GRADIENT ESTIMATES FOR NONLINEAR ELLIPTIC EQUATIONS WITH A GRADIENT-DEPENDENT NONLINEARITY

JOSHUA CHING AND FLORICA C. CÎRSTEA

ABSTRACT. In this paper, we obtain gradient estimates of the positive solutions to weighted p-Laplacian type equations with a gradient-dependent nonlinearity of the form

$$\operatorname{div}\left(|x|^\sigma |\nabla u|^{p-2}\nabla u\right) = |x|^{-\tau} u^q |\nabla u|^m \quad \text{in } \Omega^* := \Omega \setminus \{0\}. \tag{0.1}$$

Here, $\Omega \subseteq \mathbb{R}^N$ denotes a domain containing the origin with $N \geq 2$, whereas $m, q \in [0, \infty)$, $1 < p \leq N+\sigma$ and $q > \max\{p-m-1, \sigma+\tau-1\}$. The main difficulty arises from the dependence of the right-hand side of (0.1) on x, u and $|\nabla u|$, without any upper bound restriction on the power m of $|\nabla u|$. Our proof of the gradient estimates is based on a two-step process relying on a modified version of the Bernstein's method. As a by-product, we extend the range of applicability of the Liouville-type results known for (0.1).

1. Introduction and main result

A priori estimates for second-order, nonlinear elliptic and parabolic equations are of fundamental importance in geometry and partial differential equations. Independent of any knowledge of the existence of solutions, *a priori* estimates play a crucial role in establishing existence, uniqueness, regularity and other qualitative properties of solutions. For example, a key step for proving the existence of solutions for quasilinear elliptic equations is represented by local or global gradient bounds. Such *a priori* estimates lead to Harnack inequalities, Liouville theorems and compactness theorems for linear and nonlinear partial differential equations.

In this paper we derive gradient bounds for the positive solutions to a class of elliptic equations in divergence form such as

$$\operatorname{div}(|x|^\sigma |\nabla u|^{p-2}\nabla u) = |x|^{-\tau} u^q |\nabla u|^m \quad \text{in } \Omega^* := \Omega \setminus \{0\}. \tag{1.1}$$

Here, $\Omega \subseteq \mathbb{R}^N$ denotes a domain containing the origin with $N \geq 2$, while m, p, q, σ and τ are real parameters. We define k and ℓ by

$$k := m + q - p + 1 \text{ and } \ell := q + 1 - \sigma - \tau. \tag{1.2}$$

We assume throughout the following condition

$$1 < p \leq N + \sigma, \quad \min\{k, \ell\} > 0 \quad \text{and} \quad m, q \in [0, \infty). \tag{1.3}$$

By a positive solution u of (1.1) we mean a positive function $u \in C^2(\Omega^*)$ satisfying (1.1) in the classical sense. By the strong maximum principle (see Lemma 3.1), any non-negative and non-zero solution of (1.1) is positive in Ω^*.

The main result of this paper is the following theorem.

Theorem 1.1. *Let* (1.3) *hold. There exists a positive constant* $C = C(m, N, p, q, \sigma, \tau)$ *such that for any positive solution* u *of* (1.1) *and any* $r_0 > 0$ *with* $\overline{B_{2r_0}(0)} \subset \Omega$, *it holds*

$$|\nabla u(x)| \leq C|x|^{-\frac{\ell}{k}} \quad \text{for every } 0 < |x| \leq r_0. \tag{1.4}$$

In particular, if $\Omega = \mathbb{R}^N$, *then* (1.4) *holds for all* $x \in \mathbb{R}^N \setminus \{0\}$.

F.C. Cîrstea was partially supported by ARC Discovery grant number DP120102878 "Analysis of non-linear partial differential equations describing singular phenomena".

Theorem 1.1 is also applicable if $m = 0$. In this instance and other particular cases of (1.3), by assuming an upper bound on m, gradient estimates can be obtained by deriving *a priori* estimates of the solutions, then using a suitable transformation and [22, Theorem 1], see [11, Lemma 1.1] for $m = \sigma = \tau = 0$ and [8, Lemma 3.8] for $\sigma = \tau = 0 < m < 2 = p$. We generalise such estimates to (1.1) in Theorem 1.1 through a different method (akin to that in [16, 18]) without any upper bound restriction on the power m of $|\nabla u|$ in the right-hand side of (1.1).

An important tool for obtaining gradient bounds is the classical Bernstein's method, introduced by Bernstein ([3]–[5]) at the beginning of the 20th century. The basic idea is to derive a differential equation for $|\nabla u|^2$ and then apply the maximum principle. Bernstein's method was substantially developed by Ladyzhenskaya [13] and Ladyzhenskaya and Ural'tseva [14, 15] (to obtain both interior and global gradient estimates for uniformly elliptic equations) and later applied systematically to quasilinear elliptic equations by Serrin [20], Lions [16] and many others, leading to a definitive quasilinear theory as described by Gilbarg and Trudinger [12]. A weak Bernstein method was introduced by Barles [2] for fully nonlinear elliptic equations based on the approach of viscosity solutions.

We now return to equation (1.1) for a brief review of gradient estimates. *Without the factor* u^q in (1.1), by relying on the Bernstein technique, *a priori* gradient bounds were first derived by Lions [16, Theorem IV.1] for $\Delta u = |\nabla u|^m$ and recently extended by Nguyen [18, Lemma 2.2] to include equations such as $\Delta u = |x|^{-\tau}|\nabla u|^m$ in Ω^* when $m > 1 > \tau$. *A priori* universal gradient estimates for the quasilinear elliptic equation $\text{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^m$ with $m > p-1 > 0$ on a domain Ω of $\mathbb{R}^N$ $(N \geq p)$ have been obtained by Bidaut-Véron et al. [6, Proposition 2.1]. They extended their estimates to equations on complete non-compact manifolds satisfying a lower bound estimate on the Ricci curvature and used them to derive Liouville type theorems.

We aim to generalise the gradient estimates in [18, Lemma 2.2] and [6] to the *weighted* p-Laplacian type equation (1.1) in the corresponding framework of (1.3). New difficulties arise due to the introduction of a non-negative power u^q and of a weight function $|x|^{-\tau}$ in the right-hand side of (1.1). We next outline the main steps in the derivation of the gradient bounds of Theorem 1.1 for any positive solution u_1 of (1.1). Fix $x_0 \in B_{r_0}(0) \setminus \{0\}$ such that $|\nabla u_1(x_0)| > 0$. Let $\mathcal{G}$ denote the maximal connected component of $\{x \in \Omega^* : \ |\nabla u_1(x)| > 0\}$ containing x_0. We set $\rho_0 := |x_0|$ and $a_{1,1}(x) := |x|^{-\tau-\sigma}u_1^q(x)|\nabla u_1(x)|^{m+2-p}$ for $x \in \mathcal{G} \cap B_{\rho/2}(x_0)$ so that

$$\Delta u_1 = a_{1,1}(x) - \sigma\frac{\langle x, \nabla u_1\rangle}{|x|^2} - (p-2)\,\frac{\langle (D^2u_1)(\nabla u_1), \nabla u_1\rangle}{|\nabla u_1|^2} \quad \text{on } \ \mathcal{G} \cap B_{\frac{\rho}{2}}(x_0). \tag{1.5}$$

Note that the power factor u_1^q is hidden into $a_{1,1}$. Let ϕ and ω be given by (2.8) and (2.9), respectively. By "linearising" (1.5), we need to prescribe a suitable linear operator $\mathcal{L}_1[w]$ for $w \in C^2(\mathcal{G} \cap B_{\rho/2}(x_0))$ and be able to bound $\mathcal{L}_1[w_1]$ from above on ω for

$$w_1 = \phi^{2\alpha_1}|\nabla u_1|^2,$$

where α_1 is a positive constant to be conveniently chosen as $1/(2k)$. By the definition of ϕ and ω, we have that $\max_{\overline{\omega}} w_1 = w_1(x^*)$ for some $x^* \in \omega$. Since $\nabla w_1(x^*) = 0$ and $(D^2 w_1)(x^*)$ is negative definite, the definition of $\mathcal{L}_1[w_1]$ at x^* will lead to $\mathcal{L}_1[w_1](x^*) \geq 0$ (see (2.18)).

