ANISOTROPIC ELLIPTIC EQUATIONS WITH GRADIENT-DEPENDENT LOWER ORDER TERMS AND L^1 DATA

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Abstract. For every summable function f, we prove the existence of a weak solution for a general class of Dirichlet anisotropic elliptic problems in a bounded open subset Ω of $\mathbb{R}^N$. The principal part is a divergence-form nonlinear anisotropic operator $\mathcal{A}$, the prototype of which is $\mathcal{A}u = -\sum_{j=1}^N \partial_j(|\partial_j u|^{p_j-2}\partial_j u)$ with $p_j > 1$ for all $1 \le j \le N$ and $\sum_{j=1}^N (1/p_j) > 1$. As a novelty in this paper, our lower order terms involve a new class of operators $\mathfrak{B}$ such that $\mathcal{A} - \mathfrak{B}$ is bounded, coercive and pseudo-monotone from $W_0^{1,\overrightarrow{p}}(\Omega)$ into its dual, as well as a gradient-dependent nonlinearity with an "anisotropic natural growth" in the gradient and a good sign condition.

1. Introduction and main results

1.1. **Setting of the problem.** A series of papers, such as [10, 11, 13, 15, 16], deal with nonlinear elliptic problems in a bounded open subset Ω of $\mathbb{R}^N$ involving coercive, bounded, continuous and pseudo-monotone Leray–Lions type operators from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$, where $1 < p < \infty$ and $1/p + 1/p' = 1$. The prototype model for such an operator is the p-Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The techniques developed in the above-mentioned papers accommodate for a lower-order term $g(x, u, \nabla u)$ with a "natural growth" in the gradient $|\nabla u|$ and without any restriction of its growth in $|u|$. Because of the "sign-condition" on g (that is, $g(x, t, \xi)\, t \ge 0$ for a.e. $x \in \Omega$ and all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$), either $f \in L^1(\Omega)$ or $h \in W^{-1,p'}(\Omega)$ could be included.

We continue and extend the above research program by studying general anisotropic elliptic problems in a bounded and open subset Ω of $\mathbb{R}^N$ ($N \ge 2$), subject to a homogeneous boundary condition, see (1.1). We impose no smoothness assumption on the boundary of Ω. Under suitable hypotheses, we prove in Theorem 1.1 that, for every $f \in L^1(\Omega)$, the problem

$$\begin{cases} \mathcal{A}u + \Phi(x, u, \nabla u) + \Theta(x, u, \nabla u) = \mathfrak{B}u + f & \text{in } \Omega, \\ u \in W_0^{1,\overrightarrow{p}}(\Omega), \quad \Phi(x, u, \nabla u) \in L^1(\Omega) \end{cases} \tag{1.1}$$

admits solutions in an appropriate weak sense (see Section 1.2).

As a novelty in this paper, besides any $f \in L^1(\Omega)$, we can handle in (1.1) a new class of operators $\mathfrak{B}$ from $W_0^{1,\overrightarrow{p}}(\Omega)$ into $W^{-1,\overrightarrow{p}'}(\Omega)$, which we introduce in Section 1.3, as well as a gradient-dependent lower-order term $\Phi(x, u, \nabla u)$ with an "anisotropic natural

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growth" in the gradient (see (1.12) and (1.13)). We have no restriction on the growth of Φ with respect to $|u|$. This means that $m > 1$ is arbitrary in the example of (1.2).

A toy model for our results is the following

$$\mathcal{A}u = -\sum_{j=1}^{N} \partial_j(|\partial_j u|^{p_j-2}\partial_j u), \quad \Phi(u, \nabla u) = |u|^{m-2}u \left(\sum_{j=1}^{N} |\partial_j u|^{p_j} + 1\right), \tag{1.2}$$

where $m > 1$ and we always assume that

$$1 < p_j \le p_{j+1} < \infty \text{ for every } 1 \le j \le N-1 \quad \text{and} \quad p < N. \tag{1.3}$$

Here, $p := N/\sum_{j=1}^{N}(1/p_j)$ is the harmonic mean of $p_1, \ldots, p_N$. We can take $\mathfrak{B}u = h \in W^{-1,\overrightarrow{p}'}(\Omega)$. (For other models of $\mathfrak{B}$, see Example 1.7.)

For every $r > 1$, let $r' = r/(r-1)$ be the conjugate exponent of r. We set $\overrightarrow{p} = (p_1, p_2, \ldots, p_N)$ and $\overrightarrow{p}' = (p_1', p_2', \ldots, p_N')$. The anisotropic space $W_0^{1,\overrightarrow{p}}(\Omega)$ to which our solutions of (1.1) belong is defined as the closure of $C_c^\infty(\Omega)$ (the set of smooth functions with compact support in Ω) with respect to the norm

$$\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} = \sum_{j=1}^{N} \|\partial_j u\|_{L^{p_j}(\Omega)}.$$

Here, $\nabla u = (\partial_1 u, \ldots, \partial_N u)$ is the gradient of u. Since we have no smoothness assumption on $\partial\Omega$, the critical exponent p^* for the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^r(\Omega)$ is the usual critical exponent corresponding to the harmonic mean p of the p_j's, namely,

$$p^* := \frac{Np}{N-p}.$$

Thanks to the last condition in (1.3), the critical exponent p^* is well-defined (see Remark 1.6 for the anisotropic embedding theorems). We mention that if $\Omega \subset \mathbb{R}^N$ is an open bounded domain with Lipschitz boundary and (1.3) holds, then the "true" critical exponent is p_∞, defined as the maximum between p^* and p_N. Indeed, Fragalà, Gazzola and Kawohl [28] showed that the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for every $r \in [1, p_\infty]$ and compact if $r \in [1, p_\infty)$.

Our problem (1.1) features a Leray–Lions operator $\mathcal{A}$ of the form

$$\mathcal{A}u = -\operatorname{div} \mathbf{A}(x, u, \nabla u) = -\sum_{j=1}^{N} \partial_j(A_j(x, u, \nabla u)),$$

which is a divergence-form nonlinear anisotropic operator from $W_0^{1,\overrightarrow{p}}(\Omega)$ into $W^{-1,\overrightarrow{p}'}(\Omega)$. Under the coercivity, monotonicity and growth conditions in (1.10), the operator $\mathcal{A} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ is coercive, bounded, continuous and pseudo-monotone.

Unless otherwise stated, we understand that $\mathfrak{B}$ in (1.1) satisfies two properties (P_1) and (P_2) given in Section 1.3. But, unlike $\mathcal{A}$, the operator $-\mathfrak{B}$ is not coercive in general. The growth condition in the assumption (P_1) implies that $\mathcal{A} - \mathfrak{B}$ is a coercive and bounded operator from $W_0^{1,\overrightarrow{p}}(\Omega)$ into $W^{-1,\overrightarrow{p}'}(\Omega)$. The assumption (P_2) is, in some sense, in the spirit of (iii) in the Hypothesis (II) of Theorem 1 in the celebrated paper

[33] by Leray and Lions. Every operator satisfying (P_2) is strongly continuous (see Lemma 3.7) and thus pseudo-monotone (cf. [39, p. 586]).

For every $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ and for a.e. $x \in \Omega$, we define

$$\begin{aligned} &\widehat{\Phi}(u)(x) := \Phi(x, u(x), \nabla u(x)), \quad \widehat{\Theta}(u)(x) := \Theta(x, u(x), \nabla u(x)), \\ &\widehat{A}_j(u)(x) = A_j(x, u(x), \nabla u(x)) \quad \text{for every } 1 \le j \le N. \end{aligned}$$

1.2. **Main results.** The precise assumptions that appear in our main results are presented in Section 1.3. Let $f \in L^1(\Omega)$. By a *solution* of (1.1), we mean a function $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ such that $\widehat{\Phi}(u) \in L^1(\Omega)$ and for every $v \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$, we have

$$\sum_{j=1}^N \int_\Omega \widehat{A}_j(u)\, \partial_j v\, dx + \int_\Omega \widehat{\Phi}(u)\, v\, dx + \int_\Omega \widehat{\Theta}(u)\, v\, dx = \langle \mathfrak{B} u, v \rangle + \int_\Omega f\, v\, dx.$$

The brackets $\langle \cdot, \cdot \rangle$ indicate the duality between $W^{-1,\overrightarrow{p}'}(\Omega)$ and $W_0^{1,\overrightarrow{p}}(\Omega)$.

In all our main results and unless otherwise stated, we understand that $\mathfrak{B}$ satisfies (P_1) and (P_2) given in Section 1.3. The main advance in this paper is the following.

Theorem 1.1. *Let* (1.3)*,* (1.10)*,* (1.11)*,* (1.12) *and* (1.13) *hold. Then,* (1.1) *has at least a solution for every* $f \in L^1(\Omega)$*.*

Assuming (1.3), we remark that without the term Φ, one cannot expect to find solutions of (1.1) in $W_0^{1,\overrightarrow{p}}(\Omega)$ for every $f \in L^1(\Omega)$. For the isotropic case, this observation has been made, for example, in [16]. So, in our general setting, we could ask: *What makes the existence of solutions to* (1.1) *possible in* $W_0^{1,\overrightarrow{p}}(\Omega)$? The other assumptions on Φ: a "sign-condition" as in (1.12), and (1.13). We stress that were f not to appear in (1.1), we would not need (1.13) (see Theorem 1.2, where $f = 0$).

For Theorem 1.1 we encounter two obstacles: a low summability for f and, on the other hand, the unrestricted growth of Φ with respect to $|u|$. Previously mentioned works in the isotropic case provide ways to surmount one problem at a time. The function $f \in L^1(\Omega)$ can surely be approximated by $L^\infty(\Omega)$-functions f_ε in the sense that $|f_\varepsilon| \le |f|$ and $f_\varepsilon \to f$ a.e in Ω as $\varepsilon \to 0$. Also Φ could be replaced by a "nice" function Φ_ε, preserving the properties of Φ, but gaining boundedness, namely,

$$\Phi_\varepsilon(x, t, \xi) := \frac{\Phi(x, t, \xi)}{1 + \varepsilon\, |\Phi(x, t, \xi)|} \quad \text{for a.e. } x \in \Omega \text{ and all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N.$$

However, we cannot deal with both approximations for f and Φ *simultaneously.* This limitation has already been pointed out by Bensoussan and Boccardo [10] in the isotropic case. For the approximate problems involving both Φ_ε and f_ε, we would not be able to obtain that the solutions u_ε are uniformly bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$ with respect to ε.

For the above reason, we need to consider $f = 0$ first and prove Theorem 1.2, which is a crucial step in establishing Theorem 1.1, but at the same time of independent interest.

Theorem 1.2. *Let $f = 0$ and (1.3) hold. If (1.10), (1.11) and (1.12) are satisfied, then (1.1) has a solution U, which satisfies $\widehat{\Phi}(U)\, U \in L^1(\Omega)$ and*

$$\sum_{j=1}^{N} \int_\Omega \widehat{A}_j(U)\, \partial_j U\, dx + \int_\Omega \widehat{\Phi}(U)\, U\, dx + \int_\Omega \widehat{\Theta}(U)\, U\, dx = \langle \mathfrak{B} U, U \rangle.$$

By taking $f = 0$ in (1.1), we obtain a solution with better properties and under weaker assumptions than those in Theorem 1.1. We point out that because of Φ, we cannot directly apply the theory of pseudo-monotone operators to prove the existence claim in Theorem 1.2. We overcome this difficulty by considering the approximate problem

$$\begin{cases} \mathcal{A} u_\varepsilon + \widehat{\Phi}_\varepsilon(u_\varepsilon) + \widehat{\Theta}(u_\varepsilon) = \mathfrak{B} u_\varepsilon \quad \text{in } \Omega, \\ u_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega) \end{cases} \tag{1.4}$$

for which we obtain the existence of a solution u_ε.

Since $\Phi_\varepsilon + \Theta$ satisfies the same type of assumption as Θ in (1.11), that is, there exists a constant $C_\varepsilon > 0$ such that $|(\Phi_\varepsilon + \Theta)(x, t, \xi)| \leq C_\varepsilon$ for a.e. $x \in \Omega$ and for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$, the existence of u_ε follows from our next result.

Theorem 1.3. *Let (1.3), (1.10) and (1.11) hold. Then, the problem*

$$\begin{cases} \mathcal{A} u + \widehat{\Theta}(u) = \mathfrak{B} u \quad \textit{in } \Omega, \\ u \in W_0^{1,\overrightarrow{p}}(\Omega), \end{cases} \tag{1.5}$$

admits a solution, meaning that there exists a function $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ such that

$$\sum_{j=1}^{N} \int_\Omega \widehat{A}_j(u)\, \partial_j v\, dx + \int_\Omega \widehat{\Theta}(u)\, v\, dx - \langle \mathfrak{B} u, v \rangle = 0 \quad \textit{for every } v \in W_0^{1,\overrightarrow{p}}(\Omega). \tag{1.6}$$

We establish Theorem 1.3 in Section 3 via the theory of pseudo-monotone operators. We show that $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B}$ is a coercive, bounded and pseudo-monotone operator from $W_0^{1,\overrightarrow{p}}(\Omega)$ into $W^{-1,\overrightarrow{p}'}(\Omega)$, where the left-hand side of (1.6) gives $\langle \mathcal{A} u + \mathcal{P}_\Theta(u) - \mathfrak{B} u, v \rangle$ for every $u, v \in W_0^{1,\overrightarrow{p}}(\Omega)$. Hence, the existence of a solution u of (1.5) follows (see [39, p. 589]) since $W_0^{1,\overrightarrow{p}}(\Omega)$ is a real, reflexive, and separable Banach space.

Thus, for every $\varepsilon > 0$, the approximate problem (1.4) admits a solution $u_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega)$. In Lemma 4.1 we prove *a priori* estimates in $W_0^{1,\overrightarrow{p}}(\Omega)$ for the solutions u_ε, which then (up to a subsequence) converge weakly to some U in $W_0^{1,\overrightarrow{p}}(\Omega)$ and a.e. in Ω as $\varepsilon \to 0$.

We point out that in Section 6, we will be able to show that, up to a subsequence,

$$u_\varepsilon \to U \text{ (strongly) in } W_0^{1,\overrightarrow{p}}(\Omega) \quad \text{as } \varepsilon \to 0. \tag{1.7}$$

Indeed, one could adapt the approach in [11] (where an isotropic version of (1.1) was treated with $\mathfrak{B} = h \in W^{-1,\overrightarrow{p}'}(\Omega)$, $\Theta = f = 0$). This technique will be used in a forthcoming paper [17] to prove the existence of solutions for related anisotropic problems exhibiting singular anisotropic terms. However, for our purpose of including L^1 data in (1.1), we prefer to give a unified treatment of the case $f = 0$ in Theorem 1.2 and the case $f \in L^1(\Omega)$ in Theorem 1.1. We achieve this by combining and extending techniques from [10] and [13] to establish in Lemma 4.2 that, up to a subsequence of u_ε, we have

$$\nabla u_\varepsilon \to \nabla U \text{ a.e. in } \Omega \text{ and } T_k(u_\varepsilon) \to T_k(U) \text{ (strongly) in } W_0^{1,\overrightarrow{p}}(\Omega) \text{ as } \varepsilon \to 0 \tag{1.8}$$

for every integer $k \geq 1$, where $T_k(\cdot)$ is given in (2.11). Then, we can pass to the limit as $\varepsilon \to 0$ in the weak formulation of the solution u_ε and obtain that U is a solution of (1.1) with $f = 0$ (see Section 4.2). In Section 6, we improve (1.8) in the form of (1.7).

Generally speaking, the proof of Theorem 1.1, which we give in Section 5, follows a similar course to that of Theorem 1.2 in Section 4. But there are some modifications that we outline below. We approximate $f \in L^1(\Omega)$ by $L^\infty(\Omega)$-functions f_ε and in view of Example 1.7, we can apply Theorem 1.2 to obtain a solution U_ε for the problem

$$\begin{cases} \mathcal{A}U_\varepsilon + \widehat{\Phi}(U_\varepsilon) + \widehat{\Theta}(U_\varepsilon) = \mathfrak{B}U_\varepsilon + f_\varepsilon \quad \text{in } \Omega, \\ U_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega), \quad \widehat{\Phi}(U_\varepsilon) := \Phi(x, U_\varepsilon, \nabla U_\varepsilon) \in L^1(\Omega). \end{cases} \tag{1.9}$$

We emphasize that unlike in (1.4), we have $\widehat{\Phi}$ (and not $\widehat{\Phi}_\varepsilon$) in (1.9). Because of this reason, coupled with the introduction of f_ε, we need an additional assumption in the form of (1.13) below to obtain that $\{U_\varepsilon\}_\varepsilon$ is uniformly bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$ with respect to ε (see Lemma 5.1 for details). Then, extracting a subsequence, U_ε tends to some U_0 weakly in $W_0^{1,\overrightarrow{p}}(\Omega)$ and a.e. in Ω. With an almost identical argument, we gain the counterpart of (1.8), namely, up to a subsequence, $\nabla U_\varepsilon \to \nabla U_0$ a.e. in Ω and $T_k(U_\varepsilon) \to T_k(U_0)$ (strongly) in $W_0^{1,\overrightarrow{p}}(\Omega)$ as $\varepsilon \to 0$ for every integer $k \geq 1$. To conclude the proof of Theorem 1.1, it remains to pass to the limit in the weak formulation of U_ε. The change appearing here compared with the corresponding argument in Section 4.2 is the strong convergence of $\widehat{\Phi}(U_\varepsilon)$ to $\widehat{\Phi}(U_0)$ in $L^1(\Omega)$. For the latter, we adapt an argument from [13]. For details, we refer to Lemma 5.3 in Section 5.3.

