then follows easily that $\widehat{G}_2 \cong \widehat{D_{\infty 2}}$. The simplest example of such a knot is 8_3, which has Alexander polynomial $4t^2 - 9t + 4$.

6. Goodness and Theorem B

In this section we shall consider the profinite completion, as well as pro-p completions. We call G homologically good if for every finite G-module M the map $H_i(G, M) \to H_i(\widehat{G}, M)$, induced by the canonical map $G \to \widehat{G}$, is an isomorphism.

Lemma 6.1. *Let G be an abstract group of type FP_∞ and $\mathcal{T}$ be a directed set of finite index normal subgroups in G that defines the profinite topology of G. Then the following conditions are equivalent :*

a) G is homologically good;

b) for every finite G-module M we have that $\varprojlim_{U \in \mathcal{T}} H_i(U, M) = 0$ for $i \geqslant 1$;

c) for every prime p for the trivial G-module $\mathbb{F}_p$ we have that $\varprojlim_{U \in \mathcal{T}} H_i(U, \mathbb{F}_p) = 0$ for $i \geqslant 1$;

d) G is good.

Proof. a) implies b) Suppose first that G is homologically good. Then

$$\varprojlim_{U \in \mathcal{T}} H_i(U, M) \simeq \varprojlim_{U \in \mathcal{T}} H_i(\widehat{U}, M) \simeq H_i(\varprojlim_{U \in \mathcal{T}} \widehat{U}, M) = H_i(\cap_{U \in \mathcal{T}} \widehat{U}, M) = H_i(1, M) = 0$$

b) implies c) is obvious.

c) implies a) and d) Let M be a finite G-module. By substituting U with a subgroup of finite index we can assume that U acts trivially on M. Then by decomposing M as a direct sum of its p-primary components we can assume that M is p-primary for p prime.

Let $\mathcal{R}$ be a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, where all projectives are finitely generated and since by [18, Thm. 2.5] after moving from right to left modules $Tor_i^{\mathbb{Z}G}(\mathbb{Z}_p[[\widehat{G}]], \mathbb{Z}) = 0$ for $i \geqslant 1$, we obtain that $\widehat{\mathcal{R}} = \mathbb{Z}_p[[\widehat{G}]] \otimes_{\mathbb{Z}G} \mathcal{R}$ is exact, hence is a projective resolution of the trivial pro-p $\mathbb{Z}_p[[\widehat{G}]]$-module $\mathbb{Z}_p$. Note that for every finite p-primary G-module M we have that $Hom_{\mathbb{Z}G}(\mathcal{R}^{del}, M) \simeq Hom_{\mathbb{Z}_p[[\widehat{G}]]}(\widehat{\mathcal{R}}^{del}, M)$ and $M \otimes_{\mathbb{Z}G} \mathcal{R}^{del} \simeq M \otimes_{\mathbb{Z}_p[[\widehat{G}]]} \widehat{\mathcal{R}}^{del}$, where $\mathcal{R}^{del}$, $\widehat{\mathcal{R}}^{del}$ denote the deleted complexes obtained from $\mathcal{R}$ and $\widehat{\mathcal{R}}$ i.e. we substitute the modules $\mathbb{Z}$ and $\mathbb{Z}_p$ that are in dimension -1 with the zero module. Hence

$$H^i(G, M) \simeq H^i(Hom_{\mathbb{Z}G}(\mathcal{R}^{del}, M)) \simeq H^i(Hom_{\mathbb{Z}_p[[\widehat{G}]]}(\widehat{\mathcal{R}}^{del}, M)) \simeq H^i(\widehat{G}, M)$$

and the composition of the above isomorphisms is the map $H^i(\widehat{G}, M) \to H^i(G, M)$ induced by the canonical map $G \to \widehat{G}$. Thus G is good.

Similarly

$$H_i(G, M) \simeq H_i(M \otimes_{\mathbb{Z}G} \mathcal{R}^{del}) \simeq H_i(M \otimes_{\mathbb{Z}_p[[\widehat{G}]]} \widehat{\mathcal{R}}^{del}) \simeq H_i(\widehat{G}, M)$$

and the composition of the above isomorphisms is the map $H_i(G, M) \to H_i(\widehat{G}, M)$ induced by the canonical map $G \to \widehat{G}$. Thus G is homologically good.

