ON LINEAR STOCHASTIC FLOWS

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ABSTRACT. We study the existence of the stochastic flow associated to a linear stochastic evolution equation

$$\mathrm{d}X = AX\mathrm{d}t + \sum_k B_k X \mathrm{d}W_k,$$

on a Hilbert space. Our first result covers the case where A is the generator of a C_0-semigroup, and (B_k) is a sequence of bounded linear operators such that $\sum_k \|B_k\| < +\infty$. We also provide sufficient conditions for the existence of stochastic flows in the Schatten classes beyond the space of Hilbert-Schmidt operators. Some new results and examples concerning the so-called commutative case are presented as well.

CONTENTS

1. Introduction

Consider the following linear evolution equation

$$\mathrm{d}X = AX\mathrm{d}t + \sum_k B_k X \mathrm{d}W_k, \qquad t \geq s, \ X(s) = x, \tag{1.1}$$

where (W_k) is a sequence of independent standard real valued Wiener processes defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, A is the generator of a C_0-semigroup $\left(\mathrm{e}^{tA}\right)$ on a Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$ and B_k, $k = 1, \ldots$, are possibly unbounded linear operators on H. By the solution to (1.1) we understand the so-called mild solution defined as an adapted to the filtration generated by (W_k) and having continuous trajectories in H process $X_s^x(t)$, $t \geq s$, satisfying the integral equation

$$X_s^x(t) = \mathrm{e}^{(t-s)A}x + \sum_k \int_s^t \mathrm{e}^{(t-r)A} B_k X_s^x(r)\mathrm{d}W_k(r), \qquad t \geq s.$$

For more details see the books of Da Prato and Zabczyk [9] and Flandoli [10].

Let us denote by $L(H, H)$ the space of all bounded linear operators on H. We denote by $\| \cdot \|$ the operator norm on $L(H, H)$ and by $| \cdot |_H$ the norm on H. Let $\Delta := \{(s, t) \colon 0 \leq s \leq t < +\infty\}$.

Definition 1.1. We say that (1.1) defines a <u>stochastic flow</u> if there exists a mapping $\mathcal{X} \colon \Delta \times \Omega \mapsto L(H, H)$ such that:

    (i) for every $s \geq 0$ and $x \in H$, the process $\mathcal{X}(s, t; \cdot)(x)$, $t \geq s$, has continuous trajectories in H, $\mathbb{P}$-a.s.,

    (ii) for every $s \geq 0$ and $x \in H$, we have $\mathcal{X}(s, t)(x) = X_s^x(t)$, for all $t \geq s$, $\mathbb{P}$-a.s.,

    (iii) for all $0 \leq s \leq t \leq r$ and $\omega \in \Omega$, $\mathcal{X}(t, r; \omega) \circ \mathcal{X}(s, t; \omega) = \mathcal{X}(s, r; \omega)$.

Remark 1.2. Property (i) means that for each s and x, the solution $X_s^x(t) = \mathcal{X}(s, t; \cdot)(x)$ has continuous trajectories. This holds in the most of interesting cases by means of the so-called Da Prato-Kwapień-Zabczyk factorization, see e.g. [9]. In particular, the solution has the required property if (B_k) is a finite sequence of bounded operators on H. Note, that property (iii) of the stochastic flow follows from the existence and uniqueness of the solution. Finally, note that for all $0 \leq s \leq t \leq r \leq h$ and $x, y \in H$, the random variables $\mathcal{X}(s, t)(x)$ and $\mathcal{X}(r, h)(y)$ are independent. Stochastic flows with this property are known as Brownian flows.

The theory of stochastic flows for linear and nonlinear stochastic differential equations in finite dimensional spaces is well established, see for example [18]. In particular, it is known that the nonlinear SDE

$$\mathrm{d}X = b(X)\mathrm{d}t + \sigma(X)\mathrm{d}W, \qquad X(s) = x, \tag{1.2}$$

defines a (nonlinear) stochastic flow if b and σ are C^1 with bounded derivatives. The proof of this result requires a Sobolev embedding theorem. The case of additive noise, where σ

does not depend on X, is easier. Namely, let $Y_s^x(t)$, $t \geq s$, be the solution to the ordinary differential equation with random coefficients

(1.3) $$\mathrm{d}Y(t) = b\left(Y(t) + \sigma(W(t) - W(s))\right)\mathrm{d}t, \qquad Y(s) = x.$$

Then the solution X_s^x to (1.2) is given by $X_s^x(t) = Y_s^x(t) + \sigma(W(t) - W(s))$, and consequently (1.2) defines a stochastic flow if (1.3) defines a flow.

It is natural to ask a question about the existence of stochastic flow associated to a stochastic evolution equation

(1.4) $$\mathrm{d}X = (AX + F(X))\mathrm{d}t + \sum_k \sigma_k(X)\mathrm{d}W_k, \quad X(0) = x,$$

where H is an infinite-dimensional Hilbert space and A is the generator of a C_0-semigroup on H. If σ does not depend on X, then one can apply the same argument as in in the finite dimensional case. In this paper we focus on a linear equation (1.1), where $F = 0$ and $\sigma_k(x) = B_k x$. Skorokhod showed in his famous example [23] that the question of the existence of the flow for equation (1.1) is much more intricate than in finite dimensions. In Section 4 we consider a more general version of the Skorokhod example to obtain new and interesting phenomena.

In general, it is not possible to prove the existence of the flow in infinite dimensions using the Sobolev embedding theorems that are not available in infinite dimensions. However, for many nonlinear equations of the form (1.4) the corresponding flows can be obtained as continuous transformations of the flows corresponding to ordinary or partial differential equations with random coefficients (see e.g. [1, 2, 3, 4, 6, 7, 8, 12, 20]). In fact, we will use a similar method in the proof of our main result concerning linear equations. A very general approach to the question of existence of stochastic flows for non-linear stochastic partial differential equations can be found in [11].

It is tempting to obtain the existence of the flow $\mathcal{X}$ corresponding to linear equation (1.1) by solving a linear equation

$$\mathrm{d}\mathcal{X} = A\mathcal{X}\mathrm{d}t + \sum_k B_k \mathcal{X}\,\mathrm{d}W_k, \qquad t \geq s, \ \mathcal{X}(s) = I,$$

on the space of bounded operators. Unfortunately, the theory of stochastic integration on the space of all bounded linear operators on infinite dimensional Hilbert space does not exist. It is possible however to integrate in some smaller spaces, such as the Schatten classes. This approach has been applied by Flandoli [10]) in the case of Hilbert-Schmidt operators. In Section 5 of this paper we use the theory of stochastic integration on M-type 2 Banach spaces to extend the Flandoli result to the scale of Schatten classes $\mathbb{S}^p$ for all $p \in [2, \infty)$. We will show that in the diagonal case solutions in the Schatten classes can exist $\mathbb{P}$-a.s. but not in the sense of expected value. In particular, in Proposition 4.3.3 we give conditions under which $\mathrm{Tr}(\mathcal{X}(t)) < +\infty$, $\mathbb{P}$-a.s. while $\mathbb{E}\,\mathrm{Tr}(\mathcal{X}(t)) = +\infty$. It would be interesting to find more general conditions for such a behaviour of stochastic flows.

The paper is organized as follows: in the next section we prove a theorem dealing with equation (1.1) with a sequence of bounded operators B_k. We do not assume that

the operators commute, therefore there are no explicit solutions. Then in Sections 4 we examine the case of commuting B_k. In this case the solution is given in an explicit form, and this allows us to construct interesting examples that lead to new questions that remain open. In Section 5 we study the existence of stochastic flows in Schatten classes.

2. MAIN RESULTS

In this section we consider a linear stochastic equation on a separable Hilbert space H:

$$(2.1) \qquad \mathrm{d}Y = B_0 Y \mathrm{d}t + \sum_k B_k Y \mathrm{d}W_k$$

driven by a sequence of independent Wiener processes W_k. In theorems below we assume that $B_0, B_1, \ldots$ is a sequence of bounded linear operators on H such that

$$(2.2) \qquad M := \sum_k \|B_k\| < +\infty.$$

We start with a result on the existence and regularity of the flow.

