Classical $\mathcal{W}$-algebras for centralizers

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Abstract

We introduce a new family of Poisson vertex algebras $\mathcal{W}(\mathfrak{a})$ analogous to the classical $\mathcal{W}$-algebras. The algebra $\mathcal{W}(\mathfrak{a})$ is associated with the centralizer $\mathfrak{a}$ of an arbitrary nilpotent element in $\mathfrak{gl}_N$. We show that $\mathcal{W}(\mathfrak{a})$ is an algebra of polynomials in infinitely many variables and produce its free generators in an explicit form. This implies that $\mathcal{W}(\mathfrak{a})$ is isomorphic to the center at the critical level of the affine vertex algebra associated with $\mathfrak{a}$.

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1 Introduction

The *classical $\mathcal{W}$-algebras* associated with simple Lie algebras $\mathfrak{g}$ were introduced by Drinfeld and Sokolov [6] as Poisson algebras of functions on infinite-dimensional manifolds. A detailed review of the constructions of [6] and additional background of the theory can be found in a more recent work by De Sole, Kac and Valeri [3]. As shown in [3], the classical $\mathcal{W}$-algebras can be understood as Poisson vertex algebras and can be used to produce integrable hierarchies of bi-Hamiltonian equations.

The same framework of Poisson vertex algebras was used in our paper [11] to construct explicit generators of the classical $\mathcal{W}$-algebras $\mathcal{W}(\mathfrak{g})$ associated to principal nilpotent elements of simple Lie algebras $\mathfrak{g}$ in types A, B, C, D and G. The generators are given in a uniform way as certain determinants of matrices formed by elements of differential algebras associated with $\mathfrak{g}$. Another approach involving generalized quasideterminants was developed in [4] to describe generators of the classical $\mathcal{W}$-algebras in type A associated with arbitrary nilpotent elements of $\mathfrak{g}$; see also [5] for extensions of this approach to other classical Lie algebras.

The principal classical $\mathcal{W}$-algebras also emerge from a different perspective. Due to a theorem of Feigin and Frenkel [7], [8, Ch. 4], the Poisson algebra $\mathcal{W}({}^L\mathfrak{g})$ associated to the Langlands dual Lie algebra ${}^L\mathfrak{g}$ is isomorphic to the center at the critical level of the affine vertex algebra $V(\mathfrak{g})$; see also [9, Ch. 13] for a correspondence of generators in the classical types. In a recent work of Arakawa and Premet [1] a description of the center at the critical level of the affine vertex algebra $V(\mathfrak{a})$ was given for a family of centralizers $\mathfrak{a} = \mathfrak{g}^e$ of certain nilpotent elements $e \in \mathfrak{g}$. Explicit generators of the center in type A were constructed in [10].

Our goal in this paper is to introduce analogues of the classical $\mathcal{W}$-algebras for the centralizers $\mathfrak{a} = \mathfrak{g}^e$ in the case $\mathfrak{g} = \mathfrak{gl}_N$. Our classical $\mathcal{W}$-algebras $\mathcal{W}(\mathfrak{a})$ turn out to fit the general framework of Poisson vertex algebras of [3] by possessing a λ-bracket. We prove that $\mathcal{W}(\mathfrak{a})$ is an algebra of polynomials in infinitely many variables and produce a family of its free generators in an explicit form. Moreover, we show that a Miura-type map leads to a natural isomorphism between the algebra $\mathcal{W}(\mathfrak{a})$ and the center of the affine vertex algebra $V(\mathfrak{a})$ at the critical level.

2 Definition of $\mathcal{W}(\mathfrak{a})$

Suppose that $e \in \mathfrak{g} = \mathfrak{gl}_N$ is a nilpotent matrix with Jordan blocks of sizes $\lambda_1, \dots, \lambda_n$, where $\lambda_1 \leqslant \dots \leqslant \lambda_n$ and $\lambda_1 + \dots + \lambda_n = N$. The centralizer $\mathfrak{a} = \mathfrak{g}^e$ is a Lie algebra with the basis elements $E_{ij}^{(r)}$, where the range of indices is defined by the inequalities $1 \leqslant i, j \leqslant n$ and $\lambda_j - \min(\lambda_i, \lambda_j) \leqslant r < \lambda_j$, with the commutation relations

$$\big[E_{ij}^{(r)}, E_{kl}^{(s)}\big] = \delta_{kj}\, E_{il}^{(r+s)} - \delta_{il}\, E_{kj}^{(r+s)},$$

assuming that $E_{ij}^{(r)} = 0$ for $r \geqslant \lambda_j$. The formulas expressing $E_{ij}^{(r)}$ in terms of the basis elements of $\mathfrak{gl}_N$ can be found e.g. in [2] and [12]. It is clear from the relations that in the particular case $\lambda_1 = \dots = \lambda_n = p$ the Lie algebra $\mathfrak{a}$ is isomorphic to the truncated polynomial current algebra $\mathfrak{gl}_n[x]/(x^p = 0)$ (also known as the *Takiff algebra*).

Equip the Lie algebra $\mathfrak{a}$ with the invariant symmetric bilinear form $(\,|\,)$ defined by

$$(E_{ij}^{(0)}\,|\,E_{ji}^{(0)}) = \lambda_i, \tag{2.1}$$

whereas all remaining values of the form on the basis vectors are zero. It is understood that if $i \neq j$ then relation (2.1) occurs only if $\lambda_i = \lambda_j$.

Consider the *differential algebra* $\mathcal{V}(\mathfrak{a})$ which is defined as the algebra of differential polynomials in the variables $E_{ij}^{(r)}[s]$, where $s = 0, 1, 2, \dots$, while $E_{ij}^{(r)}$ ranges over the basis elements of $\mathfrak{a}$, equipped with the derivation ∂ defined by $\partial(E_{ij}^{(r)}[s]) = E_{ij}^{(r)}[s+1]$. We will regard $\mathfrak{a}$ as a subspace of $\mathcal{V}(\mathfrak{a})$ by using the embedding $X \mapsto X[0]$ for $X \in \mathfrak{a}$.