d) implies c) Fix a prime p and consider the Pontrygin duality given by $M^* = Hom_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ that induces a functorial isomorphism $H^i(G, M^*) \simeq H_i(G, M)^*$ for a finite G-module M. Then

$$(\varprojlim_{U \in \mathcal{T}} H_i(U, \mathbb{F}_p))^* \simeq \varinjlim_{U \in \mathcal{T}} H_i(U, \mathbb{F}_p)^* \simeq \varinjlim_{U \in \mathcal{T}} H^i(U, \mathbb{F}_p^*) = \varinjlim_{U \in \mathcal{T}} H^i(U, \mathbb{F}_p) = 0$$

where the last equality follows from [27, Ch. 1, sec. 2, ex.1a)]. □

A group B is called homologically p-good if for every finite pro-p $\mathbb{Z}_p[[\widehat{B}_p]]$-module M we have that the canonical map $B \to \widehat{B}_p$ induces an isomorphism $H_i(B, M) \to H_i(\widehat{B}_p, M)$.

Lemma 6.2. *Let G be a abstract group of type FP_∞, p be a fixed prime number and $\mathcal{T}$ be a directed set of p-power index normal subgroups in G that defines the pro-p topology of G. Then the following conditions are equivalent :*

a) G is homologically p-good;

b) for every finite pro-p $\mathbb{Z}_p[[\widehat{G}_p]]$-module M we have that $\varinjlim_{U \in \mathcal{T}} H_i(U, M) = 0$ for $i \geqslant 1$;

c) for the trivial G-module $\mathbb{F}_p$ we have that $\varinjlim_{U \in \mathcal{T}} H_i(U, \mathbb{F}_p) = 0$ for $i \geqslant 1$;

d) G is p-good.

Proof. The proof is an obvious modification of the proof of Lemma 6.1. □

Lemma 6.3. *Let $1 \to A \to B \to C \to 1$ be a short exact sequence of abstract groups such that both A and C are p-good, $H^i(A, M)$ is finite for any finite pro-p $\mathbb{F}_p[[\widehat{B}_p]]$-module M and $1 \to \widehat{A}_p \to \widehat{B}_p \to \widehat{C}_p \to 1$ is exact. Then B is p-good.*

Proof. Consider the LHS spectral sequence $\widehat{E}^2_{i,j} = H^i(\widehat{C}_p, H^j(\widehat{A}_p, M))$ that converges to $H^{i+j}(\widehat{B}_p, M)$, where M is a finite pro-p $\mathbb{Z}_p[[\widehat{B}_p]]$-module. Consider the LHS spectral sequence $E^2_{i,j} = H^i(C, H^j(A, M))$ that converges to $H^{i+j}(B, M)$. By the p-goodness of A and C the map $B \to \widehat{B}_p$ induces an isomorphism $\widehat{E}^2_{i,j} \to E^2_{i,j}$. By the naturality of the spectral sequence we conclude by induction on $k \geqslant 2$ that the map $B \to \widehat{B}_p$ induces an isomorphism $\widehat{E}^k_{i,j} \to E^k_{i,j}$, hence an isomorphism $\widehat{E}^\infty_{i,j} \to E^\infty_{i,j}$. Then the convergence of the spectral sequence implies that the map $B \to \widehat{B}_p$ induces an isomorphism $H^{i+j}(\widehat{B}_p, M) \to H^{i+j}(B, M)$. □

Lemma 6.4. *Any orientable surface group is good and p-good.*

Proof. The goodness is a particular case of [9, Thm. 1.3] and the p-goodness is a particular case of [15, Thm. A]. Alternatively both statements have elementary proofs using the results from the previous and this section. □

In particular, the pro-p completion of an orientable PD_2-group is a pro-p PD_2-group. The situation is somewhat different in the non-orientable case.

Lemma 6.5. *Let C be a non-orientable PD_2-group. Then $\widehat{C}_2$ is a pro-2 PD_2-group but $\widehat{C}_p$ is a free pro-p group, for every odd prime p.*

Proof. Let C^+ be the kernel of the orientation character $w : C \to \mathbb{Z}^\times$. Then $[C : C^+] = 2$, since C is non-orientable, and so $\widehat{C^+}_2$ has index 2 in $\widehat{C}_2$. Since C^+ is an orientable PD_2-group it follows that $\widehat{C}_2$ is a pro-2 Poincaré duality group of dimension 2.

Assume now that p is an odd prime,. Then $H_1(C;\mathbb{F}_p) \cong \mathbb{F}_p^r$, for some $r \geqslant 1$, and $H_2(C;\mathbb{F}_p) = 0$. Let F be the free group of rank r and $f : F \to C$ a homomorphism such that $H_1(f;\mathbb{F}_p)$ is an isomorphism. Since $H_2(f;\mathbb{F}_p)$ is also an isomorphism, f induces isomorphisms on all corresponding quotients of the p-lower central series of these groups [28]. Hence $\widehat{F}_p \cong \widehat{C}_p$. □

We return briefly to consider profinite completion, rather than pro-p completions.