Theorem 2.1. *Assume* (2.2). *Then equation* (2.1) *defines a stochastic flow* $\mathcal{Y}$ *on* H. *Moreover, for any* $T \in (0, +\infty)$, *and* $q \geq 1$,

$$(2.3) \qquad \sup_{0 \leq s \leq t \leq T} \mathbb{E} \|\mathcal{Y}(s,t)\|^q < +\infty.$$

Moreover, for every $T \in (0, +\infty)$, *and* $L = 2, 3, \ldots$, *there exists a constant* C *such that for all for* $0 \leq s \leq t \leq u \leq T$ *we have*

$$(2.4) \qquad \mathbb{E} \|\mathcal{Y}(s,t) - \mathcal{Y}(s,u)\|^{2L} \leq C \, |t-u|^{L-1}.$$

Consequently, by the Kolmogorov test, for any $s \geq 0$, $\mathbb{P}$-*a.s.* $\mathcal{Y}(s,t)$ *is a Hölder continuous* $L(H,H)$-*valued mapping of* $t \geq s$, *with exponent* $\gamma < 1/2$. *Finally, for any* $T \in (0, +\infty)$, *and* $L = 2, 3, \ldots$, *there exists a constant* C *such that for all for* $0 \leq s \leq t \leq u \leq T$ *we have*

$$(2.5) \qquad \mathbb{E} \|\mathcal{Y}(s,t) - \mathcal{Y}(u,v)\|^{2L} \leq C \left(|u-s| + |t-v|\right)^{L-1}.$$

Consequently, by the Kolmogorov test, for any $s \geq 0$, $\mathbb{P}$-*a.s.* $\mathcal{Y}(s,t)$ *is a Hölder continuous* $L(H,H)$-*valued mapping of* $(t,s) \in \Delta$, *with exponent* $\gamma < 1/2$.

Theorem 2.2. *Assume* (2.2). *Let* $\tilde{B} = \sum_{k \geq 1} B_k^2$, *and let* $\mathcal{Y}$ *be the flow defined by* (2.1). *Then for all* $0 \leq s \leq t$, $\mathcal{Y}(s,t)$ *is an invertible operator* $\mathbb{P}$-*a.s. and* $\mathcal{Z}(s,t) = (\mathcal{Y}^{-1}(s,t))^\star$ *is the flow defined by the equation*

$$(2.6) \qquad \mathrm{d}Z = \left(-B_0^\star + \tilde{B}^\star\right) Z \mathrm{d}t - \sum_k B_k^\star Z \mathrm{d}W_k, \quad Z(s) = z.$$

Theorem 2.3. *Assume* (2.2) *Then for any generator* A *of a* C_0-*semigroup on* H, *equation* (1.1) *defines stochastic flow on* H.

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Theorem 2.4. *Assume that $B_0, B_1, \ldots, B_N$ is a finite sequence of commuting generators of C_0-groups on H. Then the equation*

$$\mathrm{d}X = B_0 X \mathrm{d}t + \sum_{k=1}^{N} B_k X \mathrm{d}_S W_k, \quad X(s) = x$$

considered in the Stratonovich sense defines stochastic flow on H. Moreover,

$$\mathcal{X}(s,t) = \exp\left\{ (t-s)B_0 + \sum_{k=1}^{N} \left(W_k(t) - W_k(s) \right) B_k \right\}.$$

2.1. **Proof of Theorem 2.1.** Write $W_0(t) \equiv t$. Then (2.1) can be written as follows

$$\mathrm{d}Y = \sum_{k=0}^{+\infty} B_k Y \mathrm{d}W_k, \qquad Y(s) = x.$$

Let

$$\Theta := \bigcup_{n=1}^{+\infty} \{0, 1, \ldots\}^n.$$

Given $\alpha \in \{0, 1, \ldots\}^n$ we set $|\alpha| = n$ and $B^\alpha := B_{\alpha_1} B_{\alpha_2} \ldots B_{\alpha_n}$. Let us fix an $s \geq 0$. We define by induction the iterated stochastic integrals. If $|\alpha| = 1$ and $\alpha = (k) := k$, $k = 0, 1, \ldots$, then $I_k(t) = W_k(t) - W_k(s)$. For $\alpha \in \{0, 1, \ldots\}^n$ we set

$$I_\alpha(s,t) := \int_s^t I_{\widehat{\alpha}}(s,r) \mathrm{d}W_{\alpha_1}(r),$$

where $\widehat{\alpha} := (\alpha_2, \ldots, \alpha_n)$. Then the solution Y_s^x to (2.1) is given by

$$Y_s^x(t) = x + \sum_{n=1}^{+\infty} \sum_{\alpha:\, |\alpha|=n}^{+\infty} I_\alpha(s,t) B^\alpha x.$$

Therefore the corresponding flow $\mathcal{Y}$ exists and

$$\mathcal{Y}(s,t;\omega) = I + \sum_{n=1}^{+\infty} \sum_{\alpha:\, |\alpha|=n}^{+\infty} I_\alpha(s,t;\omega) B^\alpha$$

provided the series

$$\sum_{n=1}^{+\infty} \sum_{\alpha:\, |\alpha|=n}^{+\infty} I_\alpha(s,t;\omega) B^\alpha$$

converges $\mathbb{P}$-a.s in $L(H,H)$. We have

$$\sum_{n=1}^{+\infty} \sum_{\alpha:\, |\alpha|=n}^{+\infty} \| I_\alpha(s,t;\omega) B^\alpha \| \leq \xi(s,t;\omega),$$

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where

$$\xi(s,t;\omega) := \sum_{n=1}^{+\infty} \sum_{|\alpha|=n}^{+\infty} |I_\alpha(s,t;\omega)| \, \|B^\alpha\|.$$

Therefore the stochastic flow exists if $\xi(s,t) < +\infty$, $\mathbb{P}$-a.s. Moreover, $\|\mathcal{Y}(s,t;\omega)\| \leq 1 + \xi(s,t;\omega)$. In fact we will show that for any $T \in (0,+\infty)$ and $L = 1,2,3,\ldots$, we have

(2.7)
$$\sup_{0 \leq s \leq t \leq T} \mathbb{E}\, \xi^{2L}(s,t) < +\infty,$$

which obviously guarantees (2.3). To this end note that for $\alpha \colon |\alpha| = n \geq 1$ we have

$$\mathbb{E}\,|I_\alpha(s,t)|^{2L} \leq C_L^{2Ln} \frac{\max\{(1,(t-s)^{2Ln}\}}{n!}(t-s)^{L-1} =: \eta_{n,L}(s,t),$$

where $C_L := \frac{2L}{2L-1}$.

Let $\alpha^1, \alpha^2, \ldots, \alpha^{2L} \in \Theta$. By the Schwarz inequality we have

$$\mathbb{E} \prod_{i=1}^{2L} |I_{\alpha^i}(s,t)| \leq \prod_{i=1}^{2L} \left(\mathbb{E}\,|I_{\alpha^i}(s,t)|^{2L} \right)^{1/(2L)}.$$

Therefore

$$\mathbb{E}\,\xi^{2L}(s,t) = \mathbb{E}\left(\sum_{n=1}^{+\infty} \sum_{\alpha\colon |\alpha|=n}^{+\infty} |I_\alpha(s,t)| \, \|B^\alpha\| \right)^{2L}$$

$$\leq \sum_{n_1,\ldots n_{2L}=1}^{+\infty} \sum_{\alpha^i\colon |\alpha^i|=n_i} \prod_{i=1}^{2L} \|B^{\alpha^i}\| \, \mathbb{E} \prod_{i=1}^{2L} |I_{\alpha^i}(s,t)|$$

$$\leq \left(\sum_{n=1}^{+\infty} \sum_{\alpha\colon |\alpha|=n}^{+\infty} \|B^\alpha\| \left(\mathbb{E}\,|I_\alpha(s,t)|^{2L} \right)^{1/(2L)} \right)^{2L}$$

$$\leq \left(\sum_{n=1}^{+\infty} (\eta_{n,L}(s,t))^{1/(2L)} \sum_{\alpha\colon |\alpha|=n}^{+\infty} \|B^\alpha\| \right)^{2L}$$

Since

$$\sum_{\alpha\colon |\alpha|=n}^{+\infty} \|B^\alpha\| \leq \sum_{\alpha\colon |\alpha|=n}^{+\infty} \|B_{\alpha_1}\| \|B_{\alpha_2}\| \ldots \|B_{\alpha_n}\|$$

$$\leq \left(\sum_{k=1}^{+\infty} \|B_k\| \right)^n =: M^n,$$

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we have

$$
\begin{aligned}
\left(\mathbb{E}\,\xi^{2L}(s,t)\right)^{1/(2L)} &\leq \sum_{n=1}^{+\infty} M^n \left(\eta_{n,L}(s,t)\right)^{1/(2L)} \\
&\leq \sum_{n=1}^{+\infty} M^n C_L^n \left(\frac{\max\{1,(t-s)^{2Ln}\}}{n!}\right)^{1/(2L)} (t-s)^{\frac{1}{2}-\frac{1}{2L}} < +\infty,
\end{aligned}
$$

which gives (2.7). We will show (2.4) using calculations from the proof of (2.7). Namely, we have

$$
\begin{aligned}
\mathbb{E}\,\|\mathcal{Y}(s,t) - \mathcal{Y}(s,t+h)\|^{2L} &= \mathbb{E}\left\|\sum_{n=1}^{+\infty}\sum_{\alpha:\,|\alpha|=n}\left[I_\alpha(s,t) - I_\alpha(s,t+h)\right]B^\alpha\right\|^{2L} \\
&\leq \left(\sum_{n=1}^{+\infty}\sum_{\alpha:\,|\alpha|=n}\|B^\alpha\|\left(\mathbb{E}\,|I_\alpha(s,t) - I_\alpha(s,t+h)|^{2L}\right)^{1/(2L)}\right)^{2L}.
\end{aligned}
$$