Recently, anisotropic elliptic and parabolic problems have been widely investigated in literature. The increasing interest in nonlinear anisotropic problems is justified by their applications in many areas from image recovery and the mathematical modeling of non-Newtonian fluids to biology, where they serve as models for the propagation of epidemic diseases in heterogeneous domains (see, for example, [6] and [9]). Unfortunately, some fundamental tools available for the isotropic case cannot be extended to the anisotropic setting (such as the strong maximum principle, see [38]). Nevertheless, with a rapidly growing literature on anisotropic problems, many questions concerning existence, uniqueness and regularity of weak solutions have been solved with different techniques (see, for instance, [1, 2, 5, 7, 8, 14, 20, 21, 23–25, 28–30, 34]).

1.3. **Assumptions.** We return to problem (1.1), where $\Omega \subset \mathbb{R}^N$ is an open, bounded set. We have assumed (1.3). For every $1 \leq j \leq N$, the functions $A_j(x, t, \xi), \Theta(x, t, \xi)$ and $\Phi(x, t, \xi)$ from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$ are Carathéodory (that is, they are measurable on Ω for every $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in t, ξ for a.e. $x \in \Omega$).

Assume that there exist constants $\nu_0, \nu > 0$, nonnegative functions $\eta_j(\cdot) \in L^{p_j'}(\Omega)$ for $1 \le j \le N$ such that for a.e. $x \in \Omega$, for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $\widehat{\xi} \in \mathbb{R}^N$:

$$\left.\begin{aligned} &\sum_{j=1}^N A_j(x,t,\xi)\,\xi_j \ge \nu_0 \sum_{j=1}^N |\xi_j|^{p_j} && \text{[coercivity]},\\ &\sum_{j=1}^N \left(A_j(x,t,\xi) - A_j(x,t,\widehat{\xi})\right)\left(\xi_j - \widehat{\xi}_j\right) > 0 \quad \text{if } \xi \ne \widehat{\xi} && \text{[monotonicity]},\\ &|A_j(x,t,\xi)| \le \nu \left[\eta_j(x) + |t|^{p^*/p_j'} + \left(\sum_{i=1}^N |\xi_i|^{p_i}\right)^{1/p_j'}\right] && \text{[growth condition]} \end{aligned}\right\} \tag{1.10}$$

for every $1 \le j \le N$.

We stress that in our growth condition on A_j in (1.10), we take the greatest exponent for $|t|$ from the viewpoint of the anisotropic Sobolev inequalities. With respect to the existent literature, this attracts some modifications in the proof of pseudo-monotonicity of $\mathcal{A} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ (see Lemma 3.6).

Suppose there exist a constant $C_\Theta > 0$, a nonnegative function $c(\cdot) \in L^1(\Omega)$ and a continuous nondecreasing function $\phi : \mathbb{R} \to \mathbb{R}^+$ such that

$$|\Theta(x,t,\xi)| \le C_\Theta \tag{1.11}$$

$$\Phi(x,t,\xi)\,t \ge 0 \quad \text{[sign-condition]}, \quad |\Phi(x,t,\xi)| \le \phi(|t|)\left(\sum_{j=1}^N |\xi_j|^{p_j} + c(x)\right) \tag{1.12}$$

for a.e. $x \in \Omega$ and for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

For Theorem 1.1 only, we further assume that there exist constants $\tau, \gamma > 0$ such that

$$|\Phi(x,t,\xi)| \ge \gamma \sum_{j=1}^N |\xi_j|^{p_j} \quad \text{for all } |t| \ge \tau \tag{1.13}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^N$.

Assumptions on $\mathfrak{B}$. Let $\mathfrak{B} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ satisfy two properties:

(P_1) There exist constants $\mathfrak{C} > 0$, $\mathfrak{s} \in [1, p^*)$, $\mathfrak{a}_0 \ge 0$, $\mathfrak{b} \in (0, p_1 - 1)$ if $\mathfrak{a}_0 > 0$ and $\mathfrak{b} \in (0, p_1/p')$ if $\mathfrak{a}_0 = 0$ such that for all $u, v \in W_0^{1,\overrightarrow{p}}(\Omega)$, it holds

$$|\langle \mathfrak{B}u, v\rangle| \le \mathfrak{C}\left(1 + \|u\|^{\mathfrak{b}}_{W_0^{1,\overrightarrow{p}}(\Omega)}\right)\left(\mathfrak{a}_0 \|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)} + \|v\|_{L^{\mathfrak{s}}(\Omega)}\right). \tag{1.14}$$

(P_2) If $u_\ell \rightharpoonup u$ and $v_\ell \rightharpoonup v$ (weakly) in $W_0^{1,\overrightarrow{p}}(\Omega)$ as $\ell \to \infty$, then

$$\lim_{\ell \to \infty} \langle \mathfrak{B}u_\ell, v_\ell\rangle = \langle \mathfrak{B}u, v\rangle.$$

Remark 1.4. The case $\mathfrak{a}_0 > 0$ in (1.14) allows for $\mathfrak{B}u = h \in W^{-1,\overrightarrow{p}'}(\Omega)$ in (1.1) (see Example 1.7). As noted in [11] for the isotropic case, we cannot in general expect a solution of (1.1) to be bounded. There is a nice trade-off for taking $\mathfrak{a}_0 = 0$ in (1.14):

the range of $\mathfrak{b}$ in (1.14) can be extended to $(0, p_1/p')$ (compared to $\mathfrak{b} \in (0, p_1 - 1)$ for $\mathfrak{a}_0 > 0$).

Example of operators $\mathfrak{B}$. We first recall an anisotropic Sobolev inequality corresponding to the case $p < N$, see [37].

Lemma 1.5. *Let $N \geq 2$ be an integer. If (1.3) holds, then there exists a positive constant $\mathcal{S} = \mathcal{S}(N, \overrightarrow{p})$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq \mathcal{S} \prod_{j=1}^{N} \|\partial_j u\|_{L^{p_j}(\mathbb{R}^N)}^{1/N} \quad \textit{for all } u \in C_c^\infty(\mathbb{R}^N).$$

Remark 1.6. Let Ω be a bounded, open subset of $\mathbb{R}^N$ ($N \geq 2$). If (1.3) holds, then using a density argument and the arithmetic-geometric mean inequality, we find that

$$\|u\|_{L^{p^*}(\Omega)} \leq \mathcal{S} \prod_{j=1}^{N} \|\partial_j u\|_{L^{p_j}(\Omega)}^{1/N} \leq \frac{\mathcal{S}}{N} \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \quad \text{for all } u \in W_0^{1,\overrightarrow{p}}(\Omega). \tag{1.15}$$

Moreover, by Hölder's inequality, the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for every $s \in [1, p^*]$ and compact for every $s \in [1, p^*)$.

We next give a simple example of an operator $\mathfrak{B} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ satisfying (P_1) and (P_2).

Example 1.7. For every $u, v \in W_0^{1,\overrightarrow{p}}(\Omega)$, we define

$$\langle \mathfrak{B}u, v \rangle = \int_\Omega F\, v\, dx + \langle h, v \rangle,$$

where $F \in L^{(p^*)'}(\Omega)$ and $h \in W^{-1,\overrightarrow{p}'}(\Omega)$. Then, $\mathfrak{B}$ satisfies (P_1) and (P_2).

Structure of this paper. In Section 2 we include some convergence results to be used later in the paper. In Section 3 we prove Theorem 1.3. We dedicate Section 4 to the proof of Theorem 1.2 and Section 5 to the proof of Theorem 1.1. We conclude the paper with Section 6, where we make further comments on Theorem 1.2 by proving the strong convergence in (1.7).

2. Auxiliary results

In this section, we assume (1.3) and (1.10).

2.1. Preliminaries.

For v, w and $\{u_\varepsilon\}_\varepsilon$ in $W_0^{1,\overrightarrow{p}}(\Omega)$, we define

$$\begin{aligned} \mathcal{D}_{u_\varepsilon}(v,w)(x) &:= \sum_{j=1}^{N} \left[A_j(x, u_\varepsilon(x), \nabla v(x)) - A_j(x, u_\varepsilon(x), \nabla w(x))\right] \partial_j (v-w)(x), \\ H_{u_\varepsilon}(v,w)(x) &:= \sum_{j=1}^{N} A_j(x, u_\varepsilon(x), \nabla v(x))\, \partial_j w(x) \end{aligned} \tag{2.1}$$

for a.e. $x \in \Omega$. Hence, we can write $\mathcal{D}_{u_\varepsilon}(v,w)$ as follows

$$\mathcal{D}_{u_\varepsilon}(v,w) = H_{u_\varepsilon}(v,v) - H_{u_\varepsilon}(v,w) - H_{u_\varepsilon}(w,v) + H_{u_\varepsilon}(w,w).$$

The monotonicity assumption in (1.10) gives that

$$\mathcal{D}_{u_\varepsilon}(v,w) \geq 0 \quad \text{a.e. in } \Omega,$$

whereas the coercivity condition in (1.10) yields that

$$H_{u_\varepsilon}(v,v) \geq \nu_0 \sum_{j=1}^N |\partial_j v|^{p_j},$$

where $\nu_0 > 0$ is a constant. We thus find that

$$\mathcal{D}_{u_\varepsilon}(v,w) \geq \nu_0 \sum_{j=1}^N |\partial_j v|^{p_j} - |H_{u_\varepsilon}(v,w)| - |H_{u_\varepsilon}(w,v)|. \tag{2.2}$$

2.2. **Some convergence results.** Here, we establish Lemma 2.1, which will be used in the proof of Lemma 3.6. Further, in Section 2.3 we prove Lemma 2.2, which will be invoked in the proof of Theorem 1.2 in Section 4. To prove Lemmas 2.1 and 2.2, we adapt an argument from [15, Lemma 5], the proof of which goes back to Browder [19].

By Remark 1.6, whenever

$$u_\varepsilon \rightharpoonup u \text{ (weakly) in } W_0^{1,\overrightarrow{p}}(\Omega) \text{ as } \varepsilon \to 0, \tag{2.3}$$

we can pass to a subsequence (always relabeled $\{u_\varepsilon\}$) such that

$$u_\varepsilon \to u \ \text{ strongly in } L^r(\Omega) \text{ if } r \in [1, p^*) \ \text{ and } u_\varepsilon \to u \ \text{ a.e. in } \Omega. \tag{2.4}$$

Lemma 2.1. *Let u and $\{u_\varepsilon\}_\varepsilon$ be in $W_0^{1,\overrightarrow{p}}(\Omega)$ such that* (2.3) *holds. Suppose that*

$$\mathcal{D}_{u_\varepsilon}(u_\varepsilon, u) \to 0 \qquad \text{a.e. in } \Omega \quad \text{as } \varepsilon \to 0.$$

Then, up to a subsequence, we have

$$\nabla u_\varepsilon \to \nabla u \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0. \tag{2.5}$$

Proof. Let Z be a subset of Ω with meas $(Z) = 0$ such that for every $x \in \Omega \setminus Z$, we have $|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $|\eta_j(x)| < \infty$ for all $1 \leq j \leq N$, as well as

$$u_\varepsilon(x) \to u(x), \quad \mathcal{D}_{u_\varepsilon}(u_\varepsilon, u)(x) \to 0 \text{ as } \varepsilon \to 0. \tag{2.6}$$

For every $x \in \Omega \setminus Z$, we claim that

$$\{|\nabla u_\varepsilon(x)|\}_\varepsilon \ \text{ is uniformly bounded with respect to } \varepsilon. \tag{2.7}$$

Proof of (2.7). We fix $x \in \Omega \setminus Z$. In view of (2.2), we have

$$\mathcal{D}_{u_\varepsilon}(u_\varepsilon, u)(x) \geq \nu_0 \sum_{j=1}^N |\partial_j u_\varepsilon(x)|^{p_j} - |H_{u_\varepsilon}(u_\varepsilon, u)(x)| - |H_{u_\varepsilon}(u, u_\varepsilon)(x)|. \tag{2.8}$$

By Young's inequality, for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$\begin{aligned} |H_{u_\varepsilon}(u_\varepsilon, u)(x)| &\leq \sum_{j=1}^N \left(\delta\, |A_j(x, u_\varepsilon, \nabla u_\varepsilon)|^{p_j'} + C_\delta |\partial_j u(x)|^{p_j}\right), \\ |H_{u_\varepsilon}(u, u_\varepsilon)(x)| &\leq \sum_{j=1}^N \left(\delta\, |\partial_j u_\varepsilon(x)|^{p_j} + C_\delta |A_j(x, u_\varepsilon, \nabla u)|^{p_j'}\right). \end{aligned} \tag{2.9}$$

We use the growth condition in (1.10) to bound from above the right-hand side of each inequality in (2.9). Then, in view of (2.8), there exist positive constants C and $\widehat{C_\delta}$, both independent of ε (and only $\widehat{C_\delta}$ depending on δ), such that

$$\mathcal{D}_{u_\varepsilon}(u_\varepsilon, u)(x) \geq (\nu_0 - C\,\delta) \sum_{j=1}^{N} |\partial_j u_\varepsilon(x)|^{p_j} - \widehat{C_\delta}\, \mathfrak{g}_{u_\varepsilon}(u)(x), \tag{2.10}$$

where $\mathfrak{g}_{u_\varepsilon}(u)(x)$ is given by

$$\mathfrak{g}_{u_\varepsilon}(u)(x) = \sum_{j=1}^{N} \eta_j^{p_j'}(x) + |u_\varepsilon(x)|^{p^*} + \sum_{j=1}^{N} |\partial_j u(x)|^{p_j}.$$

Using (2.6) and choosing $\delta \in (0, \nu_0/C)$, from (2.10) we conclude (2.7). $\square$

Proof of (2.5) **concluded.**

Let $x \in \Omega \setminus Z$ be arbitrary. Define

$$\xi_\varepsilon = \nabla u_\varepsilon(x) \quad \text{and } \xi = \nabla u(x).$$

To show that $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$, it is enough to prove that any accumulation point of ξ_ε, say ξ^*, coincides with ξ. From (2.7), we have $|\xi^*| < \infty$. Using (2.6) and the continuity of $A_j(x, \cdot, \cdot)$ with respect to the last two variables for every $1 \leq j \leq N$, we find that

$$\mathcal{D}_{u_\varepsilon}(u_\varepsilon, u)(x) \to \sum_{j=1}^{N} [A_j(x, u(x), \xi^*) - A_j(x, u(x), \xi)]\,(\xi_j^* - \xi_j) \quad \text{as } \varepsilon \to 0.$$

This, jointly with (2.6) and the monotonicity condition in (1.10), gives that $\xi^* = \xi$. The proof of Lemma 2.1 is complete since $x \in \Omega \setminus Z$ is arbitrary and $\text{meas}\,(Z) = 0$. $\square$

2.3. **Strong convergence of $T_k(u_\varepsilon)$.** The main aim of this section is to prove Lemma 2.2, which will be used later to establish Lemma 4.2.

For every $k > 0$, let $T_k : \mathbb{R} \to \mathbb{R}$ be the truncation at height k, that is,

$$T_k(s) = s \quad \text{if } |s| \leq k, \quad T_k(s) = k\,\frac{s}{|s|} \quad \text{if } |s| > k. \tag{2.11}$$

Lemma 2.2. *Let $k \geq 1$ be a fixed integer. Let u and $\{u_\varepsilon\}_\varepsilon$ be in $W_0^{1,\overrightarrow{p}}(\Omega)$ such that*

$$u_\varepsilon \rightharpoonup u \text{ (weakly) in } W_0^{1,\overrightarrow{p}}(\Omega) \text{ as } \varepsilon \to 0. \tag{2.12}$$

Suppose that, up to a subsequence of $\{u_\varepsilon\}$, (depending on k, but relabeled $\{u_\varepsilon\}$)

$$\mathcal{D}_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u)) \to 0 \quad \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0, \tag{2.13}$$

where $\mathcal{D}_{u_\varepsilon}(\cdot,\cdot)$ is defined in (2.1). *Then, up to a subsequence of $\{u_\varepsilon\}$, as $\varepsilon \to 0$, we have*

$$\nabla T_k(u_\varepsilon) \to \nabla T_k(u) \text{ a.e. in } \Omega, \tag{2.14}$$

$$T_k(u_\varepsilon) \to T_k(u) \text{ (strongly) in } W_0^{1,\overrightarrow{p}}(\Omega). \tag{2.15}$$

Proof. By (2.12) and (2.13), up to a subsequence of $\{u_\varepsilon\}$, we have (2.4) and

$$\mathcal{D}_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u)) \to 0 \quad \text{a.e. in } \Omega \text{ as } \varepsilon \to 0.$$

Let Z be a subset of Ω as in the proof of Lemma 2.1, where $\mathcal{D}_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u)))$ replaces $\mathcal{D}_{u_\varepsilon}(u_\varepsilon, u)$. We follow the same argument as in Lemma 2.1 with the obvious modifications suggested by the above replacement. Then, for every $x \in \Omega \setminus Z$, we obtain

$$\begin{aligned}\mathcal{D}_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u))(x) \geq & \nu_0 \sum_{j=1}^N |\partial_j T_k(u_\varepsilon)(x)|^{p_j} \\ & - |H_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u))(x)| - |H_{u_\varepsilon}(T_k(u), T_k(u_\varepsilon))(x)|\end{aligned} \tag{2.16}$$

for every $x \in \Omega \setminus Z$. This leads to $\{|\nabla T_k(u_\varepsilon)(x)|\}_\varepsilon$ being uniformly bounded with respect to ε and also (2.14).