Introduce the *λ-bracket* on $\mathcal{V}(\mathfrak{a})$ as a linear map

$$\mathcal{V}(\mathfrak{a}) \otimes \mathcal{V}(\mathfrak{a}) \to \mathbb{C}[\lambda] \otimes \mathcal{V}(\mathfrak{a}), \qquad a \otimes b \mapsto \{a_{\lambda}\, b\}.$$

By definition, it is given by

$$\{X_{\lambda}\, Y\} = [X, Y] + (X|Y)\,\lambda \qquad \text{for} \quad X, Y \in \mathfrak{a},$$

and extended to $\mathcal{V}(\mathfrak{a})$ by sesquilinearity $(a, b \in \mathcal{V}(\mathfrak{a}))$:

$$\{\partial a_{\lambda}\, b\} = -\lambda\ \{a_{\lambda}\, b\}, \qquad \{a_{\lambda}\, \partial b\} = (\lambda + \partial)\{a_{\lambda}\, b\},$$

skewsymmetry

$$\{a_{\lambda}\, b\} = -\{b_{-\lambda-\partial}\, a\},$$

and the Leibniz rule $(a, b, c \in \mathcal{V}(\mathfrak{a}))$:

$$\{a_{\lambda}\, bc\} = \{a_{\lambda}\, b\}c + \{a_{\lambda}\, c\}b.$$

The λ-bracket defines the *affine Poisson vertex algebra* structure on $\mathcal{V}$; see [3].

Consider the following triangular decomposition of the Lie algebra $\mathfrak{a}$,

$$\mathfrak{a} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{2.2}$$

where the subalgebras are defined by

$$\mathfrak{n}_- = \text{span of } \{E_{ij}^{(r)}\,|\, i > j\}, \quad \mathfrak{n}_+ = \text{span of } \{E_{ij}^{(r)}\,|\, i < j\} \quad \text{and} \quad \mathfrak{h} = \text{span of } \{E_{ii}^{(r)}\},$$

with the superscript r ranging over all admissible values. Set $\mathfrak{p} = \mathfrak{n}_- \oplus \mathfrak{h}$ and define the projection map $\pi_{\mathfrak{p}} : \mathfrak{a} \to \mathfrak{p}$ with the kernel $\mathfrak{n}_+$. We regard $\mathcal{V}(\mathfrak{p})$ as a natural differential subalgebra of $\mathcal{V}(\mathfrak{a})$ and define the differential algebra homomorphism

$$\rho : \mathcal{V}(\mathfrak{a}) \to \mathcal{V}(\mathfrak{p})$$

as the identity map on all elements of $\mathfrak{p} \subset \mathfrak{a}$, while for the basis elements of $\mathfrak{n}_+$ we set

$$\rho(E_{ij}^{(r)}) = \begin{cases} 1 & \text{if} \quad j = i + 1 \quad \text{and} \quad r = \lambda_{i+1} - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Definition 2.1. The *classical $\mathcal{W}$-algebra* $\mathcal{W}(\mathfrak{a})$ associated with $\mathfrak{a}$ is defined by

$$\mathcal{W}(\mathfrak{a}) = \{P \in \mathcal{V}(\mathfrak{p}) \mid \rho\{X_\lambda P\} = 0 \quad \text{for all} \quad X \in \mathfrak{n}_+\}.$$

In the case $e = 0$ this agrees with the standard terminology; cf. [3].

Remark 2.2. One can define a more general family of classical $\mathcal{W}$-algebras $\mathcal{W}_A(\mathfrak{a})$ by altering the map ρ and making it dependent on a chosen set of polynomials $A_1(u), \dots, A_{n-1}(u)$ of the form

$$A_i(u) = A_i^{(\lambda_{i+1}-\lambda_i)} u^{\lambda_{i+1}-\lambda_i} + \dots + A_i^{(\lambda_{i+1}-1)} u^{\lambda_{i+1}-1}$$

for $i = 1, \dots, n-1$. Instead of (2.3) one then takes

$$\rho(E_{ij}^{(r)}) = \begin{cases} A_i^{(r)} & \text{if} \quad j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

However, it will follow from the arguments below (see the proof of Theorem 3.1) that all $\mathcal{W}$-algebras $\mathcal{W}_A(\mathfrak{a})$ are isomorphic to each other provided that the leading coefficient of each polynomial $A_i(u)$ is nonzero. For that reason we will stick to the particular choice of such polynomials $A_i(u) = u^{\lambda_{i+1}-1}$ as given in (2.3). □

The following properties are verified by using the same arguments as in the case $e = 0$; see [3, Sec. 3.2].

Lemma 2.3. (i) *The mapping*

$$(a, g) \mapsto \rho\{a_\lambda g\}, \qquad a \in \mathfrak{n}_+, \quad g \in \mathcal{V}(\mathfrak{p})$$

defines a representation of the Lie conformal algebra $\mathbb{C}[\partial]\,\mathfrak{n}_+$ on $\mathcal{V}(\mathfrak{p})$.

(ii) *The Lie conformal algebra $\mathbb{C}[\partial]\,\mathfrak{n}_+$ acts on $\mathcal{V}(\mathfrak{p})$ by conformal derivations so that*

$$\rho\{a_\lambda gh\} = \rho\{a_\lambda g\}h + \rho\{a_\lambda h\}g$$

for all $a \in \mathfrak{n}_+$ and $g, h \in \mathcal{V}(\mathfrak{p})$. □

Proposition 2.4. *The subspace $\mathcal{W}(\mathfrak{a}) \subset \mathcal{V}(\mathfrak{p})$ is a differential subalgebra. Moreover, $\mathcal{W}(\mathfrak{a})$ is a Poisson vertex algebra equipped with the λ-bracket*

$$(a, b) \mapsto \rho\{a_\lambda b\}, \qquad a, b \in \mathcal{W}(\mathfrak{a}).$$

□

3 Generators of $\mathcal{W}(\mathfrak{a})$

For an $n \times n$ matrix $A = [a_{ij}]$ with entries in a ring we will consider its *column-determinant* defined by

$$\operatorname{cdet} A = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn} \sigma \cdot a_{\sigma(1)\,1} \dots a_{\sigma(n)\,n}.$$

Equip the tensor product space $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[d_1, \dots, d_n]$ with an algebra structure such that the additional elements $d_1, \dots, d_n$ pairwise commute and satisfy the commutation relations

$$\big[d_i, X[s]\big] = \lambda_i \, X[s+1], \qquad X \in \mathfrak{p}. \tag{3.4}$$

Recall that the λ_i form an increasing sequence and combine the basis elements of the Lie algebra $\mathfrak{a}$ into polynomials in a variable u by

$$E_{ij}(u) = \begin{cases} E_{ij}^{(0)} + \dots + E_{ij}^{(\lambda_j - 1)} \, u^{\lambda_j - 1} & \text{if} \quad i \geqslant j, \\ E_{ij}^{(\lambda_j - \lambda_i)} \, u^{\lambda_j - \lambda_i} + \dots + E_{ij}^{(\lambda_j - 1)} \, u^{\lambda_j - 1} & \text{if} \quad i < j. \end{cases}$$