Since

$$
\mathbb{E}\,|I_\alpha(s,t) - I_\alpha(s,t+h)|^{2L} = \mathbb{E}\,|I_\alpha(0,h)|^{2L},
$$

and (2.7) holds, for $0 \leq s \leq t \leq u \leq T$ we have

$$
\begin{aligned}
\mathbb{E}\,\|\mathcal{Y}(s,t) - \mathcal{Y}(s,u)\|^{2L} &= \mathbb{E}\,\|\mathcal{Y}(0,0) - \mathcal{Y}(0,u-t)\|^{2L} \\
&= \mathbb{E}\,\xi^{2L}(0,u-t) \\
&\leq \left(\sum_{n=1}^{+\infty}(MC_L)^n\left(\frac{\max\{1,T^{2Ln}\}}{n!}\right)^{1/(2L)}\right)^{2L}(u-t)^{L-1}. \qquad \square
\end{aligned}
$$

We are showing now (2.5). We can assume that $0 \leq s \leq u$. Then there are three cases (i): $s \leq t \leq u \leq v$, (ii): $s \leq u \leq t \leq v$ and (iii): $s \leq u \leq v \leq t$. The first case follows from (2.4) by elementary calculations. For, we have

$$
\begin{aligned}
\mathbb{E}\,\|\mathcal{Y}(s,t) - \mathcal{Y}(u,v)\|^{2L} &\leq 2^{2L}\left(\mathbb{E}\,\|\mathcal{Y}(s,t)\|^{2L} + \mathbb{E}\,\|\mathcal{Y}(u,v)\|^{2L}\right) \\
&\leq C\left(|t-s|^{L-1} + |v-u|^{L-1}\right) \\
&\leq C\left(|t-s| + |v-u|\right)^{L-1} = C\left(t-s+v-u\right)^{L-1} \\
&\leq C\left(v-t+u-s\right)^{L-1} = C\left(|t-v| + |s-u|\right)^{L-1}.
\end{aligned}
$$

The cases (ii) and (iii) follow also from (2.3), (2.4), and the flow property of $\mathcal{Y}$. Indeed consider (ii). Then

$$
\begin{aligned}
\|\mathcal{Y}(s,t) - \mathcal{Y}(u,v)\| &= \|\mathcal{Y}(u,t) \circ \mathcal{Y}(s,u) - \mathcal{Y}(t,v) \circ \mathcal{Y}(u,t)\| \\
&= \|(\mathcal{Y}(u,t) \circ (\mathcal{Y}(s,u) - I) - (\mathcal{Y}(t,v) - I) \circ \mathcal{Y}(u,t)\| \\
&\leq \|\mathcal{Y}(u,t)\|\,\|\mathcal{Y}(s,u) - \mathcal{Y}(s,s)\| + \|\mathcal{Y}(t,v) - \mathcal{Y}(t,t)\|\,\|\mathcal{Y}(u,t)\|.
\end{aligned}
$$

Therefore applying the Schwarz inequality, and (2.3) and (2.4) for $4L$, we can find a constant C such that

$$\mathbb{E}\left\|\mathcal{Y}(s,t) - \mathcal{Y}(u,v)\right\|^{2L} \leq C\left(|s-u|^{2L-1} + |t-v|^{2L-1}\right)^{1/2} \leq C'\left(|s-u| + |t-v|\right)^{L-1}.$$

Finally in the last case (case (iii)), we have

$$
\begin{aligned}
\|\mathcal{Y}(s,t) - \mathcal{Y}(u,v)\| &= \|\mathcal{Y}(v,t) \circ \mathcal{Y}(u,v) \circ \mathcal{Y}(s,u) - \mathcal{Y}(u,v)\| \\
&= \|\mathcal{Y}(v,t) \circ \mathcal{Y}(u,v) \circ (\mathcal{Y}(s,u) - I) + (\mathcal{Y}(v,t) - I) \circ \mathcal{Y}(u,v)\| \\
&\leq \|\mathcal{Y}(v,t)\| \, \|\mathcal{Y}(u,v)\| \, \|\mathcal{Y}(s,u) - \mathcal{Y}(s,s)\| + \|\mathcal{Y}(v,t) - \mathcal{Y}(v,v)\| \, \|\mathcal{Y}(u,v)\|.
\end{aligned}
$$

Therefore applying the Schwarz inequality, and (2.3) and (2.4) for $4L$ and $8L$, respectively, we can find a constant C such that

$$\mathbb{E}\left\|\mathcal{Y}(s,t) - \mathcal{Y}(u,v)\right\|^{2L} \leq C\left(|s-u|^{2L-1} + |t-v|^{2L-1}\right)^{1/2} \leq C'\left(|s-u| + |t-v|\right)^{L-1}.$$

$\square$.

2.2. **Proof of Theorem 2.2.** From the first part we can easily deduce the invertibility of $\mathcal{Y}(s,t)$. For, let us fix $0 \leq s \leq t < +\infty$. Consider, the partition $s = t_0^n < t_1^n < \ldots t_n^n = t$ such that $t_{k+1}^n - t_k^n = \frac{t-s}{n}$. From the Hölder continuity of $\mathcal{Y}$ it follows that with probability 1, for any $\omega \in \Omega$, there is an $n(\omega)$ such that

$$\left\|\mathcal{Y}(t_k^n, t_{k+1}^n; \omega) - \mathcal{Y}(t_k^n, t_k^n; \omega)\right\| \leq \frac{1}{2}.$$

Hence, since $\mathcal{Y}(t_k^n, t_k^n; \omega) = I$, each $\mathcal{Y}(t_k^n, t_{k+1}^n; \omega)$ is invertible. Since

$$\mathcal{Y}(s,t;\omega) = \mathcal{Y}(t_{n-1}^n, t_n^n; \omega) \circ \mathcal{Y}(t_{n-2}^n, t_{n-1}^n; \omega) \circ \ldots \circ \mathcal{Y}(t_0^n, t_1^n; \omega),$$

the invertibility of $\mathcal{Y}(s,t;\omega)$ follows.

From the first part it also follows that (2.6) defines stochastic flow $\mathcal{Z}$. We will first show that $\mathcal{Z}^\star(s,t)\mathcal{Y}(s,t) = I$. To this end note that for all $z, y \in H$, and $0 \leq s \leq t$, we have

$$\langle \mathcal{Z}^\star(s,t)\mathcal{Y}(s,t)y, z \rangle_H = \langle \mathcal{Y}(s,t)y, \mathcal{Z}(s,t)z \rangle_H = \langle Y_s^y(t), Z_s^z(t) \rangle_H.$$

Next

$$
\begin{aligned}
\mathrm{d}\langle Y_s^y(t), Z_s^z(t) \rangle_H &= \langle B_0 Y_s^y(t), Z_s^z(t) \rangle_H \mathrm{d}t \\
&\quad + \langle Y_s^y(t), (-B_0^\star + \tilde{B}^\star) Z_s^z(t) \rangle_H \mathrm{d}t - \sum_k \langle B_k Y_s^y(t), B_k^\star Z_s^z(t) \rangle_H \mathrm{d}t \\
&\quad + \sum_k \langle B_k Y_s^y(t), Z_s^z(t) \rangle_H \mathrm{d}W_k(t) - \sum_k \langle Y_s^y(t), B_k^\star Z_s^z(t) \rangle_H \mathrm{d}W_k(t) \\
&= 0.
\end{aligned}
$$

Hence for all $y, z \in H$ and $0 \leq s \leq t$,

$$\langle \mathcal{Z}^\star(s,t)\mathcal{Y}(s,t)y, z \rangle_H = \langle y, z \rangle_H,$$

and therefore $\mathcal{Z}^\star(s,t)\mathcal{Y}(s,t) = I$, $\mathbb{P}$-a.s. Since $\mathcal{Y}(s,t)$ is invertible, $\mathcal{Z}^\star(s,t) = (\mathcal{Y}(s,t))^{-1}$ as required. $\square$

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2.3. **Proof of Theorem 2.3.** Recall that $(A, D(A))$ is the generator of a C_0-semigroup e^{tA}, $t \geq 0$, on a Hilbert space H. Let $A_\lambda = \lambda A(\lambda I - A)^{-1}$ be the Yosida approximation of A. Consider the following approximation of (1.1),

$$(2.8) \qquad \mathrm{d}X = A_\lambda X \mathrm{d}t + \sum_k B_k X \mathrm{d}W_k, \quad X(s) = x \in H.$$

