AN EXPLICIT FORMULA FOR A BRANCHED COVERING

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Abstract. We give an explicit formula for a 2-fold branched covering from $\mathbb{CP}^2$ to S^4, and relate it to other maps between quotients of $S^2 \times S^2$.

It is well known that the quotient of the complex projective plane $\mathbb{CP}^2$ by complex conjugation is the 4-sphere [2, 4]. (See also [1].) In the course of Massey's exposition he shows that $\mathbb{CP}^2$ is the quotient of $S^2 \times S^2$ by the involution which exchanges the factors, and that $\mathbb{RP}^2$ is similarly a quotient of $\mathbb{RP}^1 \times \mathbb{RP}^1$. Lawson has displayed very clearly the topology underlying these facts [3]. We shall use harmonic coordinates for real and complex projective spaces to give explicit formulae for some of these quotient maps. (We describe briefly the work of Massey and Kuiper at the end of this note.)

A smooth map $f : M \to N$ between closed n-manifolds is a 2-fold branched covering if M has a codimension-2 submanifold B (the branch locus), such that $f|_{M\setminus B}$ is a 2-to-1 immersion, $f|_B : B \to f(B)$ is a bijection onto a submanifold $f(B)$ (the branch set) and along B the map f looks like $(b, z) \mapsto (f(b), z^2)$ in local coordinates, with $b \in B$ and transverse complex coordinate $z \in \mathbb{C}$.

Let A be the antipodal involution of S^2, and let σ and τ be the diffeomorphisms of $S^2 \times S^2$ given by $\sigma(s, s') = (s', A(s))$ and $\tau(s, s') = (s', s)$, for $s, s' \in S^2$. Then σ and τ generate a dihedral group of order 8, since $\sigma^4 = \tau^2 = 1$ and $\tau\sigma\tau = \sigma^{-1}$.

We shall view S^2 as the unit sphere in $\mathbb{C} \times \mathbb{R}$. The stereographic projection $\gamma : S^2 \to \mathbb{CP}^1$ is given by $\gamma(z, t) = [z : 1 - t]$, for $(z, t) \in S^2$, and its inverse is

$$\gamma^{-1}([u : v]) = (\frac{2u\bar{v}}{|u|^2 + |v|^2}, \frac{|u|^2 - |v|^2}{|u|^2 + |v|^2}),$$

for $[u : v] \in \mathbb{CP}^1$. The action of the antipodal map on $\mathbb{CP}^1$ is given by

$$\gamma A \gamma^{-1}([u : v]) = [-\bar{v} : \bar{u}],$$

for $[u : v] \in \mathbb{CP}^1$.

Let α, β and $f : \mathbb{C}^2 \to \mathbb{C}^2$ be the maps given by $\alpha(w, z) = (\frac{w-z}{2}, \frac{w+z}{2})$, $\beta(w, z) = (z, z^2 - 4w)$ and $f(w, z) = (wz, w + z)$, for $(w, z) \in \mathbb{C}^2$. Then α and β are biholomorphic, and $\beta f \alpha(w, z) = (w, z^2)$. Therefore f is a 2-fold branched covering, branched over $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_2^2 = 4z_1\}$. The extension $\hat{f} : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$ given by

$$\hat{f}([u : v], [u' : v']) = [uu' : uv' + u'v : vv']$$

is a 2-fold branched covering, branched over $\hat{f}(\Delta)$, where $\Delta \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is the diagonal. (Thus $\hat{f}(\Delta)$ is the image of $\mathbb{CP}^1$ in $\mathbb{CP}^2$ under the Segre embedding.) The

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composite $\lambda = \hat{f}(\gamma \times \gamma)$ is essentially Lawson's map, giving

$$S^2 \times S^2/\langle\tau\rangle \cong \mathbb{CP}^1 \times \mathbb{CP}^1/\langle\gamma\tau\gamma^{-1}\rangle \cong \mathbb{CP}^2.$$

Let $c_n : \mathbb{CP}^n \to \mathbb{CP}^n$ be complex conjugation, for $n \geq 1$. Then $\hat{f}(c_1 \times c_1) = c_2\hat{f}$. Lawson observed that if $\theta : \mathbb{CP}^2 \to \mathbb{CP}^2$ is the linear automorphism given by

$$\theta([u : v : w]) = [(iu + w) : (1 - i)v : (u + iw)]$$

then

$$\lambda\sigma^2 = \theta^2 c_2 \lambda = \theta c_2 \theta^{-1}\lambda.$$

Hence c_2 is conjugate to a map covered by the *free* involution σ^2. (Note that $\theta^2([u : v : w]) = [w : -v : u]$, $c_2\theta c_2 = \theta^{-1}$ and $\theta^4 = id_{\mathbb{CP}^2}$.)

