KURDYKA-ŁOJASIEWICZ-SIMON INEQUALITY FOR GRADIENT FLOWS IN METRIC SPACES

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Abstract. This paper is dedicated to providing new tools and methods for studying the trend to equilibrium of gradient flows in metric spaces $(\mathfrak{M}, d)$ in the *entropy* and *metric* sense, to establish *decay rates*, *finite time of extinction*, and to characterize *Lyapunov stable equilibrium points*. More precisely, our main results are.

- Introduction of a *gradient inequality* in the metric space framework, which in the Euclidean space $\mathbb{R}^N$ is due to Łojasiewicz [Éditions du C.N.R.S., 87-89, Paris, 1963] and Kurdyka [Ann. Inst. Fourier, 48 (3), 769-783, 1998].
- Establish trend to equilibrium in the *entropy* and *metric* sense of gradient flows generated by a functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ satisfying a Kurdyka-Łojasiewicz inequality in a neighborhood of an equilibrium point of $\mathcal{E}$. In particular, sufficient conditions are given implying *decay rates* and *finite time of extinction* of gradient flows.
- Construction of a *talweg curve* in $\mathfrak{M}$ with an *optimal* growth function yielding the validity of a Kurdyka-Łojasiewicz inequality.
- Characterize *Lyapunov stable* equilibrium points of energy functionals satisfying a Kurdyka-Łojasiewicz inequality near such points.
- Characterization of the *entropy-entropy production inequality* with the Kurdyka-Łojasiewicz inequality near equilibrium points of $\mathcal{E}$.

As an application of these results, the following results are established.

- New upper bounds on the extinction time of gradient flows associated with the total variational flow.
- If the metric space $\mathfrak{M}$ is the p-Wasserstein space $\mathcal{P}_p(\mathbb{R}^N)$, $1 < p < \infty$, then new HWI-, Talagrand-, and logarithmic Sobolev inequalities are obtained for functionals $\mathcal{E}$ associated with *nonlinear diffusion* problems modeling drift, potential and interaction phenomena.

  It is shown that these inequalities are equivalent to the Kurdyka-Łojasiewicz inequality and so, imply trend to equilibrium of the gradient flows of $\mathcal{E}$ with decay rates or arrive in finite time.

Contents

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1. Introduction

The long-time asymptotic behavior of gradient flows is one fundamental task in the study of gradient systems and often studied separately from well-posedness. Given a metric space $(\mathfrak{M}, d)$, we call a curve v a *gradient flow* in $\mathfrak{M}$ if there is an *energy* functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ such that v is a 2-*curve of maximal slope of* $\mathcal{E}$ ([7], see Definition 2.12); in brief, $v : [0, +\infty) \to \mathfrak{M}$ is locally absolutely continuous with *metric derivative* $|v'| \in L^2_{loc}(0, +\infty)$ (see Definition 2.5), $\mathcal{E} \circ v$ is locally absolutely continuous, and there is a second functional $g : \mathfrak{M} \to [0, +\infty]$ called *strong upper gradient of* $\mathcal{E}$ satisfying

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(v(t))\right| \le g(v(t))\, |v'|(t) \qquad \text{for a.e. } t > 0,$$

(see Definition 2.8) and such that *energy dissipation inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(v(t)) \le -\frac{1}{2}|v'|^2(t) - \frac{1}{2}g^2(v(t)) \qquad \text{holds for a.e. } t > 0. \tag{1.1}$$

This general notion of gradient flows in metric spaces is consistent with the notion of *strong solutions* of gradient systems in the *Hilbert spaces framework* (developed in [19], cf [7, Corollary 1.4.2]):

$$v'(t) + \partial\mathcal{E}(v(t)) \ni 0 \qquad \text{for } t > 0, \tag{1.2}$$

where $\partial\mathcal{E}$ is a *sub-differential operator* of a semi-convex, proper, lower semi-continuous functional $\mathcal{E}$ on a Hilbert space (see Section 4.1).

Since the pioneering work [43] by Jordan, Kinderlehrer and Otto, we know that solutions of some diffusion equations for probability distributions can also be derived as gradient flows of a given functional $\mathcal{E}$ with respect to a differential structure induced by the p-Wasserstein space $(\mathcal{P}_p(M), W_p)$. Thus, we call this setting the *metric space framework* (see also [60, 7] and Section 4.2).

While for the Hilbert space framework, an extensive literature on determining the long-time asymptotic behavior of gradients flows already exists (for instance, see [39, 38] and references therein), it seems that the methods available for the metric space framework (cf, for instance, [12] or [61]) are mainly based on the idea in establishing global *entropy (entropy production/transport) inequalities*.

Before passing to the main results of this article, we review the concept of the classical *entropy method* as it is used, for instance, in studying the convergence to equilibrium of solutions of kinetic equations as the Boltzmann or Fokker-Planck equation (cf [12, §3], [59], [60, §9], or [21, 52, 61]). This will reveal the connection of the famous *entropy-entropy production inequality* (see (1.3) below) and the *Kurdyka-Łojasiewicz-Simon inequality* (see (1.6) below), which is the main object of this paper.

With the task to determine the asymptotic behavior for large time of gradient flows, the following three problems arise naturally.

(I) Does every gradient flow $v(t)$ trend to an equilibrium φ of $\mathcal{E}$ as $t \to +\infty$?
(II) In which sense or topology the trend of $v(t)$ to φ holds as $t \to +\infty$?
(III) What is the *decay rate* of $d(v(t), \varphi)$ as $t \to +\infty$?

Due to inequality (1.1), $\mathcal{E}$ is a *Lyapunov functional* of each of its gradient flow curve v, that is, $\mathcal{E} \circ v$ is monotonically decreasing $[0, +\infty)$. In addition, if $\mathcal{E}$ and g are both lower semicontinuous on $\mathfrak{M}$, then $\mathcal{E}$ is even a *strict Lyapunov functional*, that is, the condition $\mathcal{E} \circ v \equiv$ const. on $[t_0, +\infty)$ for some $t_0 \geq 0$ implies that $v \equiv$ const. on $[t_0, +\infty)$ (cf Proposition 2.38) and so, the *ω-limit set*

$$\omega(v) := \Big\{ \varphi \in \mathfrak{M} \,\Big|\, \text{ there is } t_n \uparrow +\infty \text{ s.t. } \lim_{n \to \infty} d(v(t_n), \varphi) = 0 \Big\}$$

is contained in the set $\mathbb{E}_g := g^{-1}(\{0\})$ of *equilibrium points* of $\mathcal{E}$ with respect to the strong upper gradient g (see Definition 2.36). Therefore, Problem (I) is positively answered.

Given that the strict Lyapunov functional $\mathcal{E}$ of a gradient flow v attains an equilibrium point at $\varphi \in \omega(v)$, one possibility to measure the discrepancy between v and the equilibrium φ is given by *relative entropy*

$$\mathcal{E}(v(t)|\varphi) := \mathcal{E}(v(t)) - \mathcal{E}(\varphi) \geq 0.$$

Thus concerning Problem (II), there are two types of *trend to equilibrium*; namely the following (*weaker* type)

(1) **Trend to equilibrium in the entropy sense**; a gradient flow v of $\mathcal{E}$ is said to *trend to equilibrium φ in the entropy sense* if

$$\mathcal{E}(v(t)|\varphi) \to 0 \qquad \text{as } t \to +\infty,$$

and the (*stronger* type)

(2) **Trend to equilibrium in the metric sense**: a gradient flow v is said to *trend to equilibrium φ of $\mathcal{E}$ in the metric sense* if

$$\lim_{t \to +\infty} d(v(t), \varphi) = 0.$$

Now, the classical *entropy method* suggests to find a strictly increasing function $\Phi \in C([0, +\infty))$ satisfying $\Phi(0) = 0$ such that $\mathcal{E}$ satisfies a (global)

entropy-entropy production/dissipation (EEP-)inequality

$$\mathbb{D}(v) \geq \Phi(\mathcal{E}(v|\varphi)) \tag{1.3}$$

for all $v \in D(\mathcal{E})$ at an equilibrium point φ. Here, the map $\mathbb{D} : \mathfrak{M} \to [0, +\infty]$ is called the *entropy production functional* and satisfies

$$\mathbb{D}(v(t)) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(v(t)) = g^2(v(t)) \qquad \text{for } t > 0 \tag{1.4}$$

for every gradient flow v of $\mathcal{E}$ with strong upper gradient g (cf Proposition 2.16 in Section 2.3 below). Combining (1.3) with (1.4), rearranging and then integrating the resulting inequality yields for each curve v generated by $\mathcal{E}$ trend to equilibrium φ in the entropy sense.