Write the column-determinant

$$D_n(u) = \text{cdet} \begin{bmatrix} d_1 + E_{11}(u) & u^{\lambda_2 - 1} & 0 & 0 & \dots & 0 \\ E_{21}(u) & d_2 + E_{22}(u) & u^{\lambda_3 - 1} & 0 & \dots & 0 \\ E_{31}(u) & E_{32}(u) & d_3 + E_{33}(u) & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & u^{\lambda_n - 1} \\ E_{n1}(u) & E_{n2}(u) & E_{n3}(u) & \dots & \dots & d_n + E_{nn}(u) \end{bmatrix}$$

as an element of $\mathcal{V}(\mathfrak{p})[u] \otimes \mathbb{C}[d_1, \dots, d_n]$ by moving the elements d_i in any monomial to the right with the use of relations (3.4). Furthermore, take the image of this element under the linear map $\mathcal{V}(\mathfrak{p})[u] \otimes \mathbb{C}[d_1, \dots, d_n] \mapsto \mathcal{V}(\mathfrak{p})[u] \otimes \mathbb{C}[d]$ which is the identity on $\mathcal{V}(\mathfrak{p})[u]$ and sends each d_i to d, where d is understood as a formal variable. The image of the column-determinant is a polynomial in d of the form

$$d^n + w_1(u) \, d^{n-1} + \dots + w_n(u), \qquad w_k(u) = \sum_r w_k^{(r)} \, u^r, \qquad w_k^{(r)} \in \mathcal{V}(\mathfrak{p}).$$

The following theorem is our main result. Its particular case for the element $e = 0$ (that is, with $\lambda_1 = \dots = \lambda_n = 1$) is [11, Thm 3.2].

Theorem 3.1. *All elements $w_k^{(r)}$ with $k = 1, \dots, n$ and*

$$\lambda_{n-k+2} + \dots + \lambda_n < r + k \leqslant \lambda_{n-k+1} + \dots + \lambda_n \tag{3.5}$$

belong to the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{a})$. Moreover, under these conditions the elements $\partial^s w_k^{(r)}$ with $s = 0, 1, \dots$ are algebraically independent and generate the algebra $\mathcal{W}(\mathfrak{a})$.

Example 3.2. For $n = 2$ we have

$$w_1(u) = E_{11}(u) + E_{22}(u), \qquad w_2(u) = E_{11}(u) E_{22}(u) - u^{\lambda_2 - 1} E_{21}(u) + \lambda_1 \partial E_{22}(u),$$

so that

$$w_1^{(r)} = E_{11}^{(r)} + E_{22}^{(r)}, \qquad r = 0, 1, \dots, \lambda_2 - 1,$$

and

$$w_2^{(r)} = \sum_{a+b=r} E_{11}^{(a)} E_{22}^{(b)} - E_{21}^{(r - \lambda_2 + 1)} + \lambda_1 E_{22}^{(r)}[1]$$

for $r = \lambda_2 - 1, \dots, \lambda_1 + \lambda_2 - 2$. □

Proof of Theorem 3.1. Due to Lemma 2.3 (i), the first part of the theorem will follow if we show that for $i = 1, \dots, n-1$ the relations

$$\rho\{E^{(t)}_{i\,i+1\,\lambda}\, w^{(r)}_k\} = 0$$

hold for all $t = \lambda_{i+1} - \lambda_i, \dots, \lambda_{i+1} - 1$, assuming the conditions on the values of k and r as stated in the theorem. Extend the $\boldsymbol{\lambda}$-bracket and the representation of the Lie conformal algebra $\mathbb{C}[\partial]\,\mathfrak{n}_+$ on $\mathcal{V}(\mathfrak{p})$ to the space $\mathcal{V}(\mathfrak{p})\otimes\mathbb{C}[d_1, \dots, d_n]$ in such a way that the action on the component $\mathbb{C}[d_1, \dots, d_n]$ is trivial. Then for any $X \in \mathfrak{n}_+$ and $Y \in \mathcal{V}(\mathfrak{p})$ we have the relations

$$\{X_{\,\boldsymbol{\lambda}}\, d_i Y\} = (d_i + \boldsymbol{\lambda}\lambda_i)\{X_{\,\boldsymbol{\lambda}}\, Y\}.$$

We will set $d_i^+ = d_i + \boldsymbol{\lambda}\lambda_i$ for brevity. For all $k \geqslant l$ we have

$$\{E^{(t)}_{i\,i+1\,\lambda}\, E_{k\,l}(u)\} = \delta_{k\,i+1}\, E^t_{i\,l}(u) - \delta_{i\,l}\, E^t_{k\,i+1}(u) + \delta_{t\,0}\,\delta_{k\,i+1}\,\delta_{i\,l}\,\boldsymbol{\lambda}\lambda_i,$$

where we set

$$E^t_{ij}(u) = E^{(t)}_{ij} + \dots + E^{(\lambda_j - 1)}_{ij}\, u^{\lambda_j - t - 1}.$$

By using Lemma 2.3 (ii) and taking the $\boldsymbol{\lambda}$-bracket of $E^{(t)}_{i\,i+1}$ with the elements of consecutive columns of the column-determinant $D_n(u)$, we get

$$\rho\{E^{(t)}_{i\,i+1\,\lambda}\, D_n(u)\} = \sum_{a=1}^{i+1} D_{n\,a}(u),$$

where $D_{n\,a}(u)$ is the column-determinant obtained from $D_n(u)$ as follows. The variables d_c should be replaced by the rule $d_c \mapsto d_c^+$ for $c = 1, \dots, a-1$ and column a in $D_n(u)$ should be replaced by a new column which we now describe.

For $a = 1, \dots, i-1$ we have

$$D_{n\,a}(u) = \text{cdet}\begin{bmatrix} d_1^+ + E_{11}(u) & u^{\lambda_2 - 1} & \dots & 0 & \dots & 0 \\ E_{21}(u) & d_2^+ + E_{22}(u) & \dots & 0 & \dots & 0 \\ E_{31}(u) & E_{32}(u) & \dots & \vdots & \dots & 0 \\ \vdots & \vdots & \dots & E^t_{i\,a}(u) & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots & \dots & u^{\lambda_n - 1} \\ E_{n1}(u) & E_{n2}(u) & \dots & 0 & \dots & d_n + E_{nn}(u) \end{bmatrix},$$

where the only nonzero entry $E^t_{i\,a}(u)$ in column a occurs in row $i+1$.