If we identify $\mathbb{C}$ with $\mathbb{R}^2$ and use *real* harmonic coordinates we may instead extend f to a map $g : \mathbb{RP}^2 \times \mathbb{RP}^2 \to \mathbb{RP}^4$, given by

$$g([r : s : t], [r' : s' : t']) = [rr' - ss' : rs' + sr' : rt' + tr' : st' + ts' : tt'].$$

This is a 2-fold branched covering, with branch locus the diagonal, and induces a diffeomorphism

$$\mathbb{RP}^2 \times \mathbb{RP}^2/(x, y) \sim (y, x) \cong \mathbb{RP}^4.$$

The map g has a lift $\tilde{g} : S^2 \times S^2 \to S^4$, given by

$$\tilde{g}((r, s, t), (r', s', t')) = \nu(rr' - ss', rs' + sr', rt' + tr', st' + ts', tt'),$$

where $\nu : \mathbb{R}^5 \setminus \{O\} \to S^4$ is the radial normalization. (If $(r, s, t), (r', s', t') \in S^2$ then the norm of $(rr' - ss', rs' + sr', rt' + tr', st' + ts', tt')$ is $\sqrt{1 + 2tt'(rr' + ss')}$.) We may also obtain $\tilde{g}$ by normalizing the map

$$((z, t), (z', t')) \mapsto (zz', zt' + tz', tt') \in \mathbb{C}^2 \times \mathbb{R} \setminus \{O\}, \quad \forall (z, t), (z', t') \in S^2.$$

This map is invariant under σ^2 and τ, and is generically 4-to-1. Hence it factors through a map $g^+ : S^2 \times S^2/\langle\sigma^2\rangle \to S^4$, and induces maps $G = \tilde{g}\lambda^{-1} : \mathbb{CP}^2 \to S^4$ and $h : S^2 \times S^2/\langle\sigma\rangle \to \mathbb{RP}^4$. The latter three maps are each 2-fold branched coverings.

The five nontrivial subgroups of the group D_8 generated by σ and τ that do *not* contain τ (namely, $\langle\sigma\tau\rangle$, $\langle\tau\sigma\rangle$, $\langle\sigma^2\rangle$, $\langle\sigma\rangle$ and $\langle\sigma\tau, \tau\sigma\rangle$) each act freely. The lattice of quotients is:

$$\begin{tikzcd}
 & & & S^2 \times S^2 \arrow[dlll, dash] \arrow[dl, dash] \arrow[dr, dash] \arrow[drrr, dash, "\lambda"] & & & \\
S^2 \times RP^2 \arrow[dr, dash] & & RP^2 \times S^2 \arrow[dl, dash] & & S^2 \times S^2/\langle\sigma^2\rangle \arrow[dlll, dash] \arrow[dl, dash] \arrow[dr, dash, "g^+"] & & \mathbb{CP}^2 \arrow[dl, dash, "G"] \\
 & RP^2 \times RP^2 \arrow[drr, dash, "g"'] & & S^2 \times S^2/\langle\sigma\rangle \arrow[d, dash, "h"'] & & S^4 \arrow[dll, dash] & \\
 & & & \mathbb{RP}^4 & & &
\end{tikzcd}$$

The unlabelled maps are double coverings, and the diagram commutes. (The part of this diagram involving the vertices $S^2 \times S^2$, $\mathbb{CP}^2$, $\mathbb{RP}^2 \times \mathbb{RP}^2$, $\mathbb{RP}^4$ and S^4 is displayed in [4].)