Proposition 6.6. *Let $1 \to A \to B \to C \to 1$ be a short exact sequence of groups such that A is an orientable surface group, B is an orientable PD_{2+m}-group and C is a good PD_m group. Then $\widehat{B}$ is a strong orientable profinite PD_{2+m}-group at p. In particular, if C is an orientable surface group (so $m = 2$) then $\widehat{B}$ is a strong orientable profinite PD_4-group.*

Proof. By Lemma 6.4 surface groups are good and by [27, Ch. 1, Sec. 2.6, Ex. 2c)] so are extensions of good groups where the bottom group is FP_∞. In particular, B is good. Let $\mathcal{T}$ be a directed set of normal subgroups of finite index in B that defines the profinite topology of B. Then by Lemma 6.1 $\varprojlim_{U\in\mathcal{T}} H_i(U, \mathbb{F}_p) = 0$ for $i \geqslant 1$. Then we can apply Theorem 2.2. □

There is a subtle point here; a "good" group need not be p-good for any prime p. The simplest example is perhaps the group $\pi_1(M(3_1))$ of §4 mentioned above. There is an exact sequence $1 \to A \to B \to C \to 1$ with $A \cong \mathbb{Z}$, $B = \pi_1(M(3_1))$ and $C \cong \mathbb{Z}$, and so B is good, by Proposition 6.6. However, $\widehat{B}_p \cong \widehat{C}_p = \widehat{\mathbb{Z}}_p$, and so B is not p-good, for any prime p.

We may now prove Theorem B.

Theorem 6.7. *Let $1 \to A \to B \to C \to 1$ be a short exact sequence of groups such that A is an orientable surface group, B is an orientable PD_{s+2}-group and C is an orientable PD_s group, where $s = 2$ or $s = 3$. Let $\mathcal{T}$ be a directed set of normal subgroups of p-power index in B that defines the pro-p topology of B. Suppose that $\varprojlim_{U\in\mathcal{T}} H_1(U \cap A, \mathbb{F}_p) = 0$ and furthermore if $s = 3$, there is an upper bound on the deficiency of the subgroups of finite index in $\widehat{C}_p$. Then*

a) if $s = 2$, then $\widehat{B}_p$ is a pro-p PD_4 group and B is p-good.

b) if $s = 3$, then $\widehat{B}_p$ is virtually a pro-p PD_k-group for some $k \in \{2, 3, 5\}$. If $k = 5$ then B and C are p-good.

If additionally B is orientable and p-good then $\widehat{B}_p$ is an orientable pro-p PD_{2+s} group.

Proof. Note that B is a PD_{s+2}-group. Let $\overline{A}$ be closure of the image of A in $\widehat{B}_p$ i.e. $\overline{A}$ is the completion of A with respect $\{U \cap A \mid U \in \mathcal{T}\}$. Then we have a short exact sequence of pro-p groups

$$1 \to \overline{A} \to \widehat{B}_p \to \widehat{C}_p \to 1.$$

By Lemma 3.4 we see that $\overline{A} \simeq \widehat{A}_p$ is an orientable pro-p PD_2-group.

If $s = 2$ then $\widehat{C}_p$ is an orientable pro-p PD_2-group. Hence $\widehat{B}_p$ is a pro-p PD_4-group.

If $s = 3$ and $\widehat{C}_p$ is infinite then by the remark after Theorem 2.1 we have two options: $\varprojlim_{U\in\mathcal{T}} H_2(V_U, \mathbb{F}_p) = 0$ or $\varprojlim_{U\in\mathcal{T}} H_2(V_U, \mathbb{F}_p) = \mathbb{F}_p$. In the former $\widehat{C}_p$ is an orientable pro-p PD_3-group and in the latter $\widehat{C}_p$ is virtually $\mathbb{Z}_p$. Since $\overline{A} = \widehat{A}_p$ is a pro-p PD_2-group we conclude that if $\widehat{C}_p$ is infinite then $\widehat{B}_p$ is a pro-p PD_5-group or virtually a pro-p PD_3-group.

The p-goodness follows from Lemma 6.3. We need that in the case $s = 3$, if $\widehat{C}_p$ is an orientable pro-p PD_3-group then C is p-good, that follows from [18, Thm. A].