The column-determinant $D_{n\,i+1}(u)$ has the same form, except that the only nonzero entry in column $i+1$ occurring in row $i+1$ equals $u^{\lambda_{i+1} - t - 1}$.

In the remaining column-determinant $D_{n\,i}(u)$ we only display columns i and $i+1$ which have the form

$$\mathrm{cdet}\begin{bmatrix} \dots & 0 & 0 & \dots \\ \dots & \vdots & \vdots & \dots \\ \dots & -u^{\lambda_{i+1}-t-1} & u^{\lambda_{i+1}-1} & \dots \\ \dots & E^{t}_{ii}(u)-E^{t}_{i+1\,i+1}(u)+\delta_{t0}\,\boldsymbol{\lambda}\,\lambda_i & d_{i+1}+E_{i+1\,i+1}(u) & \dots \\ \dots & -E^{t}_{i+2\,i+1}(u) & E_{i+2\,i+1}(u) & \dots \\ \dots & \vdots & \vdots & \dots \\ \dots & -E^{t}_{n\,i+1}(u) & E_{n\,i+1}(u) & \dots \end{bmatrix}.$$

By expanding $D_{n\,i}(u)$ along columns i and $i+1$ simultaneously, we find that a nonzero contribution from this expansion can come only from the 2×2 column-minors with entries in rows i and b with $b \geqslant i+1$. For $b > i+1$ such a minor equals

$$-\,u^{\lambda_{i+1}-t-1}\,E_{b\,i+1}(u)+u^{\lambda_{i+1}-1}\,E^{t}_{b\,i+1}(u) \tag{3.6}$$

which is zero for $t=0$ and is equal to a certain polynomial $C_b(u)$ in u of degree $\lambda_{i+1}-2$ for $t>0$. In the latter case the contribution to $D_{n\,i}(u)$ arising from this 2×2 column-minor equals an alternating sum of products of the form

$$\begin{aligned}\Big(\delta_{\sigma(1)\,1}\,d^{+}_{1}+E_{\sigma(1)\,1}(u)\Big)\dots\Big(\delta_{\sigma(i-1)\,i-1}\,d^{+}_{i-1}+E_{\sigma(i-1)\,i-1}(u)\Big)\,C_b(u)\\ \times\Big(\delta_{\sigma(i+2)\,i+2}\,d_{i+2}+E_{\sigma(i+2)\,i+2}(u)\Big)\dots\Big(\delta_{\sigma(n)\,n}\,d_n+E_{\sigma(n)\,n}(u)\Big)\end{aligned}$$

with $\sigma\in\mathfrak{S}_n$, where each occurrence of $E_{j\,j+1}(u)$ amongst $E_{\sigma(l)\,l}(u)$ should be replaced with $u^{\lambda_{j+1}-1}$, and we assume that $E_{j\,l}(u)=0$ for $l>j+1$. Write such a product as an element of $\mathcal{V}(\mathfrak{p})[u]\otimes\mathbb{C}[d_1,\dots,d_n]$ and take its image under the linear map $\mathcal{V}(\mathfrak{p})[u]\otimes\mathbb{C}[d_1,\dots,d_n]\mapsto\mathcal{V}(\mathfrak{p})[u]\otimes\mathbb{C}[d]$. Since the u-degree of $E_{j\,l}(u)$ equals λ_l-1, the resulting expression will be a polynomial in d such that the coefficient of d^{n-k} is a polynomial in u whose degree does not exceed

$$(\lambda_{n-k+1}-1)+\dots+(\lambda_{i-1}-1)+(\lambda_{i+1}-2)+(\lambda_{i+2}-1)+\dots+(\lambda_n-1)$$

if $n-k\leqslant i-1$, and does not exceed

$$(\lambda_{i+1}-2)+(\lambda_{n-k+3}-1)+\dots+(\lambda_n-1)$$

if $n-k\geqslant i$. In both bases the degree in u is less than

$$(\lambda_{n-k+2}-1)+\dots+(\lambda_n-1) \tag{3.7}$$

and so the contribution of such 2×2 column-minors to the expression $\rho\{E^{(t)}_{i\,i+1\,\lambda}\,w^{(r)}_{k}\}$ is equal to zero because by the conditions of the theorem the minimal possible value of r is given by (3.7).

Furthermore, we have

$$D_{n\,i+1}(u) = D^{+}_{1,\dots,i}(u)\, u^{\lambda_{i+1}-t-1}\, D_{i+2,\dots,n}(u),$$

where for a subset $K \subset \{1,\dots,n\}$ we denote by $D_K(u)$ the principal column-minor of $D_n(u)$ corresponding to the rows and columns labelled by the elements of K, while $D^{+}_K(u)$ is obtained from $D_K(u)$ by replacing all elements d_a with d^{+}_a. Expand $D^{+}_{1,\dots,i}(u)$ along the last row to get

$$D^{+}_{1,\dots,i}(u) = \sum_{a=1}^{i-1} (-1)^{a+i}\, D^{+}_{1,\dots,a-1}(u)\, u^{\lambda_{a+1}-1} \dots u^{\lambda_i-1}\, E_{i\,a}(u) + D^{+}_{1,\dots,i-1}(u)\, \big(d^{+}_i + E_{ii}(u)\big).$$

This shows that if $t > 0$ then $\rho\{E^{(t)}_{i\,i+1\,\lambda}\, D_n(u)\}$ can be written as the expression

$$\begin{aligned} &\sum_{a=1}^{i} (-1)^{a+i}\, D^{+}_{1,\dots,a-1}(u)\, u^{\lambda_{a+1}-1} \dots u^{\lambda_i-1} \Big(u^{\lambda_{i+1}-t-1}\, E_{i\,a}(u) - u^{\lambda_{i+1}-1}\, E^{t}_{i\,a}(u)\Big)\, D_{i+2,\dots,n}(u) \\ &\quad - D^{+}_{1,\dots,i-1}(u) \left(\Big(u^{\lambda_{i+1}-t-1}\, E_{i+1\,i+1}(u) - u^{\lambda_{i+1}-1}\, E^{t}_{i+1\,i+1}(u)\Big) - u^{\lambda_{i+1}-t-1}\, (d^{+}_i - d_{i+1})\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times D_{i+2,\dots,n}(u), \end{aligned}$$

while for the value $t = 0$ occurring in the case $\lambda_i = \lambda_{i+1}$ we have

$$\rho\{E^{(0)}_{i\,i+1\,\lambda}\, D_n(u)\} = D^{+}_{1,\dots,i-1}(u)\, u^{\lambda_{i+1}-1}\, (d_i - d_{i+1})\, D_{i+2,\dots,n}(u).$$