If the function Φ in (1.3) is known, then the entropy method has certainly the advantage that EEP-inequality (1.3) provides decay rates (Problem (III)) to the trend to equilibrium in the entropy sense. For instance, if Φ is *linear*, that is, $\Phi(s) = \lambda s$, $(s \geq 0,\ \lambda > 0)$, then EEP-inequality (1.3) implies *exponential decay rates* and if Φ is *polynomial* $\Phi(s) = Ks^{1+\alpha}$, $(\alpha \in (0, 1),\ K > 0)$, then EEP-inequality (1.3) implies that $\mathcal{E}(v(t)|\varphi)$ decays to 0 at least like $t^{-1/\alpha}$ (*polynomial decay*).

On the other hand, the existence of a Φ such that $\mathcal{E}$ satisfies a (global) EEP-inequality (1.3), requires in both the Hilbert and metric space framework additional rather strong conditions on $\mathcal{E}$; for instance, if $\mathcal{E}$ is defined on the Hilbert space $L^2(\Omega)$, (where $\Omega \subseteq \mathbb{R}^N$ is a smooth open domain with a smooth boundary), then a *Poincaré-Sobolev inequality* associated with $\mathcal{E}$ leads to EEP-inequality (1.3) (see Section 4.1.1), or if for some $\lambda > 0$, $\mathcal{E}$ is λ*-geodesically convex* on the p-Wasserstein space $\mathcal{P}_p(\Omega)$ (see Definition 2.22).

Now, to obtain trend to equilibrium in the metric sense, an **entropy-transportation (ET-)inequality**

$$d(v, \varphi) \leq \Psi(\mathcal{E}(v|\varphi)), \qquad (v \in D(\mathcal{E})), \tag{1.5}$$

can be very useful (see, for instance, [22, 60] or [3]), where $\Psi \in C[0, +\infty)$ is a strictly increasing function satisfying $\Psi(0) = 0$. If Φ and Ψ are known, then *decay estimates* to the trend to equilibrium in the metric sense can be derived by combining the two inequalities (1.3) and (1.5).

However, there are many important examples of functionals $\mathcal{E}$ that do not satisfy a *global* EEP-inequality (1.3) (see, for instance, [52, Theorem 2, p 21] or [61, p 98]). In particular, if $\mathcal{E}$ is not λ-geodesically convex for a $\lambda > 0$, then it might not have a unique global equilibrium point. But the study of trend to equilibrium of gradient flows generated by this class of functionals remain important (open) problems and requires more sophisticated tools and arguments.

Our approach to attack the above mentioned problems is via a functional inequality, which in the Hilbert space framework is known as the **Kurdyka-Łojasiewicz (KŁ-)inequality**: a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g and equilibrium point $\varphi \in D(\mathcal{E})$ is said to satisfy a *Kurdyka-Łojasiewicz inequality* on a set $\mathcal{U} \subseteq g^{-1}((0, +\infty)) \cap \big\{v \in \mathfrak{M} \,\big|\, \theta'(\mathcal{E}(\cdot|\varphi)) > 0\big\}$ if there is a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ such that

$$\theta'(\mathcal{E}(v|\varphi))\, g(v) \geq 1 \qquad \text{for all } v \in \mathcal{U}. \tag{1.6}$$

Since in the literature, EEP-inequality (1.3) is rather used from the community studying kinetic equations, while Kurdyka-Łojasiewicz inequality (1.6) is rather familiar for communities from *algebraic geometry* classifying singularities of manifolds or by groups studying evolution equations which can be written in a Hilbert space setting, it is important to stress that both communities actually work with the same inequality. To see this, recall that in the metric space framework, the entropy production functional $\mathbb{D}$ of a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g is given by (1.4), which in applications g, is usually given by the descending slope $|D^{-}\mathcal{E}|$ of $\mathcal{E}$ (see Definition 2.23). Thus, if θ satisfies, in addition, that $\theta \in C^1((0, +\infty))$ and $\lim_{s\to 0+} \theta'(s) = +\infty$, then

KŁ-inequality (1.6) *is, in fact, EEP-inequality* (1.3) *for* $\Phi(s) := \frac{1}{(\theta'(s))^2}$.

In the pioneering works [46, 47], Łojasiewicz showed that every *real-analytic* energy functional $\mathcal{E} : \mathcal{U} \to \mathbb{R}$ defined on a open subset $\mathcal{U} \subseteq \mathbb{R}^N$ satisfies near each equilibrium point $\varphi \in \mathcal{U}$ a **Łojasiewicz (Ł-)inequality**

$$|\mathcal{E}(v|\varphi)|^{1-\alpha} \le C\, \|\nabla\mathcal{E}(v)\|_{\mathbb{R}^N} \quad \text{for all } v \in \tilde{\mathcal{U}}_r := \{v \in \mathcal{U}\,|\,|v-\varphi| < r\} \tag{1.7}$$

for some exponent $\alpha \in (0, 1/2]$ and $r > 0$. In these two papers, Łojasiewicz developed a method proving that the (*local*) validity of *gradient inequality* (1.7) in a neighborhood $\tilde{\mathcal{U}}_r$ of φ is sufficient for establishing convergence to equilibrium in the metric sense. Simon [54] was the first who made Łojasiewicz's gradient-inequality (1.7) available for evolution problems formulated in an infinite dimensional Hilbert space framework and generalized Łojasiewicz's method. As an application, Simon established the long-time convergence in the metric sense of analytic solutions of semi-linear parabolic equations and of geometric flows. His ideas were further developed by many authors (see, for instance, [42, 40, 41, 39] concerning the long-time asymptotic behavior of solutions of semi-linear heat and wave equations, see [35] concerning gradient flows associated with geometric flows).

The condition $\mathcal{E}$ *being real-analytic* is rather a geometric property than a regularity property of $\mathcal{E}$. This is well demonstrated by the gradient system in $\mathbb{R}^2$ given by the *Mexican hat* functional $\mathcal{E}$ due to Palis and de Melo [51, p 14]. In this example, $\mathcal{E}$ belongs to the class $C^\infty(\mathbb{R}^2)$, but $\mathcal{E}$ admits a bounded gradient flow v with an ω-limit set $\omega(v)$ which is isomorphic to the unit circle S^1. The geometric properties of real-analytic functions and Ł-inequality (1.7) were studied systematically with tools from algebraic geometry and generalized to the class of *definable* functionals (see [58]). By introducing the concept of a *talweg curve*, Kurdyka [44] showed that every definable C^1 functional $\mathcal{E}$ on $\mathcal{U} \subseteq \mathbb{R}^N$ satisfies near every equilibrium point a KŁ-inequality (1.6) and with this, he established of every bounded definable gradient flow in $\mathbb{R}^N$, the trend to equilibrium in the metric sense.