If B is p-good and B is orientable then by Theorem 2.2 $\widehat{B}_p$ is orientable pro-p PD_{2+s} group. □

Lemma 6.8. *Let $1 \to K \to G \to D \to 1$ be a short exact sequence of groups, where G is a PD_n-group, D is a PD_{n-1}-group and $K \simeq \mathbb{Z}$. Then G is an orientable PD_n-group if and only if K is the dualizing module of D.*

Proof. Consider the LHS spectral sequence $E^2_{i,j} = H_i(D, H_j(K, \mathbb{Z}))$ that converges to $H_{i+j}(G, \mathbb{Z})$. Since $E^2_{i,j} = 0$ if $i \geqslant n$ or $j \geqslant 2$, by the convergence we conclude that $H_n(G, \mathbb{Z}) \simeq H_{n-1}(D, H_1(K, \mathbb{Z}))$. Let W be the dualizing module of D and V be the dualizing module of G. Both V and W are infinite cyclic as abelian groups but in general the corresponding actions of D and G need not be trivial. Then

$$H^0(G, V) \simeq H_n(G, \mathbb{Z}) \simeq H_{n-1}(D, H_1(K, \mathbb{Z})) \simeq H_{n-1}(D, K) \simeq H^0(D, K \otimes W).$$

Since V and $K \otimes W$ are infinite cyclic as abelian groups, we have that G is orientable $\iff$ G acts trivially on V $\iff$ $H^0(G, V) \neq 0$ $\iff$ $H^0(D, K \otimes W) \neq 0$ $\iff$ $K \otimes W$ is the trivial D-module (via the diagonal action) $\iff$ $K \simeq W$ as D-module. □

Lemma 6.9. *Suppose $p \neq 2$ and $1 \to K \to G \to C \to 1$ is a short exact sequence of groups, where the action of C via conjugation on $K = \mathbb{Z}$ is non-trivial. Then $\widehat{G}_p \simeq \widehat{C}_p$.*

Proof. Suppose that U is a normal subgroup of p-power index in G. Then for some i we have that $K^{2^i} = [K, G, \ldots, G] \subseteq \gamma_{i+1}(G) \subseteq U$, hence $K^{2^i} \subseteq U \cap K \subseteq K$. Since $[K : U \cap K]$ is a p-power that divides $2^i = [K : K^{2^i}]$, we conclude that $K = U \cap K \subseteq U$. Hence $\widehat{G}_p \simeq \widehat{C}_p$. □

7. PD_4-groups with Euler characteristic 0

In this section we shall prove Theorem C of the Introduction. (This is Theorem 7.2 below.)

Lemma 7.1. *Let G be an orientable PD_4-group. If the pro-p completion $\widehat{G}_p$ is a pro-p PD_3-group then $\chi(G) \geqslant 2$.*

Proof. We have $\beta_1(\widehat{G}_p; \mathbb{F}_p) = \beta_1(G; \mathbb{F}_p)$ and $\beta_2(\widehat{G}_p; \mathbb{F}_p) \leqslant \beta_2(G; \mathbb{F}_p)$, for any group G [27, §2.6]. Since $\widehat{G}_p$ is a pro-p PD_3-group, $\beta_1(\widehat{G}_p; \mathbb{F}_p) = \beta_2(\widehat{G}_p; \mathbb{F}_p)$, and since the image of $H^2(\widehat{G}_p; \mathbb{F}_p)$ in $H^2(G; \mathbb{F}_p)$ is self-annihilating under cup product, $\beta_2(G; \mathbb{F}_p) \geqslant 2\beta_1(G; \mathbb{F}_p)$, by the non-singularity of Poincaré duality. Hence $\chi(G) = \sum_{0 \leqslant i \leqslant 4} (-1)^i \beta_i(G; \mathbb{F}_p) = 2 - 2\beta_1(G; \mathbb{F}_p) + \beta_2(G; \mathbb{F}_p) \geqslant 2$. □

Let G be a group with a normal subgroup $A \cong \mathbb{Z}^2$. If we fix a basis for A we may identify $\mathrm{Aut}(A)$ with $\mathrm{GL}(2, \mathbb{Z})$, and conjugation in G then determines an action $\theta : G/A \to \mathrm{GL}(2, \mathbb{Z})$. We shall say that the action is orientable if its image lies in $\mathrm{SL}(2, \mathbb{Z})$. (Thus G acts orientably if and only if the induced action by $\det \theta$ on $A \wedge A \cong \mathbb{Z}$ is trivial.)