We conclude the proof of Lemma 2.2 by showing (2.15). From (2.14), we see that $\{|\partial_j T_k(u_\varepsilon) - \partial_j T_k(u)|^{p_j}\}_\varepsilon$ is a sequence of nonnegative integrable functions, converging to 0 a.e. on Ω. Thus, by Vitali's Theorem, we obtain that $\partial_j T_k(u_\varepsilon) \to \partial_j T_k(u)$ in $L^{p_j}(\Omega)$ as $\varepsilon \to 0$ for every $1 \leq j \leq N$ by proving that

$$\left\{ \sum_{j=1}^N |\partial_j T_k(u_\varepsilon)|^{p_j} \right\}_\varepsilon \quad \text{is uniformly integrable over } \Omega. \tag{2.17}$$

The claim of (2.17) follows from (2.13) and (2.16) whenever $\{H_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u))\}_\varepsilon$ and $\{H_{u_\varepsilon}(T_k(u), T_k(u_\varepsilon))\}_\varepsilon$ converge in $L^1(\Omega)$ as $\varepsilon \to 0$. We next establish that

$$H_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u)) \to \sum_{j=1}^N A_j(x, u, \nabla T_k(u))\, \partial_j T_k(u) \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{2.18}$$

Then, using Vitali's Theorem, we show that

$$H_{u_\varepsilon}(T_k(u), T_k(u_\varepsilon)) \to \sum_{j=1}^N A_j(x, u, \nabla T_k(u))\, \partial_j T_k(u) \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{2.19}$$

For the definition of $H_{u_\varepsilon}(\cdot,\cdot)$, see (2.1).

Proof of (2.18). Let $1 \leq j \leq N$ be arbitrary. We see that $\{A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon))\}_\varepsilon$ is bounded in $L^{p_j'}(\Omega)$ from the growth condition in (1.10) and the boundedness of $\{u_\varepsilon\}_\varepsilon$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ and, hence, in $L^{p^*}(\Omega)$. Moreover, the sequence $\{A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon))\}_\varepsilon$ converges to $A_j(x, u, \nabla T_k(u))$ a.e. in Ω as $\varepsilon \to 0$ using (2.14), the convergence $u_\varepsilon \to u$ a.e. in Ω (from (2.4)) and the continuity of $A_j(x,\cdot,\cdot)$ in the last two variables. Thus, up to a subsequence of $\{u_\varepsilon\}$, we infer that

$$A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon)) \rightharpoonup A_j(x, u, \nabla T_k(u)) \quad \text{(weakly) in } L^{p_j'}(\Omega) \text{ as } \varepsilon \to 0.$$

This proves (2.18).

Proof of (2.19). Using (2.14) and the continuity properties of A_j, as $\varepsilon \to 0$,

$$A_j(x, u_\varepsilon, \nabla T_k(u))\, \partial_j T_k(u_\varepsilon) \to A_j(x, u, \nabla T_k(u))\, \partial_j T_k(u) \text{ a.e. in } \Omega \tag{2.20}$$

for each $1 \le j \le N$. Observe that $\{\chi_{\{|u_\varepsilon|<k\}} |A_j(x, u_\varepsilon, \nabla T_k(u))|^{p_j'}\}_\varepsilon$ is uniformly integrable over Ω (from the growth condition of A_j in (1.10)) and

$$\partial_j T_k(u_\varepsilon) = \chi_{\{|u_\varepsilon|<k\}}\, \partial_j u_\varepsilon.$$

Thus, since $\{\partial_j u_\varepsilon\}_\varepsilon$ is bounded in $L^{p_j}(\Omega)$, it follows from Hölder's inequality that

$$\{A_j(x, u_\varepsilon, \nabla T_k(u))\, \partial_j T_k(u_\varepsilon)\}_\varepsilon \text{ is uniformly integrable over } \Omega \tag{2.21}$$

for each $1 \le j \le N$. From (2.20), (2.21) and Vitali's Theorem, we reach (2.19).

Having established (2.18) and (2.19), we need only recall (2.13) and (2.16) to conclude the proof of (2.17) and thus of (2.15). This completes the proof of Lemma 2.2. □

From Lemma 2.2 and a standard diagonal argument, we obtain the following.

Corollary 2.3. *Let* (2.12) *and* (2.13) *hold. Then, there exists a subsequence of* $\{u_\varepsilon\}_\varepsilon$*, relabeled* $\{u_\varepsilon\}_\varepsilon$*, such that*

$$\nabla u_\varepsilon \to \nabla u \ \textit{a.e. in } \Omega \textit{ and } T_k(u_\varepsilon) \to T_k(u) \ \textit{(strongly) in } W_0^{1,\overrightarrow{p}}(\Omega) \textit{ as } \varepsilon \to 0$$

for every integer $k \ge 1$.

3. Proof of Theorem 1.3

Throughout this section, we assume (1.3), (1.10) and (1.11).

From (1.11), the operator $\mathcal{P}_\Theta : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ is bounded, where we define

$$\langle \mathcal{P}_\Theta(u), v\rangle := \int_\Omega \widehat{\Theta}(u)\, v\, dx \quad \text{for every } u, v \in W_0^{1,\overrightarrow{p}}(\Omega). \tag{3.1}$$

In this section, we establish Theorem 1.3, which we recall below.

Theorem 3.1. *Let* (1.3), (1.10) *and* (1.11) *hold. Let* $\mathfrak{B} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ *satisfy* (P_1) *and* (P_2)*. Then, the problem*

$$\begin{cases} \mathcal{A}u + \Theta(x, u, \nabla u) = \mathfrak{B}u & \textit{in } \Omega,\\ u \in W_0^{1,\overrightarrow{p}}(\Omega), \end{cases} \tag{3.2}$$

admits at least a solution, namely, there exists a function $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ *such that*

$$\sum_{j=1}^N \int_\Omega \widehat{A}_j(u)\, \partial_j v\, dx + \int_\Omega \widehat{\Theta}(u)\, v\, dx - \langle \mathfrak{B}u, v\rangle = 0 \quad \textit{for every } v \in W_0^{1,\overrightarrow{p}}(\Omega). \tag{3.3}$$

For the reader's convenience and to make our presentation self-contained, we give all the details about the pseudo-monotonicity of $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$. These computations could be of interest also in the corresponding isotropic case for which the details are usually scattered in the literature.

Before giving the proof of Theorem 3.1, we recall a few concepts that we need in the sequel (see, for example, [18] and [39, p. 586]).

Definition 3.2. An operator $\mathcal{P}: W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ is called

(a_1) *monotone (strictly monotone)* if $\langle \mathcal{P}u - \mathcal{P}v, u - v\rangle \geq 0$ for every $u, v \in W_0^{1,\overrightarrow{p}}(\Omega)$ (with equality if and only if $u = v$);

(a_2) *pseudo-monotone* if the convergence $u_\ell \rightharpoonup u$ (weakly) in $W_0^{1,\overrightarrow{p}}(\Omega)$ as $\ell \to \infty$ and $\limsup_{\ell\to\infty} \langle \mathcal{P}u_\ell, u_\ell - u\rangle \leq 0$ imply that

$$\langle \mathcal{P}u, u - w\rangle \leq \liminf_{\ell\to\infty} \langle \mathcal{P}u_\ell, u_\ell - w\rangle \quad \text{for all } w \in W_0^{1,\overrightarrow{p}}(\Omega);$$

(a_3) *strongly continuous*[1] if $u_\ell \rightharpoonup u$ (weakly) in $W_0^{1,\overrightarrow{p}}(\Omega)$ as $\ell \to \infty$ implies that $\mathcal{P}u_\ell \to \mathcal{P}u$ in $W^{-1,\overrightarrow{p}'}(\Omega)$ as $\ell \to \infty$;

(a_4) *coercive* if $\langle \mathcal{P}u, u\rangle / \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \to \infty$ as $\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \to \infty$;

(a_5) *of M type*[2] if $u_\ell \rightharpoonup u$ (weakly) in $W_0^{1,\overrightarrow{p}}(\Omega)$ as $\ell \to \infty$, together with $\mathcal{P}u_\ell \rightharpoonup g$ (weakly) in $W^{-1,\overrightarrow{p}'}(\Omega)$ as $\ell \to \infty$ and $\limsup_{\ell\to\infty} \langle \mathcal{P}u_\ell, u_\ell\rangle \leq \langle g, u\rangle$, imply that

$$g = \mathcal{P}u \quad \text{and} \quad \langle \mathcal{P}u_\ell, u_\ell\rangle \to \langle g, u\rangle \text{ as } \ell \to \infty.$$

Proposition 3.3. *Every strongly continuous operator $\mathcal{P}: W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ is pseudo-monotone. Every bounded operator $\mathcal{P}: W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ of M type is pseudo-monotone. The sum of two pseudo-monotone operators is pseudo-monotone.*

3.1. **Proof of Theorem 3.1.** We outline the main ideas in the proof of Theorem 3.1. Let $\mathcal{P}_\Theta$ be defined by (3.1). In view of (3.3), the existence of a solution to (3.2) follows when the operator $\mathcal{A}+\mathcal{P}_\Theta-\mathfrak{B}: W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ is surjective. Since $W_0^{1,\overrightarrow{p}}(\Omega)$ is a real, reflexive, and separable Banach space, it is known (see, for instance, [39, p. 589]) that $\mathcal{A}+\mathcal{P}_\Theta-\mathfrak{B}: W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ is surjective whenever it is bounded, coercive and pseudo-monotone. In Lemma 3.9, we establish all these properties for $\mathcal{A}+\mathcal{P}_\Theta-\mathfrak{B}$.

The most difficult property to prove is the pseudo-monotonicity for $\mathcal{A}+\mathcal{P}_\Theta-\mathfrak{B}: W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$. We next indicate the main steps in the proof. In Lemma 3.6, we show that $\mathcal{A}+\mathcal{P}_\Theta: W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ is pseudo-monotone. Since the latter is a bounded operator, we conclude that it is pseudo-monotone by showing that it is of M type (see Proposition 3.3). Now, the properties (P_1) and (P_2) for $\mathfrak{B}: W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ ensure that it is strongly continuous (see Lemma 3.7) and, hence, pseudo-monotone by Proposition 3.3. Then, as a sum of pseudo-monotone operators, we obtain that $\mathcal{A}+\mathcal{P}_\Theta-\mathfrak{B}: W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ is pseudo-monotone. We proceed with the details.

Lemma 3.4. *Let* (1.3), (1.10) *and* (1.11) *hold. The mappings $\widehat{\Theta}: W_0^{1,\overrightarrow{p}}(\Omega) \to L^{(p^*)'}(\Omega)$ and $\widehat{A}_j: W_0^{1,\overrightarrow{p}}(\Omega) \to L^{p_j'}(\Omega)$ are continuous for each $1 \leq j \leq N$.*

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[1] Some authors (see, for instance, Showalter [36, p. 36]) use the terminology *completely continuous* instead of strongly continuous.

[2] Some authors (see, for example, Le Dret [32, p. 232]) use the terminology *sense 1 pseudomonotone* instead of M type.

Proof. Let $1 \le j \le N$ be arbitrary. By the growth condition of A_j in (1.10), there exist a constant $C > 0$ and a nonnegative function $\eta_j \in L^{p_j'}(\Omega)$ such that for all $u \in W_0^{1,\overrightarrow{p}}(\Omega)$,

$$|\widehat{A}_j(u)|^{p_j'} \le C \left(\eta_j^{p_j'} + |u|^{p^*} + \sum_{i=1}^{N} |\partial_i u|^{p_i} \right) \in L^1(\Omega). \tag{3.4}$$

Since the embeddings $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $L^\infty(\Omega) \hookrightarrow L^{(p^*)'}(\Omega)$ are continuous, from (3.4) and (1.11), we infer that $\widehat{A}_j : W_0^{1,\overrightarrow{p}}(\Omega) \to L^{p_j'}(\Omega)$ and $\widehat{\Theta} : W_0^{1,\overrightarrow{p}}(\Omega) \to L^{(p^*)'}(\Omega)$ are well-defined. To prove the continuity of these mappings, we let $u_n \to u$ (strongly) in $W_0^{1,\overrightarrow{p}}(\Omega)$ as $n \to \infty$. Hence, $u_n \to u$ (strongly) in $L^{p^*}(\Omega)$ and $\partial_i u_n \to \partial_i u$ (strongly) in $L^{p_i}(\Omega)$ as $n \to \infty$ for every $1 \le i \le N$. Now, using (3.4) with u_n instead of u, we obtain that $\{|\widehat{A}_j(u_n)|^{p_j'}\}_{n\ge 1}$ is uniformly integrable over Ω. By passing to a subsequence $\{u_{n_k}\}_{k\ge1}$ of $\{u_n\}$, we have $u_{n_k} \to u$ and $\nabla u_{n_k} \to \nabla u$ a.e. in Ω as $k \to \infty$. Since A_j and Θ are Carathéodory functions, we have $\widehat{\Theta}(u_{n_k}) \to \widehat{\Theta}(u)$ and $\widehat{A}_j(u_{n_k}) \to \widehat{A}_j(u)$ a.e. in Ω as $k \to \infty$. Then, by (1.11) and the Dominated Convergence Theorem, we find that $\widehat{\Theta}(u_{n_k}) \to \widehat{\Theta}(u)$ in $L^{(p^*)'}(\Omega)$. By Vitali's Theorem, we see that $\widehat{A}_j(u_{n_k}) \to \widehat{A}_j(u)$ in $L^{p_j'}(\Omega)$ as $k \to \infty$. Since the limits $\widehat{\Theta}(u)$ and $\widehat{A}_j(u)$ are independent of the subsequence $\{u_{n_k}\}_{k\ge1}$, we conclude that $\widehat{\Theta}(u_n) \to \widehat{\Theta}(u)$ in $L^{(p^*)'}(\Omega)$ and $\widehat{A}_j(u_n) \to \widehat{A}_j(u)$ in $L^{p_j'}(\Omega)$ as $n \to \infty$. Hence, $\widehat{A}_j : W_0^{1,\overrightarrow{p}}(\Omega) \to L^{p_j'}(\Omega)$ and $\widehat{\Theta} : W_0^{1,\overrightarrow{p}}(\Omega) \to L^{(p^*)'}(\Omega)$ are continuous. This completes the proof. □

Lemma 3.5. *Let* (1.3), (1.10) *and* (1.11) *hold. Then,* $\mathcal{A}+\mathcal{P}_\Theta : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ *is a bounded, coercive and continuous operator.*

Proof. The boundedness of the operator $\mathcal{A}+\mathcal{P}_\Theta : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ is a consequence of the growth condition of A_j in (1.10), coupled with (1.11). The coercivity of $\mathcal{A}+\mathcal{P}_\Theta$ follows readily from (1.11) and the coercivity assumption in (1.10). Moreover, by Hölder's inequality and the continuity of the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, we find a positive constant C such that, for every $u_1, u_2 \in W_0^{1,\overrightarrow{p}}(\Omega)$,

$$\begin{aligned}
&\|(\mathcal{A}+\mathcal{P}_\Theta)(u_1) - (\mathcal{A}+\mathcal{P}_\Theta)(u_2)\|_{W^{-1,\overrightarrow{p}'}(\Omega)} \\
&\le \sup_{\substack{v \in W_0^{1,\overrightarrow{p}}(\Omega),\\ \|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le 1}} \Big(\sum_{j=1}^{N} \int_\Omega |\widehat{A}_j(u_1) - \widehat{A}_j(u_2)||\partial_j v|\, dx + \int_\Omega |\widehat{\Theta}(u_1) - \widehat{\Theta}(u_2)||v|\, dx \Big) \\
&\le \sum_{j=1}^{N} \|\widehat{A}_j(u_1) - \widehat{A}_j(u_2)\|_{L^{p_j'}(\Omega)} + C\, \|\widehat{\Theta}(u_1) - \widehat{\Theta}(u_2)\|_{L^{(p^*)'}(\Omega)}.
\end{aligned}$$

Hence, using Lemma 3.4, we conclude the continuity of $\mathcal{A}+\mathcal{P}_\Theta : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$. This finishes the proof of Lemma 3.5. □

Lemma 3.6. *Let* (1.3), (1.10) *and* (1.11) *hold. Then,* $\mathcal{A}+\mathcal{P}_\Theta : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ *is a pseudo-monotone operator.*