Let $X_s^{x,\lambda}$ be the solution to (2.8). Recall that X_s^x is the solution to (1.1). It is easy to show that

$$(2.9) \qquad \lim_{\lambda \to +\infty} \mathbb{E} \left| X_s^{x,\lambda}(t) - X_s^x(t) \right|_H^2 = 0, \qquad \forall t \geq s.$$

Define $B_0 = \frac{1}{2} \sum_{k=1}^{+\infty} B_k^2$. Consider the following linear stochastic differential equation.

$$(2.10) \qquad \mathrm{d}Y = B_0 Y \mathrm{d}t + \sum_k B_k Y \mathrm{d}W_k, \qquad Y(s) = x \in H.$$

From the first part of the theorem we know that (2.8) and (2.10) define stochastic flows, $\mathcal{X}_\lambda$ and $\mathcal{Y}$, respectively. Consider the following equation with random coefficients

$$(2.11) \qquad \mathrm{d}\mathcal{G}(t) = \mathcal{Y}(s,t)^{-1} \left(A_\lambda - B_0 \right) \mathcal{Y}(s,t)\mathcal{G}(t)\mathrm{d}t, \quad \mathcal{G}(s) = I.$$

Taking into account Hölder continuity of $\mathcal{Y}$ one can see that the solution $\mathcal{G}_\lambda$ exists and

$$\mathcal{G}_\lambda(t,s) = \exp \left\{ \int_s^t \mathcal{Y}(s,r)^{-1} \left(A_\lambda - B_0 \right) \mathcal{Y}(s,r)\mathrm{d}r \right\}.$$

Moreover, see [24], there is a strongly continuous two parameter evolution system $\mathcal{G}(s,t)$ of bounded linear operators on H, such that for any $x \in H$,

$$(2.12) \qquad \lim_{\lambda \to +\infty} \mathcal{G}_\lambda(s,t)x = \mathcal{G}(s,t)x, \qquad \mathbb{P} - a.s.$$

We will show that (1.1) defines stochastic flow and $\mathcal{X}(s,t) = \mathcal{Y}(s,t)\mathcal{G}(s,t)$. Taking into account (2.9) and (2.12) it is enough to show that $X_s^{x,\lambda}(t) = \mathcal{Y}(s,t)\mathcal{G}_\lambda(s,t)x$, $t \geq s$, that is that $\mathcal{Y}(s,t)\mathcal{G}_\lambda(s,t)x$, $t \geq s$, solves (2.8). Clearly $\mathcal{Y}(s,s)\mathcal{G}_\lambda(s,s)x = x$. Next, for any $y \in H$ we have

$$\mathrm{d}\mathcal{Y}^\star(s,t)y = \mathcal{Y}^\star(s,t)B_0^\star y\mathrm{d}t + \sum_{k=1}^{+\infty} \mathcal{Y}^\star(s,t)B_k^\star y\mathrm{d}W_k(t).$$

Therefore

$$\begin{aligned}
\mathrm{d} \langle \mathcal{G}_\lambda(s,t)x, \mathcal{Y}^\star(s,t)y \rangle_H &= \left\langle \mathcal{Y}(s,t)^{-1} \left(A_\lambda - B_0 \right) \mathcal{Y}(s,t)\mathcal{G}_\lambda(s,t)x, \mathcal{Y}^\star(s,t)y \right\rangle_H \\
&\quad + \left\langle \mathcal{G}_\lambda(s,t)x, \mathcal{Y}^\star(s,t)B_0^\star y \right\rangle_H \mathrm{d}t \\
&\quad + \sum_{k=1}^{+\infty} \left\langle \mathcal{G}_\lambda(s,t)x, \mathcal{Y}^\star(s,t)B_k^\star y \right\rangle_H \mathrm{d}W_k(t),
\end{aligned}$$

and consequently we have the desired conclusion

$$\mathrm{d}\mathcal{Y}(s,t)\mathcal{G}_\lambda(s,t)x = A_\lambda \mathcal{Y}(s,t)\mathcal{G}_\lambda(s,t)x\mathrm{d}t + \sum_{k=1}^{+\infty} B_k \mathcal{Y}(s,t)\mathcal{G}_\lambda(s,t)x\mathrm{d}W_k(t). \qquad \square$$

2.4. **Proof of Theorem 2.4.** This part is well known. We present it only for complete presentation.

Remark 2.5. The trick used in the proof of Theorem 2.3 is well-known in finite dimensional case and in some infinite dimensional cases and is known as the Doss–Sussman transformation.

There are important and interesting questions: (i) whether the flow $\mathcal{X}$ defined by (1.1) is Hölder continuous $L(H, H)$-valued mapping of $(s, t) \in \Delta$, (ii) if the flow is invertible. Clearly, see the proof of Theorem 2.1(ii), (i) implies (ii). Unfortunately, if A is unbounded it is probably impossible that the flow is continuous in the operator norm. Let us recall that the semigroup e^{tA} generated by A is continuous in the operator norm if and only if A is bounded. In the case when e^{tA}, $t > 0$, are compact it is however possible that e^{tA} is continuous in the operator norm for $t > 0$. Therefore, we can expect that in some cases the flow is continuous in the operator norm on the open set $0 < s < t < +\infty$, see Section 5. Finally note that if the flow $\mathcal{X}$ is invertible, than formally $\mathcal{Z} = (\mathcal{X}^{-1})^*$ is the flow for the equation

$$\mathrm{d}Z = \left(-A^\star + \sum_k \left(B_k^2 \right)^\star \right) Z \mathrm{d}t - \sum_k B_k^\star Z \mathrm{d}W_k.$$

Unfortunately, only in the case when A generates a group, $-A^\star$ generates a C_0-semigroup, and therefore the equation is often ill posed.

Example 2.6. This example is an extension of a model that is important in the theory of random evolution of spins in ferromagnetic materials, see [12].

Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded domain. For a sequence $g_k \in C\left(\mathcal{O}, \mathbb{R}^3\right)$ we define operators

$$B_k \colon L^2\left(\mathcal{O}; \mathbb{R}^3\right) \to L^2\left(\mathcal{O}; \mathbb{R}^3\right), \quad B_k x(\xi) = x(\xi) \times g_k(\xi), \quad \xi \in \mathcal{O},$$

where $\times$ stands for vector product in $\mathbb{R}^3$. It is easy to see that each B_k is skew-symmetric, that is $B_k^* = -B_k$. Assume that $\sum_k \|g_k\|_\infty < +\infty$, hence $\sum_k \|B_k\| < +\infty$. Then, by Theorem 2.1, the stochastic differential equation

$$\mathrm{d}Y = \sum_{k=1}^{+\infty} B_k Y \mathrm{d}W_k,$$

defines stochastic flow $\mathcal{Y}$ on $\mathbb{L}^2$ and, by Theorem 2.3, $\mathcal{Z}(s, t) = (\mathcal{Y}^{-1}(s, t))^\star$ is the flow defined by the equation

$$\mathrm{d}Z = \tilde{B} Z \mathrm{d}t + \sum_k B_k Z \mathrm{d}W_k.$$

where $\tilde{B} = \sum_{k \geq 1} B_k^2$.

3. Nonlinear case

The method based on the Doss–Susmann transformation can be generalized to some non-linear equations, see [11, 16, 17]. Assume that $B_0, \ldots B_N \in C^1(\mathbb{R})$ is a finite sequence of functions with bounded derivatives. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$ and let $(A, D(A))$

be the generator of a C_0-semigroup on $H = L^2(\mathcal{O})$. Consider the following SPDE and SDE in the Stratonovich sense

$$(3.1) \qquad \mathrm{d}X = [AX + B_0(X)]\,\mathrm{d}t + \sum_{k=1}^{N} B_k(X)\mathrm{d}_SW_k, \qquad X(s) = x \in L^2(\mathcal{O}),$$

$$(3.2) \qquad \mathrm{d}y = B_0(y)\mathrm{d}t + \sum_{k=1}^{N} B_k(y)\mathrm{d}_SW_k(t), \qquad y(s) = \xi \in \mathbb{R}.$$

By the classical theory of SDEs, see e.g. [15, 18], (3.2) defines stochastic flow $\eta(s,t)$, $0 \leq s \leq t$, of diffeomorphism of $\mathbb{R}$. Consider now the following stochastic evolution equation on the Hilbert space $L^2(\mathcal{O})$,

$$(3.3) \qquad \mathrm{d}Y = B_0(Y)\mathrm{d}t + \sum_{k=1}^{N} B_k(Y)\mathrm{d}_SW_k, \qquad Y(s) = x \in L^2(\mathcal{O}).$$