On the affine piece $U_2 = \{[z_0 : z_1 : z_2] \mid z_2 \neq 0\} \cong \mathbb{C}^2$ we have

$$f^{-1}([p:q:1]) = ([\frac{1}{2}(q \pm \sqrt{q^2-4p}) : 1], [\frac{1}{2}(q \mp \sqrt{q^2-4p}) : 1]).$$

Hence

$$G([p:q:1]) = \nu(4p, 2p\bar{q} - 2q, p\bar{p} + 1 - \frac{1}{2}q\bar{q} - \frac{1}{2}|q^2 - 4p|)$$

on U_2. Homogenizing this formula gives

$$G([p:q:r]) = \nu(4p\bar{r}, 2p\bar{q} - 2q\bar{r}, p\bar{p} + r\bar{r} - \frac{1}{2}q\bar{q} - \frac{1}{2}|q^2 - 4pr|).$$

The argument of ν is nonzero when $(p,q,r) \neq 0$, and its length is the square root of an homogeneous quartic polynomial in the real and imaginary parts of the harmonic coordinates of $\mathbb{CP}^2$. Thus G is a real analytic function, and it is 2-to-1 on a dense open subset of its domain. Its essential structure is most easily seen after using θ to make a linear change of coordinates. Let $\eth = G\theta$. Then $\eth c_2 = \eth$, and $\eth$ is a 2-fold branched covering, with branch locus $\mathfrak{Re}(\mathbb{CP}^2) \cong \mathbb{RP}^2$, the set of real points of $\mathbb{CP}^2$. The complement of $\mathfrak{Re}(\mathbb{CP}^2)$ in $\mathbb{CP}^2$ is simply connected, and so $\pi_1(S^4 \setminus \eth(\mathbb{RP}^2)) = Z/2Z$, since c_2 acts freely on $\mathbb{CP}^2 \setminus \mathfrak{Re}(\mathbb{CP}^2)$. Thus S^4 is the quotient of $\mathbb{CP}^2$ by complex conjugation [2, 4].

Remark. The main step in [4] used a result on fixed point sets of involutions of symmetric products to obtain a diffeomorphism $\mathbb{RP}^2 \times \mathbb{RP}^2/(x,y) \sim (y,x) \cong \mathbb{RP}^4$. Our contribution has been the explicit branched covering $g : \mathbb{RP}^2 \times \mathbb{RP}^2 \to \mathbb{RP}^4$, and the subsequent formula for G. The argument in [2] was very different. Let $\eta : \mathbb{C}^3 \to \mathbb{R}^6$ be the function given by

$$\eta(z_1, z_2, z_3) = (|z_1|^2, |z_2^2|, |z_3^2|, \mathfrak{Re}(z_2\bar{z_3}), \mathfrak{Re}(z_3\bar{z_1}), \mathfrak{Re}(z_1\bar{z_2})).$$

Then $\eta(\zeta v) = \eta(v)$ for all $v \in \mathbb{C}^3$ and $\zeta \in S^1$, so $\eta|_{S^5}$ factors through $\mathbb{CP}^2 = S^5/S^1$. Moreover, $\eta(\bar{v}) = \eta(v)$ for all $v \in \mathbb{C}^3$, so $\eta|_{S^5}$ factors through $\mathbb{CP}^2/\langle c_2\rangle$. Kuiper then showed that $\eta(S^5)$ lies in the affine hyperplane defined by $x_1 + x_2 + x_3 = 1$, and is the boundary of the convex hull of the Veronese embedding of $\mathbb{RP}^2$ in $\mathbb{R}^5$. Hence η induces a PL homeomorphism from $\mathbb{CP}^2/\langle c_2\rangle$ to S^4.

References

[1] Akbulut, S. and Kirby, R.C. Branched covers of surfaces in 4-manifolds, Math. Ann. 252 (1979/80), 111–131.

[2] Kuiper, N. The quotient space of $CP(2)$ by complex conjugation is the 4-sphere, Math. Ann. 208 (1974), 175–177.

[3] Lawson, T. Splitting S^4 on RP^2 via the branched cover of CP^2 over S^4, Proc. Amer. Math. Soc. 86 (1982), 328–330.

[4] Massey, W. The quotient space of the complex projective plane under conjugation is the 4-sphere, Geom. Dedicata 2 (1973), 371–374.

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