Proof. Since we have already remarked the boundedness of the operator $\mathcal{A}+\mathcal{P}_\Theta$, it is enough to show that it is of M type (see Proposition 3.3). To this end, suppose that there exist u, $\{u_\ell\}_{\ell\geq 1}$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ and $g\in W^{-1,\overrightarrow{p}'}(\Omega)$ such that

$$u_\ell \rightharpoonup u \text{ (weakly) in } W_0^{1,\overrightarrow{p}}(\Omega) \text{ as } \ell\to\infty, \tag{3.5}$$

$$(\mathcal{A}+\mathcal{P}_\Theta)(u_\ell) \rightharpoonup g \text{ (weakly) in } W^{-1,\overrightarrow{p}'}(\Omega) \text{ as } \ell\to\infty, \tag{3.6}$$

$$\limsup_{\ell\to\infty} \langle (\mathcal{A}+\mathcal{P}_\Theta)(u_\ell), u_\ell\rangle \leq \langle g,u\rangle. \tag{3.7}$$

We prove that

$$g=(\mathcal{A}+\mathcal{P}_\Theta)(u), \tag{3.8}$$

$$\langle (\mathcal{A}+\mathcal{P}_\Theta)(u_\ell), u_\ell\rangle \to \langle g,u\rangle \quad \text{as } \ell\to\infty. \tag{3.9}$$

From (3.5) and the compactness of the embedding $W_0^{1,\overrightarrow{p}}(\Omega)\hookrightarrow L^p(\Omega)$, we obtain that, up to a subsequence,

$$u_\ell\to u \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \tag{3.10}$$

Moreover, using (3.4) with u replaced by u_ℓ, we get that $\widehat{A}_j(u_\ell)$ is bounded in $L^{p_j'}(\Omega)$ for every $1\leq j\leq N$. Hence, in view of (1.11), there exist $\mu\in L^{p'}(\Omega)$ and $g_j\in L^{p_j'}(\Omega)$ for $1\leq j\leq N$ so that, up to a further subsequence of $\{u_\ell\}$ (denoted by $\{u_\ell\}$), we have

$$\begin{aligned} &\widehat{\Theta}(u_\ell)\rightharpoonup \mu \quad \text{(weakly) in } L^{p'}(\Omega) \quad \text{as } \ell\to\infty,\\ &\widehat{A}_j(u_\ell)\rightharpoonup g_j \quad \text{(weakly) in } L^{p_j'}(\Omega) \quad \text{as } \ell\to\infty \end{aligned} \tag{3.11}$$

for every $1\leq j\leq N$. Thus, using the reflexivity of $W_0^{1,\overrightarrow{p}}(\Omega)$ and (3.6), we find that

$$\langle g,v\rangle = \lim_{\ell\to\infty}\langle (\mathcal{A}+\mathcal{P}_\Theta)(u_\ell),v\rangle = \sum_{j=1}^N\int_\Omega g_j\,\partial_j v\,dx + \int_\Omega \mu\, v\,dx \tag{3.12}$$

for every $v\in W_0^{1,\overrightarrow{p}}(\Omega)$. From (3.10) and (3.11), we infer that

$$\lim_{\ell\to\infty}\int_\Omega \widehat{\Theta}(u_\ell)\,u_\ell\,dx = \int_\Omega \mu\,u\,dx. \tag{3.13}$$

From (3.7), (3.12) and (3.13), we obtain that

$$\begin{aligned} \limsup_{\ell\to\infty}\langle (\mathcal{A}+\mathcal{P}_\Theta)(u_\ell),u_\ell\rangle &= \limsup_{\ell\to\infty}\left(\sum_{j=1}^N\int_\Omega \widehat{A}_j(u_\ell)\,\partial_j u_\ell\,dx + \int_\Omega \widehat{\Theta}(u_\ell)\,u_\ell\,dx\right)\\ &= \limsup_{\ell\to\infty}\sum_{j=1}^N\int_\Omega \widehat{A}_j(u_\ell)\,\partial_j u_\ell\,dx + \int_\Omega \mu\,u\,dx\\ &\leq \langle g,u\rangle = \sum_{j=1}^N\int_\Omega g_j\,\partial_j u\,dx + \int_\Omega \mu\,u\,dx, \end{aligned} \tag{3.14}$$

that is,

$$\limsup_{\ell\to\infty}\sum_{j=1}^N\int_\Omega \widehat{A}_j(u_\ell)\,\partial_j u_\ell\,dx \le \sum_{j=1}^N\int_\Omega g_j\,\partial_j u\,dx. \tag{3.15}$$

Our next aim is to show that

$$\liminf_{\ell\to\infty}\sum_{j=1}^N\int_\Omega \widehat{A}_j(u_\ell)\,\partial_j u_\ell\,dx \ge \sum_{j=1}^N\int_\Omega g_j\,\partial_j u\,dx. \tag{3.16}$$

The proof of (3.16) is a bit different from the classical one in the isotropic case since in our growth condition on A_j in (1.10), we have taken the greatest exponent for $|t|$ from the viewpoint of the anisotropic Sobolev inequalities. Let us emphasize what is new compared with the classical proof. Let $1\le j\le N$ be arbitrary. Since $u_\ell\to u$ a.e. in Ω and A_j is a Carathéodory function, we see that

$$A_j(x,u_\ell,\nabla u)\to A_j(x,u,\nabla u)\quad \text{a.e. in } \Omega. \tag{3.17}$$

The growth condition in (1.10) gives a constant $C>0$ and a nonnegative function $\eta_j\in L^{p_j'}(\Omega)$ such that

$$|A_j(x,u_\ell,\nabla u)|^{p_j'}\le C\left(\eta_j^{p_j'}+|u_\ell|^{p^*}+\sum_{i=1}^N|\partial_i u|^{p_i}\right) \tag{3.18}$$

for every $\ell\ge 1$. Because the power of $|u_\ell|$ in the right-hand side of (3.18) is p^*, the critical exponent, the compactness of the embedding $W_0^{1,\overrightarrow{p}}(\Omega)\hookrightarrow L^{p^*}(\Omega)$ fails, in general. Hence, we cannot claim anymore that $\{|A_j(x,u_\ell,\nabla u)|^{p_j'}\}_{\ell\ge1}$ is uniformly integrable over Ω. Thus, we cannot apply Vitali's theorem to deduce the strong convergence of $A_j(x,u_\ell,\nabla u)$ to $A_j(x,u,\nabla u)$ in $L^{p_j'}(\Omega)$ as $\ell\to\infty$. However, if we fix $k\ge1$, then by the growth condition in (1.10), we infer that

$$\{|A_j(x,u_\ell,\nabla u_\ell)|^{p_j'}\,\chi_{\{|u_\ell|\le k\}}\}_{\ell\ge1}\quad\text{is uniformly integrable over } \Omega.$$

Then, since $\chi_{\{|u_\ell|\le k\}}\to\chi_{\{|u|\le k\}}$ as $\ell\to\infty$, from (3.17) and Vitali's theorem, we get

$$A_j(x,u_\ell,\nabla u)\,\chi_{\{|u_\ell|\le k\}}\to A_j(x,u,\nabla u)\,\chi_{\{|u|\le k\}}\ \text{strongly in } L^{p_j'}(\Omega)\text{ as }\ell\to\infty \tag{3.19}$$

We now return to the proof of (3.16) with modifications suggested by (3.19). To prove (3.16), it suffices to show that for every integer $k\ge1$, we have

$$\liminf_{\ell\to\infty}\sum_{j=1}^N\int_\Omega \widehat{A}_j(u_\ell)\,\partial_j u_\ell\,dx \ge \sum_{j=1}^N\int_\Omega g_j\,(\partial_j u)\,\chi_{\{|u|\le k\}}dx. \tag{3.20}$$

Indeed, by letting $k\to\infty$ in (3.20) and applying the Dominated Convergence Theorem, we arrive at (3.16).

Proof of (3.20). Fix an integer $k\ge1$. The coercivity condition in (1.10) yields that

$$\sum_{j=1}^N \widehat{A}_j(u_\ell)\,\partial_j u_\ell \ge \sum_{j=1}^N \widehat{A}_j(u_\ell)\,(\partial_j u_\ell)\,\chi_{\{|u_\ell|\le k\}}. \tag{3.21}$$

For the right-hand side of (3.21), we use the monotonicity condition in (1.10), that is,

$$\begin{aligned}\sum_{j=1}^N \widehat{A}_j(u_\ell)\,(\partial_j u_\ell)\,\chi_{\{|u_\ell|\le k\}} \geq &\sum_{j=1}^N \widehat{A}_j(u_\ell)\,(\partial_j u)\,\chi_{\{|u_\ell|\le k\}}\\ &+\sum_{j=1}^N A_j(x,u_\ell,\nabla u)\,(\partial_j u_\ell-\partial_j u)\,\chi_{\{|u_\ell|\le k\}}.\end{aligned} \tag{3.22}$$

Let $1 \le j \le N$ be arbitrary. By the Dominated Convergence Theorem, we have $(\partial_j u)\,\chi_{\{|u_\ell|\le k\}} \to (\partial_j u)\,\chi_{\{|u|\le k\}}$ strongly in $L^{p_j}(\Omega)$ as $\ell\to\infty$. Recall from (3.11) that $\widehat{A}_j(u_\ell) \rightharpoonup g_j$ (weakly) in $L^{p_j'}(\Omega)$ as $\ell\to\infty$. Hence, as $\ell\to\infty$, we have

$$\widehat{A}_j(u_\ell)\,(\partial_j u_\ell)\,\chi_{\{|u_\ell|\le k\}} \to g_j\,(\partial_j u)\chi_{\{|u|\le k\}} \text{ strongly in } L^1(\Omega). \tag{3.23}$$

Since $\partial_j u_\ell \rightharpoonup \partial_j u$ (weakly) in $L^{p_j}(\Omega)$ as $\ell\to\infty$, using (3.19), we gain the following

$$A_j(x,u_\ell,\nabla u)\,(\partial_j u_\ell-\partial_j u)\,\chi_{\{|u_\ell|\le k\}} \to 0 \quad \text{strongly in } L^1(\Omega) \text{ as } \ell\to\infty. \tag{3.24}$$

In light of (3.23) and (3.24), we see that

$$\sum_{j=1}^N \int_\Omega \widehat{A}_j(u_\ell)\,(\partial_j u)\,\chi_{\{|u_\ell|\le k\}} + \sum_{j=1}^N \int_\Omega A_j(x,u_\ell,\nabla u)\,(\partial_j u_\ell-\partial_j u)\,\chi_{\{|u_\ell|\le k\}}$$

converges as $\ell\to\infty$ to the right-hand side of (3.20). Using this convergence, jointly with the inequalities in (3.21) and (3.22), we conclude the proof of (3.20).

As mentioned above, from (3.20) we obtain (3.16).

Inequalities (3.15) and (3.16) ensure that

$$\lim_{\ell\to\infty}\sum_{j=1}^N \int_\Omega \widehat{A}_j(u_\ell)\,\partial_j u_\ell\,dx = \sum_{j=1}^N \int_\Omega g_j\,\partial_j u\,dx. \tag{3.25}$$

Using this fact into (3.14), we conclude the proof of (3.9).

It remains to establish (3.8). From (3.25), we also see that

$$\sum_{j=1}^N \int_\Omega [A_j(x,u_\ell,\nabla u_\ell)-A_j(x,u_\ell,\nabla u)]\,(\partial_j u_\ell-\partial_j u)\,\chi_{\{|u_\ell|\le k\}}\,dx \to 0 \quad \text{as } \ell\to\infty. \tag{3.26}$$

By (3.26) and the monotonicity condition in (1.10), we infer that

$$\sum_{j=1}^N [A_j(x,u_\ell,\nabla u_\ell)-A_j(x,u_\ell,\nabla u)]\,(\partial_j u_\ell-\partial_j u) \to 0 \text{ a.e in } \{|u_\ell|\le k\} \text{ as } \ell\to\infty.$$

By a standard diagonal argument, we can find a subsequence of $\{u_\ell\}$ (still denoted by $\{u_\ell\}$) such that the above convergence holds for every $k\ge 1$. This implies that

$$\sum_{j=1}^N [A_j(x,u_\ell,\nabla u_\ell)-A_j(x,u_\ell,\nabla u)]\,(\partial_j u_\ell-\partial_j u) \to 0 \text{ a.e. in } \Omega \text{ as } \ell\to\infty.$$

In the notation of Section 2.1, this means that

$$\mathcal{D}_{u_\ell}(u_\ell, u) \to 0 \quad \text{a.e. in } \Omega \text{ as } \ell \to \infty.$$

Thus, by Lemma 2.1, we infer that, up to a subsequence,

$$\nabla u_\ell \to \nabla u \quad \text{a.e. in } \Omega \text{ as } \ell \to \infty. \tag{3.27}$$

Since Φ and A_j (with $1 \le j \le N$) are Carathéodory functions, from (3.27), we find that

$$\widehat{\Theta}(u_\ell) \to \widehat{\Theta}(u) \quad \text{and} \quad \widehat{A}_j(u_\ell) \to \widehat{A}_j(u) \text{ a.e. in } \Omega \text{ as } \ell \to \infty.$$

Using this fact, jointly with (3.11), we obtain that

$$\mu = \widehat{\Theta}(u) \quad \text{and} \quad g_j = \widehat{A}_j(u) \text{ for every } 1 \le j \le N. \tag{3.28}$$

From (3.12) and (3.28), we conclude that

$$\langle g, v\rangle = \sum_{j=1}^{N} \int_\Omega \widehat{A}_j(u)\, \partial_j v\, dx + \int_\Omega \widehat{\Theta}(u)\, v\, dx = \langle \mathcal{A}u, v\rangle + \langle \mathcal{P}_\Theta, v\rangle$$

for every $v \in W_0^{1,\overrightarrow{p}}(\Omega)$. This proves that $g = (\mathcal{A} + \mathcal{P}_\Theta)\, u$, namely, (3.8) holds.

In conclusion, by satisfying the M type condition in Definition 3.2, the operator $\mathcal{A}+\mathcal{P}_\Theta$ turns out to be pseudo-monotone. The proof of Lemma 3.6 is now complete. □

Lemma 3.7. *Every operator $\mathfrak{B} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ satisfying (P_1) and (P_2) is bounded and strongly continuous.*

Proof. The boundedness of $\mathfrak{B}$ is an easy consequence of our assumption (P_1). Indeed, by (P_1), we have $1 \le \mathfrak{s} < (p^*)'$, giving the compactness of the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{\mathfrak{s}}(\Omega)$ (see Remark 1.6). Moreover, there exists a positive constant $\mathfrak{C}_1$, depending only on $\mathfrak{a}_0$, $\mathfrak{C}$, N, $\overrightarrow{p}$ and meas$\,(\Omega)$, such that

$$|\langle \mathfrak{B}u, v\rangle| \le \mathfrak{C}_1 \left(1 + \|u\|^{\mathfrak{b}}_{W_0^{1,\overrightarrow{p}}(\Omega)}\right) \|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}$$

for every $u, v \in W_0^{1,\overrightarrow{p}}(\Omega)$. Thus, $\mathfrak{B} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ is a bounded operator.

To go on, we show that every operator $\mathfrak{B} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ satisfying (P_2) is strongly continuous. This means that if $u_\ell \rightharpoonup u$ (weakly) in $W_0^{1,\overrightarrow{p}}(\Omega)$ as $\ell \to \infty$, then $\mathfrak{B}u_\ell \to \mathfrak{B}u$ in $W^{-1,\overrightarrow{p}'}(\Omega)$ as $\ell \to \infty$. Assume by contradiction that there exist $\varepsilon_0 > 0$ and a subsequence of $\{u_\ell\}$ (relabeled $\{u_\ell\}$) such that

$$\sup_{\substack{v \in W_0^{1,\overrightarrow{p}}(\Omega),\\ \|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le 1}} |\langle \mathfrak{B}u_\ell - \mathfrak{B}u, v\rangle| > \varepsilon_0 \quad \text{for every } \ell \ge 1.$$

Hence, there also exists $\{v_\ell\}$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ with $\|v_\ell\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le 1$ such that

$$|\langle \mathfrak{B}u_\ell - \mathfrak{B}u, v_\ell\rangle| > \varepsilon_0 \quad \text{for all } \ell \ge 1. \tag{3.29}$$

By the boundedness of $\{v_\ell\}$ in $W_0^{1,\overrightarrow{p}}(\Omega)$, up to a subsequence, we have

$$v_\ell \rightharpoonup v \;\; \text{(weakly) in } W_0^{1,\overrightarrow{p}}(\Omega) \;\; \text{as } \ell \to \infty. \tag{3.30}$$

Since $\mathfrak{B}u \in W^{-1,\overrightarrow{p}'}(\Omega)$, from (3.30) we infer that

$$\langle \mathfrak{B}u, v_\ell \rangle \to \langle \mathfrak{B}u, v \rangle \text{ as } \ell \to \infty. \tag{3.31}$$

From (P_2) and (3.31), we obtain that $|\langle \mathfrak{B}u_\ell - \mathfrak{B}u, v_\ell\rangle| \to 0$ as $\ell \to \infty$, which is in contradiction with (3.29). Thus, $\mathfrak{B}$ is strongly continuous, completing the proof. □

To prove Lemma 3.9 below, we need an iterated version of Young's inequality.

Lemma 3.8 (Young's inequality)**.** *Let $N \geq 2$ be an integer. Assume that $\beta_1, \ldots, \beta_N$ are positive numbers and $1 < R_k < \infty$ for each $1 \leq k \leq N-1$. If $\sum_{k=1}^{N-1}(1/R_k) < 1$, then for every $\delta > 0$, there exists a positive constant C_δ (depending on δ) such that*

$$\prod_{k=1}^{N} \beta_k \leq \delta \sum_{k=1}^{N-1} \beta_k^{R_k} + C_\delta\, \beta_N^{R_N},$$

where we define $R_N = \left[1 - \sum_{k=1}^{N-1}(1/R_k)\right]^{-1}$.