Write the expression $\rho\{E^{(t)}_{i\,i+1\,\lambda}\, D_n(u)\}$ as an element of $\mathcal{V}(\mathfrak{p})[u] \otimes \mathbb{C}[d_1,\dots,d_n]$ and take its image under the linear map $\mathcal{V}(\mathfrak{p})[u] \otimes \mathbb{C}[d_1,\dots,d_n] \mapsto \mathcal{V}(\mathfrak{p})[u] \otimes \mathbb{C}[d]$. Similar to the argument above with elements (3.6), we can see that the image is a polynomial in d such that the coefficient of d^{n-k} is a polynomial in u whose degree is less than the number in (3.7). This completes the proof of the first part of the theorem.

As a next step, we will show that the elements $\partial^s w^{(r)}_k$ are algebraically independent. We will adapt the corresponding argument used in the proof of [3, Theorem 3.14] and introduce the differential polynomial degree on $\mathcal{V}(\mathfrak{p})$ by setting the degree of $X[s]$ to be equal to $s+1$ for any nonzero $X \in \mathfrak{p}$. The minimal degree components of the elements $w^{(r)}_k$ with $r = q + \lambda_{n-k+2} + \dots + \lambda_n - k + 1$ are given by the formulas

$$w^{(r)}_k = (-1)^{k-1} \sum_{p=k}^{n} E^{(q+\lambda_n-\lambda_{n-k+1}+\dots+\lambda_{p+1}-\lambda_{p-k+2})}_{p\,p-k+1} + \text{ higher degree terms}$$

for $q = 0, 1, \dots, \lambda_{n-k+1} - 1$. By applying s times the derivation ∂ to both sides we get the respective minimal degree components of the elements $\partial^s w^{(r)}_k$. Their algebraic independence now follows from the observation that all elements of the form

$$\sum_{p=k}^{n} E^{(q+\lambda_n-\lambda_{n-k+1}+\dots+\lambda_{p+1}-\lambda_{p-k+2})}_{p\,p-k+1}[s]$$

with $s = 0, 1, \dots$ are algebraically independent.

Finally, we will show that the algebra $\mathcal{W}(\mathfrak{a})$ is generated by the elements $\partial^s w_k^{(r)}$. Expanding $D_n(u)$ along the last row we get the expression

$$D_n(u) = D_{1,\dots,n-1}(u)\Big(d_n + E_{nn}(u)\Big) + \sum_{k=2}^{n} (-1)^{k-1} D_{1,\dots,n-k}(u)\, u^{\lambda_{n-k+2}-1} \dots u^{\lambda_n - 1}\, E_{n\,n-k+1}(u).$$

Hence for $k = 1, \dots, n$ and $r = q + \lambda_{n-k+2} + \dots + \lambda_n - k + 1$ we can write

$$w_k^{(r)} = (-1)^{k-1} E_{n\,n-k+1}^{(q)} + R, \qquad q = 0, 1, \dots, \lambda_{n-k+1} - 1, \tag{3.8}$$

where R is a polynomial in the variables of the form $E_{nm}^{(p)}[s]$ with $m = n-k+2, \dots, n$ and the variables $E_{lm}^{(p)}[s]$ with $n > l \geqslant m \geqslant 1$.

Now suppose that an element $P \in \mathcal{V}(\mathfrak{p})$ belongs to the subalgebra $\mathcal{W}(\mathfrak{a}) \subset \mathcal{V}(\mathfrak{p})$. Applying relations (3.8) consecutively with $k = n, n-1, \dots, 1$ we can write P as a polynomial in the new variables $\partial^s w_k^{(r)}$ with $k = 1, \dots, n$ and $s \geqslant 0$ satisfying conditions (3.5), together with $E_{lm}^{(p)}[s]$ with $n > l \geqslant m \geqslant 1$. Take $X = E_{in}^{(\lambda_n - t - 1)}$ for $i \in \{1, \dots, n-1\}$ and $t \in \{0, 1, \dots, \lambda_i - 1\}$ in Definition 2.1 and observe that the images of the $\boldsymbol{\lambda}$-brackets of X with the new variables under the map ρ are equal to zero, except for the variables $E_{n-1\,i}^{(t)}[s]$. Since

$$\{E_{in}^{(\lambda_n - t - 1)}\,{}_{\boldsymbol{\lambda}}\, E_{n-1\,i}^{(t)}[s]\} = -(\boldsymbol{\lambda} + \partial)^s\, E_{n-1\,n}^{(\lambda_n - 1)},$$

by applying Lemma 2.3 (ii) we find that the property $\rho\{X_{\boldsymbol{\lambda}}\, P\} = 0$ implies the relation

$$\sum_{s \geqslant 0} \boldsymbol{\lambda}^s \frac{\partial P}{\partial E_{n-1\,i}^{(t)}[s]} = 0$$

which means that P does not depend on the variables $E_{n-1\,i}^{(t)}[s]$ with $i = 1, \dots, n-1$. Repeating the same argument for the elements $X \in \mathfrak{n}_+$ of the form $X = E_{ij}^{(\lambda_j - t - 1)}$ with $j = n-1, \dots, 2$ and $i \in \{1, \dots, j-1\}$, we may conclude that P does not depend on the variables $E_{lm}^{(p)}[s]$ with $n > l \geqslant m \geqslant 1$ so that P is a polynomial in the variables $\partial^s w_k^{(r)}$, as required. $\square$

4 Miura map and center at the critical level

Here we will show that the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{a})$ is isomorphic to the center at the critical level of the affine vertex algebra $V(\mathfrak{a})$; cf. [7], [8, Ch. 4]. We will do this by relying on the work [10] and providing an explicit correspondence between the generators of both algebras. We will use a Miura-type map on the $\mathcal{W}$-algebra side and a Harish-Chandra-type isomorphism on the vertex algebra side.