The construction of $\mathcal{L}_1$ is a critical step, which becomes significantly more difficult than for the Laplacian type equations treated in [16, 18]. For our more intricate weighted p-Laplacian type equation (1.1), the nonlinearity depends on the unknown u and its gradient, and also on the space variable x. By denoting $z_1(x) = |\nabla u_1(x)|^2$ for every $x \in \mathcal{G} \cap B_{\rho/2}(x_0)$ and

$$\mathcal{A}_1[w] := -\Delta w - \sigma\frac{\langle x, \nabla w\rangle}{|x|^2} - (p-2)\,\frac{\langle (D^2w)(\nabla u_1), \nabla u_1\rangle}{|\nabla u_1|^2},$$

our operator $\mathcal{L}_1[w]$ is defined by

$$\mathcal{L}_1[w] := \mathcal{A}_1[w] + (m+2-p)\, a_{1,1} \frac{\langle \nabla u_1, \nabla w \rangle}{z_1} - M \frac{\langle \nabla z_1, \nabla w \rangle}{z_1}. \tag{1.6}$$

Here, M is a sufficiently large constant (as in Lemma 3.4). Compared with [16, 18] (where $p = 2$ and $\sigma = q = 0$), our operator $\mathcal{L}_1$ in (1.6) introduces the extra term $-M \frac{\langle \nabla z_1, \nabla w \rangle}{z_1}$, a trick inspired by the work of Bidaut-Véron et al. [6, Proposition 2.1]. We mention that Bernstein's method is adapted differently in [6] than in this paper. In Lemma 3.4, we bound $\mathcal{L}_1[w_1]$ from above to get positive constants $d_i(m, N, p, q, \sigma, \tau)$ for $i = 1, 2, 3$ such that

$$\mathcal{L}_1[w_1] \leq \left[d_1 \frac{z_1}{\phi |x|^2} - d_2 \left(a_{1,1}(x)\right)^2 - d_3 \frac{|\nabla z_1|^2}{z_1} \right] \phi^{2\alpha_1} \leq d_1 \phi^{\frac{1}{k}-1} \frac{z_1}{|x|^2} - d_2 u_1^{2q} \phi^{\frac{1}{k}} \frac{z_1^{m+2-p}}{|x|^{2(\tau+\sigma)}} \tag{1.7}$$

for all $x \in \omega$ since $\alpha_1 = 1/(2k)$. The right-hand side of (1.7) shows that we cannot proceed further without an intermediate estimate that relates the gradient of the solution to the solution itself. This is done in Lemma 2.1 by an appropriate log transform of u_1 in (2.2) to obtain a new function u_2 satisfying (2.6). Up to a constant, this transformation combines the powers of u_1 with powers of $|\nabla u_1|$ (see the definition of h_2 in (2.3)) into the exponential term e^{-ku_2}. As a result, we can similarly modify the Bernstein method as for u_1 to obtain a gradient estimate for u_2 using $\mathcal{L}_2$ in (2.7) applied to w_2 given by (2.11). Hence, we can derive the intermediate estimate (2.1) in Lemma 2.1, which employed in (1.7) leads to (2.17). Since $\mathcal{L}_1[w_1](x^*) \geq 0$, by letting $x = x^*$ in (2.17), we finally reach the main estimate in (1.4) for the solution u_1 of (1.1).

Structure of the paper. In Section 2, we provide the main ingredients in the proof of Theorem 1.1. We postpone the technicalities of the proof to Section 3. By applying Theorem 1.1, we obtain (i) a Liouville-type result in Corollary 2.3, which improves through a different method the corresponding results in Farina and Serrin [10, Theorems 2,3] and (ii) *a priori* estimates of the positive solutions of (1.1) in Corollary 2.4.

2. Proof of the main result

As explained in the introduction, a crucial step in establishing (1.4) is the estimate in (2.1) relating the gradient of an arbitrary positive solution u_1 of (1.1) to the solution u_1 itself.

Lemma 2.1. *Let* (1.3) *hold. There exists a positive constant* $C_1 = C_1(m, N, p, q, \sigma, \tau)$ *such that for every positive solution* u_1 *of* (1.1) *and any* $r_0 > 0$ *with* $\overline{B_{2r_0}(0)} \subset \Omega$, *we have that*

$$|\nabla u_1(x)| \leq C_1 \frac{u_1(x)}{|x|} \quad \text{for all } 0 < |x| \leq r_0. \tag{2.1}$$

Proof. Fix $x_0 \in \mathbb{R}^N$ with $0 < |x_0| \leq r_0$. To prove (1.4) and (2.1) at $x = x_0$ for a positive solution u_1 of (1.1), we assume that $|\nabla u_1(x_0)| > 0$. Let $\mathcal{G}$ denote the maximal connected component of the set $\{x \in \Omega \setminus \{0\} : \ |\nabla u_1(x)| > 0\}$ containing x_0. We set $\rho_0 := |x_0|$. Let $C_0 > 0$ be a large constant such that $C_0 u_1(x) < 1$ for every $\rho_0/2 \leq |x| \leq 3\rho_0/2$. We define

$$u_2(x) := -\log(C_0 u_1(x)) \quad \text{for every } |x| \in (\rho_0/2, 3\rho_0/2). \tag{2.2}$$

We set $z_j(x) = |\nabla u_j(x)|^2 > 0$ for $j = 1, 2$ and $x \in \mathcal{G} \cap B_{\rho_0/2}(x_0)$. For any $t > 0$, we denote

$$f(x,t) := |x|^{-\tau-\sigma} t^{\frac{m+2-p}{2}} \text{ for } x \in \mathbb{R}^N \setminus \{0\} \text{ and } h_j(t) := \begin{cases} t^q & \text{if } j = 1, \\ -C_0^{-k} e^{-kt} & \text{if } j = 2. \end{cases} \tag{2.3}$$

For $j=1,2$ and $x\in\mathcal{G}\cap B_{\rho_0/2}(x_0)$, let $a_{0,j,w}(x)$, $a_{1,j,w}(x)$ and $\mathcal{A}_j[w]$ be given by

$$\begin{cases} a_{0,j,w}(x) := (j-1)(p-1)\langle \nabla u_j, \nabla w\rangle, \quad a_{1,j,w}(x) := h_j(u_j) f(x,z_j)\dfrac{\langle \nabla u_j, \nabla w\rangle}{z_j}, \\ \mathcal{A}_j[w](x) := -\Delta w(x) + a_{2,w}(x) + a_{3,j,w}(x), \text{ where we define} \\ a_{2,w}(x) := -\sigma \dfrac{\langle x, \nabla w\rangle}{|x|^2}, \quad a_{3,j,w}(x) := -(p-2)\dfrac{\langle (D^2 w)(\nabla u_j), \nabla u_j\rangle}{z_j}. \end{cases} \tag{2.4}$$