Lemma 3.9. *Let* (1.3), (1.10) *and* (1.11) *hold. Assume that $\mathfrak{B} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ satisfies (P_1) and (P_2). Then, $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B}$ is a bounded, coercive and pseudo-monotone operator from $W_0^{1,\overrightarrow{p}}$ into $W^{-1,\overrightarrow{p}'}(\Omega)$.*

Proof. From Lemmas 3.5 and 3.7, we find that $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B}$ is a bounded operator from $W_0^{1,\overrightarrow{p}}(\Omega)$ into $W^{-1,\overrightarrow{p}'}(\Omega)$. We now show that it is also coercive, namely,

$$\frac{\langle \mathcal{A}u + \mathcal{P}_\Theta(u) - \mathfrak{B}u, u\rangle}{\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)}} \to \infty \quad \text{as} \quad \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \to \infty.$$

For every $u, v \in W_0^{1,\overrightarrow{p}}(\Omega)$, by the coercivity condition in (1.10) and (1.11), we have

$$\langle \mathcal{A}u + \mathcal{P}_\Theta(u) - \mathfrak{B}u, u\rangle \geq \nu_0 \sum_{j=1}^{N} \|\partial_j u\|_{L^{p_j}(\Omega)}^{p_j} - C_\Theta \|u\|_{L^1(\Omega)} - |\langle \mathfrak{B}u, u\rangle|. \tag{3.32}$$

We claim that for every $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$\langle \mathcal{A}u + \mathcal{P}_\Theta(u) - \mathfrak{B}u, u\rangle \geq (\nu_0 - N\delta) \sum_{j=1}^{N} \|\partial_j u\|_{L^{p_j}(\Omega)}^{p_j} - C_\delta \tag{3.33}$$

for every $u \in W_0^{1,\overrightarrow{p}}(\Omega)$.

Proof of (3.33). From (1.14), we have

$$|\langle \mathfrak{B}u, u\rangle| \leq \mathfrak{C} \left(1 + \|u\|^{\mathfrak{b}}_{W_0^{1,\overrightarrow{p}}(\Omega)}\right) \left(\mathfrak{a}_0 \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} + \|u\|_{L^{\mathfrak{s}}(\Omega)}\right)$$

for all $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, where $\mathfrak{C} > 0$, $\mathfrak{s} \in [1, p^*)$, $\mathfrak{a}_0 \geq 0$, $\mathfrak{b} \in (0, p_1 - 1)$ if $\mathfrak{a}_0 > 0$ and $\mathfrak{b} \in (0, p_1/p')$ if $\mathfrak{a}_0 = 0$. We will use the continuity of the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^r(\Omega)$ with $r \in [1, p^*]$. We distinguish two cases, depending on whether or not $\mathfrak{a}_0$ is positive.

I) If $\mathfrak{a}_0 > 0$, then for positive constants C_1 and C_2, we find that

$$C_\Theta \|u\|_{L^1(\Omega)} + |\langle \mathfrak{B}u, u\rangle| \leq C_1 \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} + C_2 \|u\|^{\mathfrak{b}+1}_{W_0^{1,\overrightarrow{p}}(\Omega)} \tag{3.34}$$

for every $u \in W_0^{1,\overrightarrow{p}}(\Omega)$. Recall the assumption $\mathfrak{b}+1 < p_1 = \min_{1\leq j\leq N} p_j$. Hence, using (3.34) into (3.32), jointly with Young's inequality, we conclude (3.33).

II) We now assume $\mathfrak{a}_0 = 0$. Then, from the anisotropic Sobolev inequality in (1.15), there exists a positive constant C such that

$$C_\Theta \|u\|_{L^1(\Omega)} + |\langle \mathfrak{B}u, u\rangle| \leq C \left(\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} + \sum_{j=1}^N \|\partial_j u\|^{\mathfrak{b}}_{L^{p_j}(\Omega)} \prod_{k=1}^N \|\partial_k u\|^{\frac{1}{N}}_{L^{p_k}(\Omega)} \right) \tag{3.35}$$

for all $u \in W_0^{1,\overrightarrow{p}}(\Omega)$. For every $\delta > 0$ and $1 \leq j \leq N$, by Lemma 3.8, we find a positive constant $C_{\delta,j}$, depending on δ, such that

$$C\, \|\partial_j u\|^{\mathfrak{b}}_{L^{p_j}(\Omega)} \prod_{k=1}^N \|\partial_k u\|^{\frac{1}{N}}_{L^{p_k}(\Omega)} \leq \delta \sum_{k\in\{1,\dots,N\}\setminus\{j\}} \|\partial_k u\|^{p_k}_{L^{p_k}(\Omega)} + C_{\delta,j} \|\partial_j u\|^{\alpha_j}_{L^{p_j}(\Omega)}, \tag{3.36}$$

for all $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, where α_j is given by

$$\alpha_j = \frac{p' p_j (N\mathfrak{b}+1)}{N p_j + p'}.$$

Since $\mathfrak{a}_0 = 0$, the hypothesis $\mathfrak{b} < p_1/p'$ yields that $\alpha_j < p_j$ for every $1 \leq j \leq N$. Thus, by Young's inequality, for every $\varepsilon > 0$, there exists a constant $C_{\varepsilon,\delta} > 0$ such that

$$C\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} + \sum_{j=1}^N C_{\delta,j} \|\partial_j u\|^{\alpha_j}_{L^{p_j}(\Omega)} \leq \varepsilon \sum_{j=1}^N \|\partial_j u\|^{p_j}_{L^{p_j}(\Omega)} + C_{\varepsilon,\delta} \tag{3.37}$$

for all $u \in W_0^{1,\overrightarrow{p}}(\Omega)$. Hence, from (3.36) and (3.37), we see that the left-hand side of (3.35) is bounded above by $[(N-1)\,\delta + \varepsilon] \sum_{j=1}^N \|\partial_j u\|^{p_j}_{L^{p_j}(\Omega)} + C_{\varepsilon,\delta}$. Using this fact in the right-hand side of (3.32), we conclude the proof of (3.33).

It is now clear that by choosing $\delta > 0$ small enough, the inequality in (3.33) yields the coercivity of the operator $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$.

Finally, Proposition 3.3 ensures that $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ is pseudo-monotone as a sum of pseudo-monotone operators (see Lemmas 3.6 and 3.7). □

4. Proof of Theorem 1.2

Here, we assume (1.3), (1.10), (1.11) and (1.12), whereas $\mathfrak{B} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ satisfies (P_1) and (P_2). For every $\varepsilon > 0$, we define $\Phi_\varepsilon(x,t,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ as follows

$$\Phi_\varepsilon(x,t,\xi) := \frac{\Phi(x,t,\xi)}{1 + \varepsilon\, |\Phi(x,t,\xi)|}$$

for a.e. $x \in \Omega$ and all $(t,\xi) \in \mathbb{R}\times\mathbb{R}^N$. For $\varepsilon > 0$ fixed, Φ_ε satisfies the same properties as Φ, that is, the sign-condition and the growth condition in (1.12). Moreover, Φ_ε becomes a bounded function, namely, for a.e. $x \in \Omega$ and every $(t,\xi) \in \mathbb{R}\times\mathbb{R}^N$,

$$\Phi_\varepsilon(x,t,\xi)\, t \geq 0, \quad |\Phi_\varepsilon(x,t,\xi)| \leq \min\{|\Phi(x,t,\xi)|, 1/\varepsilon\}. \tag{4.1}$$

We consider approximate problems to (1.1) with $f = 0$ and Φ replaced by Φ_ε, that is,

$$\begin{cases} \mathcal{A}u_\varepsilon + \Phi_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) + \Theta(x, u_\varepsilon, \nabla u_\varepsilon) = \mathfrak{B}u_\varepsilon \quad \text{in } \Omega, \\ u_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega). \end{cases} \tag{4.2}$$

As in Theorem 3.1, by a solution of (4.2), we mean a function $u_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega)$ such that

$$\sum_{j=1}^N \int_\Omega \widehat{A}_j(u_\varepsilon)\, \partial_j v\, dx + \int_\Omega \widehat{\Phi}_\varepsilon(u_\varepsilon)\, v\, dx + \int_\Omega \widehat{\Theta}(u_\varepsilon)\, v\, dx = \langle \mathfrak{B}u_\varepsilon, v\rangle \tag{4.3}$$

for every $v \in W_0^{1,\overrightarrow{p}}(\Omega)$, where for convenience we define

$$\widehat{\Phi}_\varepsilon(u_\varepsilon)(x) := \Phi_\varepsilon(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) \quad \text{for a.e. } x \in \Omega.$$

Lemma 4.1. *For every $\varepsilon > 0$, there exists a solution u_ε for (4.2). Moreover, we have*

(*a*) *For a positive constant C, independent of ε, it holds*

$$\|u_\varepsilon\|_{W_0^{1,\overrightarrow{p}}(\Omega)} + \int_\Omega \widehat{\Phi}_\varepsilon(u_\varepsilon)\, u_\varepsilon\, dx \le C. \tag{4.4}$$

(*b*) *There exists $U \in W_0^{1,\overrightarrow{p}}(\Omega)$ such that, up to a subsequence of $\{u_\varepsilon\}$,*

$$u_\varepsilon \rightharpoonup U \ \textit{(weakly) in } W_0^{1,\overrightarrow{p}}(\Omega) \quad \textit{and} \quad u_\varepsilon \to U \ \textit{a.e. in } \Omega \textit{ as } \varepsilon \to 0. \tag{4.5}$$

Proof. Let $\varepsilon > 0$ be arbitrary. From (4.1), we see that $\Phi_\varepsilon + \Theta$ satisfies the same assumptions as Θ in Section 3. So, Theorem 3.1 applies with $\mathcal{P}_\Theta$ replaced by $\mathcal{P}_{\Theta,\varepsilon}$, where

$$\langle \mathcal{P}_{\Theta,\varepsilon}(u), v\rangle := \int_\Omega \left(\widehat{\Theta}(u) + \widehat{\Phi}_\varepsilon(u)\right) v\, dx \quad \text{for every } u, v \in W_0^{1,\overrightarrow{p}}(\Omega).$$

This means that (4.2) admits at least a solution $u_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega)$ for every $\varepsilon > 0$.

(*a*) Recall from (3.33) that for every $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$\langle \mathcal{A}u + \mathcal{P}_\Theta(u) - \mathfrak{B}u, u\rangle \ge (\nu_0 - N\delta) \sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}^{p_j} - C_\delta \tag{4.6}$$

for every $u \in W_0^{1,\overrightarrow{p}}(\Omega)$. Now, by taking $v = u_\varepsilon$ in (4.3), we derive that

$$\langle \mathcal{A}u_\varepsilon + \mathcal{P}_\Theta(u_\varepsilon) - \mathfrak{B}u_\varepsilon, u_\varepsilon\rangle + \int_\Omega \widehat{\Phi}_\varepsilon(u_\varepsilon)\, u_\varepsilon\, dx = 0. \tag{4.7}$$

Thus, letting $u = u_\varepsilon$ in (4.6) and using (4.7), jointly with (4.1), we arrive at

$$(\nu_0 - N\delta) \sum_{j=1}^N \|\partial_j u_\varepsilon\|_{L^{p_j}(\Omega)}^{p_j} \le (\nu_0 - N\delta) \sum_{j=1}^N \|\partial_j u_\varepsilon\|_{L^{p_j}(\Omega)}^{p_j} + \int_\Omega \widehat{\Phi}_\varepsilon(u_\varepsilon)\, u_\varepsilon\, dx \le C_\delta.$$

By choosing $\delta \in (0, \nu_0/N)$, we readily conclude the assertion of (4.4).

(*b*) From (4.4) and the reflexivity of $W_0^{1,\overrightarrow{p}}(\Omega)$, we infer that, up to a subsequence, u_ε converges weakly to some U in $W_0^{1,\overrightarrow{p}}(\Omega)$. Then, we conclude (4.5) by using Remark 1.6, which implies that, up to a subsequence, $u_\varepsilon \to U$ (strongly) in $L^\kappa(\Omega)$ if $\kappa \in [1, p^*)$ and $u_\varepsilon \to U$ a.e. in Ω. This completes the proof of Lemma 4.1. □

For the remainder of this section, u_ε and U have the meaning in Lemma 4.1. For every $k>0$, the truncation T_k at height k is defined by (2.11). Moreover, we define

$$G_k(s) = s - T_k(s) \quad \text{for every } s \in \mathbb{R}. \tag{4.8}$$

In particular, we have $G_k = 0$ on $[-k,k]$ and $t\,G_k(t) \geq 0$ for every $t \in \mathbb{R}$.

4.1. **Strong convergence of $T_k(u_\varepsilon)$.** For $v, w \in W_0^{1,\overrightarrow{p}}(\Omega)$ and a.e. $x \in \Omega$, we define $\mathcal{D}_{u_\varepsilon}(v,w)(x)$ as in (2.1), namely,

$$\mathcal{D}_{u_\varepsilon}(v,w)(x) := \sum_{j=1}^{N} \left[A_j(x, u_\varepsilon(x), \nabla v(x)) - A_j(x, u_\varepsilon(x), \nabla w(x))\right] \partial_j (v-w)(x). \tag{4.9}$$

For any fixed integer $k \geq 1$, we obtain $\mathcal{D}_{u_\varepsilon}(T_k(u_\varepsilon), T_k(U))$ by replacing v and w in (4.9) by $T_k(u_\varepsilon)$ and $T_k(U)$, respectively. For simplicity, we write $\mathcal{D}_{\varepsilon,k}(x)$ instead of $\mathcal{D}_{u_\varepsilon}(T_k(u_\varepsilon), T_k(U))(x)$, that is,

$$\mathcal{D}_{\varepsilon,k}(x) := \sum_{j=1}^{N} \left[A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon)) - A_j(x, u_\varepsilon, \nabla T_k(U))\right] \partial_j (T_k(u_\varepsilon) - T_k(U)). \tag{4.10}$$

Lemma 4.2. *There exists a subsequence of $\{u_\varepsilon\}$, relabeled $\{u_\varepsilon\}$, such that*

$$\nabla u_\varepsilon \to \nabla U \text{ a.e. in } \Omega \text{ and } T_k(u_\varepsilon) \to T_k(U) \text{ (strongly) in } W_0^{1,\overrightarrow{p}}(\Omega) \text{ as } \varepsilon \to 0 \tag{4.11}$$

for every integer $k \geq 1$.

Proof. Recall that $\{u_\varepsilon\}$ satisfies (4.5) in Lemma 4.1. By a standard diagonal argument, it suffices to show that for every integer $k \geq 1$, there exists a subsequence $\{u_\varepsilon\}$ (depending on k and relabeled $\{u_\varepsilon\}$) satisfying

$$\nabla T_k(u_\varepsilon) \to \nabla T_k(u) \text{ a.e. in } \Omega \quad \text{and} \quad T_k(u_\varepsilon) \to T_k(u) \text{ (strongly) in } W_0^{1,\overrightarrow{p}}(\Omega). \tag{4.12}$$

Moreover, in light of Lemma 2.2, we conclude (4.12) by showing that, for every integer $k \geq 1$, there exists a subsequence of $\{u_\varepsilon\}$ (depending on k and relabeled $\{u_\varepsilon\}$) such that

$$\mathcal{D}_{\varepsilon,k} \to 0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{4.13}$$

Let $k \geq 1$ be fixed. As noted in Section 2.1, the monotonicity assumption in (1.10) yields that $\mathcal{D}_{\varepsilon,k} \geq 0$ a.e. in Ω. Hence, to prove (4.13), it suffices to show that (up to a subsequence of $\{u_\varepsilon\}$), we have

$$\limsup_{\varepsilon \to 0} \int_\Omega \mathcal{D}_{\varepsilon,k}(x)\, dx \leq 0. \tag{4.14}$$

We define $z_{\varepsilon,k}$ as follows

$$z_{\varepsilon,k} := T_k(u_\varepsilon) - T_k(U).$$

We observe that

$$\partial_j z_{\varepsilon,k}\, \chi_{\{|u_\varepsilon| \geq k\}} = -\partial_j T_k(U)\, \chi_{\{|u_\varepsilon| \geq k\}} = -\partial_j U\, \chi_{\{|u_\varepsilon| \geq k\}}\, \chi_{\{|U| < k\}}.$$

Moreover, we see that

$$\chi_{\{|u_\varepsilon| \geq k\}}\, \chi_{\{|U| < k\}} \to 0 \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0. \tag{4.15}$$

By the Dominated Convergence Theorem, for every $1 \leq j \leq N$, we have

$$\partial_j U \, \chi_{\{|u_\varepsilon| \geq k\}} \, \chi_{\{|U|<k\}} \to 0 \quad \text{(strongly) in } L^{p_j}(\Omega) \ \text{ as } \varepsilon \to 0. \tag{4.16}$$