Theorem 7.2. *Let $1 \to A \to B \to C \to 1$ be a short exact sequence of groups such that $A \simeq \mathbb{Z}^2$, B is an orientable PD_4 group and C is an orientable PD_2-group. Then one of the following holds:*

a) $\widehat{B}_p$ is an orientable pro-p PD_4-group and B is p-good;

b) $\widehat{B}_p \cong \widehat{C}_p$, and the image of A in $\widehat{B}_p$ is trivial.

Proof. Let $[B, A]$ be the normal subgroup generated by commutators $gag^{-1}a^{-1}$ for $g \in B$ and $a \in A$. Let $\Gamma^1 A = [B, A] + pA$ and $\Gamma^{k+1}A = [B, \Gamma^k A] + p^{k+1}A$, for all $k \geqslant 1$. These subgroups have finite index in A and are normal in B. The quotient $A/\Gamma^1 A$ is central in $B/\Gamma^1 A$, and B acts on the finite p-group $A/\Gamma^k A$ through a finite p-group, for all $k \geqslant 1$. Hence B has a normal subgroup U of p-power index such that $A < U$ and $A/\Gamma^k A$ is central in $U/\Gamma^k A$. The quotient U/A is an orientable PD_2-group and $A/\Gamma^k A$ is a finite abelian group of exponent dividing p^k. On applying Lemma 3.1 several times, we see that the class in $H^2(U/A; A/\Gamma^k A)$ of the central extension

$$0 \to A/\Gamma^k A \to U/\Gamma^k A \to U/A \to 1$$

restricts to 0 in a normal subgroup V/A of p-power index. Hence $V/\Gamma^k A \cong (V/A) \times (A/\Gamma^k A)$. An argument by induction on nilpotency class shows that $\cap_{k\geqslant 1}\Gamma^k A$ has trivial image in every finite quotient of G which is a p-group. It follows that $\cap_{k\geqslant 1}\Gamma^k A$ is the kernel of the pro-p completion homomorphism from B to $\widehat{B}_p$.

Fix an isomorphism $A \cong \mathbb{Z}^2$, and let $\theta : B \to \mathrm{GL}(2, \mathbb{Z})$ be the action of B an A induced by conjugation in G. Let $\theta_p : B \to \mathrm{GL}(2, \mathbb{F}_p)$ be the *mod-p* reduction of θ.

If $\theta_p(g) - I$ is not invertible (for some $g \in B$) then $\theta_p(g)$ has 1 as an eigenvalue. Since B and C are orientable the action of B on A is orientable. Hence both eigenvalues of $\theta_p(g)$ are 1, since they are mutually inverse, and so $(\theta_p(g) - I)^2 = 0$.

Suppose first that this holds for all $g \in B$. Then we may assume that $\theta_p(B) \leqslant U(2, \mathbb{F}_p)$, the subgroup of upper unitriangular matrices [23, 8.1.10]. Thus A has a basis $\{e_1, e_2\}$ such that $[B, e_1] \leqslant pA$ and $[B, e_2] + pA \leqslant \mathbb{Z}e_1 + pA$. Hence $\Gamma^1 A \leqslant \mathbb{Z}e_1 + pA$. Define subgroups $[B,_s e_1]$ inductively by setting $[B,_1 e_1] = [B, e_1]$ and $[B,_{s+1} e_1] = [B, [B,_s e_1]]$ for $s \geqslant 1$. Then by induction on k we have $\Gamma^k A \leqslant p^k A + \Sigma_{h+j=k-1}\mathbb{Z}p^h[B,_j e_1]$, $[B,_{2k-1} e_1] \leqslant p^k A$ and $[B,_{2k} e_1] \leqslant \mathbb{Z}p^k e_1 + p^{k+1}A \leqslant p^k A$. Thus $\Gamma^2 kA \leqslant p^k A$, for all $k \geqslant 1$, and so $\cap_{k\geqslant 1}\Gamma^k A = 1$. In this case $\overline{A} \cong \widehat{\mathbb{Z}}_p^2$ and so $\widehat{B}_p$ is a pro-p PD_4-group. Since $\beta_1(\widehat{B}_p; \mathbb{F}_p) = \beta_1(B; \mathbb{F}_p)$ and $\chi(\widehat{B}_p) = 0 = \chi(B)$, it follows that $\beta_2(\widehat{B}; \mathbb{F}_p) = \beta_2(B; \mathbb{F}_p)$. It then follows easily from Lemma 3.5 and the nonsingularity of Poincaré duality that B is p-good.