When $w=u_j$ in a_{i,j,u_j} for $i=0,1,3$, we simply write $a_{i,j}$. For symmetry of notation, we also use $a_{2,j}$ instead of a_{2,u_j}. In particular, since $\nabla z_j = 2(D^2 u_j)(\nabla u_j)$, we have

$$a_{0,j} = (j-1)(p-1) z_j, \quad a_{1,j}(x) = h_j(u_j) f(x,z_j), \quad a_{3,j} = -\frac{(p-2)}{2}\frac{\langle \nabla z_j, \nabla u_j\rangle}{z_j}. \tag{2.5}$$

Then, u_j (with $j=1,2$) satisfies the equation

$$\Delta u_j = \sum_{i=0}^{3} a_{i,j}(x) \quad \text{for all } x\in \mathcal{G}\cap B_{\rho_0/2}(x_0). \tag{2.6}$$

Next, for $j=1,2$ and $x\in\mathcal{G}\cap B_{\rho_0/2}(x_0)$, we introduce the operator for $w\in C^2(\mathcal{G}\cap B_{\rho_0/2}(x_0))$

$$\mathcal{L}_j[w] := \mathcal{A}_j[w] - M\frac{\langle \nabla z_j, \nabla w\rangle}{z_j} + 2a_{0,j,w} + (m+2-p)\, a_{1,j,w}, \tag{2.7}$$

where $M = M(m,N,p,q,\sigma,\tau) > 0$ denotes a large constant (see (3.40) in Lemma 3.4).

Let $\eta\in C_c^\infty(\mathbb{R}^N)$ be such that $0\le \eta\le 1$, $\operatorname{Supp}(\eta)\subset B_{1/2}(0)$ and $\eta\equiv 1$ in $B_{1/3}(0)$. We define

$$\phi(x) = \eta(\rho_0^{-1}(x-x_0)) \quad \text{for all } x\in\mathbb{R}^N. \tag{2.8}$$

Let ω denote the following open set

$$\omega := \mathcal{G}\cap \{x\in B_{\rho_0/2}(x_0):\ (x-x_0)/\rho_0 \in \operatorname{Int}(\operatorname{Supp}(\eta))\}. \tag{2.9}$$

Note that there exists a positive constant $c'=c'(N)$ such that

$$|D^2\phi| \le c'\rho_0^{-2} \quad \text{and} \quad |\nabla\phi| \le c'\rho_0^{-1}\phi^{\frac{1}{2}} \quad \text{for every } x\in\omega. \tag{2.10}$$

The aim of Lemma 3.2 in Section 3 is to compute $\mathcal{L}_j[w_j]$ for w_j given by

$$w_j := \phi^{2\alpha_j} z_j \ \text{ with } \ \alpha_1 = 1/(2k) \ \text{ and } \ \alpha_2 = 1/2. \tag{2.11}$$

Then, in Lemma 3.4, we obtain an upper bound estimate of $\mathcal{L}_j[w_j]$, proving that there exist positive constants $d_i = d_i(m,N,p,q,\sigma,\tau)$ with $i=0,1,2,3$ such that for $j=1,2$, we have

$$\mathcal{L}_j[w_j] \le \left[-d_0(j-1)z_j^2 + d_1\frac{z_j}{\phi|x|^2} - d_2\,(a_{1,j}(x))^2 - d_3\frac{|\nabla z_j|^2}{z_j}\right]\phi^{2\alpha_j} \text{ for all } x\in\omega. \tag{2.12}$$

Since $\alpha_2 = 1/2$ and $w_2 = \phi z_2$, for $j=2$, we obtain that

$$\mathcal{L}_2[w_2] \le z_2\left(-d_0\, w_2 + d_1|x|^{-2}\right) \quad \text{for all } x\in\omega. \tag{2.13}$$

Using that $w_2|_{\partial\omega}=0$, there exists $x^*\in\omega$ such that $\max_{x\in\overline{\omega}} w_2(x) = w_2(x^*) > 0$. Then, since $\nabla w_2(x^*)=0$ and $(D^2 w_2)(x^*)$ is negative definite, by the Min-max theorem, we find that

$$\mathcal{L}_2[w_2](x^*) = \mathcal{A}_2[w_2](x^*) = \left[-\Delta w_2 - (p-2)\frac{\langle (D^2 w_2)(\nabla u_2), \nabla u_2\rangle}{|\nabla u_2|^2}\right](x^*) \ge 0. \tag{2.14}$$

Indeed, if $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_N \le 0$ denote the eigenvalues of $(D^2 w_2)(x^*)$, we have

$$\mathcal{A}_2[w_2](x^*) \ge \begin{cases} -(p-1)\lambda_1 - \sum_{i=2}^{N} \lambda_i & \text{if } 1 < p \le 2, \\ -\sum_{i=1}^{N-1} \lambda_i - (p-1)\lambda_N & \text{if } 2 < p < \infty. \end{cases}$$

Letting $x = x^*$ in (2.13) and using (2.14), we arrive at

$$w_2(x^*) \le (d_1/d_0)\,|x^*|^{-2}, \tag{2.15}$$

where d_0 and d_1 are positive constants depending only on m, N, p, q, σ and τ. Recall that $w_2(x^*) = \max_{x\in\overline{\omega}} w_2(x)$ and $|x^*| \ge |x_0|/2$. Since $\eta \equiv 1$ on $B_{1/3}(0)$, we have $\phi(x_0) = 1$. Hence, from (2.2) and (2.15), we obtain that

$$\frac{|\nabla u_1(x_0)|^2}{(u_1(x_0))^2} = |\nabla u_2(x_0)|^2 = \phi(x_0)|\nabla u_2(x_0)|^2 \le w_2(x^*) \le \frac{4d_1}{d_0|x_0|^2}.$$

This proves the assertion of (2.1) for $x = x_0$ arbitrary in $\Omega \setminus \{0\}$ with $|\nabla u_1(x_0)| \ne 0$. The proof of Lemma 2.1 is thus complete. □

Proof of Theorem 1.1 *completed.* By taking $j = 1$ and $\alpha_1 = 1/(2k)$ in (2.12), we find that

$$\mathcal{L}_1[w_1] \le d_1\phi^{\frac{1}{k}-1}|x|^{-2}z_1 - d_2 u_1^{2q}\phi^{\frac{1}{k}}|x|^{-2(\tau+\sigma)}z_1^{m+2-p} \quad \text{for all } x \in \omega. \tag{2.16}$$

In (2.16) we use Lemma 2.1 and $w_1 = \phi^{1/k}z_1$ to conclude that

$$\mathcal{L}_1[w_1] \le \phi^{\frac{1}{k}-1}|x|^{-2}z_1\left(d_1 - C_1^{-2q}d_2|x|^{2\ell}w_1^k\right) \quad \text{for all } x \in \omega, \tag{2.17}$$

where $C_1 > 0$ is the constant appearing in (2.1), while k and ℓ are given by (1.2).