On the other hand, from the growth condition on A_j in (1.10) and the *a priori* estimates in Lemma 4.1, we infer that $\{A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon))\}_\varepsilon$ and $\{A_j(x, u_\varepsilon, \nabla T_k(U))\}_\varepsilon$ are bounded in $L^{p_j'}(\Omega)$ and, hence, up to a subsequence of $\{u_\varepsilon\}$, they converge weakly in $L^{p_j'}(\Omega)$ for each $1 \leq j \leq N$. This, jointly with (4.16), gives that

$$\Xi_{j,\varepsilon,k}(x) := [A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon)) - A_j(x, u_\varepsilon, \nabla T_k(U))] \, \partial_j U \chi_{\{|u_\varepsilon| \geq k\}} \, \chi_{\{|U|<k\}}$$

converges to 0 in $L^1(\Omega)$ as $\varepsilon \to 0$ for every $1 \leq j \leq N$. It follows that

$$\int_\Omega \mathcal{D}_{\varepsilon,k}(x) \, \chi_{\{|u_\varepsilon| \geq k\}} \, dx = - \sum_{j=1}^N \int_\Omega \Xi_{j,\varepsilon,k}(x) \, dx \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus, to conclude (4.14), it remains to show that

$$\limsup_{\varepsilon \to 0} \int_\Omega \mathcal{D}_{\varepsilon,k}(x) \, \chi_{\{|u_\varepsilon|<k\}} \, dx \leq 0. \tag{4.17}$$

Proof of (4.17). We define $\varphi_\lambda : \mathbb{R} \to \mathbb{R}$ as follows

$$\varphi_\lambda(t) = t \exp\left(\lambda t^2\right) \quad \text{for every } t \in \mathbb{R}.$$

We choose $\lambda = \lambda(k) > 0$ large such that $4\nu_0^2 \, \lambda > \phi^2(k)$, where ϕ appears in the growth assumption on Φ, see (1.12). This choice of λ ensures that

$$\lambda t^2 - \frac{\phi(k)}{2\nu_0}|t| + \frac{1}{4} > 0 \quad \text{for every } t \in \mathbb{R}. \tag{4.18}$$

Then, in view of (4.18), we have

$$\varphi_\lambda'(t) - \frac{\phi(k)}{\nu_0} \, |\varphi_\lambda(t)| > \frac{1}{2} \quad \text{for all } t \in \mathbb{R}. \tag{4.19}$$

For $v \in W_0^{1,\overrightarrow{p}}(\Omega)$, we define

$$\mathcal{E}_{\varepsilon,k}(v) = \sum_{j=1}^N \int_\Omega A_j(x, u_\varepsilon, \nabla v) \partial_j z_{\varepsilon,k} \left[\varphi_\lambda'(z_{\varepsilon,k}) - \frac{\phi(k)}{\nu_0} \, |\varphi_\lambda(z_{\varepsilon,k})| \right] \chi_{\{|u_\varepsilon|<k\}} \, dx.$$

Returning to the definition of $\mathcal{D}_{\varepsilon,k}$ in (4.10) and using (4.19), we arrive at

$$\frac{1}{2} \int_\Omega \mathcal{D}_{\varepsilon,k}(x) \, \chi_{\{|u_\varepsilon|<k\}} \, dx \leq \mathcal{E}_{\varepsilon,k}(T_k(u_\varepsilon)) - \mathcal{E}_{\varepsilon,k}(T_k(U)). \tag{4.20}$$

Since $T_k(u_\varepsilon) = u_\varepsilon$ on the set $\{|u_\varepsilon| < k\}$, in light of (4.20), we complete the proof of (4.17) by showing that

$$\lim\nolimits_{\varepsilon \to 0} \mathcal{E}_{\varepsilon,k}(T_k(U)) = 0, \tag{4.21}$$

$$\limsup\nolimits_{\varepsilon \to 0} \mathcal{E}_{\varepsilon,k}(u_\varepsilon) \leq 0. \tag{4.22}$$

Proof of (4.21). Indeed, for each $1 \leq j \leq N$, the growth condition on A_j in (1.10) gives a nonnegative function $F_j \in L^{p_j'}(\Omega)$ such that on the set $\{|u_\varepsilon| < k\}$, we have

$|A_j(x,u_\varepsilon,\nabla T_k(U))|\leq F_j$ for every $\varepsilon>0$. Since $|z_{\varepsilon,k}|\leq 2k$, it follows that there exists a constant $C_k>0$ such that

$$\left|\varphi_\lambda'(z_{\varepsilon,k})-\frac{\phi(k)}{\nu_0}\,|\varphi_\lambda(z_{\varepsilon,k})|\right|\leq C_k.$$

On the other hand, for each $1\leq j\leq N$, we have

$$\partial_j z_{\varepsilon,k}\,\chi_{\{|u_\varepsilon|<k\}}=\partial_j z_{\varepsilon,k}+\partial_j U\,\chi_{\{|U|<k\}}\chi_{\{|u_\varepsilon|\geq k\}}.$$

This, together with (4.16) and the weak convergence of $\partial_j z_{\varepsilon,k}$ to 0 in $L^{p_j}(\Omega)$ as $\varepsilon\to 0$, implies that $\partial_j z_{\varepsilon,k}\,\chi_{\{|u_\varepsilon|<k\}}$ converges weakly to 0 in $L^{p_j}(\Omega)$ as $\varepsilon\to 0$. Hence, we have

$$|\mathcal{E}_{\varepsilon,k}(T_k(U))|\leq C_k\sum_{j=1}^N\int_\Omega F_j\,|\partial_j z_{\varepsilon,k}|\,\chi_{\{|u_\varepsilon|<k\}}\,dx\to 0\quad\text{as }\varepsilon\to 0,$$

which proves (4.21).

Proof of (4.22). From (4.5), we have

$$z_{\varepsilon,k}\to 0\text{ a.e. in }\Omega\text{ and }z_{\varepsilon,k}\rightharpoonup 0\text{ (weakly) in }W_0^{1,\overrightarrow{p}}(\Omega)\quad\text{as }\varepsilon\to 0.$$

Since $|z_{\varepsilon,k}|\leq 2k$ a.e. in Ω, we get $\varphi_\lambda(z_{\varepsilon,k})\in W_0^{1,\overrightarrow{p}}(\Omega)\cap L^\infty(\Omega)$. Moreover,

$$\varphi_\lambda(z_{\varepsilon,k})\to 0\ \text{ a.e. in }\Omega\text{ and }\varphi_\lambda(z_{\varepsilon,k})\rightharpoonup 0\ \text{ (weakly) in }W_0^{1,\overrightarrow{p}}(\Omega)\text{ as }\varepsilon\to 0.\tag{4.23}$$

Observe that $u_\varepsilon\,z_{\varepsilon,k}\geq 0$ on the set $\{|u_\varepsilon|\geq k\}$, which gives that

$$\widehat{\Phi}_\varepsilon(u_\varepsilon)\,\varphi_\lambda(z_{\varepsilon,k})\,\chi_{\{|u_\varepsilon|\geq k\}}\geq 0.$$

Thus, by testing (4.3) with $v=\varphi_\lambda(z_{\varepsilon,k})$, we obtain that

$$\begin{aligned}\langle\mathcal{A}u_\varepsilon,\varphi_\lambda(z_{\varepsilon,k})\rangle+\int_\Omega\widehat{\Phi}_\varepsilon(u_\varepsilon)\,\varphi_\lambda(z_{\varepsilon,k})\,\chi_{\{|u_\varepsilon|<k\}}\,dx\leq&\langle\mathfrak{B}u_\varepsilon,\varphi_\lambda(z_{\varepsilon,k})\rangle\\&-\int_\Omega\widehat{\Theta}(u_\varepsilon)\,\varphi_\lambda(z_{\varepsilon,k})\,dx.\end{aligned}\tag{4.24}$$

To simplify exposition, we now introduce some notation:

$$\begin{aligned}&X_k(\varepsilon):=\phi(k)\int_\Omega\left[\frac{1}{\nu_0}\sum_{j=1}^N\widehat{A}_j(u_\varepsilon)\,\partial_j(T_kU)+c(x)\right]|\varphi_\lambda(z_{\varepsilon,k})|\,\chi_{\{|u_\varepsilon|<k\}}\,dx,\\&Y_k(\varepsilon):=\sum_{j=1}^N\int_\Omega\widehat{A}_j(u_\varepsilon)\,\partial_jU\,\varphi_\lambda'(z_{\varepsilon,k})\,\chi_{\{|U|<k\}}\,\chi_{\{|u_\varepsilon|\geq k\}}\,dx.\end{aligned}\tag{4.25}$$

We rewrite the first term in the left-hand side of (4.24) as follows

$$\langle\mathcal{A}u_\varepsilon,\varphi_\lambda(z_{\varepsilon,k})\rangle=\sum_{j=1}^N\int_\Omega\widehat{A}(u_\varepsilon)\partial_j z_{\varepsilon,k}\,\varphi_\lambda'(z_{\varepsilon,k})\,\chi_{\{|u_\varepsilon|<k\}}\,dx-Y_k(\varepsilon).\tag{4.26}$$

The coercivity condition in (1.10) and the growth condition of Φ in (1.12) imply that

$$|\widehat{\Phi}_\varepsilon(u_\varepsilon)|\,\chi_{\{|u_\varepsilon|<k\}} \leq \phi(k)\left[\frac{1}{\nu_0}\sum_{j=1}^N \widehat{A}_j(u_\varepsilon)\partial_j u_\varepsilon + c(x)\right]\chi_{\{|u_\varepsilon|<k\}}. \tag{4.27}$$

In the right-hand side of (4.27) we replace $\partial_j u_\varepsilon$ by $\partial_j z_{\varepsilon,k} + \partial_j T_k(U)$, then we multiply the inequality by $|\varphi_\lambda(z_{\varepsilon,k})|$ and integrate over Ω with respect to x. It follows that the second term in the left-hand side of (4.24) is at least

$$-\frac{\phi(k)}{\nu_0}\sum_{j=1}^N \int_\Omega \widehat{A}_j(u_\varepsilon)\partial_j z_{\varepsilon,k}\,|\varphi_\lambda(z_{\varepsilon,k})|\,\chi_{\{|u_\varepsilon|<k\}}\,dx - X_k(\varepsilon).$$

Using this fact, as well as (4.26), in (4.24), we see that $\mathcal{E}_{\varepsilon,k}(u_\varepsilon)$ satisfies the estimate

$$\mathcal{E}_{\varepsilon,k}(u_\varepsilon) \leq X_k(\varepsilon) + Y_k(\varepsilon) + \langle \mathfrak{B}u_\varepsilon, \varphi_\lambda(z_{\varepsilon,k})\rangle - \int_\Omega \widehat{\Theta}(u_\varepsilon)\,\varphi_\lambda(z_{\varepsilon,k})\,dx, \tag{4.28}$$

where $X_k(\varepsilon)$ and $Y_k(\varepsilon)$ are defined in (4.25).

To conclude the proof of (4.22), it suffices to show that each term in the right-hand side of (4.28) converges to 0 as $\varepsilon \to 0$. Recall that $\varphi_\lambda(z_{\varepsilon,k}) \in W_0^{1,\overrightarrow{p}}(\Omega)\cap L^\infty(\Omega)$ satisfies (4.23). Thus, using (1.11) and the property (P_2) of $\mathfrak{B}$, we get that the third, as well as the fourth, term in the right-hand side of (4.28) converges to zero as $\varepsilon \to 0$.

We next look at $X_k(\varepsilon)$. In view of the pointwise convergence in (4.23), we infer from the Dominated Convergence Theorem that

$$c(x)|\varphi_\lambda(z_{\varepsilon,k})|\,\chi_{\{|u_\varepsilon|<k\}} \to 0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{4.29}$$

Next, up to a subsequence of $\{u_\varepsilon\}$, we find that $\widehat{A}_j(u_\varepsilon)$ converges weakly in $L^{p_j'}(\Omega)$ as $\varepsilon \to 0$ for every $1 \leq j \leq N$ using the boundedness of $\widehat{A}_j : W_0^{1,\overrightarrow{p}}(\Omega) \to L^{p_j'}(\Omega)$ (see Lemma 3.5). Hence, $\sum_{j=1}^N \widehat{A}_j(u_\varepsilon)\,\partial_j U$ converges in $L^1(\Omega)$ as $\varepsilon \to 0$. Then, there exists a nonnegative function $F \in L^1(\Omega)$ (independent of ε) such that, up to a subsequence of $\{u_\varepsilon\}$, we have

$$\left|\sum_{j=1}^N \widehat{A}_j(u_\varepsilon)\,\partial_j U\right| \leq F \quad \text{a.e. in } \Omega \text{ for every } \varepsilon > 0. \tag{4.30}$$

We can now again use the Dominated Convergence Theorem to conclude that

$$\sum_{j=1}^N \widehat{A}_j(u_\varepsilon)\,\partial_j T_k(U)\,|\varphi_\lambda(z_{\varepsilon,k})|\,\chi_{\{|u_\varepsilon|<k\}} \to 0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{4.31}$$

From (4.29) and (4.31), we find that $\lim_{\varepsilon\to 0} X_k(\varepsilon) = 0$. Since $|\varphi_\lambda'(z_{\varepsilon,k})|$ is bounded above by a constant independent of ε (but dependent on k), we can use a similar argument, based on (4.15) and (4.30), to conclude that, up to a subsequence of $\{u_\varepsilon\}$, we have $\lim_{\varepsilon\to 0} Y_k(\varepsilon) = 0$. This finishes the proof of the convergence to zero of the right-hand side of (4.28) as $\varepsilon \to 0$. Consequently, the proof of (4.22), and thus of (4.17), is complete. This ends the proof of Lemma 4.2. $\square$

4.2. **Passing to the limit.** From now on, the meaning of $\{u_\varepsilon\}_\varepsilon$ is given by Lemma 4.2. Using Lemma 4.1, we prove in Lemma 4.4 that U is a solution of (1.1) with $f = 0$ and, moreover, U satisfies all the properties stated in Theorem 1.2. Besides (4.11), the other fundamental property that allows us to pass to the limit as $\varepsilon \to 0$ in (4.3) for every $v \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ is the following convergence

$$\widehat{\Phi}_\varepsilon(u_\varepsilon) \to \widehat{\Phi}(U) \ \ \text{(strongly) in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{4.32}$$

The proof of (4.32) is the main objective of our next result.

Lemma 4.3. *We have* $\widehat{\Phi}(U)\, U^j \in L^1(\Omega)$ *for* $j = 0, 1$ *and* (4.32) *holds.*

Proof. We show using Fatou's Lemma that $\widehat{\Phi}(U)\, U \in L^1(\Omega)$, which we then use to derive that also $\widehat{\Phi}(U) \in L^1(\Omega)$. Indeed, from the pointwise convergence

$$u_\varepsilon \to U \text{ and } \nabla u_\varepsilon \to \nabla U \ \ \text{a.e. in } \Omega \ \ \text{as } \varepsilon \to 0,$$

jointly with the fact that $\Phi(x, t, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, we infer that $\widehat{\Phi}(u_\varepsilon)$ converges to $\widehat{\Phi}(U)$ a.e. in Ω as $\varepsilon \to 0$. Then, we have

$$\widehat{\Phi}_\varepsilon(u_\varepsilon)\, u_\varepsilon \to \widehat{\Phi}(U)\, U \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0. \tag{4.33}$$

Using (4.33) and that $\{\widehat{\Phi}_\varepsilon(u_\varepsilon)\, u_\varepsilon\}_\varepsilon$ is a sequence of nonnegative functions that is uniformly bounded in $L^1(\Omega)$ with respect to ε (from Lemma 4.1), by Fatou's Lemma we conclude that

$$\widehat{\Phi}(U)\, U \in L^1(\Omega).$$

This, together with the growth condition in (1.12), yields that $\widehat{\Phi}(U) \in L^1(\Omega)$. Indeed, for any $M > 0$, on the set $\Omega \cap \{|U| \le M\}$, we have

$$|\widehat{\Phi}(U)| \le \phi(M) \left(\sum_{j=1}^N |\partial_j U|^{p_j} + c(x) \right) \in L^1(\Omega).$$

In turn, on the set $\Omega \cap \{|U| > M\}$, it holds

$$|\widehat{\Phi}(U)| \le M^{-1}\, \widehat{\Phi}(U)\, U \in L^1(\Omega).$$

Hence, it follows that $\widehat{\Phi}(U) \in L^1(\Omega)$.

To finish the proof of Lemma 4.3, it remains to establish (4.32).