Denote by $\mathcal{V}(\mathfrak{h})$ the differential subalgebra of $\mathcal{V}(\mathfrak{p})$, generated by the elements $E_{ii}^{(r)}$ with $i = 1, \dots, n$ and $r = 0, \dots, \lambda_i - 1$. Let

$$\varphi : \mathcal{V}(\mathfrak{p}) \to \mathcal{V}(\mathfrak{h})$$

denote the homomorphism of differential algebras defined on the generators as the projection $\mathfrak{p} \to \mathfrak{h}$ with the kernel $\mathfrak{n}_-$.

Proposition 4.1. *The restriction of the homomorphism φ to the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{a})$ is injective.*

Proof. It will be sufficient to verify that the images of all generators $\partial^s w_k^{(r)} \in \mathcal{W}(\mathfrak{a})$ with $k = 1, \dots, n$ and $s \geqslant 0$ satisfying conditions (3.5) are algebraically independent in $\mathcal{V}(\mathfrak{h})$. The images $\overline{w}_k^{(r)} = \varphi(w_k^{(r)})$ are found by writing the product

$$\big(d_1 + E_{11}(u)\big) \dots \big(d_n + E_{nn}(u)\big)$$

as an element of the vector space $\mathcal{V}(\mathfrak{h})[u] \otimes \mathbb{C}[d_1, \dots, d_n]$ and taking its image under the linear map $\mathcal{V}(\mathfrak{h})[u] \otimes \mathbb{C}[d_1, \dots, d_n] \mapsto \mathcal{V}(\mathfrak{h})[u] \otimes \mathbb{C}[d]$ which is the identity on $\mathcal{V}(\mathfrak{h})[u]$ and sends each d_k to d. The image is a polynomial in d of the form

$$d^n + \overline{w}_1(u)\, d^{n-1} + \dots + \overline{w}_n(u), \qquad \overline{w}_k(u) = \sum_r \overline{w}_k^{(r)}\, u^r.$$

Introduce a grading on $\mathcal{V}(\mathfrak{h})$ by setting the degree of $X[s]$ to be equal to s for any nonzero $X \in \mathfrak{h}$. The proposition will follow if we show that the minimal degree components $\partial^s v_k^{(r)}$ of the respective elements $\partial^s \overline{w}_k^{(r)}$ are algebraically independent. These components are found by the formulas

$$\partial^s v_k^{(r)} = \sum_{i_1 < \dots < i_k} \ \sum_{r_1 + \dots + r_k = r} \ \sum_{s_1 + \dots + s_k = s} \frac{s!}{s_1! \dots s_k!}\, E_{i_1 i_1}^{(r_1)}[s_1] \dots E_{i_k i_k}^{(r_k)}[s_k]. \tag{4.9}$$

We will verify that the differentials of these polynomials are linearly independent. Introduce particular orderings on the set of the variables $E_{ii}^{(r)}[s]$ and on the set of polynomials $\partial^s v_k^{(r)}$ as follows. First, if $s < s'$ then we set $E_{ii}^{(r)}[s] \prec E_{jj}^{(p)}[s']$ and $\partial^s v_k^{(r)} \prec \partial^{s'} v_l^{(p)}$ for all admissible values of the remaining parameters. Formulas (4.9) imply that if the orderings within the sets of elements with a given s are chosen consistently for different values of s, then the Jacobian matrix will have a block-diagonal form with identical diagonal blocks. Therefore, it will be sufficient to consider the Jacobian matrix corresponding to the elements with $s = 0$. In this case formula (4.9) gives

$$v_k^{(r)} = \sum_{i_1 < \dots < i_k} \ \sum_{r_1 + \dots + r_k = r} E_{i_1 i_1}^{(r_1)} \dots E_{i_k i_k}^{(r_k)}. \tag{4.10}$$

Now list the variables in a particular order:

$$E_{nn}^{(\lambda_n - 1)}, \dots, E_{11}^{(\lambda_1 - 1)}, E_{nn}^{(\lambda_n - 2)}, \dots, E_{11}^{(\lambda_1 - 2)}, \dots, E_{nn}^{(0)}, \dots, E_{11}^{(\lambda_1 - \lambda_n)},$$

assuming that any variables with negative superscripts are excluded from the list. This means, in particular, that the last segment of the sequence has the form $E_{nn}^{(0)}, \dots, E_{l+1\, l+1}^{(0)}$, if for certain index l we have $\lambda_l < \lambda_{l+1} = \dots = \lambda_n$. Similarly, list the polynomials by

$$v_1^{(\lambda_n - 1)}, \dots, v_n^{(\lambda_n + \dots + \lambda_1 - n)}, v_1^{(\lambda_n - 2)}, \dots, v_n^{(\lambda_n + \dots + \lambda_1 - n - 1)}, \dots, v_1^{(0)}, \dots, v_n^{(\lambda_{n-1} + \dots + \lambda_1 - n + 1)},$$

where a polynomial $v_k^{(r)}$ is excluded from the list if conditions (3.5) do not hold. Note that the corresponding Jacobian matrix is square of size $N = \lambda_1 + \dots + \lambda_n$. Introduce multiplicities $q_1, \dots, q_l$ of the decreasing sequence $(\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$ by

$$\lambda_n = \dots = \lambda_{n-q_1+1} > \lambda_{n-q_1} = \dots = \lambda_{n-q_1-q_2+1} > \dots > \lambda_{q_l} = \dots = \lambda_1,$$

so that the q_i are positive integers with $q_1 + \dots + q_l = n$. First consider the upper left block of the Jacobian matrix of size $n \times n$. By formulas (4.10), the elements

$$v_1^{(\lambda_n - 1)}, \dots, v_{q_1}^{(\lambda_n + \dots + \lambda_{n-q_1+1} - q_1)}$$

are the respective elementary symmetric polynomials in the variables

$$E_{n\,n}^{(\lambda_n - 1)}, \dots, E_{n-q_1+1\, n-q_1+1}^{(\lambda_{n-q_1+1} - 1)}.$$