Let $x^* \in \omega$ be such that $\max_{x\in\overline{\omega}} w_1(x) = w_1(x^*) > 0$. As before, we arrive at

$$\mathcal{L}_1[w_1](x^*) = \mathcal{A}_1[w_1](x^*) = \left[-\Delta w_1 - (p-2)\frac{\langle (D^2 w_1)(\nabla u_1), \nabla u_1\rangle}{|\nabla u_1|^2}\right](x^*) \ge 0. \tag{2.18}$$

Letting $x = x^*$ in (2.17) and using (2.18), we arrive at

$$w_1(x^*) \le \left(\frac{d_1}{d_2}\right)^{\frac{1}{k}} C_1^{\frac{2q}{k}}|x^*|^{-\frac{2\ell}{k}}, \tag{2.19}$$

where C_1, d_1 and d_2 are positive constants depending only on m, N, p, q, σ and τ. Recall that $w_1(x^*) = \max_{x\in\overline{\omega}} \phi(x)|\nabla u_1(x)|^2$ and $|x^*| \ge |x_0|/2$. Since $\phi(x_0) = 1$, from (2.19), we obtain that

$$|\nabla u_1(x_0)|^2 = (\phi(x_0))^{1/k}|\nabla u_1(x_0)|^2 \le w_1(x^*) \le C^2|x_0|^{-\frac{2\ell}{k}},$$

where $C := \left(2^\ell C_1^q\sqrt{d_1/d_2}\right)^{1/k}$ is a positive constant depending only on m, N, p, q, σ and τ. This proves the assertion of (1.4) for $x = x_0$ arbitrary in $\Omega \setminus \{0\}$ with $|\nabla u_1(x_0)| \ne 0$. The proof of our Theorem 1.1 is now finished. □

Lemma 2.2. *Let* (1.3) *hold and* Ω_1 *be any domain in* $\mathbb{R}^N$. *There exists a positive constant* $C_1 = C_1(m, N, p, q, \sigma, \tau)$ *such that any positive solution* u *of* (1.1) *in* Ω_1 *satisfies*

$$|\nabla u(x)| \le \frac{C_1 u(x)}{\operatorname{dist}(x, \partial\Omega_1)} \quad \text{and} \quad |\nabla u(x)| \le C_1 \operatorname{dist}(x, \partial\Omega_1)^{-\frac{\ell}{k}} \quad \text{for all } x \in \Omega_1. \tag{2.20}$$

Proof. The claim follows by taking $\rho_0 = \operatorname{dist}(x_0, \partial\Omega_1)$ rather than $\rho_0 = |x_0|$ in the proofs of Theorem 1.1 and Lemma 2.1. □

The Liouville-type results in [10, Theorem 2,3] are improved by the following.

Corollary 2.3 (Liouville-type theorem). *Let* (1.3) *hold. Any* $C^1(\mathbb{R}^N)$ *positive solution of* (1.1) *in* $\mathbb{R}^N$ *must be identically constant.*

Proof. We follow [16, Corollary IV.2]. Fix $x_1 \in \mathbb{R}^N$. Let $R > 0$ be arbitrary and $\Omega_1 = B_R(x_1)$ in Lemma 2.2. Then for C_1 as in Lemma 2.2, we find that $|\nabla u(x_1)| \le C_1 R^{-\frac{\ell}{k}}$. Letting $R \to \infty$, we obtain that $|\nabla u(x_1)| = 0$. Since x_1 was arbitrary, we conclude the claim. □

Corollary 2.4 (A priori estimates of solutions). *Let* (1.3) *hold and* u *be an arbitrary positive solution of* (1.1). *If* $k > \ell$, *then* $u \in L^\infty_{\rm loc}(\Omega)$. *If* $k \le \ell$, *then there exists a positive constant* c_1 *depending only on* m, N, p, q, σ *and* τ *such that for all* $0 < |x| < r_0$ *with* $\overline{B_{2r_0}(0)} \subset \Omega$, *it holds*

$$u(x) \le \begin{cases} \max\limits_{\partial B_{r_0}(0)} u + c_1 \log(r_0/|x|) & \text{if } k = \ell; \\ \max\limits_{\partial B_{r_0}(0)} u + c_1 |x|^{1-\frac{\ell}{k}} & \text{if } k < \ell. \end{cases} \tag{2.21}$$

Proof. The proof follows similarly to that in [6, Section 2.2]. Fix $x \in B_{r_0}(0) \setminus \{0\}$ and let $X = r_0 x/|x|$. Using Theorem 1.1, we find that

$$|u(x) - u(X)| \le |x - X| \int_0^1 |(\nabla u)(tx + (1-t)X)|\, dt \le C|x - X| \int_0^1 (t|x| + (1-t)r_0)^{-\frac{\ell}{k}}\, dt.$$

The conclusion follows by integration. □

For a different proof of the second inequality in (2.21) in the case $\sigma = \tau = 0 < m < 2 = p$, we refer to Ching and Cîrstea [8, Lemma 3.4], where a comparison with a suitable boundary blow-up super-solution is used. Unlike the case $m = 0$, it has been observed in [8, Remark 3.5] that the term $\max_{\partial B_{r_0}(0)} u$ arising in the estimate (2.21) is due to the introduction of the gradient factor $|\nabla u|^m$ in (1.1) and cannot be removed.

3. Auxiliary results

In Lemma 3.1 we prove that the strong maximum principle is applicable for the non-negative solutions of (1.1) when (1.3) holds. The proof of Lemma 2.1 was essentially based on the estimate of (2.12), which follows from Lemma 3.4 and Remark 3.5. We present here the proof of Lemma 3.4, which is quite intricate and relies on Lemmas 3.2 and 3.3.

Lemma 3.1 (Strong maximum principle). *Assume that* (1.3) *holds. For any non-negative solution* u *of* (1.1), *either* $u > 0$ *in* Ω^* *or* $u \equiv 0$ *in* Ω^*.

Proof. Let $\varepsilon_0 > 0$ be small so that $B_{\varepsilon_0}(0) \subset \Omega$. For every $x \in \Omega \setminus \{0\}$, we can find $R > \varepsilon$ and $\varepsilon \in (0, \varepsilon_0)$ such that $x \in \Omega_{R,\varepsilon}$, where $\Omega_{R,\varepsilon} := (\Omega \cap B_R(0)) \setminus \overline{B_\varepsilon(0)}$. Hence, the claim follows by checking that the strong maximum principle holds in $\Omega_{R,\varepsilon}$ for every $R > \varepsilon$ and any $\varepsilon \in (0, \varepsilon_0)$. We use [19, Theorem 5.4.1] for (5.4.1) on $\Omega_{R,\varepsilon}$, namely

$$\partial_{x_j} \left\{ a_{ij}(x,u) A(|\nabla u|) \partial_{x_j} u \right\} + B(x, u, \nabla u) \le 0$$

with $a_{ij}(x,u) = |x|^\sigma \delta_{ij}$, where δ_{ij} denotes the Kronecker delta, $A(t) = t^{p-2}$ for any $t \ge 0$ and

$$B(x, z, \boldsymbol{\xi}) = -|x|^{-\tau} z^q |\boldsymbol{\xi}|^m \quad \text{for } x \in \Omega_{R,\varepsilon},\ z \ge 0 \ \text{ and } \boldsymbol{\xi} \in \mathbb{R}^N.$$

We have $A \in C^1(\mathbb{R}^+)$ and $\lim_{t \searrow 0} tA'(t)/A(t) = p - 2 > -1$ so that (A1)' and (5.4.3) in [19] hold. It is easy to check (A2) in [19, p. 3]. We also have (5.4.4) using [19, Remark 3, p. 117]. It remains to check (B1) and (F2) in [19, p. 107]. Since $k > 0$ from (1.3), we can find s such that $s > \max\{(p-1)/q, 1\}$ and $ms' > p-1$, where s' denotes the Hölder conjugate of s, that is $s' := s/(s-1)$. By Young's inequality, we have

$$z^q |\boldsymbol{\xi}|^m \le \frac{z^{qs}}{s} + \frac{|\boldsymbol{\xi}|^{ms'}}{s'} \le \frac{z^{qs}}{s} + \frac{|\boldsymbol{\xi}|^{p-1}}{s'} \quad \text{for all } z \in \mathbb{R}^+ \text{ and } \boldsymbol{\xi} \in \mathbb{R}^N \text{ with } |\boldsymbol{\xi}| \le 1.$$

Hence, Condition (B1) holds with $\Phi(|\boldsymbol{\xi}|) = |\boldsymbol{\xi}| A(|\boldsymbol{\xi}|) = |\boldsymbol{\xi}|^{p-1}$, $\kappa = \max(R^{-\tau}, \varepsilon^{-\tau})/s'$ and $f(z) = (\max(R^{-\tau}, \varepsilon^{-\tau})/s)\, z^{qs}$ satisfying (F2). We can now apply Theorem 5.4.1 in [19] to conclude the proof of Lemma 3.1. □

In the rest of this section, we work in the framework and notation of Lemma 2.1. Our main aim is to prove Lemma 3.4, which gives an estimate from above for $\mathcal{L}_j[w_j]$, where the operator $\mathcal{L}_j[w]$ and w_j are defined in (2.7) and (2.11), respectively. An important ingredient is Lemma 3.2 in which we evaluate $\mathcal{L}_j[w_j]$, see (3.2). For this purpose, we introduce the following.