If $\theta_p(g) - I$ is invertible in $\mathrm{GL}(2, \mathbb{F}_p)$ for some $g \in C$ then $A = [B, A] + pA$. Hence $A = [B, A] + p^k A$ for all $k \geqslant 1$, by the Burnside Basis theorem [23, 5.3.2] (equivalently, by Nakayama's Lemma), applied to the finite p-group $A/p^k A$. Hence $\cap_{k\geqslant 1}\Gamma^k A = A$, so $\overline{A} = 1$ and $\widehat{B}_p \cong \widehat{C}_p$ is a pro-p PD_2-group. □

If C is a non-orientable PD_2-group then B is orientable if and only if the determinant of the action is the orientation character of C). In this case the above argument goes through with little change for $p = 2$.

Remark Suppose $p \neq 2$ and $1 \to K \to G \to C \to 1$ be a short exact sequence of groups, where C is an orientable surface group and the action of C via conjugation on $K = \mathbb{Z}$ is non-trivial. Thus G is a non-orientable PD_3-group. Consider $B = S \times G$, where $S = \mathbb{Z}$. Then for $A = S \times K$ we have the short exact sequence of groups $1 \to A \to B \to C \to 1$ with A and C orientable surface groups, but by Lemma 6.8 B is not orientable. Then by Lemma 6.9 $\widehat{B}_p \simeq \widehat{S}_p \times \widehat{G}_p = \mathbb{Z}_p \times \widehat{C}_p$ is an orientable pro-p PD_3-group and $\overline{A} \simeq \mathbb{Z}_p$. This is an example of a group that is not orientable and does not satisfy the conclusions of Theorem C but the only assumption of Theorem C it fails is the orientability one.

Taking products of aspherical 3-manifolds exemplifying case (d) of Theorem 2.1 with the circle gives examples illustrating part (b) of Theorem C. Let $M = M(K)$, where K is a non-trivial fibred knot. Then $E = M \times S^1$ is an aspherical 4-manifold with fundamental group $G \times \mathbb{Z}$, and E fibres over the torus T with fibre F. If B is a surface of genus $h > 1$ and $f : B \to T$ is a degree-1 map then the total space $E_f = N \times_T B$ of the pullback of the fibration $N \to T$ over f is aspherical, and $\pi = \pi_1(E_f)$ is an extension of $\rho = \pi_1(B)$ by ϕ. It is easy to see that $\phi \leqslant \gamma_k\pi$ for all k, and so $\widehat{\pi}_p \cong \widehat{\rho}_p$, for all primes p.

If L is the 2-component boundary link obtained by Whitehead doubling each component of the Hopf link then $M(L) \times S^1$ is an aspherical orientable 4-manifold, but does not fibre over a surface. The pro-p completion of $G \times \mathbb{Z}$ is $\widehat{F(2)}_p \times \widehat{\mathbb{Z}}_p$, which has cohomological p-dimension 2, but is not a Demuškin group.

Note that the hypothesis "$\varprojlim_{U \in \mathcal{T}} H_1(U \cap A, \mathbb{F}_p) = 0$" of Theorem B does not hold for these examples.

8. Dimension drop

An orientable PD_n-group G has dimension drop k on pro-p completion if $\widehat{G}_p$ is a pro-p PD_{n-k}-group. There are aspherical closed orientable n-manifolds N such that $\pi_1(N)$ has dimension drop k (for all primes p), for all $n \geqslant 2$ and $2 \leqslant k \leqslant n$, except when $n = k = 5$ [12]. This exception reflects the fact that 5 is not in the additive semigroup generated by 3 and 4, dimensions in which aspherical homology spheres are known. We shall fill this gap below. However, whether there are any examples of dimension drop 1 remains an open question.

Let X be a compact 4-manifold whose boundary components are diffeomorphic to the 3-torus T^3. A *Dehn filling* of a component Y of ∂X is the adjunction of $T^2 \times D^2$ to X via a diffeomorphism $\partial(T^2 \times D^2) \cong Y$. If the interior of X has a complete hyperbolic metric then "most" systems of Dehn fillings on some or all of the boundary components give manifolds which admit metrics of non-positive curvature, and the fundamental groups of the cores of the solid tori $T^2 \times D^2$ map injectively to the fundamental group of the filling of X, by the Gromov-Thurston 2π-Theorem. (Here "most" means "excluding finitely many fillings of each boundary component". See [2].)