Then (3.3) defines stochastic flow $\mathcal{Y}$ on $L^2(\mathcal{O})$ and

$$\mathcal{Y}(s,t)(x)(\xi) = \eta(s,t)(x(\xi)), \qquad x \in L^2(\mathcal{O}), \quad \xi \in \mathcal{O}.$$

Let $G_s^x(t)$ be the solution to the evolution equation with random coefficients

$$\frac{\mathrm{d}}{\mathrm{d}t}G_s^x(t) = \mathcal{U}(s,t,G_s^x(t)), \qquad G_s^x(s) = x,$$

where

$$\mathcal{U}(s,t,y) = [D\mathcal{Y}(s,t)(y)]^{-1} A\mathcal{Y}(s,t)(y).$$

Above, $D\mathcal{Y}(s,t)(y)$ is the derivative with respect to initial condition y. Clearly we need to assume that $\mathcal{Y}(s,t)(G_s^x(t))$ is in the domain of A, and $D\mathcal{Y}(s,t)(G_s^x(t))$ is an invertible operator. Then by the Itô–Vencel formula

$$X_s^x(t) = \mathcal{Y}(s,t)(G_s^x(t)).$$

In [11], this method was applied to equations of the type

$$\mathrm{d}X = \mathcal{A}(X)\mathrm{d}t + \sum_{k=1}^{N} B_k X\mathrm{d}_SW_k,$$

where $\mathcal{A}$ is a monotone operator, and (B_k) are first order differential linear operators.

4. Diagonal and commutative case

In this section we will first consider an extension of the Skorokhod example [23]. Let (e_k) be an orthonormal basis of an infinite-dimensional Hilbert space H and let (σ_k) and (α_k) be sequences of real numbers. We assume that $\sigma_k \geq 0$. For every $k \geq 1$ we define bounded linear operators $B_k = \sigma_k e_k \otimes e_k$, and a possibly unbounded operator

$$A = \sum_{j=1}^{+\infty} \alpha_j e_j \otimes e_j$$

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with the domain

$$D(A) = \left\{ x \in H \colon \sum_{j=1}^{+\infty} \alpha_j^2 \langle x, e_j \rangle_H^2 < +\infty \right\}.$$

Let us recall that in the Skorokhod example $\alpha_k = 0$, $k = 1, 2, \ldots$, and $\sigma_k = \sigma$ does not depend on k.

Proposition 4.1. *Consider* (1.1) *with* $A, B_1, \ldots$ *as above. Then the following holds.*
(i) For each initial value $x \in H$, (1.1) *has a square integrable solution* X_s^x *in* H *if and only if*

$$\text{(4.1)} \qquad\qquad \sup_k \left(2\alpha_k + \sigma_k^2 \right) < +\infty.$$

(ii) Assume (4.1). *Then* (1.1) *defines stochastic flow* $\mathcal{X}$ *if and only if*

$$\text{(4.2)} \qquad \rho(s, t) := \sup_k \left\{ \left[\alpha_k - \frac{\sigma_k^2}{2} \right] \sqrt{t - s} + \sigma_k \sqrt{2 \log k} \right\} < +\infty, \quad \forall\, 0 \le s < t.$$

Proof. Define

$$\zeta_k(s, t; \omega) := \exp \left\{ \sigma_k \left(W_k(t; \omega) - W_k(s; \omega) \right) + \left(\alpha_k - \frac{\sigma_k^2}{2} \right) (t - s) \right\}.$$

Clearly random variable $\zeta_k(s, t)$, $k = 1 \ldots$, are independent and

$$\text{(4.3)} \qquad \mathbb{E}\,\zeta_k(s, t) = e^{\alpha_k(t-s)} \qquad \text{and} \qquad \mathbb{E}\,\zeta_k^2(s, t) = e^{(2\alpha_k + \sigma_k^2)(t-s)}.$$

Consider a finite dimensional subspace $V = \mathrm{linspan}\{e_{i_1}, \ldots, e_{i_M}\}$ of H. Then for any $x \in V$, the solution exists and

$$\text{(4.4)} \qquad\qquad X_s^x(t) = \sum_{k=1}^{+\infty} \zeta_k(s, t) \langle x, e_k \rangle_H e_k.$$

Next, note that for any $x \in H$, if X_s^x is a solution, then it has to be of form (4.4). For, if $\Pi\colon H \mapsto \mathrm{linspan}\{e_{i_1}, \ldots, e_{i_M}\}$ is a projection then $\Pi X_s^x = X_s^{\Pi x}$. Therefore, the first claim follows from (4.3). Next, (1.1) defines a stochastic flow $\mathcal{X}$ if and only if

$$\mathbb{P} \left\{ \sup_{|x|_H \le 1} |X_s^x(t)|_H < +\infty \right\} = 1.$$

Clearly,

$$\mathbb{P} \left\{ \sup_{|x|_H \le 1} |X_s^x(t)|_H < +\infty \right\} = \mathbb{P} \left\{ \sup_k \zeta(s, t) < +\infty \right\}$$

$$= \mathbb{P} \left\{ \sup_k \left[\sigma_k \left(W_k(t) - W_k(s) \right) + \left(\alpha_k - \frac{\sigma_k^2}{2} \right) (t - s) \right] < +\infty \right\}.$$

Therefore the desired conclusion follows from the fact that if (Z_k) is a sequence of independent $\mathcal{N}(0,1)$ random variables, then

$$\limsup_{k \to +\infty} \frac{Z_k}{\sqrt{2 \log k}} = 1, \qquad \mathbb{P} - a.s.$$

$\square$

Remark 4.2. Assume (4.1) and (4.2). Let $\mathcal{X}$ be the stochastic flow. Then

$$\mathbb{E} \operatorname{Tr} \mathcal{X}(s,t) = \sum_{k=1}^{+\infty} \mathbb{E} \zeta_k(s,t) = \sum_{k=1}^{+\infty} e^{\alpha_k(t-s)}.$$

Note that (4.1) and $\sum_{k=1}^{+\infty} e^{\alpha_k(t-s)} < +\infty$ imply (4.2).

4.1. **Beyond integrability.** As above, (σ_k) is a sequence of nonnegative real numbers. Assume now that $\alpha_k = 0$. Thus

$$\zeta_k(s,t) = \exp\left\{ -\frac{\sigma_k^2}{2}(t-s) + \sigma_k \left(W_k(t) - W_k(s)\right) \right\}.$$

Let us assume that

(4.5) $\qquad \rho(s,t) := \sup_k \left\{ -\frac{\sigma_k^2}{2}\sqrt{t-s} + \sigma_k \sqrt{2 \log k} \right) < +\infty, \quad \forall 0 \le s < t.$

We do not assume however (4.1). Then

$$\mathcal{X}(s,t;\omega)(x) = \sum_{k=1}^{+\infty} \zeta_k(s,t;\omega)\langle x, e_k \rangle_H e_k, \qquad 0 \le s \le t, \ x \in H,$$

is well defined, and since $\sup_k \zeta_k(s,t) < +\infty$, $\mathcal{X} \colon \Delta \times \Omega \mapsto L(H,H)$. Obviously $\mathcal{X}(s,t)$ is a symmetric positive definite operator with eigenvectors (e_k) and eigenvalues $(\zeta_k(s,t))$. Note that it can happen that $\mathbb{E}\, |\mathcal{X}(s,t)(x)|_H^2 = +\infty$. Moreover, if $\alpha_k = 0$ then necessarily $\mathbb{E} \operatorname{Tr} (\mathcal{X}(s,t)) = +\infty$.