Proof of (4.32). Since $\widehat{\Phi}_\varepsilon(u_\varepsilon) \to \widehat{\Phi}(U)$ a.e. in Ω as $\varepsilon \to 0$ and $\widehat{\Phi}(U) \in L^1(\Omega)$, by Vitali's Theorem, it suffices to show that $\{\widehat{\Phi}_\varepsilon(u_\varepsilon)\}_\varepsilon$ is uniformly integrable over Ω. We next check this fact. For every $M > 0$, we define

$$D_{\varepsilon,M} := \{|u_\varepsilon| \le M\} \quad \text{and} \quad E_{\varepsilon,M} := \{|u_\varepsilon| > M\}.$$

For every $x \in D_{\varepsilon,M}$, using the growth condition of Φ in (1.12), we find that

$$|\widehat{\Phi}_\varepsilon(u_\varepsilon)(x)| \le |\widehat{\Phi}(u_\varepsilon)(x)| \le \phi(M) \left(\sum_{j=1}^N |\partial_j T_M(u_\varepsilon)|^{p_j} + c(x) \right).$$

Let ω be any measurable subset of Ω. It follows that

$$\int_{\omega\cap D_{\varepsilon,M}} |\widehat{\Phi}_\varepsilon(u_\varepsilon)|\,dx \le \phi(M)\left(\sum_{j=1}^N \|\partial_j(T_M u_\varepsilon)\|^{p_j}_{L^{p_j}(\omega)} + \int_\omega c(x)\,dx\right).$$

On the other hand, using (4.4) in Lemma 4.1, we see that

$$\int_{\omega\cap E_{\varepsilon,M}} |\widehat{\Phi}_\varepsilon(u_\varepsilon)|\,dx \le \frac{1}{M}\int_{\omega\cap E_{\varepsilon,M}} \widehat{\Phi}_\varepsilon(u_\varepsilon)\,u_\varepsilon\,dx \le \frac{C}{M},$$

where $C>0$ is a constant independent of ε and ω. Consequently, we find that

$$\int_\omega |\widehat{\Phi}_\varepsilon(u_\varepsilon)|\,dx \le \phi(M)\left(\sum_{j=1}^N \|\partial_j(T_M u_\varepsilon)\|^{p_j}_{L^{p_j}(\omega)} + \int_\omega c(x)\,dx\right) + \frac{C}{M}. \tag{4.34}$$

Lemma 4.2 yields that $\partial_j T_M(u_\varepsilon) \to \partial_j T_M(U)$ (strongly) in $L^{p_j}(\Omega)$ as $\varepsilon \to 0$ for every $1 \le j \le N$. Since $c(\cdot) \in L^1(\Omega)$ (see our assumption (1.12)), from (4.34) we deduce the uniform integrability of $\{\widehat{\Phi}_\varepsilon(u_\varepsilon)\}_\varepsilon$ over Ω. Then, we conclude the proof of (4.32) by Vitali's Theorem. This ends the proof of Lemma 4.3. $\square$

By Lemma 4.3, to finish the proof of Theorem 1.2, we need to show the following.

Lemma 4.4. *The function U is a solution of* (1.1) *with $f=0$ and, moreover,*

$$\sum_{j=1}^N \int_\Omega \widehat{A}_j(U)\,\partial_j U\,dx + \int_\Omega \widehat{\Phi}(U)\,U\,dx + \int_\Omega \widehat{\Theta}(U)\,U\,dx = \langle \mathfrak{B}U, U\rangle. \tag{4.35}$$

Proof. Fix $v \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ arbitrary. Since u_ε is a solution of (4.2), we have

$$\sum_{j=1}^N \int_\Omega \widehat{A}_j(u_\varepsilon)\,\partial_j v\,dx + \int_\Omega \widehat{\Phi}_\varepsilon(u_\varepsilon)\,v\,dx + \int_\Omega \widehat{\Theta}(u_\varepsilon)\,v\,dx = \langle \mathfrak{B}u_\varepsilon, v\rangle. \tag{4.36}$$

By Lemma 4.3, the second term in the left-hand side of (4.36) converges to $\int_\Omega \widehat{\Phi}(U)\,v$ as $\varepsilon \to 0$, whereas the right-hand side of (4.36) converges to $\langle \mathfrak{B}U, v\rangle$ based on the weak convergence of u_ε to U in $W_0^{1,\overrightarrow{p}}(\Omega)$ as $\varepsilon \to 0$. Using (4.5) in Lemma 4.1 and the convergence of ∇u_ε to ∇U a.e. in Ω as $\varepsilon \to 0$, we find that

$$\widehat{\Theta}(u_\varepsilon) \to \widehat{\Theta}(U) \quad \text{and} \quad \widehat{A}_j(u_\varepsilon) \to \widehat{A}_j(U) \text{ a.e. in } \Omega \text{ for } 1 \le j \le N. \tag{4.37}$$

Thus, in light of (1.11), and the Dominated Convergence Theorem, we obtain that

$$\int_\Omega \widehat{\Theta}(u_\varepsilon)\,v\,dx \to \int_\Omega \widehat{\Theta}(U)\,v\,dx \quad \text{as } \varepsilon \to 0.$$

Since $\{\widehat{A}_j(u_\varepsilon)\}_\varepsilon$ is uniformly bounded in $L^{p_j'}(\Omega)$ with respect to ε, we observe from (4.37) that (up to a subsequence)

$$\widehat{A}_j(u_\varepsilon) \rightharpoonup \widehat{A}_j(U) \text{ (weakly) in } L^{p_j'}(\Omega) \text{ as } \varepsilon \to 0$$

for each $1 \le j \le N$. It follows that

$$\sum_{j=1}^N \int_\Omega \widehat{A}_j(u_\varepsilon)\,\partial_j v\,dx \to \sum_{j=1}^N \int_\Omega \widehat{A}_j(U)\,\partial_j v\,dx \quad \text{as } \varepsilon \to 0.$$

By letting $\varepsilon \to 0$ in (4.36), we conclude that

$$\sum_{j=1}^N \int_\Omega \widehat{A}_j(U)\,\partial_j v\,dx + \int_\Omega \widehat{\Phi}(U)\,v\,dx + \int_\Omega \widehat{\Theta}(U)\,v\,dx = \langle \mathfrak{B}U, v\rangle \tag{4.38}$$

for every $v \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$. Hence, U is a solution of (1.1) with $f = 0$.

It remains to prove (4.35). Since U may not be in $L^\infty(\Omega)$, we cannot directly use $v = U$ in (4.38). Nevertheless, for every $k > 0$, we have $T_k(U) \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$. Hence, by taking $v = T_k(U)$ in (4.38), we see that

$$\langle \mathcal{A}U, T_k(U)\rangle + \int_\Omega \widehat{\Phi}(U)\,T_k(U)\,dx + \int_\Omega \widehat{\Theta}(U)\,T_k(U)\,dx = \langle \mathfrak{B}U, T_k(U)\rangle. \tag{4.39}$$

Notice that $\|T_k(U)\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le \|U\|_{W_0^{1,\overrightarrow{p}}(\Omega)}$ for all $k > 0$. Moreover, for every $1 \le j \le N$, we have $\partial_j(T_k(U)) \to \partial_j U$ a.e. in Ω as $k \to \infty$ so that

$$T_k(U) \rightharpoonup U \text{ (weakly) in } W_0^{1,\overrightarrow{p}}(\Omega) \text{ as } k \to \infty.$$

Since $\mathcal{A}U$ and $\mathfrak{B}U$ belong to $W^{-1,\overrightarrow{p}'}(\Omega)$, it follows that

$$\lim_{k\to\infty} \langle \mathcal{A}U, T_k(U)\rangle = \langle \mathcal{A}U, U\rangle \quad \text{and} \quad \lim_{k\to\infty} \langle \mathfrak{B}U, T_k(U)\rangle = \langle \mathfrak{B}U, U\rangle.$$

Recalling that $\widehat{\Phi}(U)\,U \in L^1(\Omega)$ and (1.11) holds, from the Dominated Convergence Theorem, we can pass to the limit $k \to \infty$ in (4.39) to obtain (4.35). □

5. Proof of Theorem 1.1

Suppose for the moment only (1.3), (1.10), (1.11) and (1.12). Overall, to prove Theorem 1.1, we follow similar arguments to those developed for proving Theorem 1.2 in Section 4. But there are several differences that appear when introducing a function $f \in L^1(\Omega)$ in the equation (1.1). We first approximate f by a "nice" function $f_\varepsilon \in L^\infty(\Omega)$ with the properties that

$$|f_\varepsilon| \le |f| \text{ a.e. in } \Omega \text{ and } f_\varepsilon \to f \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0. \tag{5.1}$$

Then, by the Dominated Convergence Theorem, we find that

$$f_\varepsilon \to f \text{ (strongly) in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{5.2}$$

For example, for every $\varepsilon > 0$, we could take

$$f_\varepsilon(x) := \frac{f(x)}{1 + \varepsilon |f(x)|} \quad \text{for a.e. } x \in \Omega.$$

This approximation is done so that we can apply Theorem 1.2 for the problem generated by (1.1) with f_ε in place of f. Then such an approximate problem admits at least a solution U_ε, namely,

$$\begin{cases} \mathcal{A}U_\varepsilon + \widehat{\Phi}(U_\varepsilon) + \widehat{\Theta}(U_\varepsilon) = \mathfrak{B}U_\varepsilon + f_\varepsilon & \text{in } \Omega, \\ U_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega), \quad \widehat{\Phi}(U_\varepsilon) \in L^1(\Omega). \end{cases} \tag{5.3}$$

To see this, we return to Example 1.7, which shows that if

$$\langle \mathfrak{B}_\varepsilon u, v\rangle = \langle \mathfrak{B}u, v\rangle + \int_\Omega f_\varepsilon \, v\, dx \quad \text{for every } u, v \in W_0^{1,\overrightarrow{p}}(\Omega), \tag{5.4}$$

then $\mathfrak{B}_\varepsilon : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ satisfies (P_1) and (P_2). By Theorem 1.2 applied for $\mathfrak{B}_\varepsilon$ instead of $\mathfrak{B}$, we obtain a solution U_ε for (5.3). This means that

$$\sum_{j=1}^N \int_\Omega \widehat{A}_j(U_\varepsilon)\, \partial_j v\, dx + \int_\Omega \widehat{\Phi}(U_\varepsilon)\, v\, dx + \int_\Omega \widehat{\Theta}(U_\varepsilon)\, v\, dx = \langle \mathfrak{B}U_\varepsilon, v\rangle + \int_\Omega f_\varepsilon \, v\, dx \tag{5.5}$$

for every $v \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$. However, unlike Theorem 1.2, to obtain that U_ε is uniformly bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$ with respect to ε, we need an additional hypothesis, namely, (1.13), which we formulate below for convenience:
there exist positive constants τ and γ such that for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^N$

$$|\Phi(x, t, \xi)| \geq \gamma \sum_{j=1}^N |\xi_j|^{p_j} \quad \text{for all } |t| \geq \tau. \tag{5.6}$$

For the rest of this section, besides (1.3), (1.10), (1.11) and (1.12), we also assume (5.6). To avoid repetition, we understand that all the computations in Section 4 are done here replacing u_ε, U and Φ_ε by U_ε, U_0 and Φ, respectively. We only stress the differences that appear compared with the developments in Section 4.

5.1. ***A priori* estimates.** In Lemma 4.1 we gave *a priori* estimates for the solution u_ε to (4.2), corresponding to the problem (1.1) with $f = 0$ and Φ_ε instead of Φ. We next obtain *a priori* estimates for U_ε solving (5.3), that is, (1.1) with f_ε instead of f.

Lemma 5.1. *Let U_ε be a solution of* (5.3).
(a) *For a positive constant C, independent of ε, we have*

$$\|U_\varepsilon\|_{W_0^{1,\overrightarrow{p}}(\Omega)} + \int_\Omega |\widehat{\Phi}(U_\varepsilon)|\, dx \leq C. \tag{5.7}$$

(b) *There exists $U_0 \in W_0^{1,\overrightarrow{p}}(\Omega)$ such that, up to a subsequence of $\{U_\varepsilon\}$,*

$$U_\varepsilon \rightharpoonup U_0 \ (\textit{weakly}) \textit{ in } W_0^{1,\overrightarrow{p}}(\Omega), \quad U_\varepsilon \to U_0 \textit{ a.e. in } \Omega \textit{ as } \varepsilon \to 0. \tag{5.8}$$

Proof. There are new ideas coming into play because of the introduction of f_ε and working with Φ in (5.3) (rather than Φ_ε). Hence, we provide the details.

(a) Let $\tau > 0$ be as in (5.6). The choice of f_ε gives that $\|f_\varepsilon\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$. As a test function in (5.5), we take $v = T_\tau(U_\varepsilon)$, which belongs to $W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$. Since

$\partial_j T_\tau(U_\varepsilon) = \chi_{\{|U_\varepsilon|<\tau\}}\, \partial_j U_\varepsilon$ a.e. in Ω for every $1 \le j \le N$, using also the sign-condition of Φ in (1.12), we obtain that

$$\sum_{j=1}^{N} \int_\Omega \widehat{A}_j(U_\varepsilon)\, \partial_j U_\varepsilon\, \chi_{\{|U_\varepsilon|<\tau\}}\, dx + \tau \int_\Omega |\widehat{\Phi}(U_\varepsilon)|\, \chi_{\{|U_\varepsilon|\ge\tau\}}\, dx \tag{5.9}$$

is bounded above by

$$|\langle \mathfrak{B} U_\varepsilon, T_\tau(U_\varepsilon)\rangle| + \tau \left(\|f\|_{L^1(\Omega)} + C_\Theta \,\mathrm{meas}\,(\Omega)\right). \tag{5.10}$$

By virtue of (5.6) and the coercivity condition in (1.10), we see that the quantity in (5.9) is bounded below by

$$\nu_0 \sum_{j=1}^{N} \int_\Omega |\partial_j U_\varepsilon|^{p_j}\, \chi_{\{|U_\varepsilon|<\tau\}}\, dx + \tau\gamma \sum_{j=1}^{N} \int_\Omega |\partial_j U_\varepsilon|^{p_j}\, \chi_{\{|U_\varepsilon|\ge\tau\}}\, dx.$$

If we define $c_0 := \min\{\nu_0, \tau\gamma\}$, then $c_0 > 0$ and the above estimates lead to

$$c_0 \sum_{j=1}^{N} \int_\Omega |\partial_j U_\varepsilon|^{p_j}\, dx \le |\langle \mathfrak{B} U_\varepsilon, T_\tau(U_\varepsilon)\rangle| + \tau \left(\|f\|_{L^1(\Omega)} + C_\Theta \,\mathrm{meas}\,(\Omega)\right). \tag{5.11}$$

From (1.14) in the assumption (P_1), for every $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, we have

$$|\langle \mathfrak{B} u, T_\tau(u)\rangle| \le \mathfrak{C}(1 + \|u\|^{\mathfrak{b}}_{W_0^{1,\overrightarrow{p}}(\Omega)})(\mathfrak{a}_0 \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} + \|u\|_{L^{\mathfrak{s}}(\Omega)}), \tag{5.12}$$

where $\mathfrak{C} > 0$, $\mathfrak{s} \in [1, p^*)$, $\mathfrak{a}_0 \ge 0$, $\mathfrak{b} \in (0, p_1 - 1)$ if $\mathfrak{a}_0 > 0$ and $\mathfrak{b} \in (0, p_1/p')$ if $\mathfrak{a}_0 = 0$. With an argument similar to that in Lemma 3.9, we can deduce that

$$\frac{c_0 \displaystyle\sum_{j=1}^{N} \int_\Omega |\partial_j u|^{p_j}\, dx - |\langle \mathfrak{B} u, T_\tau(u)\rangle|}{\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)}} \to \infty \text{ as } \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \to \infty.$$

This fact, jointly with (5.11), implies that there exists a constant $C_0 > 0$ such that

$$\|U_\varepsilon\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le C_0 \quad \text{for every } \varepsilon > 0. \tag{5.13}$$

By letting $u = U_\varepsilon$ in (5.12) and using (5.13), we find a constant $C_1 > 0$ such that

$$|\langle \mathfrak{B} U_\varepsilon, T_\tau(U_\varepsilon)\rangle| \le C_1 \quad \text{for every } \varepsilon > 0. \tag{5.14}$$

Since the sum over $j = 1, \ldots, N$ in (5.9) is nonnegative, the remaining term in (5.9) is bounded above by the quantity in (5.10). Then, using (5.14), we arrive at

$$\int_\Omega |\widehat{\Phi}(U_\varepsilon)|\, \chi_{\{|U_\varepsilon|\ge\tau\}}\, dx \le C_1 \tau^{-1} + \|f\|_{L^1(\Omega)} + C_\Theta \,\mathrm{meas}\,(\Omega) := C_2. \tag{5.15}$$

Now, from the boundedness of $\{U_\varepsilon\}$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ (see (5.13)) and the growth condition on Φ in (1.12), we obtain a positive constant C_3 such that

$$\int_\Omega |\widehat{\Phi}(U_\varepsilon)|\, \chi_{\{|U_\varepsilon|\le\tau\}}\, dx \le C_3 \quad \text{for every } \varepsilon > 0. \tag{5.16}$$

Putting together the estimates in (5.13), (5.15) and (5.16), we conclude (5.7).

(b) The assertion in (5.8) follows from (5.13), jointly with the reflexivity of $W_0^{1,\overrightarrow{p}}(\Omega)$ and the compactness of the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^r(\Omega)$ for $r \in [1, p^*)$ (see Remark 1.6). The proof of Lemma 5.1 is now complete. □

5.2. **Strong convergence of $T_k(U_\varepsilon)$.** The game plan is closely related to that in Section 4.1. As mentioned before, when adapting the calculations, we need to replace u_ε, U and $\mathfrak{B}$ in Section 4 by U_ε, U_0 and $\mathfrak{B}_\varepsilon$, respectively. The counterpart of Lemma 4.2 holds so that we obtain the following.

Lemma 5.2. *There exists a subsequence of $\{U_\varepsilon\}_\varepsilon$, relabeled $\{U_\varepsilon\}_\varepsilon$, such that*

$$\nabla U_\varepsilon \to \nabla U_0 \textit{ a.e. in } \Omega \textit{ and } T_k(U_\varepsilon) \to T_k(U_0) \textit{ (strongly) in } W_0^{1,\overrightarrow{p}}(\Omega) \textit{ as } \varepsilon \to 0$$

for every positive integer k.