First versions of *local* and *global* KŁ-inequality (1.6) for proper, lower semicontinuous, and (semi-)convex functionals $\mathcal{E} : H \to (-\infty, +\infty]$ on a Hilbert space H were introduced by Bolte et al. [16] (see also [24]). They adapted Kurdyka's notion of a talweg curve to the Hilbert space framework and characterized the validity of a KŁ-inequality (1.6) with the existence of a talweg. In addition, first formulations of KŁ-inequality (1.6) in a metric space framework were also given in [16] (see also the recent work [15]).

In this paper, we introduce *local* and *global* KŁ-inequalities (1.6) for proper functionals $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ defined on a metric space $(\mathfrak{M}, d)$ (see Definition 3.2 in Section 3.1). Our definition here is slightly different to the one in [16, 15], but consistent with one in the Hilbert space framework given, for instance, by [24]. This enables us to provide new fine tools for determining the trend to equilibrium in both the *entropy sense* (in Section 2.5) and the *metric sense* (in Section 3.5) of gradient flows in $\mathfrak{M}$. More precisely, we show in Theorem 3.5 that if $\mathcal{E}$ is bounded from below and satisfies a KŁ-inequality (1.6) on a set $\mathcal{U}$ then every gradient flow v of $\mathcal{E}$ satisfying $v([t_0, +\infty)) \subseteq \mathcal{U}$ for some $t_0 \geq 0$ has finite length. In Section 3.3, we study sufficient conditions implying that a functional $\mathcal{E}$ on $\mathfrak{M}$ satisfies KŁ-inequality (1.6) near an equilibrium point φ of $\mathcal{E}$. In particular, Theorem 3.11 provides optimal conditions on the *talweg curve* in $\mathfrak{M}$ ensuring the validity of a KŁ-inequality (1.6) by $\mathcal{E}$ near φ. In Section 3.4, we characterize the (local) validity of a KŁ-inequality (1.6) by a functional $\mathcal{E}$ with the existence of a talweg curve. We adapt Łojasiewicz's and Kurdyka's convergence method from [46, 47] and [44] to the metric space framework (Section 3.5) and establish the trend to equilibrium in the metric sense of every gradient flow of $\mathcal{E}$ (Theorem 3.17). We define Łojasiewicz's inequality (1.7) for proper functionals $\mathcal{E}$ on $\mathfrak{M}$ (see Definition 3.4) and deduce from it *decay rates* of the trend to equilibrium in the metric and entropy sense, and give upper bounds on the extinction time of gradient flows (Theorem 3.20). Note, these results are consistent with the Hilbert space framework (cf, for instance, [40, 27, 16]). Section 3.7 is concerned with the characterization of *Lyapunov stable* equilibrium points φ under the assumption that $\mathcal{E}$ satisfies a KŁ-inequality (1.6) in a neighborhood of φ, and in Section 3.8, we demonstrate that KŁ-inequality (1.6) is equivalent to a (generalized) ET-inequalities (1.5).

Before outlining some applications to the theory developed in Section 4.1 of this paper, we briefly review the example of the *linear Fokker-Planck equation*

$$\partial_t v = \Delta v + \nabla \cdot (v \nabla V) \qquad \text{on } M \times (0, +\infty), \tag{1.8}$$

where M a complete C^2-Riemannian manifold of dimension $N \geq 1$, $\mathcal{L}^N$ is the standard volume measure on M, and $V \in C^2(M)$ is a given potential. It was demonstrated in [43] (see also [60]) that if one fixes a *reference* probability measure $\nu = v_\infty \mathcal{L}^N \in \mathcal{P}_2(M)$ with $v_\infty = e^{-V}$, then solutions v of equation (1.8) can be written as the the probability distribution $v(t)$ of the gradient flows $\mu(t) = v(t)\, \mathcal{L}^N$ generated by the *Boltzmann H-functional*

$$\mathcal{H}(\mu|\nu) := \int_M \tfrac{\mathrm{d}\mu}{\mathrm{d}\nu} \log \tfrac{\mathrm{d}\mu}{\mathrm{d}\nu} \, \mathrm{d}\nu \qquad \text{for every } \mu = v\mathcal{L}^N \in \mathcal{P}_2(M). \tag{1.9}$$

Since

$$\mathcal{H}(\mu|\nu) = \int_M v \log v \, \mathrm{d}x + \int_M v\, V \, \mathrm{d}x =: \mathcal{E}(v),$$

one sees that EEP-inequality (1.3) is the **logarithmic Sobolev inequality**

$$\mathcal{H}(\mu|\nu) \leq \frac{1}{2\lambda} \mathcal{I}(\mu|\nu), \tag{1.10}$$

for some $\lambda > 0$, which holds true if $D^2(V) + \mathrm{Ric} \geq \lambda$ due to [14]. The functional $\mathcal{I}(\mu|\nu)$ in (1.10) is called the *relative Fisher information* of μ with respect to

ν and coincides with the entropy production

$$\mathbb{D}(\mu) = \int_M \left|\nabla \log \tfrac{\mathrm{d}\mu}{\mathrm{d}\nu}\right|^2 \mathrm{d}\mu = 4\int_M \left|\nabla \sqrt{\tfrac{\mathrm{d}\mu}{\mathrm{d}\nu}}\right|^2 \mathrm{d}\nu =: \mathcal{I}(\mu|\nu).$$

Thus, in other words, the logarithmic Sobolev inequality (1.10) is actually KŁ-inequality (1.6) for $\theta(s) = 2\,c|s|^{-\frac{1}{2}}s$. Moreover, ET-inequality (1.5) is, in fact, **Talagrand's entropy transportation inequality** (cf [49])

$$W_2(\mu,\nu) \le \sqrt{\tfrac{2}{\lambda}\mathcal{H}(\mu|\nu)}. \tag{1.11}$$

Thanks to Corollary 3.32 of this paper, Talagrand's inequality (1.11) and the logarithmic Sobolev inequality (1.10) are equivalent to each other.

Section 4 is dedicated to applications: Section 4.1 is concerned with the Hilbert space framework and Section 4.2 with the Wasserstein framework. Section 4.1.2 and 4.1.3 provide two examples of gradient flows associated with the *total variational flow*. These examples illustrate that an KŁ-inequality (1.6) involving known growth functions θ provide upper bounds on the extinction time of gradient flows. In Section 4.2.1, we establish new (generalized) *entropy transportation inequalities*, *logarithmic Sobolev inequalities* and *HWI-inequalities* (see Theorem 4.15 and cf [49, 3, 2] and [29]) associated with proper, lower semicontinuous energy functionals $\mathcal{E}$ defined on a p-Wasserstein spaces $\mathcal{P}_{p,d}(\Omega)$, ($\Omega \subseteq \mathbb{R}^N$ an open set and $1 < p < \infty$). The probability densities of the gradient flows of these functionals $\mathcal{E}$ are solutions of parabolic doubly nonlinear equations. We prove exponential decay rates and finite time of extinction of those gradient flows (Corollary 4.19). We note that all results in Section 4 provide new insights in the study of the examples.

During the preparation of this paper, we got aware of the recent work [15] by Blanchet and Bolte. They show in the case $\lambda = 0$ that for proper, lower semicontinuous and λ-*geodesically convex* functionals $\mathcal{E}$ on $\mathcal{P}_2(\mathbb{R}^N)$, a (local) Łojasiewicz -inequality is equivalent to a (local and generalized) ET-inequality (1.5). Then, as an application of this, they establish the equivalence between Talagrand's inequality (1.11) and the logarithmic Sobolev inequality (1.10) for the Boltzmann H-functional (1.9) on $\mathbb{R}^N$ for $V = 0$. The method in [15] is based on a similar idea to ours (cf Theorem 3.28), but our results presented in Section 3.8 are concerned with a much general framework. In our forthcoming paper, we show how these results help us to study the geometry of metric measure length space with a Ricci-curvature bound in the sense of Sturm [55, 56] and Lott-Villani [48].