Proposition 4.3. *Assume* (4.5). *Then the following holds:*

(i) *If $l \in [0, +\infty]$ is an accumulation point of the sequence (σ_k), then either $l = 0$ or $l = +\infty$.*

(ii) *If $\sigma_k \to 0$ then $\sup_k \sigma_k \sqrt{\log k} < +\infty$. In particular, $\mathbb{P}$-a.s. the sequence $(\zeta_k(s,t))$ of eigenvalues of $\mathcal{X}(s,t)$ has accumulation points different from zero, hence $\mathcal{X}(s,t)$ is not compact $\mathbb{P}$-a.s.*

(iii) *If $\sigma_k \to +\infty$ then $\frac{\sigma_k}{\sqrt{\log k}} \to +\infty$ and $\mathcal{X}(s,t)$ is Trace class $\mathbb{P}$-a.s.*

Proof. Statement (i) is obvious as $\sigma_k \ge 0$. The first part of (ii) is obvious. Assume that $\sigma_k \to 0$ and that $\sup_k \sigma_k \sqrt{\log k} < +\infty$. We have to show that $\mathbb{P}$-a.s. the sequence

$$\exp\left\{ \sigma_k \left(W_k(t) - W_k(s)\right) \right\}, \qquad k = 1, \dots$$

has accumulation points different from zero, or equivalently that

$$\liminf_{k \to +\infty} \sigma_k \left(W_k(t) - W_k(s)\right) > -\infty, \qquad \mathbb{P} - a.s.$$

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Since
$$\liminf_{k\to+\infty} \frac{W_k(t) - W_k(s)}{\sqrt{2\log k}} = -\sqrt{t-s}, \qquad \mathbb{P}-a.s.$$

we have
$$\liminf_{k\to+\infty} \sigma_k \left(W_k(t) - W_k(s) \right) = -\sqrt{t-s} \limsup_{k\to+\infty} \sigma_k \sqrt{2\log k} > -\infty, \qquad \mathbb{P}-a.s.$$

We are proving now the last statement od the proposition. Condition (4.5) can be rewritten in the form
$$\sup_k \sigma_k^2 \left(\frac{\sqrt{2\log k}}{\sigma_k} - \frac{\sqrt{t-s}}{2} \right) < +\infty, \qquad \forall\, 0 \le s < t.$$

Since $\sigma_k \to +\infty$, we have
$$\limsup_{k\to+\infty} \frac{\sqrt{2\log k}}{\sigma_k} < \frac{\sqrt{t-s}}{2}, \qquad \forall\, 0 \le s < t,$$

which leads to the desired conclusion that $\frac{\sigma_k}{\sqrt{\log k}} \to +\infty$.

We will use the Kolmogorov three series theorem to show that
$$\operatorname{Tr} \mathcal{X}(s,t) = \sum_{k=1}^{+\infty} \zeta_k(s,t) < +\infty, \qquad \mathbb{P}-a.s.$$

Let us fix s and t. Define $Y_k = \zeta_k(s,t)\chi_{[0,1]}\left(\zeta_k(s,t)\right)$. We need to show that

(4.6)
$$\sum_{k=1}^{+\infty} \mathbb{P}\left(\zeta_k(s,t) > 1\right) < +\infty,$$

(4.7)
$$\sum_{k=1}^{+\infty} \mathbb{E}\, Y_k < +\infty,$$

(4.8)
$$\sum_{k=1}^{+\infty} \operatorname{Var} Y_k < +\infty$$

Denoting by Z a normal $\mathcal{N}(0,1)$ random variable and putting $b = \frac{\sqrt{t-s}}{2}$ we obtain

$$\sum_{k=1}^{+\infty} \mathbb{P}\left(\zeta_k(s,t) > 1\right) = \sum_{k=1}^{+\infty} \mathbb{P}\left(Z > b\sigma_k\right)$$

$$\le c + \frac{1}{\sqrt{2\pi}} \sum_{k=2}^{+\infty} \frac{1}{b\sigma_k} e^{-b^2\sigma_k^2/2} = c + \frac{1}{\sqrt{2\pi}} \sum_{k=2}^{+\infty} \frac{1}{b\sigma_k} e^{-\delta_k \log k}$$

$$\le c + C \sum_{k=2}^{+\infty} \frac{1}{k^{\delta_k}}$$

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with $\delta_k = \frac{b^2 \sigma_k^2}{2 \log k}$. Since $\delta_k \to +\infty$, (4.6) follows. Let $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathfrak{F}_t} = \zeta_k(s,t)$, where $\mathfrak{F}_t = \sigma(W_k(r), k = 1, \ldots, r \leq t)$. Then $W_k^{\mathbb{Q}}(r) := W_k(r) - \sigma_k r$, $r \leq t$, are independent Wiener processes under $\mathbb{Q}$ and

$$\log \zeta_k(s,t) = \frac{1}{2}\sigma_k^2(t-s) + \sigma_k\left(W_k^{\mathbb{Q}}(t) - W_k^{\mathbb{Q}}(s)\right).$$

Therefore,

$$\mathbb{E}\, Y_k = \mathbb{E}\, \zeta_k(s,t)\chi_{[0,1]}\left(\zeta_k(s,t)\right) = \mathbb{Q}\left(\zeta_k(s,t) \leq 1\right) = \mathbb{Q}\left(Z > \frac{1}{2}\sigma_k\sqrt{t-s}\right)$$

and by the same arguments as above we find that (4.7) holds as well. Finally, since

$$\mathbb{E}\, Y_k^2 = \mathbb{E}\, \zeta_k^2(s,t)\chi_{[0,1]}\left(\zeta_k(s,t)\right) \leq \mathbb{E}\, Y_k,$$

condition (4.8) is satisfied and the proof is complete. $\qquad\square$

4.2. The case of commuting operators.
Let us recall that in the Skorokhod example $A = 0$ and $B_j = e_j \otimes e_j$ are commuting operators. We have

$$
\begin{aligned}
X_s^x(t) &= \sum_{j=1}^{+\infty} \exp\left\{W_j(t) - W_j(s) - \frac{t-s}{2}\right\} e_j \otimes e_j(x) \\
&= \lim_{N \to +\infty} \sum_{j=1}^{N} \exp\left\{W_j(t) - W_j(s) - \frac{t-s}{2}\right\} e_j \otimes e_j(x) \\
&= \lim_{N \to +\infty} \exp\left\{\sum_{j=1}^{N} e_j \otimes e_j\left(W_j(t) - W_j(s) - \frac{t-s}{2}\right)\right\}(x).
\end{aligned}
$$

The convergence is not uniform in x. In the last line we calculate the exponent of a bounded operator

$$\sum_{j=1}^{N} e_j \otimes e_j\left(W_j(t) - W_j(s) - \frac{t-s}{2}\right).$$

A proof the following simple generalization of Proposition 4.1 is left to the reader. Recall that a sequence of bounded operators S_n converges to a bounded operator S <u>strongly</u> if $S_n x \to S x$ for any $x \in H$.

Proposition 4.4. *Assume that (B_k) is an infinite sequence of bounded commuting operators on a Hilbert space H. Then:*

(i) *For any $x \in H$ and $s \geq 0$ there is a square integrable solution X_s^x to (1.1) if and only if for all $0 \leq s \leq t$, the sequence $\exp\left\{(t-s)\sum_{k=1}^{n} B_k^2\right\}$ converges strongly as $n \to +\infty$. Moreover,*

$$X_s^x(t) = \lim_{n \to +\infty} \exp\left\{\sum_{k=1}^{n} B_k\left(W_k(t) - W_k(s)\right) + B_0(t-s) - \frac{1}{2}\sum_{k=1}^{n} B_k^2\,(t-s)\right\} x,$$

where the limit is in $L^2(\Omega, \mathfrak{F}, \mathbb{P}; H)$.

(ii) (1.1) *generates a stochastic flow if and only if for all $0 \leq s \leq t$, $\mathbb{P}$-a.s.*

$$\exp\left\{\sum_{k=1}^{n} B_k\left(W_k(t) - W_k(s)\right) - \frac{1}{2}\sum_{k=1}^{n} B_k^2\left(t - s\right)\right\}$$

converges as $n \to +\infty$ in the operator norm.

4.3. System of multiplication operators.
In this section we assume that

$$H = L^2(\mathbb{R}^d, \vartheta(\xi)\mathrm{d}\xi),$$

where the weight $\vartheta \colon \mathbb{R}^d \mapsto (0, +\infty)$ is a measurable function. Let W be a Wiener process taking values in H. Then, see see e.g. [9],

(4.9)
$$W = \sum_{k} W_k e_k\,,$$

where (W_k) are independent real-valued Wiener processes, and $\{e_k\}$ is an orthonormal basis of the Reproducing Hilbert Kernel Space (RHKS for short) of W.

Consider the equation

(4.10)
$$\mathrm{d}X(t) = X(t)\mathrm{d}W(t), \qquad X(s) = x,$$

Taking into account (4.9) we can write (4.10) in the form

(4.11)
$$\mathrm{d}X(t) = \sum_{k} B_k X \mathrm{d}W_k(t), \qquad X(s) = x,$$

where B_k are multiplication operators; $B_k h = h e_k$, $h \in H$. Clearly a multiplication operator $h \mapsto he$ is bounded on H if and only if $e \in L^\infty(\mathbb{R}^d)$. Note that bounded multiplication operators commute and are symmetric.