It is well-known that the corresponding $q_1 \times q_1$ Jacobian matrix is non-degenerate which is equivalent to the algebraic independence of the elementary symmetric polynomials. Similarly, each of the next q_2 elements

$$v_{q_1+1}^{(\lambda_n + \dots + \lambda_{n-q_1} - q_1 - 1)}, \dots, v_{q_1+q_2}^{(\lambda_n + \dots + \lambda_{n-q_1-q_2+1} - q_1 - q_2)}$$

is the respective elementary symmetric polynomial in the variables

$$E_{n-q_1\, n-q_1}^{(\lambda_{n-q_1} - 1)}, \dots, E_{n-q_1-q_2+1\, n-q_1-q_2+1}^{(\lambda_{n-q_1-q_2+1} - 1)}$$

multiplied by the product $E_{n\,n}^{(\lambda_n - 1)} \dots E_{n-q_1+1\, n-q_1+1}^{(\lambda_{n-q_1+1} - 1)}$. Therefore, the next $q_2 \times q_2$ diagonal block of the Jacobian matrix is also non-degenerate. A similar structure is clearly retained by all remaining $q_i \times q_i$ diagonal blocks. In particular, each of the last q_l elements

$$v_{n-q_l+1}^{(\lambda_n + \dots + \lambda_{q_l} - n + q_l - 1)}, \dots, v_n^{(\lambda_n + \dots + \lambda_1 - n)}$$

is the respective elementary symmetric polynomial in the variables $E_{q_l q_l}^{(\lambda_{q_l} - 1)}, \dots, E_{1\,1}^{(\lambda_1 - 1)}$ multiplied by the product of the first $n - q_l$ variables $E_{n\,n}^{(\lambda_n - 1)} \dots E_{q_l+1\, q_l+1}^{(\lambda_{q_l+1} - 1)}$.

By formulas (4.10), each of the next $\lambda_1 - 1$ diagonal $n \times n$ blocks of the Jacobian matrix exhibits a similar structure. Namely, for each $i = 1, \dots, l$ their $q_i \times q_i$ diagonal sub-blocks coincide with the respective $q_i \times q_i$ sub-block in the first block of size $n \times n$.

As a next step, consider the submatrix of the Jacobian matrix of size $(n - q_l) \times (n - q_l)$ corresponding to the subsequence of variables

$$E_{n\,n}^{(\lambda_n - \lambda_1 - 1)}, \dots, E_{q_l+1\, q_l+1}^{(\lambda_{q_l+1} - \lambda_1 - 1)}$$

and the polynomials

$$v_1^{(\lambda_n - \lambda_1 - 1)}, \dots, v_{n-q_l}^{(\lambda_n + \dots + \lambda_{q_l+1} - \lambda_1 - n + q_l)}.$$

It follows from (4.10) that this submatrix has a block-triangular form with the diagonal blocks of sizes $q_i \times q_i$ for $i = 1, \dots, l-1$ which coincide with the respective sub-blocks of the $n \times n$ submatrix considered above. The argument continues in the same way for the remaining variables and polynomials, and the same observation applies to the remaining part of the Jacobian matrix. Since all diagonal blocks are non-degenerate we may conclude that so is the full matrix. $\square$

Remark 4.2. (i) The above argument provides another proof of the algebraic independence of the generators $\partial^s w_k^{(r)}$ of the algebra $\mathcal{W}(\mathfrak{a})$ given in Theorem 3.1.

(ii) Proposition 4.1 allows one to regard the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{a})$ as a differential subalgebra of $\mathcal{V}(\mathfrak{h})$ generated by the elements $\overline{w}_k^{(r)}$ with $k = 1, \dots, n$ and r satisfying conditions (3.5). It would be interesting to find an intrinsic characterization of this subalgebra. This could possibly involve some version of *screening operators* like in the case $e = 0$; see [8, Ch. 8]. In this case the embedding $\mathcal{W}(\mathfrak{a}) \hookrightarrow \mathcal{V}(\mathfrak{h})$ is the *Miura map*, *loc. cit.* □

Now we recall the definition of the center at the critical level associated with the centralizer $\mathfrak{a}$ from [1] and a construction of its generators [10]. The affine Kac–Moody algebra $\widehat{\mathfrak{a}}$ is the central extension

$$\widehat{\mathfrak{a}} = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1},$$

where $\mathfrak{a}[t, t^{-1}]$ is the Lie algebra of Laurent polynomials in t with coefficients in $\mathfrak{a}$. For any $r \in \mathbb{Z}$ and $X \in \mathfrak{g}$ we will write $X[r] = X\, t^r$. The commutation relations of the Lie algebra $\widehat{\mathfrak{a}}$ have the form

$$\big[X[r], Y[s]\big] = [X, Y][r+s] + r\,\delta_{r,-s}\langle X, Y\rangle\,\mathbf{1}, \qquad X, Y \in \mathfrak{a},$$

and the element $\mathbf{1}$ is central in $\widehat{\mathfrak{a}}$. Here the invariant symmetric bilinear form $\langle\ ,\ \rangle$ is different from (2.1) and defined by the formulas

$$\big\langle E_{ii}^{(0)}, E_{jj}^{(0)}\big\rangle = \min(\lambda_i, \lambda_j) - \delta_{ij}\big(\lambda_1 + \dots + \lambda_{i-1} + (n-i+1)\lambda_i\big),$$

and if $\lambda_i = \lambda_j$ for some $i \neq j$ then

$$\big\langle E_{ij}^{(0)}, E_{ji}^{(0)}\big\rangle = -\big(\lambda_1 + \dots + \lambda_{i-1} + (n-i+1)\lambda_i\big),$$

whereas all remaining values of the form on the basis vectors are zero. The *vacuum module at the critical level* over $\widehat{\mathfrak{a}}$ is the quotient

$$V(\mathfrak{a}) = \mathrm{U}(\widehat{\mathfrak{a}})/\mathrm{I},$$

where I is the left ideal of $\mathrm{U}(\widehat{\mathfrak{a}})$ generated by $\mathfrak{a}[t]$ and the element $\mathbf{1} - 1$. The vacuum module is a vertex algebra and its *center* is defined as the subspace

$$\mathfrak{z}(\widehat{\mathfrak{a}}) = \{v \in V(\mathfrak{a}) \mid \mathfrak{a}[t]\, v = 0\}.$$

The center is a commutative associative algebra which can be regarded as a subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$. This subalgebra is invariant with respect to the *translation operator* T which is the derivation of the algebra $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$ whose action on the generators is given by

$$T : X[r] \mapsto -r\, X[r-1], \qquad X \in \mathfrak{a}, \quad r < 0.$$

By [1, Thm 1.4], there exists a *complete set of Segal–Sugawara vectors* $S_1, \dots, S_N \in \mathfrak{z}(\widehat{\mathfrak{a}})$, which means that all translations $T^r S_l$ with $r \geqslant 0$ and $l = 1, \dots, N$ are algebraically independent and any element of $\mathfrak{z}(\widehat{\mathfrak{a}})$ can be written as a polynomial in the shifted vectors; that is,

$$\mathfrak{z}(\widehat{\mathfrak{a}}) = \mathbb{C}[T^r S_l \mid l = 1, \dots, N, \ \ r \geqslant 0].$$

In the case $e = 0$ this reduces to the Feigin–Frenkel theorem in type A [7, 8].