We continue this paper with Section 2 by summarizing some important notions and results from [7] of the theory of gradient flows in metric spaces.

2. A brief primer on gradient flows in metric spaces

Throughout this section $(\mathfrak{M}, d)$ denotes a complete metric space provided nothing more specifically was mentioned and $-\infty \le a < b \le +\infty$.

2.1. Metric derivative of curves in metric spaces.

Here, $1 \le p \le \infty$. We begin with the following definition.

Definition 2.1. A curve $v : (a,b) \to \mathfrak{M}$ is said to belong to the class $AC^p(a,b;\mathfrak{M})$ if there exists an $m \in L^p(a,b)$ satisfying

$$d(v(s),v(t)) \le \int_s^t m(r)\,\mathrm{d}r \qquad \text{for all } a < s \le t < b. \tag{2.1}$$

Similarly, a curve $v : (a,b) \to \mathfrak{M}$ belongs to the class $AC^p_{loc}(a,b;\mathfrak{M})$ if there is an $m \in L^p_{loc}(a,b)$ satisfying (2.1).

Notation 2.2. In the case $p = 1$, we simply write $AC(a,b;\mathfrak{M})$ for the class $AC^1(a,b;\mathfrak{M})$ and $AC_{loc}(a,b;\mathfrak{M})$ instead of $AC^1_{loc}(a,b;\mathfrak{M})$.

Remark 2.3. Since $L^p_{loc}(a,b) \subseteq L^1_{loc}(a,b)$, the class $AC^p_{loc}(a,b;\mathfrak{M})$ is included in $AC_{loc}(a,b;\mathfrak{M})$. Every curve v of the class $AC(a,b;\mathfrak{M})$ is absolutely continuous on the open interval (a,b) and hence, uniformly continuous on (a,b). By the completeness of $\mathfrak{M}$, if $a > -\infty$, then there is an element $v(a+) \in \mathfrak{M}$ such that $\lim_{t\to a+} v(t) = v(a+)$ exists in $\mathfrak{M}$. Analogously, if $b < +\infty$, then there is $v(b-) \in \mathfrak{M}$ such that $\lim_{t\to b-} v(t) = v(b-)$ exists in $\mathfrak{M}$.

Proposition 2.4 ([7, Theorem 1.1.2], **Metric derivative**). *For every curve $v \in AC^p(a,b;\mathfrak{M})$, the limit*

$$|v'|(t) := \lim_{s\to t} \frac{d(v(s),v(t))}{|s-t|} \tag{2.2}$$

exists for a.e. $t \in (a,b)$ and $|v'| \in L^p(a,b)$ satisfies (2.1). Moreover, among all functions $m \in L^p(a,b)$ satisfying (2.1), one has $|v'|(t) \le m(t)$ for a.e. $t \in (a,b)$.

Definition 2.5. For a curves $v \in AC^p(a,b;\mathfrak{M})$, one calls the function $|v'| \in L^p(a,b)$ given by (2.2) the *metric derivative* of v.

Lemma 2.6 ([7, Lemma 1.1.4], **Arc-length reparametrization**). *Let $v \in AC(a,b;\mathfrak{M})$ with length*

$$\gamma = \gamma(v) := \int_a^b |v'|(t)\,\mathrm{d}t. \tag{2.3}$$

Then there exists an increasing absolutely continuous map

$$s : (a,b) \to [0,\gamma] \qquad \textit{satisfying } s(0+) = 0,\ s(b-) = \gamma,$$

and a curve $\hat{v} \in AC^\infty(0,\gamma;\mathfrak{M})$ such that

$$v(t) = \hat{v}(s(t)), \quad |\hat{v}'| = 1.$$

2.2. **Strong upper gradients of $\mathcal{E}$.** In this subsection, we introduce the first main tool for establishing the existence gradient flows in metric spaces.

Definition 2.7. A functional $\mathcal{E} : \mathfrak{M} \to (-\infty,\infty]$ is called *proper* if there is a $v \in \mathfrak{M}$ such that $\mathcal{E}(v) < +\infty$ and the set $D(\mathcal{E}) := \{v \in \mathfrak{M}\,|\,\mathcal{E}(v) < +\infty\}$ is called the *effective domain* of $\mathcal{E}$.

Definition 2.8 (**Strong upper gradient**). For a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty,\infty]$, a proper functional $g : \mathfrak{M} \to [0,+\infty]$ is called a *strong upper gradient of $\mathcal{E}$* if for every curve $v \in AC(0,+\infty;\mathfrak{M})$, the composition function $g \circ v : (0,+\infty) \to [0,\infty]$ is Borel-measurable and

$$|\mathcal{E}(v(t)) - \mathcal{E}(v(s))| \le \int_s^t g(v(r))\,|v'|(r)\,\mathrm{d}r \qquad \text{for all } a < s \le t < b. \tag{2.4}$$

Remark 2.9.

(1) We always assume that the effective domain $D(g)$ of a strong upper gradient g of $\mathcal{E}$ is a subset of the effective domain $D(\mathcal{E})$ of $\mathcal{E}$.
(2) In Definition 2.8, it is not assumed that

$$(g \circ v)\,|v'| \in L^1_{loc}(a,b). \tag{2.5}$$

Thus, the value of the integral in (2.4) might be infinity.

The following notion is taken from [8].

Definition 2.10. For a given functional $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$, the *local Lipschitz constant* at $u \in D(\mathcal{E})$ is given by

$$|D\mathcal{E}|(u) := \limsup_{v \to u} \frac{|\mathcal{E}(v) - \mathcal{E}(u)|}{d(v,u)}$$

and $|D\mathcal{E}|(u) := +\infty$ for every $u \in \mathfrak{M} \setminus D(\mathcal{E})$. Similarly, the *ascending slope* $|D^+\mathcal{E}| : \mathfrak{M} \to [0, +\infty]$ of $\mathcal{E}$ is defined by

$$|D^+\mathcal{E}|(u) := \begin{cases} \limsup\limits_{v \to u} \dfrac{[\mathcal{E}(v) - \mathcal{E}(u)]^+}{d(v,u)} & \text{if } u \in D(\mathcal{E}), \\ +\infty & \text{if otherwise} \end{cases}$$

and the *descending slope* $|D^-\mathcal{E}| : \mathfrak{M} \to [0, +\infty]$ of $\mathcal{E}$ is given by

$$|D^-\mathcal{E}|(u) := \begin{cases} \limsup\limits_{v \to u} \dfrac{[\mathcal{E}(v) - \mathcal{E}(u)]^-}{d(v,u)} & \text{if } u \in D(\mathcal{E}), \\ +\infty & \text{if otherwise.} \end{cases}$$

We come back to the descending slope $|D^-\mathcal{E}|$ in Section 2.4. Our next proposition follows immediately from Definition 2.8. We state this standard result for later use.

Proposition 2.11. *If $g : \mathfrak{M} \to [0, +\infty]$ is a strong upper gradient of a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$, then for every $v \in AC_{loc}(a, b; \mathfrak{M})$ satisfying (2.5), the composition function $\mathcal{E} \circ v : (a, b) \to (-\infty, \infty]$ is locally absolutely continuous and*

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(v(t)) \right| \le g(v(t))\,|v'|(t) \qquad \textit{for almost every } t \in (a, b). \tag{2.6}$$

2.3. ***p*-Gradient flows in metric spaces.** In this subsection, let $1 < p < \infty$ and $p' := \frac{p}{p-1}$ be the *Hölder conjugate* of p and we focus on the case $a = 0$, $n = +\infty$.

We begin by recalling the following definition taken from [34] (see also [7]).