Proposition 4.5. *Assume that $e_k \in L^\infty(\mathbb{R}^d)$. Then:*

(i) *For any $x \in H$ and $s \geq 0$ there exists a square integrable solution X_s^x to (4.11) if and only if $\sum_{k=1}^{+\infty} e_k^2 \in L^\infty(\mathbb{R}^d)$. Moreover,*

$$X_s^x(t) = L^2 - \lim_{n \to +\infty} \exp\left\{\sum_{k=1}^{n}\left(W_k(t) - W_k(s)\right)e_k - \frac{t - s}{2}\sum_{k=1}^{n} e_k^2\right\}x.$$

(ii) *Assume that $\sum_{k=1}^{+\infty} e_k^2 \in L^\infty(\mathbb{R}^d)$. Then (4.11) defines a stochastic flow if and only if*

$$\mathbb{P}\left\{\operatorname*{ess\,sup}_{\xi \in \mathbb{R}^d}\sum_{k} e_k(\xi)W_k(t) < +\infty\right\} = 1,$$

equivalently iff the process $W(t) = \sum_{k} W_k(t)e_k$ lives in $L^\infty(\mathbb{R}^d)$; that is

$$\mathbb{P}\left(W(t) \in L^\infty(\mathbb{R}^d)\right) = 1.$$

(iii) *If $\sum_{k}\log\sqrt{k}|e_k| \in L^\infty(\mathbb{R}^d)$, then (4.11) defines stochastic flow on H.*

Proof. The first part follows from Proposition 4.4. The second part is a reformulation of the second part of Proposition 4.4. The las part follows from the law of iterated logarithm. □

Example 4.6. Assume that $W = W(t, \xi)$ is a spatially homogeneous Wiener process on $\mathbb{R}^d$, see e.g. [9]. Then $W(t) = \sum_k e_k W_k$, where $e_k = \widehat{f_k \mu}$, $\{f_k\}$ is a orthonormal basis of $L^2_{(s)}(\mathbb{R}^d, \mu)$ and μ is the spectral measure of W. The sum over finite or infinite number of indices k. We can assume that $e_k \in L^\infty(\mathbb{R}^d)$ by choosing suitable f_k. Then

$$\sum_k e_k^2 = \sum_k \left|\widehat{f_k \mu}\right|^2 = \mu(\mathbb{R}^d).$$

Therefore, (4.11) has a square integrable solution if and only if μ is finite, that is W is a random field. The condition $\mathbb{P}\left(W(t) \in L^\infty(\mathbb{R}^d)\right)$ wich is if and only if condition for the existence of stochastic flow holds only in some degenerated cases. Namely assume that

$$W(t, \xi) = \sum_{k=1}^{+\infty} a_k \left(W_k(t) \cos\langle \xi, \eta_k \rangle + \tilde{W}_k(t) \sin\langle \xi, \eta_k \rangle\right),$$

where $\{\eta_k\} \subset \mathbb{R}^d$, $(a_k) \in l^2$, and W_k and $\tilde{W}_k$, $k \in \mathbb{N}$, are independent real-valued Brownian motions. Then $W = W(t, \xi)$ is a spatially homogeneous Wiener process. Taking into account that

$$\limsup_{k \to +\infty} \frac{W_k(t)}{\sqrt{2 \log k}} = \sqrt{t},$$

we see the equation defines stochastic flow on any weighted L^2-space if $\sum_k |a_k|\sqrt{\log k} < +\infty$.

Example 4.7. Assume that W is a Brownian sheet on $[0, L)^{d+1}$ where $L \in [0, +\infty]$. To be more precisely $W(t, \xi_1, \xi_2, \ldots, \xi_d)$ is a Gaussian random field on $[0, L)^{d+1}$ with the covariance

$$\mathbb{E}\, W(t, \xi_1, \ldots, \xi_d) W(s, \eta_1, \ldots, \eta_d) = t \wedge s \prod_{k=1}^{d} \xi_k \wedge \eta_k.$$

Then

$$f_k = \frac{\partial^d}{\partial \xi_1, \ldots \partial \xi_d} e_k$$

is an orthonormal basis of $L^2([0, L)^d)$, respectively. Hence

$$\sum_k e_k^2(\xi) = \sum_k \langle \chi_{[0,\xi_1] \times \ldots \times [0,\xi_d]}, f_k \rangle^2 = \xi_1 \xi_2 \ldots \xi_d.$$

Therefore (4.11) has a square integrable solution (in $L^2([0, T)^d)$ if and only if $L < \infty$. Clearly, Brownian sheet has continuous trajectories, and therefore for arbitrary $T < +\infty$ and $L < +\infty$ we have $\mathbb{P}\left\{\sup_{0 \le t < T, \xi \in [0,L)^d} W(t, \xi) < +\infty\right\} = 1$. Hence if (4.11) is considered on a bounded domain the equation defines stochastic flow. Let us observe that the stochastic flow can be also well defined, but not square integrable, if $L = +\infty$. Indeed, the stochastic flow exists, if and only if

(4.12) $$\mathbb{P}\left\{\sup_{\xi \in [0,+\infty)^d} \left(W(t, \xi_1, \ldots, \xi_d) - \frac{t}{2}\xi_1 \xi_2, \ldots \xi_d\right) < +\infty\right\} = 1.$$

5. Equations in Schatten classes

The problem of the existence of the stochastic flow in the finite dimensional case $H = \mathbb{R}^d$ is simple. The flow $\mathcal{X}(s,t)$ takes valued in the space of bounded linear operators $L(H,H)$ that can be identified with the space of $d \times d$ matrices $M(d \times d)$. We have the following SDE for $\mathcal{X}$ in the space $M(d \times d)$;

$$(5.1) \qquad \mathrm{d}\mathcal{X} = A\mathcal{X}\mathrm{d}t + (\mathrm{d}\mathcal{W})\mathcal{X}, \qquad \mathcal{X}(s,s) = I,$$

where $\mathcal{W} := \sum_{k=1}^{d} B_k W_k$ and I is the identity matrix. By a standard fixed point argument (5.1) has a unique global solution.

In infinite dimensional case, even if (B_k) is a finite sequence of bounded linear operators, the situation is different. There is no proper theory of stochastic integration in the space $L(H,H)$ if H is infinite-dimensional. One can overcome this difficulty by replacing $L(H,H)$ with a smaller space of operators, such as the Hilbert–Schmidt or, more generally, the Schatten classes of operators.

5.1. **Main result.**
Let us recall that for every $p \in [1, +\infty)$ the Schatten class $\mathbb{S}^p$ of compact operators $K \colon H \to H$ is a Banach space endowed with the norm

$$\|K\|_p = \left(\sum_{k=1}^{+\infty} \lambda_k \left(K^\star K \right)^{p/2} \right)^{1/p} < +\infty,$$

where $\lambda_k(K^\star K)$ stands for the k-th eigenvalue of $K^\star K$. For every $p \in [1, +\infty)$ the space $\mathbb{S}^p$ is a separable Banach space.

Lemma 5.1. *For every $p \geq 2$ the space $\mathbb{S}^p$ is an M-type 2 Banach space.*

Proof. By Propositions 5.4.2 in [13] the space $\mathbb{S}^p$ is a UMD space for every $p \in (1, +\infty)$. By Proposition 7.1.11 in [14], $\mathbb{S}^p$ has type 2 for $p \in [2, +\infty)$, and by Proposition 4.3.13 in [13], M-type 2 property follows. $\qquad \square$

Lemma 5.1 ensures that if $p \geq 2$, then $\mathbb{S}^p$ is a right space for stochastic integration, for more details see e.g. [5]. The definition of the Wiener process given below uses the fact that $L(H,H) = (\mathbb{S}^1)^\star$.

Definition 5.2. Let $B_k \in L(H,H)$ for $k \geq 1$. We call $\mathcal{W} = \sum_{k=1}^{+\infty} W_k B_k$ a <u>cylindrical Wiener process</u> on $L(H,H)$ if for every $K \in \mathbb{S}^1$ the process

$$W^K(t) = \sum_{k=1}^{+\infty} \mathrm{Tr}\left(K B_k \right) W_k(t),$$

is a real-valued Wiener process, and there is a constant C independent of K such that

$$(5.2) \qquad \mathbb{E}\left| W^K(t) \right|^2 = t \sum_{k=1}^{+\infty} \left(\mathrm{Tr}\left(K B_k \right) \right)^2 \leq Ct\|K\|_1^2 < +\infty.$$

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Note that condition (5.2) holds if $\sum_k \|B_k\|^2_{L(H,H)} < \infty$. Indeed, we have (see Theorem 3.1 in [22])

$$|\mathrm{Tr}\,(KB_k)| \leq \|KB_k\|_1 \leq \|B_k\|_{L(H,H)} \|K\|_1$$

and the claim follows.

Lemma 5.3. *Let $(A, D(A))$ be the generator of a C_0-semigroup (e^{tA}) on H. For $T \in L(H,H)$ we define*

$$(5.3) \qquad\qquad \mathcal{S}(t)T = \mathrm{e}^{tA} \circ T, \quad t \geq 0,$$

and denote $\mathcal{S} = \{\mathcal{S}(t);\, t \geq 0\}$. Then:
(a) $\mathcal{S} = (\mathcal{S}(t))$ defines a semigroup (but in general not a C_0-semigroup) of bounded operators on $L(H,H)$.
(b) For every $p \in [1, +\infty)$ we have $\mathcal{S}(t)\mathbb{S}^p \subset \mathbb{S}^p$ and $\mathcal{S}$ defines a C_0-semigroup on $\mathbb{S}^p$.