Notation. We define $\mathcal{Q}_j(x)$, $\Theta_j(x)$ and $\mathcal{X}_j(x)$ for $j = 1, 2$ and $x \in \omega$ as follows:

$$\begin{cases} \mathcal{Q}_j(x) := (\sigma + \tau) \dfrac{\langle x, \nabla u_j \rangle}{|x|^2} + \alpha_j\, (m + 2 - p)\, \dfrac{\langle \nabla u_j, \nabla \phi \rangle}{\phi}, \\ \Theta_j(x) := \Psi_j(x) - 4\alpha_j \dfrac{\langle \nabla \phi, \nabla z_j \rangle}{\phi}, \text{ where } \Psi_j := -\dfrac{2\alpha_j\, z_j}{\phi} \left((2\alpha_j - 1) \dfrac{|\nabla \phi|^2}{\phi} + \Delta \phi \right), \\ \mathcal{X}_j(x) := \Theta_j - M \dfrac{\langle \nabla z_j, \nabla w_j \rangle}{z_j \phi^{2\alpha_j}} + \dfrac{a_{2,w_j} + a_{3,j,w_j}}{\phi^{2\alpha_j}} - 2 \left\langle \nabla \left(a_{2,j} + a_{3,j} \right), \nabla u_j \right\rangle. \end{cases} \tag{3.1}$$

Lemma 3.2. *Let* (1.3) *hold. Then, for every $x \in \omega$ and $j = 1, 2$, the following holds:*

$$\mathcal{L}_j[w_j] = \left[\mathcal{X}_j - 2|D^2 u_j|^2 + 4\alpha_j a_{0,j} \frac{\langle \nabla u_j, \nabla \phi \rangle}{\phi} + 2 \left(\mathcal{Q}_j - \frac{h_j'(u_j)}{h_j(u_j)} z_j \right) a_{1,j} \right] \phi^{2\alpha_j}, \tag{3.2}$$

where $a_{0,j}$ and $a_{1,j}$ are given in (2.5).

Proof. By the definition of $\mathcal{L}_j[w]$ in (2.7), we have

$$\mathcal{L}_j[w_j] = -\Delta w_j - M \frac{\langle \nabla z_j, \nabla w_j \rangle}{z_j} + 2a_{0,j,w_j} + (m + 2 - p) a_{1,j,w_j} + a_{2,w_j} + a_{3,j,w_j}. \tag{3.3}$$

Since $\Delta z_j = 2|D^2 u_j|^2 + 2\langle \nabla(\Delta u_j), \nabla u_j \rangle$, by using (2.6), we find that

$$\Delta z_j = 2|D^2 u_j|^2 + 2 \sum_{i=0}^{3} \langle \nabla a_{i,j}, \nabla u_j \rangle \quad \text{for all } x \in \omega. \tag{3.4}$$

Recall that $w_j = \phi^{2\alpha_j} z_j$. Hence, by the product rule and (3.4), for all $x \in \omega$, we obtain that

$$-\Delta w_j = (\Theta_j - \Delta z_j)\, \phi^{2\alpha_j} = \left(\Theta_j - 2|D^2 u_j|^2 - 2 \sum_{i=0}^{3} \langle \nabla a_{i,j}, \nabla u_j \rangle \right) \phi^{2\alpha_j}. \tag{3.5}$$

Using (3.5) in (3.3), we get that

$$\mathcal{L}_j[w_j] = \left(\mathcal{X}_j - 2|D^2 u_j|^2 + \mathcal{E}_{0,j} + \mathcal{E}_{1,j} \right) \phi^{2\alpha_j} \quad \text{in } \omega, \tag{3.6}$$

where $\mathcal{E}_{0,j}$ and $\mathcal{E}_{1,j}$ are defined in ω as follows

$$\mathcal{E}_{0,j} := 2 \frac{a_{0,j,w_j}}{\phi^{2\alpha_j}} - 2\langle \nabla a_{0,j}, \nabla u_j \rangle \text{ and } \mathcal{E}_{1,j} := (m + 2 - p)\, \frac{a_{1,j,w_j}}{\phi^{2\alpha_j}} - 2\langle \nabla a_{1,j}, \nabla u_j \rangle. \tag{3.7}$$

We now evaluate the terms $\mathcal{E}_{0,j}$ and $\mathcal{E}_{1,j}$ for $j = 1, 2$. We use that

$$\langle \nabla u_j, \nabla w_j \rangle = \left(2\alpha_j z_j \frac{\langle \nabla u_j, \nabla \phi \rangle}{\phi} + \langle \nabla z_j, \nabla u_j \rangle \right) \phi^{2\alpha_j} \quad \text{in } \omega. \tag{3.8}$$

Hence, for every $x \in \omega$, we have

$$\mathcal{E}_{0,j}(x) = 4\alpha_j a_{0,j} \frac{\langle \nabla u_j, \nabla \phi \rangle}{\phi} \quad \text{and} \quad \mathcal{E}_{1,j}(x) = 2\left(\mathcal{Q}_j - \frac{h_j'(u_j)}{h_j(u_j)} z_j \right) a_{1,j}. \tag{3.9}$$

Using (3.9) into (3.6), we reach (3.2). This ends the proof of Lemma 3.2. □

Lemma 3.3. *Let* (1.3) *hold. For every $M > 3|p-2|/2$, there exist positive constants $\beta_{1,j}$ and $\beta_{2,j}$, depending on m, N, p, q, σ and M such that*

$$\mathcal{X}_j(x) \le \beta_{1,j} \frac{z_j}{|x|^2 \phi} - \beta_{2,j} \frac{|\nabla z_j|^2}{z_j} \quad \textit{for all } x \in \omega \textit{ and } j = 1, 2. \tag{3.10}$$

Proof. Let $c' > 0$ be as in (2.10). For any $M > 3|p-2|/2$, we fix $\varepsilon \in (0, \min_{j=1,2} (3\alpha_j c')^{-1})$ such that $\beta_{2,j} > 0$ for $j = 1, 2$, where we define

$$\beta_{2,j} := (1 - 3\alpha_j c' \varepsilon) M - 6\alpha_j c'(1 + |p-2|)\varepsilon - 3|p-2|/2. \tag{3.11}$$

Claim: *There exists $\bar{c}_j > 0$ depending only on m, N, p, q, σ such that for all $x \in \omega$*

$$\mathcal{X}_j(x) \le \bar{c}_j \frac{z_j}{|x|^2 \phi} - \left(M - \frac{3|p-2|}{2} \right) \frac{|\nabla z_j|^2}{z_j} + 2\alpha_j (M + 2 + 2|p-2|) \frac{|\nabla z_j||\nabla \phi|}{\phi}. \tag{3.12}$$