Definition 2.12. For a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ with strong upper gradient g, a curve $v \in AC_{loc}(0, +\infty; \mathfrak{M})$ is called a *p-curve of maximal slope of $\mathcal{E}$* if $\mathcal{E} \circ v : (0, +\infty) \to \mathbb{R}$ is non-increasing and

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(v(t)) \le -\tfrac{1}{p'} g^{p'}(v(t)) - \tfrac{1}{p}|v'|^p(t) \qquad \text{for almost every } t \in (0, +\infty). \tag{2.7}$$

Some authors (cf [8]) call Definition 2.12 the *metric formulation of a gradient flow*. Since we agree with this idea, we rather use the following notion.

Definition 2.13 (**p-gradient flows in $\mathfrak{M}$**)**.** For a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ with strong upper gradient g, a curve $v \in AC_{loc}(0, +\infty; \mathfrak{M})$ is called a *p-gradient flow of $\mathcal{E}$* or also a *p-gradient flow in $\mathfrak{M}$* if v is a p-curve of maximal slope of $\mathcal{E}$ with respect to g. For the sake of convenience, we call each 2-gradient flow in $\mathfrak{M}$, simply, *gradient flow*. Further, a p-gradient flow v of $\mathcal{E}$ has *initial value* $v_0 \in \mathfrak{M}$ if $v(0+) = v_0$.

We will also need the following more general version of p-gradient flows of $\mathcal{E}$ (cf [16, Definition 15] in the Hilbert space framework).

Definition 2.14 (**piecewise p-gradient flows in $\mathfrak{M}$**)**.** For $0 < T \le +\infty$, we call a curve $v : [0, T) \to \mathfrak{M}$ a *piecewise p-gradient flow* in $\mathfrak{M}$ if there is an energy functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ and a strictly increasing sequence $(t_n)_{n\ge0}$ such that $\mathcal{P} = \{0 = t_0, t_1, t_2, \dots\}$ is a countable partition of $[0, T)$ and for which the curve $v_n : (0, t_n) \to \mathfrak{M}$ defined by

$$v_n(t) := v(t + t_{n-1}) \qquad \text{for every } t \in [0, t_n),$$

is a p-gradient flow of $\mathcal{E}$ on $(0, t_n)$ for every $n \ge 1$, and for every i, $j \ge 1$ with $i \ne j$, the two intervals $\mathcal{E}(v_i([0, t_i]))$ and $\mathcal{E}(v_j([0, t_j]))$ have at most one intersection point.

Remark 2.15. We note that for every piecewise p-gradient flows $v : [0, T) \to \mathfrak{M}$, there is a sequence $(t_n)_{n\ge0}$ such that $\mathcal{P} = \{0 = t_0, t_1, t_2, \dots\}$ is a countable partition of $[0, T)$ for which

$$v \in AC_{loc}(t_{n-1}, t_n; \mathfrak{M}) \qquad \text{for every } n \ge 1.$$

Thus, by Remark 2.3, the right-hand and left-hand side limits $v(t_{n-1}+) \in \mathfrak{M}$ and $v(t_n-) \in \mathfrak{M}$ exist, but v is, in general, only a piecewise continuous curve, that is, for the partition $\mathcal{P}$ of $[0, T)$, one has that $v_{|[t_{n-1},t_n]} \in C([t_{n-1}, t_n]; \mathfrak{M})$.

For the sake of completeness, we give the following characterization of *p-gradient flows*, which is scattered in the literature. But we omit its easy proof.

Proposition 2.16. *Let $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ be proper functional with strong upper gradient g, and $v \in AC_{loc}(0, +\infty; \mathfrak{M})$. Then the following statements are equivalent.*

(1) v is a p-gradient flow of $\mathcal{E}$.

(2) $\mathcal{E} \circ v : (0, +\infty) \to \mathbb{R}$ is non-increasing and for all $0 < s < t < \infty$,

$$\mathcal{E}(v(s)) - \mathcal{E}(v(t)) \ge \tfrac{1}{p} \int_s^t |v'|^p(r)\,\mathrm{d}r + \tfrac{1}{p'} \int_s^t g^{p'}(v(r)),\, dr. \tag{2.8}$$

(3) $\mathcal{E} \circ v : (0, +\infty) \to \mathbb{R}$ is non-increasing and energy dissipation equality

$$\mathcal{E}(v(s)) - \mathcal{E}(v(t)) = \tfrac{1}{p} \int_s^t |v'|^p(r)\,\mathrm{d}r + \tfrac{1}{p'} \int_s^t g^{p'}(v(r))\, dr \tag{2.9}$$

holds for all $0 < s < t < +\infty$.

Moreover, for every p-gradient flow v of $\mathcal{E}$, one has that

$$v \in AC^p_{loc}(0, +\infty; \mathfrak{M}), \qquad g \circ v \in L^{p'}_{loc}(0, +\infty; \mathbb{R}),$$

and for almost every $t \in (0, +\infty)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(v(t)) = -|v'|^p(t) = -g^{p'}(v(t)) = -(g \circ v)(t) \cdot |v'|(t). \tag{2.10}$$

In particular, if $v(0+) \in D(\mathcal{E})$, then (2.9) holds for all $0 = s < t < +\infty$, and for every $T > 0$, one has $|v'| \in L^p(0,T)$ and $g \circ v \in L^{p'}(0,T)$.

Remark 2.17. Note, for every p-gradient flow v of $\mathcal{E}$, one has that

$$\mathcal{E}(v(s)) - \mathcal{E}(v(t)) = \int_s^t |v'|^p(r)\,\mathrm{d}r = \int_s^t g^{p'}(v(r))\,dr \tag{2.11}$$

for all $0 < s < t < +\infty$ due to (2.9) and (2.10).

Now, the existence of p-gradient flows in metric spaces needs additional assumptions on the functional $\mathcal{E}$ such as coercivity and lower semi-continuity. This was well elaborated in the book [7] and the existence of such curves was proved by using the concept of *minimizing movements* (due to De Giorgi [33], see also [5]). We briefly sketch the idea of this method.

For a sequence $\tau = \{\tau_n\}_{n\ge 1}$ with $\tau_n := t_\tau^n - t_\tau^{n-1} > 0$ of finite length

$$|\tau| := \sup_{n\ge 1} \tau_n \qquad \text{and such that} \qquad \lim_{n\to\infty} t_\tau^n = \sum_{k=1}^{\infty} \tau_k = +\infty,$$

one assigns a *partition*

$$\mathcal{P}_\tau := \Big\{0 = t_\tau^0 < t_\tau^1 < \cdots < t_\tau^n < \cdots\Big\} \qquad \text{of the interval } [0,\infty).$$

Then, for given $U_\tau^0 \in \mathfrak{M}$, consider the *recursive scheme*

$$\begin{cases} \text{if } U_\tau^1, U_\tau^2, \dots, U_\tau^{n-1} \text{ are known, then} \\ \text{find } U_\tau^n \in \operatorname{argmin}\left(\Phi_p(\tau_n, U_\tau^{n-1}; \cdot)\right), \end{cases} \tag{2.12}$$

where the proper functional $\Phi_p(\tau_n, U_\tau^{n-1}; \cdot) : \mathfrak{M} \to (-\infty, \infty]$ is given by

$$\Phi_p(\tau_n, U_\tau^{n-1}; U) = \frac{1}{p\tau_n^{p-1}} d^p(U_\tau^{n-1}, U) + \mathcal{E}(U) \quad \text{for every } U \in \mathfrak{M} \tag{2.13}$$

and $\operatorname{argmin}\left(\Phi_p(\tau_n, U_\tau^{n-1}; \cdot)\right)$ is the set of all global minimizers of it. Suppose for every sequence τ and $U_\tau^0 \in \mathfrak{M}$, there is a sequence $\{U_\tau^n\}_{n\ge 1} \subseteq \mathfrak{M}$ solving (2.12). Then, one interpolates the values $\{U_\tau^n\}_{n\ge 0}$ by the piecewise constant function $\overline{U}_\tau : \mathfrak{M} \to \mathbb{R}$ defined by (cf [30] in the Banach space framework)

$$\overline{U}_\tau(t) := U_\tau^0\, \mathbb{1}_{\{t=0\}}(t) + \sum_{n=1}^{\infty} U_\tau^n\, \mathbb{1}_{(t_\tau^{n-1}, t_\tau^n]}(t) \qquad \text{for every } t \ge 0. \tag{2.14}$$