To produce a complete set of Segal–Sugawara vectors, we will use the extended Lie algebra $\widehat{\mathfrak{a}} \oplus \mathbb{C}\tau_1 \oplus \dots \oplus \mathbb{C}\tau_n$ where the additional elements $\tau_1, \dots, \tau_n$ pairwise commute and satisfy the commutation relations

$$\big[\tau_i, X[r]\big] = -r\,\lambda_i\,X[r-1], \qquad \big[\tau_i, \mathbf{1}\big] = 0.$$

We will identify the universal enveloping algebra $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}] \oplus \mathbb{C}\tau_1 \oplus \dots \oplus \mathbb{C}\tau_n\big)$ with the tensor product space $V(\mathfrak{a}) \otimes \mathbb{C}[\tau_1, \dots, \tau_n]$. For all $i, j \in \{1, \dots, n\}$ introduce polynomials in a variable z with coefficients in this algebra by

$$E_{ij}(z) = \begin{cases} E_{ij}^{(0)}[-1] + \dots + E_{ij}^{(\lambda_j - 1)}[-1]\, z^{\lambda_j - 1} & \text{if} \quad i \geqslant j, \\ E_{ij}^{(\lambda_j - \lambda_i)}[-1]\, z^{\lambda_j - \lambda_i} + \dots + E_{ij}^{(\lambda_j - 1)}[-1]\, z^{\lambda_j - 1} & \text{if} \quad i < j. \end{cases}$$

Combine them into the $n \times n$ matrix, calculate the column-determinant

$$\operatorname{cdet} \begin{bmatrix} \tau_1 + E_{11}(z) & E_{12}(z) & \dots & E_{1n}(z) \\ E_{21}(z) & \tau_2 + E_{22}(z) & \dots & E_{2n}(z) \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1}(z) & E_{n2}(z) & \dots & \tau_n + E_{nn}(z) \end{bmatrix}$$

and write it as an element of $V(\mathfrak{a})[z] \otimes \mathbb{C}[\tau_1, \dots, \tau_n]$. Furthermore, take the image of this element under the linear map $V(\mathfrak{a})[z] \otimes \mathbb{C}[\tau_1, \dots, \tau_n] \mapsto V(\mathfrak{a})[z] \otimes \mathbb{C}[\tau]$ which is the identity on $V(\mathfrak{a})[z]$ and sends τ_k to τ, where τ is understood as a formal variable. The image of the column-determinant is a polynomial in τ of the form

$$\tau^n + \phi_1(z)\,\tau^{n-1} + \dots + \phi_n(z), \qquad \phi_k(z) = \sum_r \phi_k^{(r)}\, z^r.$$

We will use the following form of the main result of [10]: the coefficients $\phi_k^{(r)}$ with $k = 1, \dots, n$ and r satisfying (3.5) belong to the center $\mathfrak{z}(\widehat{\mathfrak{a}})$ of the vertex algebra $V(\mathfrak{a})$. Moreover, they form a complete set of Segal–Sugawara vectors for the Lie algebra $\mathfrak{a}$.

Denote by $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)_0$ the zero weight component of the algebra $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$ with respect to the adjoint action of the abelian subalgebra of $\mathfrak{a}$ spanned by $E_{11}^{(0)}, \dots, E_{nn}^{(0)}$. Recall the triangular decomposition (2.2) of $\mathfrak{a}$ and observe that the projection

$$\mathfrak{f} : \mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)_0 \to \mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big)$$

to the first summand in the direct sum decomposition

$$\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)_0 = \mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big) \oplus \Big(\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)_0 \cap \mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)\, t^{-1}\mathfrak{n}_+[t^{-1}]\Big)$$

is an algebra homomorphism. Note that $\mathfrak{z}(\widehat{\mathfrak{a}})$ is a subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)_0$.

Proposition 4.3. *The restriction of the homomorphism $\mathfrak{f}$ to the subalgebra $\mathfrak{z}(\widehat{\mathfrak{a}})$ is injective.*

Proof. The images of the generators of the algebra $\mathfrak{z}(\widehat{\mathfrak{a}})$ under the homomorphism $\mathfrak{f}$ are readily found from their definition. Namely, write the product

$$\big(\tau_1 + E_{11}(z)\big) \dots \big(\tau_n + E_{nn}(z)\big)$$

as an element of the vector space $\mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big)[z] \otimes \mathbb{C}[\tau_1, \dots, \tau_n]$ and take its image under the linear map $\mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big)[z] \otimes \mathbb{C}[\tau_1, \dots, \tau_n] \mapsto \mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big)[z] \otimes \mathbb{C}[\tau]$. The image is a polynomial in τ of the form

$$\tau^n + \overline{\phi}_1(z)\tau^{n-1} + \dots + \overline{\phi}_n(z), \qquad \overline{\phi}_k(z) = \sum_r \overline{\phi}_k^{(r)} z^r.$$

The proposition will follow if we show that the elements $T^s \overline{\phi}_k^{(r)} \in \mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big)$, where $s \geqslant 0$ and $k = 1, \dots, n$ with r satisfying conditions (3.5), are algebraically independent. However, we have an isomorphism of differential algebras

$$\mathcal{V}(\mathfrak{h}) \to \mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big), \qquad X[s] \mapsto s!\, X[-s-1]$$

for $s \geqslant 0$ and $X \in \mathfrak{h}$, so that the derivation ∂ corresponds to T. Under this isomorphism, we have $\overline{w}_k^{(r)} \mapsto \overline{\phi}_k^{(r)}$. It remains to note that the required property was already established in the proof of Proposition 4.1 for the elements $\partial^s \overline{w}_k^{(r)}$. □

By the arguments used in the proof of Proposition 4.3 we have the following isomorphism; cf. [8, Theorem 8.1.5].

Corollary 4.4. *We have a differential algebra isomorphism*

$$\mathfrak{z}(\widehat{\mathfrak{a}}) \to \mathcal{W}(\mathfrak{a}), \qquad \phi_k^{(r)} \mapsto w_k^{(r)}, \qquad T \mapsto \partial,$$

where $k = 1, \dots, n$ and r satisfies (3.5). □

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