Assume that the Claim has been proved. From (2.10) and Young's inequality with ε, we get

$$\frac{|\nabla z_j||\nabla \phi|}{\phi} \le \frac{3c'}{2} \frac{|\nabla z_j|}{|x|\phi^{1/2}} \le \frac{3c'}{2} \left(\varepsilon \frac{|\nabla z_j|^2}{z_j} + \frac{1}{4\varepsilon} \frac{z_j}{|x|^2 \phi} \right) \quad \text{for every } x \in \omega. \tag{3.13}$$

Using (3.13) into (3.12), we reach (3.10) with $\beta_{2,j}$ given by (3.11) and $\beta_{1,j}$ defined by

$$\beta_{1,j} := \bar{c}_j + 3\alpha_j c'(M + 2 + 2|p-2|)/(4\varepsilon). \tag{3.14}$$

Proof of Claim. If we define $\mathcal{E}_{2,j}$ and $\mathcal{E}_{3,j}$ by

$$\mathcal{E}_{2,j} := \frac{a_{2,w_j}}{\phi^{2\alpha_j}} - 2\langle \nabla a_{2,j}, \nabla u_j \rangle \quad \text{and} \quad \mathcal{E}_{3,j} := \frac{a_{3,j,w_j}}{\phi^{2\alpha_j}} - 2\langle \nabla a_{3,j}, \nabla u_j \rangle,$$

then the definition of $\mathcal{X}_j$ in (3.1) yields that

$$\mathcal{X}_j = \Theta_j - M \frac{\langle \nabla z_j, \nabla w_j \rangle}{z_j \phi^{2\alpha_j}} + \mathcal{E}_{2,j} + \mathcal{E}_{3,j}. \tag{3.15}$$

From (3.15), we will derive (3.12) by bounding from above $\Theta_j - (M\langle \nabla z_j, \nabla w_j \rangle)/(z_j \phi^{2\alpha_j})$, as well as $\mathcal{E}_{2,j}$ and $\mathcal{E}_{3,j}$ in (3.17), (3.19) and (3.23), respectively.

By the definition of Θ_j in (3.1) and $(1/z_j)\phi^{-2\alpha_j} \nabla w_j = (1/z_j)\nabla z_j + (2\alpha_j/\phi)\nabla \phi$, we find that

$$\Theta_j - M \frac{\langle \nabla z_j, \nabla w_j \rangle}{z_j \phi^{2\alpha_j}} = \Psi_j - M \frac{|\nabla z_j|^2}{z_j} - 2\alpha_j (M+2) \frac{\langle \nabla z_j, \nabla \phi \rangle}{\phi} \quad \text{in } \omega. \tag{3.16}$$

From (2.10) and $|\Delta \phi| \le \sqrt{N} |D^2 \phi|$, we have $|\Delta \phi| \le 9\sqrt{N} c'/(4|x|^2)$ in ω. Using (3.16) and denoting $\widehat{c}_j := (9\alpha_j c'/2) \left(\sqrt{N} + c'|2\alpha_j - 1| \right)$, we arrive at

$$\Theta_j - M \frac{\langle \nabla z_j, \nabla w_j \rangle}{z_j \phi^{2\alpha_j}} \le \widehat{c}_j \frac{z_j}{|x|^2 \phi} - M \frac{|\nabla z_j|^2}{z_j} + 2\alpha_j (M+2) \frac{|\nabla z_j||\nabla \phi|}{\phi}. \tag{3.17}$$

Next, we will bound $\mathcal{E}_{2,j}$ and $\mathcal{E}_{3,j}$ from above. Using the identity

$$\left\langle \nabla \left(\frac{\langle x, \nabla u_j \rangle}{|x|^2} \right), \nabla u_j \right\rangle = \frac{z_j}{|x|^2} + \frac{\langle x, \nabla z_j \rangle}{2|x|^2} - \frac{2\langle x, \nabla u_j \rangle^2}{|x|^4},$$

we obtain that

$$\mathcal{E}_{2,j}(x) = 2\sigma \left[\left(1 - 2\frac{\langle x, \nabla u_j \rangle^2}{|x|^2 z_j} \right) \phi - \alpha_j \langle x, \nabla \phi \rangle \right] \frac{z_j}{|x|^2 \phi}. \tag{3.18}$$

Therefore, using the constant c' in (2.10), we find that

$$\mathcal{E}_{2,j}(x) \leq 3|\sigma| \left(2 + c'\alpha_j\right) \frac{z_j}{|x|^2 \phi} \quad \text{for every } x \in \omega. \tag{3.19}$$

Now, using the identity that

$$D^2(\xi\zeta) \equiv \xi D^2 \zeta + \zeta D^2 \xi + (\nabla \zeta)(\nabla \xi)^\intercal + (\nabla \xi)(\nabla \zeta)^\intercal$$

for any two ξ, ζ twice continuously differentiable functions, we find that

$$D^2 w_j = \phi^{2\alpha_j} D^2 z_j + z_j D^2(\phi^{2\alpha_j}) + (\nabla \phi^{2\alpha_j})(\nabla z_j)^\intercal + (\nabla z_j)(\nabla \phi^{2\alpha_j})^\intercal \quad \text{in } \omega.$$

Since the chain rule implies that

$$D^2(\phi^{2\alpha_j}) = 2\alpha_j(2\alpha_j - 1)\phi^{2\alpha_j - 2}(\nabla \phi)(\nabla \phi)^\intercal + 2\alpha_j \phi^{2\alpha_j - 1} D^2 \phi,$$

we arrive at

$$\frac{a_{3,j,w_j}}{\phi^{2\alpha_j}} = a_{3,j,z_j} - 4\alpha_j(p-2)\frac{\langle \nabla u_j, \nabla \phi \rangle \langle \nabla u_j, \nabla z_j \rangle}{z_j \phi} + \Upsilon_j, \tag{3.20}$$

where we define Υ_j by

$$\Upsilon_j := -\frac{2\alpha_j(p-2)}{\phi} \left[\langle (D^2\phi)(\nabla u_j), \nabla u_j \rangle + (2\alpha_j - 1)\frac{\langle \nabla \phi, \nabla u_j \rangle^2}{\phi} \right].$$

In view of (2.10), by taking $c''_j = 9\alpha_j|p-2|c'(1 + c'|2\alpha_j - 1|)/2$, we have

$$\Upsilon_j(x) \leq c''_j \frac{z_j}{|x|^2 \phi} \quad \text{for all } x \in \omega. \tag{3.21}$$

Using the identity that

$$\left\langle \nabla \left(\frac{\langle \nabla z_j, \nabla u_j \rangle}{z_j} \right), \nabla u_j \right\rangle = -\frac{\langle \nabla z_j, \nabla u_j \rangle^2}{z_j^2} + \frac{|\nabla z_j|^2}{2z_j} + \frac{\langle (D^2 z_j)(\nabla u_j), \nabla u_j \rangle}{z_j},$$

for all $x \in \omega$, we find that

$$-2\langle \nabla a_{3,j}, \nabla u_j \rangle = -a_{3,j,z_j} + (p-2)\left(\frac{1}{2} - \frac{\langle \nabla z_j, \nabla u_j \rangle^2}{z_j |\nabla z_j|^2} \right) \frac{|\nabla z_j|^2}{z_j}. \tag{3.22}$$