Definition 2.18. For a given proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ and $v_0 \in \mathfrak{M}$, a curve $v : [0,\infty) \to \mathfrak{M}$ is called a *minimizing movement* of $\mathcal{E}$ starting from v_0 if for every sequence τ, there is a *discrete solution* $\overline{U}_\tau$ given by (2.14) such that

$$\lim_{|\tau|\to 0} \mathcal{E}(U_\tau^0) = \mathcal{E}(u_0),\ \limsup_{|\tau|\to 0} d(U_\tau^0, u_0) < \infty,\ \overline{U}_\tau(t) \to u(t) \quad \text{for all } t \in [0,\infty),$$

where the limit is with respect a topology $\mathcal{T}$ on $\mathfrak{M}$ for which the metric d of $\mathfrak{M}$ is sequentially lower semicontinuous.

We also need to introduce the notion of the *relaxed slope* of a functional $\mathcal{E}$, (or, also called the *sequentially lower semicontinuous envelope* of $|D^-\mathcal{E}|$).

Definition 2.19. For a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$, the *relaxed slope* $|\partial^- \mathcal{E}| : \mathfrak{M} \to [0, +\infty]$ of $\mathcal{E}$ is defined by

$$|\partial^- \mathcal{E}|(u) = \inf \left\{ \liminf_{n\to\infty} |D^- \mathcal{E}|(u_n) \;\middle|\; u_n \to u,\; \sup_{n\in\mathbb{N}} \{ d(u_n, u), \mathcal{E}(u_n)\} < \infty \right\}$$

for every $u \in D(\mathcal{E})$.

The following generation result of p-gradient flows in metric spaces is sufficient for our purpose, where we choose the weak topology σ to be the topology induced by the metric d of $\mathfrak{M}$.

Theorem 2.20 ([7, Corollary 2.4.12])**.** *Let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper lower semicontinuous function on $\mathfrak{M}$ satisfying*

(i) the functional Φ_p given by (2.13) satisfies the "convexity condition": there is a $\lambda \in \mathbb{R}$ such that for every v_0, $v_1 \in D(\mathcal{E})$, there is a curve γ with $\gamma(0) = v_0$, $\gamma(1) = v_1$ and for all $0 < \tau < \frac{1}{\lambda^-}$,

$$v \mapsto \Phi_p(\tau, v_0, v) \qquad \text{is } (\tau^{-1} + \lambda)\text{-convex on } \gamma,$$

(ii) for every $\alpha \in \mathbb{R}$, every sequence $(v_n)_{n\geq 1} \subseteq E_\alpha := \{v \in \mathfrak{M} \,|\, \mathcal{E}(v) \leq \alpha\}$ with $\sup_{n,m} d(v_n, v_m) < +\infty$ admits a convergent subsequence in $\mathfrak{M}$,

(iii) for every $\alpha \in \mathbb{R}$, the descending slope $|D^- \mathcal{E}|$ is lower semicontinuous on the sub-level set E_α,

where $\lambda^- = \max\{0, -\lambda\}$ and $\frac{1}{\lambda^-} := +\infty$ if $\lambda^- = 0$. Then, for every $v_0 \in D(\mathcal{E})$ there is a p-gradient flow of $\mathcal{E}$ with $v(0+) = v_0$.

2.4. **Geodesically convex functionals.** An important class of functionals $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ satisfying sufficient conditions to guarantee the existence of p-gradient flows of $\mathcal{E}$ is given by the ones that are *convex along constant speed geodesics* (see Theorem 2.31 below). To be more precise, we recall the following definition (cf [7]).

Definition 2.21. For $\lambda \in \mathbb{R}$, a functional $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ is called *λ-convex along a curve* $\gamma : [0, 1] \to \mathfrak{M}$ (we write $\gamma \subseteq \mathfrak{M}$) if

$$\mathcal{E}(\gamma(t)) \leq (1-t)\mathcal{E}(\gamma(0)) + t\mathcal{E}(\gamma(1)) - \tfrac{\lambda}{2} t(1-t) d^2(\gamma(0), \gamma(1)) \tag{2.15}$$

for all $t \in [0, 1]$, and $\mathcal{E}$ is called *convex along a curve* $\gamma \subseteq \mathfrak{M}$ if $\mathcal{E}$ is λ-convex along γ for $\lambda = 0$.

Definition 2.22 (**constant speed geodesics and λ-geodesic convexity**)**.** A curve $\gamma : [0, 1] \to \mathfrak{M}$ is said to be a *(constant speed) geodesic* connecting two points $v_0, v_1 \in \mathfrak{M}$ if $\gamma(0) = v_0$, $\gamma(1) = v_1$ and

$$d(\gamma(s), \gamma(t)) = (t - s) d(v_0, v_1) \quad \text{for all } s,\, t \in [0, 1] \text{ with } s \leq t.$$

A metric space $(\mathfrak{M}, d)$ with the property that for every two elements v_0, $v_1 \in \mathfrak{M}$, there is at least one constant speed geodesic $\gamma \subseteq \mathfrak{M}$ connecting v_0 and v_1 is called a *length space*. Given $\lambda \in \mathbb{R}$, a functional $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ is called *λ-geodesically convex* if for every v_0, $v_1 \in D(\mathcal{E})$, there is a constant speed geodesic $\gamma \subseteq \mathfrak{M}$ connecting v_0 and v_1 such that $\mathcal{E}$ is λ-convex along γ and $\mathcal{E}$ is called *geodesically convex* if for every v_0, $v_1 \in \mathfrak{M}$, there is a constant speed geodesic $\gamma \subseteq \mathfrak{M}$ connecting v_0 and v_1 and $\mathcal{E}$ is convex along γ.

The descending slope $|D^-\mathcal{E}|$ of a λ-geodesically convex functional $\mathcal{E}$ admits several important properties, which we recall now for later use.

Proposition 2.23 ([7, Theorem 2.4.9 & Corollary 2.410]). *For $\lambda \in \mathbb{R}$, let $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ be a proper λ-geodesically convex functional on a length space $\mathfrak{M}$. Then, the following statements hold.*

(1) The descending slope $|D^-\mathcal{E}|$ of $\mathcal{E}$ has the representation

$$|D^-\mathcal{E}|(u) = \sup_{v\neq u}\left[\frac{\mathcal{E}(u)-\mathcal{E}(v)}{d(u,v)} + \frac{\lambda}{2}d(u,v)\right]^+ \qquad \text{for every } u \in D(\mathcal{E}). \tag{2.16}$$

(2) For every $u \in D(\mathcal{E})$, $|D^-\mathcal{E}|(u)$ is the smallest $L \geq 0$ satisfying

$$\mathcal{E}(u) - \mathcal{E}(v) \leq L\, d(v,u) - \frac{\lambda}{2}d(v,u)^2 \qquad \text{for every } v \in \mathfrak{M}.$$

In particular,

$$\mathcal{E}(u) - \mathcal{E}(v) \leq |D^-\mathcal{E}|(u)\, d(v,u) - \frac{\lambda}{2}d^2(v,u) \qquad \text{for every } v \in \mathfrak{M}. \tag{2.17}$$

(3) If $\mathcal{E}$ is lower semicontinuous, then the descending slope $|D^-\mathcal{E}|$ of $\mathcal{E}$ is a strong upper gradient of $\mathcal{E}$, the relaxed slope $|\partial^-\mathcal{E}| = |D^-\mathcal{E}|$. In particular, $|D^-\mathcal{E}|$ is lower semicontinuous.