Theorem 8.1. *There are aspherical closed 5-manifolds with perfect fundamental group.*

Proof. Let $M = S^4 \setminus 5T^2$ be the complete hyperbolic 4-manifold with finite volume and five cusps considered in [13] and [22], and let $\overline{M}$ be a compact core, with interior diffeomorphic to M. Then $H_1(\overline{M}; \mathbb{Z}) \cong \mathbb{Z}^5$, $\chi(\overline{M}) = 2$ and the boundary components of $\overline{M}$ are all diffeomorphic to the 3-torus T^3. There are infinitely many quintuples of Dehn fillings of the components of $\partial\overline{M}$ such that the resulting closed 4-manifold is an aspherical homology 4-sphere [22]. Let $\widehat{M}$ be one such closed 4-manifold, and let $N \subset \widehat{M}$ be the compact 4-manifold obtained by leaving one boundary component of X unfilled. We may assume that the interior of N has a non-positively curved metric, and so N is aspherical. The Mayer-Vietoris sequence for $M = N \cup T^2 \times D^2$ gives an isomorphism

$$H_1(T^3; \mathbb{Z}) \cong H_1(N; \mathbb{Z}) \oplus H_1(T^2; \mathbb{Z}).$$

Let $\{x, y, z\}$ be a basis for $H_1(T^3;\mathbb{Z})$ compatible with this splitting. Thus x represents a generator of $H_1(N;\mathbb{Z})$ and maps to 0 in the second summand, while $\{y, z\}$ has image 0 in $H_1(N;\mathbb{Z})$ but generates the second summand. Since the subgroup generated by $\{y, z\}$ maps injectively to $\pi_1(\widehat{M})$ [2], the inclusion of ∂N into N is π_1-injective. Let ϕ be the automorphism of $\partial N = T^3$ which swaps the generators x and y, and let $P = N \cup_\phi N$. Then P is aspherical and $\chi(P) = 2\chi(N) = 4$. A Mayer-Vietoris calculation gives $H_1(P;\mathbb{Z}) = 0$, and so $\pi = \pi_1(P)$ is perfect and $H^2(P;\mathbb{Z}) \cong \mathbb{Z}^2$.

Let e generate a direct summand of $H^2(\pi;\mathbb{Z}) = H^2(P;\mathbb{Z})$, and let E be the total space of the S^1-bundle over P with Euler class e. Then E is an aspherical 5-manifold, and $G = \pi_1(E)$ is the central extension of $\pi_1(P)$ by $\mathbb{Z}$ corresponding to $e \in H^2(\pi_1(P);\mathbb{Z})$. The Gysin sequence for the bundle (with coefficients in $\mathbb{F}_p$) has a subsequence

$$0 \to H^1(E;\mathbb{F}_p) \to H^0(P;\mathbb{F}_p) \to H^2(P;\mathbb{F}_p) \to H^2(E;\mathbb{F}_p) \to \dots$$

in which the *mod-p* reduction of e generates the image of $H^0(P;\mathbb{F}_p)$. Since e is indivisible this image is nonzero, for all primes p. Therefore $H^1(G;\mathbb{F}_p) = H^1(E;\mathbb{F}_p) = 0$, for all p, and so G is perfect. □

We may use such groups to complete the results of [12].

Theorem 8.2. *For each $r \geqslant 0$ and $n \geqslant \max\{r + 2, 3\}$ there is an aspherical closed n-manifold with fundamental group π such that $\pi/\pi' \cong \mathbb{Z}^r$ and $\pi' = \pi''$.*

Proof. Let Σ be an aspherical homology 3-sphere (such as the Brieskorn 3-manifold $\Sigma(2, 3, 7)$) and let P and E be as in Theorem 8.1. Taking suitable products of copies of Σ, P, E and S^1 with each other realizes all the possibilities with $n \geqslant r + 3$, for all $r \geqslant 0$.

Let $M = M(K)$ be the 3-manifold obtained by 0-framed surgery on a nontrivial prime knot K with Alexander polynomial $\Delta(K) = 1$ (such as the Kinoshita-Terasaka knot 11_{n42}). Then M is aspherical, since K is nontrivial [7], and if $\mu = \pi_1(M)$ then $\mu/\mu' \cong \mathbb{Z}$ and μ' is perfect, since $\Delta(K) = 1$. Hence products $M \times (S^1)^{r-1}$ give examples with $n = r + 2$, for all $r \geqslant 1$. □

In particular, the dimension hypotheses in Theorem 6.3 of [12] may be simplified, so that it now asserts:

Let $m \geqslant 3$ and $r \geqslant 0$. Then there is an aspherical closed $(m + r)$-manifold M with fundamental group $G = K \times \mathbb{Z}^r$, where $K = K'$. If $m \neq 4$ we may assume that $\chi(M) = 0$, and if $r > 0$ this must be so.

This is best possible, as no PD_1- or PD_2-group is perfect, and no perfect PD_4-group H has $\chi(H) = 0$.