By adding (3.20) and (3.22), then using (3.21), we infer that

$$\mathcal{E}_{3,j}(x) \leq \frac{3|p-2|}{2} \frac{|\nabla z_j|^2}{z_j} + c''_j \frac{z_j}{|x|^2 \phi} + 4\alpha_j |p-2| \frac{|\nabla \phi||\nabla z_j|}{\phi} \quad \text{for all } x \in \omega. \tag{3.23}$$

If $\bar{c}_j := \widehat{c}_j + 3|\sigma|(2 + c'\alpha_j) + c''_j$, then by adding (3.17), (3.19) and (3.23), we conclude (3.12). This proves the Claim, thus completing the proof of Lemma 3.3. □

Lemma 3.4. *Let* (1.3) *hold. Fix* $\theta \in \mathbb{R}$ *such that* $0 < \theta < \min(k/(2(p-1)), 1/N)$. *There exists a positive constant* $M = M(m, N, p, q, \sigma, \tau)$ *such that for* $i = 0, 1, 2, 3$ *we can find positive constants* $d_i = d_i(m, N, p, q, \sigma, \tau)$ *so that for* $j = 1, 2$, *we have*

$$\mathcal{L}_j[w_j] \leq \left[-d_0(j-1)z_j^2 + d_1 \frac{z_j}{\phi |x|^2} - d_2 a_{1,j}^2(x) - d_3 \frac{|\nabla z_j|^2}{z_j} - 2\mathcal{W}_j(x) \right] \phi^{2\alpha_j}, \tag{3.24}$$

for every $x \in \omega$ *and* $\mathcal{W}_j$ *is given by*

$$\mathcal{W}_j(x) := h'_j(u_j) z_j f(x, z_j) + 2\theta a_{0,j}(x)\, a_{1,j}(x) \quad \text{for every } x \in \omega. \tag{3.25}$$

Proof. Recall that $\alpha_1 = 1/(2k)$, $\alpha_2 = 1/2$, and $\beta_{1,j}, \beta_{2,j}$ for $j = 1, 2$ are given respectively by (3.14) and (3.11). We fix $\varepsilon > 0$ small (see (3.39)) and $M > 3|p-2|/2$ large (see (3.40)) both depending only on m, N, p, q, σ, τ. From Lemmas 3.2 and 3.3, using (3.10) in (3.2), we find that

$$\frac{\mathcal{L}_j[w_j]}{\phi^{2\alpha_j}} \leq \beta_{1,j} \frac{z_j}{|x|^2\phi} - \beta_{2,j} \frac{|\nabla z_j|^2}{z_j} - 2\left(|D^2 u_j|^2 - 2\theta a_{0,j} a_{1,j}\right) + \mathcal{Z}_j - 2\mathcal{W}_j \quad \text{in } \omega, \tag{3.26}$$

where, we define $\mathcal{Z}_j(x)$ for all $x \in \omega$ and $j = 1, 2$ by

$$\mathcal{Z}_j(x) := \ 4\alpha_j a_{0,j} \frac{\langle \nabla u_j, \nabla \phi \rangle}{\phi} + 2\mathcal{Q}_j(x) a_{1,j}. \tag{3.27}$$

We will estimate $\left(|D^2 u_j|^2 - 2\theta a_{0,j} a_{1,j}\right)$ from below in Claim 1 and $\mathcal{Z}_j$ from above in Claim 2. These estimates will be used in (3.26) to obtain (3.37), from which (3.24) follows immediately.

Claim 1: *For all $x \in \omega$ and $j = 1, 2$, we have that*

$$|D^2 u_j|^2 - 2\theta a_{0,j} a_{1,j} \geq \left[(1-4\varepsilon)(a_{0,j}^2 + a_{1,j}^2) - \frac{\sigma^2 z_j}{\varepsilon |x|^2} - \frac{(p-2)^2}{4\varepsilon} \frac{|\nabla z_j|^2}{z_j}\right] \theta. \tag{3.28}$$

Proof of Claim 1: Given arbitrary $b_i \in \mathbb{R}$ for $i \in \{0, 1, 2, 3\}$, we have

$$\left(\sum_{i=0}^{3} b_i\right)^2 - 2b_0 b_1 \geq (1-4\varepsilon)(b_0^2 + b_1^2) - \frac{1}{\varepsilon} b_2^2 - \frac{1}{\varepsilon} b_3^2. \tag{3.29}$$

Indeed, (3.29) follows since for $(i, j) \in \{0, 1\} \times \{2, 3\}$, Young's inequality with ε implies

$$2b_i b_j \geq -2|b_i||b_j| \geq -\left(2\varepsilon b_i^2 + \frac{1}{2\varepsilon} b_j^2\right).$$

We observe that

$$a_{2,j}^2(x) \leq \frac{\sigma^2 z_j}{|x|^2} \quad \text{and} \quad a_{3,j}^2(x) \leq \frac{(p-2)^2}{4} \frac{|\nabla z_j|^2}{z_j} \quad \text{for every } x \in \omega. \tag{3.30}$$

Since $\theta \in (0, 1/N)$, from the inequality $|D^2 u_j|^2 \geq (\Delta u_j)^2/N$ in ω for $j = 1, 2$ and (2.6), we find

$$|D^2 u_j|^2 - 2\theta a_{0,j} a_{1,j} \geq \theta \left[\left(\sum_{i=0}^{3} a_{i,j}\right)^2 - 2a_{0,j} a_{1,j}\right]. \tag{3.31}$$

Taking $b_i = a_{i,j}$ for $i = 0, 1, 2, 3$ in (3.29), then using (3.30) and (3.31), we reach (3.28).

Claim 2: *With $\mathcal{Z}_j$ defined in* (3.27)*, we have*

$$\mathcal{Z}_j \leq \widetilde{c}_j \varepsilon a_{1,j}^2 + 6\alpha_j (j-1)(p-1)c' \varepsilon z_j^2 + \left(\frac{\widetilde{c}_j}{4\varepsilon} + \frac{3\alpha_j (j-1)(p-1)c'}{2\varepsilon}\right) \frac{z_j}{|x|^2 \phi} \tag{3.32}$$

for all $x \in \omega$, where $\widetilde{c}_j := 2|\sigma + \tau| + 3\alpha_j |m + 2 - p| c'$.

Proof of Claim 2: From (2.9) and (2.10), there exists a constant $c' = c'(N) > 0$ such that

$$|\nabla \phi(x)| \leq \frac{3c'}{2|x|} (\phi(x))^{\frac{1}{2}} \quad \text{for all } x \in \omega. \tag{3.33}$$

Thus, we find that

$$|\mathcal{Q}_j(x)| \leq \frac{|\sigma + \tau| z_j^{\frac{1}{2}}}{|x|} + \frac{3\alpha_j |m + 2 - p| c' z_j^{\frac{1}{2}}}{2|x| \phi^{\frac{1}{2}}} \leq \frac{\widetilde{c}_j}{2} \frac{z_j^{\frac{1}{2}}}{|x| \phi^{\frac{1}{2}}} \quad \text{for every } x \in \omega. \tag{3.34}$$

Using Young's inequality for $\chi_j = |a_{1,j}|$ or $\chi_j = z_j$ with $j = 1, 2$, we get

$$\chi_j \frac{z_j^{\frac{1}{2}}}{|x|\phi^{\frac{1}{2}}} \leq \varepsilon \chi_j^2 + \frac{z_j}{4\varepsilon |x|^2 \phi} \quad \text{for all } x \in \omega. \tag{3.35}$$

Hence, using (3.33), (3.34) and (3.35), we find for every $x \in \omega$

$$\begin{cases} 4\alpha_j a_{0,j} \dfrac{|\langle \nabla u_j, \nabla \phi \rangle|}{\phi} \leq 3(j-1)(p-1)\alpha_j c' \left(2\varepsilon z_j^2 + \dfrac{1}{2\varepsilon} \dfrac{z_j}{|x|^2 \phi} \right), \\ 2|\mathcal{Q}_j(x) a_{1,j}(x)| \leq \widetilde{c}_j |a_{1,j}(x)| \dfrac{z_j^{\frac{1}{2}}}{\phi^{\frac{1}{2}} |x|} \leq \widetilde{c}_j \left(\varepsilon a_{1,j}^2(x) + \dfrac{z_j}{4\varepsilon |x|^2 \phi} \right). \end{cases} \tag{3.36}$$

Thus, we obtain (3.32) from (3.36) as claimed.