Throughout this paper, we use the following notion.

Definition 2.24. For a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ and $\varphi \in D(\mathcal{E})$, we call the functional $\mathcal{E}(\cdot|\varphi) : \mathfrak{M} \to (-\infty, \infty]$ defined by

$$\mathcal{E}(v|\varphi) = \mathcal{E}(v) - \mathcal{E}(\varphi) \qquad \text{for every } \varphi \in \mathfrak{M}$$

the *relative entropy* or *relative energy* of $\mathcal{E}$ with respect to φ.

Due to Proposition 2.23, we have that the following result holds for λ-geodesically functionals $\mathcal{E}$ on a length space $\mathfrak{M}$ (cf [16, Proposition 42] in the Hilbert space framework).

Proposition 2.25. *For $\lambda \in \mathbb{R}$, let $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ be a proper, lower semicontinuous and λ-geodesically convex functional on a length space $\mathfrak{M}$ and for $\varphi \in D(\mathcal{E})$, $R > 0$, let $\mathcal{D} \subseteq [\mathcal{E}(\cdot, |\varphi)]^{-1}((-\infty, R])$ a nonempty compact set. Then, the function $s_{\mathcal{D}} : (-\infty, R] \to [0, +\infty]$ given by*

$$s_{\mathcal{D}}(r) = \inf_{\hat{v}\in\mathcal{D}\cap[\mathcal{E}=r+\mathcal{E}(\varphi)]} |D^-\mathcal{E}|(\hat{v}) \qquad \text{for every } r \in (-\infty, R] \tag{2.18}$$

is lower semicontinuous.

Proof. For $\alpha \geq 0$, let $(r_n)_{n\geq 1} \subseteq (-\infty, R]$ be a sequence converging to some $r \leq R$ such that $s_D(r_n) \leq \alpha$ for every $n \geq 1$. Further, let $\varepsilon > 0$. Then, there is a sequences $(v_n)_{n\geq 1} \subseteq \mathcal{D} \cap \mathcal{E}^{-1}(\{r_n + \mathcal{E}(\varphi)\})$ satisfying

$$|D^-\mathcal{E}|(v_n) < s_{\mathcal{D}}(r_n) - \varepsilon \qquad \text{for every } n \geq 1.$$

Since $\mathcal{D}$ is compact, there is an element $v \in \mathcal{D}$ such that after possibly passing to a subsequence $v_n \to v$ in $\mathfrak{M}$. Since $\lim_{n\to+\infty} \mathcal{E}(v_n) = \lim_{n\to+\infty} r_n + \mathcal{E}(\varphi) = r + \mathcal{E}(\varphi)$ and since $\mathcal{E}$ is lower semicontinuous, $\mathcal{E}(v) \leq r + \mathcal{E}(\varphi)$. Moreover, since $|D^-\mathcal{E}|$ is lower semicontinuous (Proposition 2.23), $v \in D(|D^-\mathcal{E}|)$ and

$$|D^-\mathcal{E}|(v) \leq \liminf_{n\to\infty} |D^-\mathcal{E}|(v_n) \leq \alpha - \varepsilon.$$

On the other hand, by (2.17),

$$\mathcal{E}(v_n) \le \mathcal{E}(v) + |D^-\mathcal{E}|(v)\, d(v_n, v) - \frac{\lambda}{2} d^2(v_n, v) \qquad \text{for every } n \ge 1.$$

Sending $n \to +\infty$ in this inequality and using that $\lim_{n\to+\infty} \mathcal{E}(v_n) = r + \mathcal{E}(\varphi)$, it follows that

$$r + \mathcal{E}(\varphi) - \mathcal{E}(v) = 0.$$

Therefore, for every $\varepsilon > 0$, there is at least one

$$v \in \mathcal{D} \subseteq \mathcal{E}^{-1}(\{r + \mathcal{E}(\varphi)\}) \quad \text{ such that } \quad |D^-\mathcal{E}|(v) \le \alpha - \varepsilon,$$

implying that $s_{\mathcal{D}}(r) \le \alpha$. □

Notation 2.26. For $\varphi \in \mathfrak{M}$ and $r > 0$, we denote by

$$B(\varphi, r) := \Big\{ v \in \mathfrak{M} \,\Big|\, d(v, \varphi) < r \Big\}$$

the *open ball in* $\mathfrak{M}$ *of radius* r *and centered at* φ.

Definition 2.27. Given a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$, an element $\varphi \in D(\mathcal{E})$ is called a *local minimum* of $\mathcal{E}$ if there is an $r > 0$ such that

$$\mathcal{E}(\varphi) \le \mathcal{E}(v) \qquad \text{for all } v \in B(\varphi, r), \tag{2.19}$$

and φ is said to be a *global minimum* of $\mathcal{E}$ if (2.19) holds for all $v \in \mathfrak{M}$. Moreover, we denote by $\mathrm{argmin}(\mathcal{E})$ the set of all global minimizers φ of $\mathcal{E}$.

Our next proposition shows that for λ-geodesically convex functionals $\mathcal{E}$ with $\lambda \ge 0$, every *local* minimum is a global one and the set all *points of equilibrium* of $\mathcal{E}$ (see Definition 2.36 below) coincides with the set of global minimizers $\mathrm{argmin}(\mathcal{E})$ of $\mathcal{E}$.

Proposition 2.28 (Fermat's rule). *Let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper functional. Then the following statements hold.*

(1) *If $\varphi \in D(\mathcal{E})$ is local minimum of $\mathcal{E}$, then $|D^-\mathcal{E}|(\varphi) = 0$.*
(2) *If for $\lambda \ge 0$, $\mathcal{E}$ is λ-geodesically convex, then*

$$\mathit{argmin}(\mathcal{E}) = \Big\{ \varphi \in D(|D^-\mathcal{E}|) \,\Big|\, |D^-\mathcal{E}|(\varphi) = 0 \Big\}.$$

(3) *If $\mathcal{E}$ is λ-geodesically convex for $\lambda > 0$, then $\mathcal{E}$ admits at most one minimum.*

Proof. Claim (1) is a direct consequence of the definition of the descending slope $|D^-\mathcal{E}|$ of $\mathcal{E}$ (cf Definition 2.10). To see that claim (2) holds, it remains to show that every $\varphi \in D(|D^-\mathcal{E}|)$ satisfying $|D^-\mathcal{E}|(\varphi) = 0$ is a global minimum of $\mathcal{E}$. But this follows from the characterization (2.16) of $|D^-\mathcal{E}|$ in Proposition 2.23. Claim (3) is a consequence of inequality [7, (2.4.20) in Lemma 2.4.13], but for the sake of completeness, we shell briefly sketch the proof here. To see this, let $\varphi \in D(\mathcal{E})$ be a minimizer of $\mathcal{E}$ and suppose that $\psi \in D(\mathcal{E})$ is another minimizer of $\mathcal{E}$. By hypothesis, there is a constant speed geodesic $\gamma : [0, 1] \to \mathfrak{M}$ such that $\gamma(0) = \varphi$ and $\gamma(1) = \psi$ and $\mathcal{E}$ is λ-convex along γ. Thus,

$$0 \le \frac{\mathcal{E}(\gamma(t)) - \mathcal{E}(\gamma(0))}{t} \le \mathcal{E}(\gamma(1)) - \mathcal{E}(\gamma(0)) - \frac{\lambda}{2}(1-t) d^2(\gamma(0), \gamma(1))$$

for every $t \in (0,1)$. We fix $t \in (0,1)$. Then, since $\lambda > 0$ and $\varphi = \gamma(0)$ is a minimizer of $\mathcal{E}$, the last inequality implies that

$$\frac{\lambda}{2}(1-t)d^2(\gamma(0),\gamma(1)) \le \mathcal{E}(\gamma(1)) - \mathcal{E}(\gamma(0)) \le 0$$

implying that $\varphi = \gamma(0) = \gamma(1) = \psi$. □

For proper functionals $\mathcal{E}$ which are λ-geodesically convex for $\lambda > 0$ and admit a (global) minimizer $\varphi \in \mathfrak{M}$, every gradient flow v of $\mathcal{E}$ trends exponentially to φ in the *metric sense* (see Corollary 3.21, cf [7, Theorem 2.4.14] and [22]). This stability result is due to inequality (2.20) in our next proposition.