As observed above, there are no known examples of dimension drop 1. No PD_n-group with $n \leqslant 3$ has such a dimension drop on any p-profinite completion. (This is clear if $n \leqslant 2$, and follows from Theorem 4.4 if $n = 3$.) Hence we may focus on the first undecided case, $n = 4$.

In seeking possible examples of dimension drop 1 in the pro-p completion of a PD_n-group, the most convenient candidates are groups whose lower central series terminates after finitely many steps. A finitely generated nilpotent group ν of Hirsch length h has a maximal finite normal subgroup $T(\nu)$, with quotient a PD_h-group. Moreover, $\nu/T(\nu)$ has nilpotency class $< h$, and is residually a finite p-group for all p, by Theorem 4 of [25, Chapter 1]. Thus the pro-p completion of ν is a pro-p PD_h-group. for all p prime to the order of $T(\nu)$.

If $\gamma_k G/\gamma_{k+1} G$ is finite, of exponent e, say, then so are all subsequent subquotients of the lower central series, by Proposition 11 of Chapter 1 of [25]. Thus if G is a PD_4-group such that $G/\gamma_3 G$

has Hirsch length 3 and $\gamma_3 G/\gamma_4 G$ is finite then, setting $\nu = G/\gamma_3 G$, the canonical projection to $\nu/T(\nu)$ induces isomorphisms on pro-p completions, for almost all primes p. Taking products of one such group with copies of $\mathbb{Z}$ would give similar examples with dimension drop 1 in all higher dimensions.

Let G be the fundamental group of a closed orientable 4-manifold which is the total space of a bundle with base and fibre aspherical closed orientable surfaces. Thus there is an epimorphism $f : G \to C$ with kernel A, where A and C are orientable PD_2-groups. The projection f induces an epimorphism $\hat{f} : \widehat{G}_p \to \widehat{C}_p$ of pro-p completions. Let K be the kernel of the canonical homomorphism from G to $\widehat{G}_p$. The kernel of $\widehat{f}$ is the closure of the image of A, and so is topologically finitely generated. If $\widehat{G}_p$ is a pro-p PD_3-group then $\mathrm{Ker}(\widehat{f}) \cong \mathbb{Z}_p$ [17, Cor. 4]. Hence A/K is finitely generated and abelian of rank 1. An immediate consequence is that $\beta_1(G; \mathbb{F}_p) = \beta_1(C; \mathbb{F}_p)$ or $\beta_1(C; \mathbb{F}_p) + 1$. This condition is not satisfied by most such surface bundle groups G, as $\beta_1(G; \mathbb{F}_p)$ may be as large as $\beta_1(A; \mathbb{F}_p) + \beta_1(C; \mathbb{F}_p)$. There are no such bundles with base or fibre the torus, by Lemma 7.1.

We make one further observation, related to Lemma 7.1. If G is an orientable PD_4-group and $\widehat{G}_p$ is a pro-p Poincaré duality group of dimension 3 then the canonical homomorphism from $H^3(\widehat{G}_p; \mathbb{F}_p)$ to $H^3(G; \mathbb{F}_p)$ is trivial. For $H^1(\widehat{G}_p; \mathbb{F}_p) \neq 0$ and so there are classes $\alpha \in H^1(\widehat{G}_p; \mathbb{F}_p) = H^1(F; \mathbb{F}_p)$ and $\beta \in H^2(\widehat{G}_p; \mathbb{F}_p) < H^2(G; \mathbb{F}_p)$ such that $\alpha \cup \beta$ generates $H^3(\widehat{G}_p; \mathbb{F}_p)$, by Poincaré duality for $\widehat{G}_p$. If this has nonzero image in $H^3(G; \mathbb{F}_p)$ then there is a $\gamma \in H^1(\widehat{G}_p; \mathbb{F}_p) = H^1(F; \mathbb{F}_p)$ such that $\alpha \cup \beta \cup \gamma \neq 0$ in $H^4(G; \mathbb{F}_p)$. But this cup product is in the image of $H^4(\widehat{G}_p; \mathbb{F}_p)$, which is 0. An equivalent formulation of this condition is that inflation from $H^3(G/U; \mathbb{F}_p)$ to $H^3(G; \mathbb{F}_p)$ is trivial for every normal subgroup U of p-power index in G. In particular, taking $U = G'X^p(G)$ (where $X^p(G)$ is the verbal subgroup generated by all pth powers) we see that the image of $\wedge^3 H^1(G; \mathbb{F}_p)$ in $H^3(G; \mathbb{F}_p)$ must be 0.

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