Proof. Since $\mathbb{S}^p$ is an ideal, we have $\mathcal{S}(t)\mathbb{S}^p \subset \mathbb{S}^p$ and the operator $\mathcal{S}(t)\colon \mathbb{S}^p \to \mathbb{S}^p$ is bounded. To prove strong continuity of $\mathcal{S}$ on $\mathbb{S}^p$, let us recall that for every $p \in [1, +\infty)$ the space $\mathbb{S}^p$ is the closure of the space of finite rank operators in the $\mathbb{S}^p$-norm. Let $t > 0$ and $T \in \mathbb{S}^p$ be fixed. There exists a sequence (T_n) of finite rank operators, such that $\|T_n - T\|_p \to 0$. Choose n such that for $t \leq 1$,

$$\|T - T_n\|_p + \left\|\mathrm{e}^{tA}\,(T - T_n)\right\|_p < \varepsilon\,.$$

Since

$$\|\mathcal{S}(t)T - T\|_p \leq \|\mathcal{S}(t)\,(T - T_n)\|_p + \|\mathcal{S}(t)T_n - T_n\|_p + \|T - T_n\|_p$$
$$\leq \varepsilon + \|\mathcal{S}(t)\,(T - T_n)\|_p$$

and

$$\lim_{t \to 0} \|\mathcal{S}(t)T_n - T_n\|_p = 0\,,$$

the strong continuity follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let A be the generator of a C_0-semigroup (e^{tA}) on H and let $\mathcal{A}$ be the generator of the semigroup $\mathcal{S}$ defined in Lemma 5.3. Let W be a cylindrical Wiener process on $L(H,H)$. Consider the following stochastic equation on $\mathbb{S}^p$, where $p \geq 2$,

$$(5.4) \qquad\qquad \mathrm{d}\mathcal{X} = \mathcal{A}\mathcal{X}\mathrm{d}t + (\mathrm{d}\mathcal{W})\mathcal{X}, \qquad \mathcal{X}(s) = I.$$

Definition 5.4. *Let $p \geq 1$. We will say that a process $\mathcal{X}(s, \cdot; \cdot)\colon [s, +\infty) \times \Omega \mapsto L(H,H)$ with continuous paths in $L(H,H)$ is an $\mathbb{S}^p$-valued solution to (5.4) if $\mathcal{X}(s, \cdot; \cdot)\colon (s, +\infty)\times \mapsto \mathbb{S}^p$ is measurable and adapted,*

$$\mathbb{E}\int_s^T \|\mathcal{X}(t)\|_p^2 \mathrm{d}t < +\infty, \qquad \text{for any } T \in (s, +\infty),$$

and

$$\mathcal{X}(t) = \mathcal{S}(t-s)I + \int_s^t \mathcal{S}(t-r)(\mathrm{d}\mathcal{W}(r))\mathcal{X}(r) \qquad \text{for } t \geq s,\ \mathbb{P}\text{-a.s.}.$$

We have the following result.

Proposition 5.5. *If $\mathcal{X}$ is a solution to* (5.4) *with $\mathcal{A}$ as above, then $\mathcal{X}$ is the flow corresponding to* (1.1).

Theorem 5.6. *Assume that there exist constants $\gamma < 1/2$ and $C > 0$ such that*

(5.5)
$$\|S(t)\|_p \leq \frac{C}{t^\gamma}, \quad t \leq 1.$$

Assume that $\mathcal{W} = \sum_k B_k W_k$ is a cylindrical Wiener process on $L(H, H)$ such that $\sum_k \|B_k\|^2 < +\infty$ and $\mathbb{P}(\mathcal{W}(t) \in L(H, H)) = 1$. Then for any $s \geq 0$, (5.4) *has a unique solution in $\mathbb{S}^p$. Moreover, $(s, +\infty) \ni t \mapsto \mathcal{X}(s, t) \in \mathbb{S}^p$ is continuous $\mathbb{P}$-a.s.*

Proof. Let us fix $0 \leq s < T < +\infty$. Let Ψ be the space of all adapted measurable processes $\psi \colon (s, T] \times \Omega \mapsto \mathbb{S}^p$ such that

$$\mathbb{E} \int_s^T \|\psi(t)\|_p^2 \mathrm{d}t < +\infty.$$

On Ψ consider the family of equivalent norms

$$|||\psi|||_\beta := \left(\mathbb{E} \int_s^T \mathrm{e}^{-\beta t} \|\psi(t)\|_p^2 \mathrm{d}t \right)^{1/2}, \qquad \beta \geq 0.$$

Note that, as (e^{tA}) is C_0 on H there is a constant $C < +\infty$ such that for $t \in (s, T]$, and $\psi \in \Psi$,

$$\sum_k \mathbb{E} \int_s^t \|S(t-r) B_k \psi(r)\|_p^2 \mathrm{d}r \leq \sum_k \mathbb{E} \int_s^t \|\mathrm{e}^{(t-r)A}\|^2 \|B_k\|^2 \|\psi(r)\|_p^2 \mathrm{d}r$$

$$\leq C \sum_k \|B_k\|^2 \mathbb{E} \int_s^t \|\psi(r)\|_p^2 \mathrm{d}r < +\infty.$$

Therefore, by assumption (5.5), the mapping

$$\mathcal{I}\psi(t) := S(t-s) I + \int_s^t S(t-r)(\mathrm{d}\mathcal{W}(r)) \psi(r), \qquad \psi \in \Psi, \quad t \in (s, T],$$

is well-defined from Ψ into Ψ. For β large enough $\mathcal{I}$ is contraction on $(\Psi, ||| \cdot |||_\beta)$. For we have

$$|||\mathcal{I}(\psi) - \mathcal{I}(\phi)|||^2 \leq C' \sum_k \|B_k\|^2 \mathbb{E} \int_s^T \mathrm{e}^{-\beta t} \int_s^t \|\psi(r) - \phi(r)\|_p^2 \mathrm{d}r \mathrm{d}t$$

$$\leq C'' \mathbb{E} \int_s^T \mathrm{e}^{-\beta r} \|\psi(r) - \phi(r)\|_p^2 \int_r^T \mathrm{e}^{-\beta(t-r)} \mathrm{d}t \mathrm{d}r$$

$$\leq C'' \frac{1}{\beta} |||\psi - \phi|||_\beta^2.$$

Thus by the Banach fixed point theorem there is an $\mathcal{X}(s, \cdot) \in \Psi$ such that $\mathcal{I}(\mathcal{X}(s, \cdot)) = \mathcal{X}(s, \cdot)$. What is left is to show that for any $\psi \in \Psi$, the stochastic integral

$$\int_s^t \mathcal{S}(t - r)(\mathrm{d}\mathcal{W}(r))\psi(r), \qquad t \geq s,$$

has continuous paths in $\mathbb{S}^p$. Since there exists an $\alpha > 0$ such that

$$\int_s^T (t - s)^{-\alpha} \|\mathcal{S}(t - s)I\|_p^2 \mathrm{d}t < +\infty,$$

the desired continuity follows from Burkholder inequality by a standard modification of the Da Prato–Kwapien–Zabczyk factorization method. $\square$

Example 5.7. Assume that A is diagonal $A = -\sum_k \alpha_k e_k \otimes e_k$, where (e_k) is an orthonormal basis of H, and $\alpha_k \geq 0$ are real numbers. Then A generates a C_0-semigroup e^{tA}, $t \geq 0$, on H. Moreover, $\mathrm{e}^{tA} \in \mathbb{S}^p$ for $t > 0$ if and only if

$$\|\mathrm{e}^{tA}\|_p = \left(\sum_k \mathrm{e}^{-p\lambda_k t} \right)^{1/p} < +\infty.$$

In particular, we can consider the heat semigroup generated by a Dirichlet Laplacian Δ in a bounded subdomain of $\mathbb{R}^2$ with sufficiently smooth boundary. The eigenvalues of Δ have asymptotics $\lambda_{kn} \sim k^2 + n^2$, hence

$$\sum_{k,n} \mathrm{e}^{-pt\left(k^2 + n^2\right)} \sim \frac{1}{2tp},$$

which yields

$$\|\mathrm{e}^{t\Delta}\|_p \sim \left(\frac{1}{2tp} \right)^{1/p}.$$

Therefore, condition (5.5) is satisfied if and only if $p > 2$. If $B_k \colon H \to H$ are bounded operators, such that $\sum_k \|B_k\|^2 < +\infty$, then Theorem 5.6 ensures the existence of the stochastic flow in $\mathbb{S}_p$ for any $p > 2$, but the Hilbert–Schmidt theory developed in [10] cannot be directly applied. Note however that in [10], B_k can be unbounded.

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