Proof of Lemma 3.4 completed: By Claims 1 and 2, using (3.28) and (3.32) in (3.26), we get

$$\mathcal{L}_j[w_j] \leq \left[-d_0(j-1) z_j^2 + d_{1,j} \frac{z_j}{\phi |x|^2} - d_{2,j} a_{1,j}^2(x) - d_{3,j} \frac{|\nabla z_j|^2}{z_j} - 2\mathcal{W}_j(x) \right] \phi^{2\alpha_j}, \tag{3.37}$$

where the constants $d_{1,j}, d_{2,j}, d_{3,j}$ and d_0 are given by

$$\begin{cases} d_{1,j} = \beta_{1,j} + \dfrac{1}{\varepsilon} \left(2\theta\sigma^2 + \dfrac{\widetilde{c}_j}{4} + \dfrac{3\alpha_j (j-1)(p-1)c'}{2} \right), \quad d_{2,j} = 2\theta - (8\theta + \widetilde{c}_j)\varepsilon, \\ d_{3,j} = \beta_{2,j} - \dfrac{\theta (p-2)^2}{2\varepsilon}, \quad d_0 = (p-1) \left[2\theta(p-1) - (3c' + 8\theta(p-1))\varepsilon \right]. \end{cases} \tag{3.38}$$

Now, we fix $\varepsilon > 0$ depending only on m, N, p, q, σ, τ small enough such that

$$d_0 > 0, \quad 1 - 3\alpha_j c' \varepsilon > 0, \quad \text{and } d_{2,j} > 0 \quad \text{for } j = 1, 2. \tag{3.39}$$

Then we choose $M > 3|p-2|/2$ depending only on m, N, p, q, σ, τ such that

$$d_{3,j} > 0, \quad \text{where } d_{3,j} \text{ is given by (3.38) for } j = 1, 2. \tag{3.40}$$

By taking $d_1 = \max_{j=1,2} d_{1,j}$ and $d_i = \min_{j=1,2} d_{i,j}$ for $i = 2, 3$, we conclude (3.24) from (3.37). □

Remark 3.5. In Lemma 3.4, we have $\mathcal{W}_j(x) \geq 0$ for every $x \in \omega$ and for $j = 1, 2$. Indeed, when $j = 1$, we recall that $a_{0,j}(x) = 0$ and $h_j'(u_j) \geq 0$ for every $x \in \omega$. When $j = 2$, we use that $h_2'(t) = -k h_2(t) \geq 0$ and our choice of θ ensures that

$$h_2'(u_2) + 2\theta(p-1) h_2(u_2) = (2\theta(p-1) - k)\, h_2(u_2) > 0. \tag{3.41}$$

References

[1] A. Attouchi, 'Gradient estimate and a Liouville theorem for a p-Laplacian evolution equation with a gradient nonlinearity', *Differential Integral Equations* 29 (2016), 137–150.

[2] G. Barles, 'A weak Bernstein method for fully nonlinear elliptic equations', *Differential Integral Equations* 4 (1991), 241–262.

[3] S.N. Bernstein, 'Sur la généralisation du problème de Dirichlet. I', *Math. Ann.* 62 (1906), 153–271.

[4] S.N. Bernstein, 'Méthode générale pour la resolution du problème de Dirichlet', *C. R. Acad. Sci. Paris* 144 (1907), 1025–1027.

[5] S.N. Bernstein, 'Sur la généralisation du problème de Dirichlet. II', *Math. Ann.* 69 (1910), 82–136.

[6] M.F. Bidaut-Véron, M. Garcia-Huidobro and L. Véron, 'Local and global properties of solutions of quasilinear Hamilton-Jacobi equations', *J. Funct. Anal.* 267 (2014), 3294–3331.

[7] T.-Y. Chang and F.C. Cîrstea, 'Singular solutions for divergence-form elliptic equations involving regular variation theory: Existence and classification', *Ann. Inst. H. Poincaré Anal. Non Linéaire* 34 (2017), 1483–1506.

[8] J. Ching and F.C. Cîrstea, 'Existence and classification of singular solutions to nonlinear elliptic equations with a gradient term', *Anal. PDE* 8 (2015), 1931–1962.

[9] F.C. Cîrstea and Y. Du, 'Isolated singularities for weighted quasilinear elliptic equations', *J. Funct. Anal.* 259 (2010), 174–202.

[10] A. Farina and J. Serrin, 'Entire solutions of completely coercive quasilinear elliptic equations, II', *J. Differential Equations* 250 (2011), 4409–4436.

[11] A. Friedman and L. Véron, 'Singular solutions of some quasilinear elliptic equations', *Arch. Rational Mech. Anal.* 96 (1986), 359–387.

[12] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Grundlehren der Math. Wissenschaften 224 (Springer, Berlin, 1983).

[13] O.A. Ladyzhenskaya, 'Solution of the first boundary problem in the large for quasi-linear parabolic equations', *Trudy Moskov. Mat. Obsc.* 7 (1958), 149–177 [Russian]

[14] O.A. Ladyzhenskaya and N.N. Ural'tseva, 'Quasilinear elliptic equations and variational problems with several independent variables', *Uspehi Mat. Nauk.* 16 (1961) 19–90 [Russian]. English translation in *Russian Math. Surveys* 16 (1961), 17–92.

[15] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Moscow, Izdat. "Nauka", 1964 [Russian]. English Translation: New York, Academic Press 1968. 2nd Russian ed. 1973.

[16] P.L. Lions, 'Quelques remarques sur les problèmes elliptiques quasilinéaires du second ordre', *J. Analyse Math.* 45 (1985), 234–254.

[17] M. Marcus and P.-T. Nguyen, 'Elliptic equations with nonlinear absorption depending on the solution and its gradient', *Proc. London Math. Soc. (3)* 111 (2015), 205–239.

[18] P.-T. Nguyen, 'Isolated singularities of positive solutions of elliptic equations with weighted gradient term', *Anal. PDE* 9 (2016), 1671–1691.

[19] P. Pucci and J. Serrin, *The Maximum Principle*, Progress in Nonlinear Differential Equations and their Applications 73 (Birkhäuser, Basel, 2007).

[20] J. Serrin, 'The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables', *Philos. Trans. Roy. Soc. London Ser. A* 264 (1969), 413–496.

[21] H. Song, J. Yin and Z. Wang, 'Isolated singularities of positive solutions to the weighted p-Laplacian', *Calc. Var. Partial Differential Equations* 55 (2016), https://doi.org/10.1007/s00526-016-0971-1

[22] P. Tolksdorf, 'Regularity for a more general class of quasilinear elliptic equations', *J. Differential Equations* 51 (1984), 126–150.

[23] N.S. Trudinger, 'On Harnack type inequalities and their application to quasilinear elliptic equations', *Comm. Pure Appl. Math.* 20 (1967), 721–747.

School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia
E-mail address: `J.Ching@maths.usyd.edu.au`
E-mail address: `florica.cirstea@sydney.edu.au`