Proof. The computations in Section 4.1 can be carried out with Φ instead of Φ_ε since the upper bounds used for $|\Phi_\varepsilon|$ were derived from those satisfied by $|\Phi|$ and the sign-condition of Φ is the same as for Φ_ε (see (4.1)). A small change arises in the proof of (4.22) because of the introduction of f_ε in (5.3). Using the definition of $\mathfrak{B}_\varepsilon$ in (5.4), the inequalities in (4.24) and (4.28) must be read with $\mathfrak{B}_\varepsilon$ instead of $\mathfrak{B}$. We note that $\langle \mathfrak{B}_\varepsilon U_\varepsilon, \varphi_\lambda(z_{\varepsilon,k})\rangle$ is the sum between $\langle \mathfrak{B} U_\varepsilon, \varphi_\lambda(z_{\varepsilon,k})\rangle$ and $\int_\Omega f_\varepsilon\, \varphi_\lambda(z_{\varepsilon,k})\, dx$. The latter term, like the former, converges to 0 as $\varepsilon \to 0$. The new claim regarding the convergence to zero of $\int_\Omega f_\varepsilon\, \varphi_\lambda(z_{\varepsilon,k})\, dx$ follows from the Dominated Convergence Theorem using (5.1), $|\varphi_\lambda(z_{\varepsilon,k})| \le 2k \exp(4\lambda k^2)$ and $\varphi_\lambda(z_{\varepsilon,k}) \to 0$ a.e. in Ω as $\varepsilon \to 0$. The remainder of the proof of (4.22) carries over easily to our setting. □

5.3. **Passing to the limit.** We aim to pass to the limit as $\varepsilon \to 0$ in (5.5) to obtain that U_0 is a solution of (1.1). Since f_ε satisfies (5.2) and $U_\varepsilon \rightharpoonup U_0$ (weakly) in $W_0^{1,\overrightarrow{p}}(\Omega)$ as $\varepsilon \to 0$, we readily have the convergence of the right-hand side of (5.5) to $\langle \mathfrak{B} U_0, v\rangle + \int_\Omega f\, v\, dx$ for every $v \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$. Moreover, because of the convergence $\nabla U_\varepsilon \to \nabla U_0$ a.e. in Ω, we can use the same argument as in Lemma 4.4 to deduce that

$$\int_\Omega \widehat{\Theta}(u_\varepsilon)\, v\, dx \to \int_\Omega \widehat{\Theta}(U_0)\, v\, dx \quad \text{as } \varepsilon \to 0,$$
$$\sum_{j=1}^N \int_\Omega \widehat{A}_j(U_\varepsilon)\, \partial_j v\, dx \to \sum_{j=1}^N \int_\Omega \widehat{A}_j(U_0)\, \partial_j v\, dx \quad \text{as } \varepsilon \to 0$$

for every $v \in W_0^{1,\overrightarrow{p}}(\Omega)$. What is here different compared with Section 4.2 is the proof of the convergence

$$\widehat{\Phi}(U_\varepsilon) \to \widehat{\Phi}(U_0) \;\; \text{(strongly) in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{5.17}$$

To conclude that U_0 is a solution of (1.1), it remains to justify (5.17). So, instead of Lemma 4.3, we establish the following.

Lemma 5.3. *We have $\widehat{\Phi}(U_0) \in L^1(\Omega)$ and* (5.17) *holds.*

Proof. We infer that $\widehat{\Phi}(U_0) \in L^1(\Omega)$ from Fatou's Lemma based on the boundedness of the second term in the left-hand side of (5.7) and the poinwise convergence

$$|\widehat{\Phi}(U_\varepsilon)| \to |\widehat{\Phi}(U_0)| \;\; \text{a.e. in } \Omega \;\; \text{as } \varepsilon \to 0. \tag{5.18}$$

The assertion of (5.18) follows from Lemma 5.2, the pointwise convergence in (5.8) and the continuity of $\Phi(x,\cdot,\cdot)$ in the last two variables.

Proof of (5.17). We will use Vitali's Theorem. To this end, taking into account (5.18), we need to show that $\{\widehat{\Phi}(U_\varepsilon)\}_\varepsilon$ is uniformly integrable over Ω. We can only partially imitate the proof of the uniform integrability of $\{\widehat{\Phi}_\varepsilon(u_\varepsilon)\}_\varepsilon$ in Lemma 4.3. Fix $M>1$ arbitrary. For any measurable subset ω of Ω, using the growth condition of Φ in (1.12), we find that

$$\int_\omega |\widehat{\Phi}(U_\varepsilon)|\,\chi_{\{|U_\varepsilon|\le M\}}\,dx \le \phi(M)\left(\sum_{j=1}^N \|\partial_j T_M(U_\varepsilon)\|^{p_j}_{L^{p_j}(\omega)} + \|c\|_{L^1(\omega)}\right). \tag{5.19}$$

Since $\partial_j T_M(U_\varepsilon)\to\partial_j T_M(U_0)$ (strongly) in $L^{p_j}(\Omega)$ as $\varepsilon\to0$ for every $1\le j\le N$ and $c(\cdot)\in L^1(\Omega)$, we see that the right-hand side of (5.19) is as small as desired uniformly in ε when the measure of ω is small.

We next bound from above $\int_\omega|\widehat{\Phi}(U_\varepsilon)|\,\chi_{\{|U_\varepsilon|>M\}}\,dx$. This is where the modification appears since we don't have anymore that $\{\widehat{\Phi}(U_\varepsilon)\,U_\varepsilon\}_\varepsilon$ is uniformly bounded in $L^1(\Omega)$ with respect to ε. We adapt an approach from [13]. In (5.5) we take

$$v=T_1(G_{M-1}(U_\varepsilon)),$$

which belongs to $W_0^{1,\overrightarrow{p}}(\Omega)\cap L^\infty(\Omega)$. Then, using (1.11), the coercivity condition in (1.10) and the sign-condition of Φ in (1.12), we obtain the estimate

$$\int_\Omega |\widehat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|>M\}}\,dx \le \int_\Omega (|f_\varepsilon|+C_\Theta)\,\chi_{\{|U_\varepsilon|\ge M-1\}}\,dx + |\langle \mathfrak{B}U_\varepsilon, T_1(G_{M-1}(U_\varepsilon))\rangle|. \tag{5.20}$$

Now, up to a subsequence of $\{U_\varepsilon\}$, from (5.8), we have

$$T_1(G_{M-1}(U_\varepsilon))\rightharpoonup T_1(G_{M-1}(U_0)) \text{ (weakly) in } W_0^{1,\overrightarrow{p}}(\Omega) \text{ as } \varepsilon\to0.$$

Using this in (5.20), jointly with (5.1) and the property (P_2) for $\mathfrak{B}$, we find that

$$\limsup_{\varepsilon\to0}\int_\Omega |\widehat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|>M\}}\,dx \le \int_\Omega (|f|+C_\Theta)\,\chi_{\{|U_0|\ge M-1\}}\,dx + |\langle \mathfrak{B}U_0, T_1(G_{M-1}(U_0))\rangle|.$$

Recall that $f\in L^1(\Omega)$ and $\mathfrak{B}$ satisfies the growth condition in (1.14). Since we have

$$\partial_j\, T_1(G_{M-1}(U_0)) = \chi_{\{M-1<|U_0|<M\}}\,\partial_j U_0 \quad \text{a.e. in } \Omega$$

for every $1\le j\le N$, from the above inequality, we infer that

$$\int_\omega |\widehat{\Phi}(U_\varepsilon)|\,\chi_{\{|U_\varepsilon|>M\}}\,dx$$

is small, uniformly in ε and ω, when M is sufficiently large. Thus, using also the comments after (5.19), we conclude the uniform integrability of $\{\widehat{\Phi}(U_\varepsilon)\}_\varepsilon$ over Ω. Hence, (5.17) follows from Vitali's Theorem, based on (5.18). The proof of Lemma 5.3 is now complete. □

By letting $\varepsilon\to0$ in (5.5), we conclude that U_0 is a solution of (1.1). This ends the proof of Theorem 1.1. □

6. Strong convergence of u_ε in Theorem 1.2

We show here that in the setting of Theorem 1.2, up to a subsequence of $\{u_\varepsilon\}$, not only the assertions of Lemma 4.2 hold, but also the strong convergence in (1.7), that is,

$$u_\varepsilon \to U \text{ (strongly) in } W_0^{1,\overrightarrow{p}}(\Omega) \quad \text{as } \varepsilon \to 0. \tag{6.1}$$

We establish (6.1) in Lemma 6.2, based on Lemma 6.1 below. For every $k \geq 1$, we define

$$L_k := \frac{|\langle \mathfrak{B} U, G_k(U)\rangle| + C_\Theta \|G_k(U)\|_{L^1(\Omega)}}{\nu_0}. \tag{6.2}$$

Lemma 6.1. *For every integer $k \geq 1$, up to a subsequence of $\{u_\varepsilon\}$, we have*

$$\limsup_{\varepsilon\to 0} \|G_k(u_\varepsilon)\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \leq \sum_{j=1}^N L_k^{1/p_j}. \tag{6.3}$$

Proof. Let $k \geq 1$ be a fixed integer. Since $G_k(u_\varepsilon) = u_\varepsilon - T_k(u_\varepsilon)$ and

$$\partial_j T_k(u_\varepsilon) = \partial_j u_\varepsilon\, \chi_{\{|u_\varepsilon|<k\}} \text{ for every } 1 \leq j \leq N,$$

from the coercivity assumption in (1.10), we see that

$$\begin{aligned} \langle \mathcal{A} u_\varepsilon, G_k(u_\varepsilon)\rangle &= \sum_{j=1}^N \int_{\{|u_\varepsilon|>k\}} \widehat{A}_j(u_\varepsilon)\, \partial_j u_\varepsilon\, dx \\ &\geq \nu_0 \sum_{j=1}^N \int_{\{|u_\varepsilon|>k\}} |\partial_j u_\varepsilon|^{p_j}\, dx = \nu_0 \sum_{j=1}^N \|\partial_j G_k(u_\varepsilon)\|^{p_j}_{L^{p_j}(\Omega)}. \end{aligned}$$

Using (4.1) and $t\, G_k(t) \geq 0$ for every $t \in \mathbb{R}$, we observe that $G_k(t)\, \widehat{\Phi}_\varepsilon(t) \geq 0$ for all $t \in \mathbb{R}$. Then, by testing (4.3) with $v = G_k(u_\varepsilon)$ and using (1.11), we find that

$$\begin{aligned} \langle \mathcal{A} u_\varepsilon, G_k(u_\varepsilon)\rangle &\leq \langle \mathcal{A} u_\varepsilon, G_k(u_\varepsilon)\rangle + \int_\Omega G_k(u_\varepsilon)\, \widehat{\Phi}_\varepsilon(u_\varepsilon)\, dx \\ &= \langle \mathfrak{B} u_\varepsilon, G_k(u_\varepsilon)\rangle - \int_\Omega \widehat{\Theta}(u_\varepsilon)\, G_k(u_\varepsilon)\, dx \\ &\leq |\langle \mathfrak{B} u_\varepsilon, G_k(u_\varepsilon)\rangle| + C_\Theta \int_\Omega |G_k(u_\varepsilon)|\, dx. \end{aligned}$$

From (4.5), the boundedness of $\{u_\varepsilon\}$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ and Remark 1.6, we can pass to a subsequence of $\{u_\varepsilon\}$ (relabeled $\{u_\varepsilon\}$) such that as $\varepsilon \to 0$

$$\begin{aligned} &T_k(u_\varepsilon) \to T_k(U) \text{ a.e. in } \Omega \text{ and } T_k(u_\varepsilon) \rightharpoonup T_k(U) \text{ (weakly) in } W_0^{1,\overrightarrow{p}}(\Omega), \\ &G_k(u_\varepsilon) \to G_k(U) \text{ a.e. in } \Omega \text{ and } G_k(u_\varepsilon) \rightharpoonup G_k(U) \text{ (weakly) in } W_0^{1,\overrightarrow{p}}(\Omega), \\ &G_k(u_\varepsilon) \to G_k(U) \text{ strongly in } L^r(\Omega) \text{ with } 1 \leq r < p^*. \end{aligned}$$

Hence, using the property (P_2), we derive that

$$\lim_{\varepsilon\to 0} \langle \mathfrak{B} u_\varepsilon, G_k(u_\varepsilon)\rangle = \langle \mathfrak{B} U, G_k(U)\rangle \quad \text{and} \quad \lim_{\varepsilon\to 0} \|G_k(u_\varepsilon)\|_{L^1(\Omega)} = \|G_k(U)\|_{L^1(\Omega)}.$$

Consequently, for every $1 \le j \le N$, we have

$$\limsup_{\varepsilon\to 0} \|\partial_j(G_k(u_\varepsilon))\|_{L^{p_j}(\Omega)} \le \left(\frac{|\langle \mathfrak{B}U, G_k(U)\rangle| + C_\Theta \|G_k(U)\|_{L^1(\Omega)}}{\nu_0}\right)^{1/p_j} = L_k^{1/p_j}.$$

This establishes the inequality in (6.3), completing the proof of Lemma 6.1. □

Lemma 6.2. *Up to a subsequence of $\{u_\varepsilon\}_\varepsilon$, relabeled $\{u_\varepsilon\}_\varepsilon$, we have* (6.1).

Proof. Recall that $\{u_\varepsilon\}_\varepsilon$ stands for a sequence $\{u_{\varepsilon_\ell}\}_{\ell\ge 1}$ with $\varepsilon_\ell \searrow 0$ as $\ell \to \infty$. By Lemmas 4.1 and 6.1, as well as from the proof of Lemma 4.2, we get that for any given integer $k \ge 1$, there exists a subsequence of $\{u_\varepsilon\}_\varepsilon$ that depends on k, say $\{u^{(k)}_{\varepsilon_\ell}\}_{\ell\ge1}$, for which (6.3) and (4.12) hold with $u^{(k)}_{\varepsilon_\ell}$ in place of $\{u_\varepsilon\}$. This means that

$$\begin{aligned} &\limsup_{\ell\to\infty} \|G_k(u^{(k)}_{\varepsilon_\ell})\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le \sum_{j=1}^N L_k^{1/p_j}, \\ &\lim_{\ell\to\infty} \|T_k\left(u^{(k)}_{\varepsilon_\ell}\right) - T_k(U)\|_{W_0^{1,\overrightarrow{p}}(\Omega)} = 0. \end{aligned} \tag{6.4}$$

We proceed inductively with respect to k, at each step $(k+1)$ selecting the subsequence $\{u^{(k+1)}_{\varepsilon_\ell}\}_{\ell\ge1}$ from $\{u^{(k)}_{\varepsilon_\ell}\}_{\ell\ge1}$, the subsequence of $\{u_\varepsilon\}$ with the properties in (6.4). Then, $\{u^{(\ell)}_{\varepsilon_\ell}\}_{\ell\ge k}$ is a subsequence of $\{u^{(j)}_{\varepsilon_\ell}\}_{\ell\ge1}$ for every $1 \le j \le k$. Hence, by a standard diagonal argument, there exists a subsequence of $\{u_\varepsilon\}_\varepsilon$, that is, $\{u^{(\ell)}_{\varepsilon_\ell}\}_\ell$, relabeled $\{u_\varepsilon\}_\varepsilon$, such that (6.3) and (4.12) hold for every $k \ge 1$, namely

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \|G_k(u_\varepsilon)\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le \sum_{j=1}^N L_k^{1/p_j}, \\ &\lim_{\varepsilon\to 0} \|T_k(u_\varepsilon) - T_k(U)\|_{W_0^{1,\overrightarrow{p}}(\Omega)} = 0. \end{aligned} \tag{6.5}$$

Using the weak convergence of $G_k(u_\varepsilon)$ to $G_k(U)$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ as $\varepsilon \to 0$, we see that

$$\|G_k(U)\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le \liminf_{\varepsilon\to 0} \|G_k(u_\varepsilon)\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le \sum_{j=1}^N L_k^{1/p_j}. \tag{6.6}$$

We now complete the proof of (6.1). From the definition of G_k in (4.8), we find that

$$\|u_\varepsilon - U\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le \|G_k(u_\varepsilon)\|_{W_0^{1,\overrightarrow{p}}(\Omega)} + \|G_k(U)\|_{W_0^{1,\overrightarrow{p}}(\Omega)} + \|T_k(u_\varepsilon) - T_k(U)\|_{W_0^{1,\overrightarrow{p}}(\Omega)}.$$

Then, in view of (6.5) and (6.6), for every $k \ge 1$, we obtain that

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon - U\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le 2\sum_{j=1}^N L_k^{1/p_j}. \tag{6.7}$$

Remark that L_k (defined in (6.2)) converges to 0 as $k \to \infty$ since $G_k(U) \rightharpoonup 0$ (weakly) in $W_0^{1,\overrightarrow{p}}(\Omega)$ and $G_k(U) \to 0$ (strongly) in $L^1(\Omega)$ as $k \to \infty$. Hence, by letting $k \to \infty$ in (6.7), we obtain (6.1). This ends the proof of Lemma 6.2. □

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