Proposition 2.29 ([7, Lemma 2.4.8 & Lemma 2.4.13]). *For $\lambda > 0$, let $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ be a proper, lower semicontinuous, λ-geodesically convex functional on a length space $(\mathfrak{M}, d)$. Then,*

$$\left(\mathcal{E}(v) - \inf_{\hat{v}\in\mathfrak{M}} \mathcal{E}(\hat{v})\right) \le \frac{1}{2\lambda}\,|D^-\mathcal{E}|^2(v) \qquad \text{for all } v \in D(\mathcal{E}). \tag{2.20}$$

In addition, if $\mathcal{E}$ satisfying coercivity condition:

$$\text{there is } u_* \in D(\mathcal{E}),\ r_* > 0 \text{ s.t. } m_* := \inf_{u\in\mathfrak{M}:d(u,u_*)\le r_*} \mathcal{E}(v) > -\infty, \tag{2.21}$$

then there is a unique minimizer $\varphi \in D(\mathcal{E})$ of $\mathcal{E}$ and

$$\frac{\lambda}{2}d^2(v,\varphi) \le \mathcal{E}(v|\varphi) \le \frac{1}{2\lambda}\,|D^-\mathcal{E}|^2(v) \qquad \text{for all } v \in D(\mathcal{E}). \tag{2.22}$$

Remark 2.30. Under the hypotheses of Proposition 2.29, and if $\mathcal{E}$ has a global minimizer φ, then inequality (2.20) is, in fact, **entropy-entropy production inequality** (1.3) for *linear* $\Phi(s) = 2\lambda s$ (cf [61, 62]). On the other hand, inequality (2.20) can also be seen as a **Kurdyka-Łojasiewicz inequality** (1.6) for $\theta(s) = \frac{\sqrt{2}}{\sqrt{\lambda}}|s|^{-\frac{1}{2}}s$ or as a **Łojasiewicz-Simon inequality** (1.7) with exponent $\alpha = \frac{1}{2}$ (see Definition 3.4, and compare with [44, 54, 42, 40, 25, 16, 24]).

For λ-geodesically convex functionals, we have the following existence result.

Theorem 2.31 ([7, Corollary 2.4.11]). *Let $1 < p < +\infty$ and for $\lambda \in \mathbb{R}$ (respectively, for $\lambda = 0$ if $p \neq 2$), let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper, lower semicontinuous, λ-geodesically convex functional on a length space $(\mathfrak{M}, d)$. Further, suppose for every $c \in \mathbb{R}$, the sub-level set*

$$E_c := \Big\{ v \in \mathfrak{M} \,\Big|\, \mathcal{E}(v) \le c \Big\} \qquad \textit{is compact in } \mathfrak{M},$$

Then with respect to the strong upper gradient $|D^-\mathcal{E}|$ of $\mathcal{E}$, for every $v_0 \in D(\mathcal{E})$, there is a p-gradient flow v of $\mathcal{E}$ with $v(0+) = v_0$.

2.5. **Some notions from the theory of dynamical systems.** Here, we summarize some important notions and results from the theory of dynamical systems in metric spaces [38], which we adapt to our more general framework.

Notation 2.32. For a curve $v : [0, +\infty) \to \mathfrak{M}$ and $t_0 \ge 0$, we denote by

$$\mathcal{I}_{t_0}(v) = \big\{ v(t) \,\big|\, t \ge t_0 \big\}$$

the *image* in $\mathfrak{M}$ of the restriction $v_{|[t_0,+\infty)}$ of v on $[t_0, +\infty)$; and by $\overline{\mathcal{I}}_{t_0}(v)$ the closure of $\mathcal{I}_{t_0}(v)$ in $\mathfrak{M}$.

We start by recalling the classical notion of ω-limit sets.

Definition 2.33. For a curve $v \in C((0,\infty);\mathfrak{M})$, the set

$$\omega(v) := \Big\{\varphi \in \mathfrak{M}\,\Big|\,\text{ there is } t_n \uparrow +\infty \text{ s.t. } \lim_{n\to\infty} v(t_n) = \varphi \text{ in } \mathfrak{M}\Big\}$$

is called the *ω-limit set* of v.

The ω-limit set $\omega(v)$ of continuous curves in $\mathfrak{M}$ has the following properties.

Proposition 2.34. *For a curve $v \in C([0,\infty);\mathfrak{M})$ the following holds.*

(1) *If for some $t_0 \geq 0$, the curve v has a relative compact image $\mathcal{I}_{t_0}(v)$ in $\mathfrak{M}$, then the ω-limit set $\omega(v)$ is non-empty.*

(2) *If there is a $\varphi \in \mathfrak{M}$ such that $\lim_{t\to\infty} v(t) = \varphi$ in $\mathfrak{M}$,*

then $\omega(v) = \{\varphi\}$.

(3) *If for $t_0 \geq 0$, v has relative compact image $\mathcal{I}_{t_0}$ in $\mathfrak{M}$ and $\varphi \in \mathfrak{M}$, then $\omega(v) = \{\varphi\}$ if and only if $\lim_{t\to\infty} v(t) = \varphi$ in $\mathfrak{M}$.*

We omit the proof of these statements since they are standard (cf [26]).

With our next definition we generalizes the classical notion of *Lyapunov functions* (cf [38]).

Definition 2.35. For a curve $v \in AC_{loc}(0,\infty;\mathfrak{M})$, a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty,\infty]$ is called a *Lyapunov function* of v if $\mathcal{E}$ is non-increasing along the trajectory $\{v(t)\,|\,t>0\}$. Moreover, a Lyapunov function $\mathcal{E}$ of v is called a *strict Lyapunov function* of v if $\mathcal{E}\circ v \equiv C$ on $[t_0,\infty)$ for some $t_0 \geq 0$ implies that v is constant on $[t_0,\infty)$.

In addition, we need to introduce the notion of equilibrium points.

Definition 2.36. An element $\varphi \in \mathfrak{M}$ is called an *equilibrium point* (or also *critical point*) of a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty,\infty]$ with strong upper gradient g if $\varphi \in D(g)$ and $g(\varphi) = 0$. We denote by $\mathbb{E}_g = g^{-1}(\{0\})$ the set of all equilibrium points of $\mathcal{E}$ with respect to strong upper gradient g.

Remark 2.37. By Proposition 2.28, if $\mathcal{E}$ is a λ-geodesically convex energy functionals $\mathcal{E}$ with $\lambda \geq 0$, then the set of equilibrium points

$$\mathbb{E}_g = \operatorname{argmin}(\mathcal{E}) \qquad \text{for the strong upper gradient } g = |D^-\mathcal{E}|.$$

Our next proposition describes the *standard Lyapunov entropy method* for p-gradient flows v of a proper functional $\mathcal{E}$ on a metric space $\mathfrak{M}$.

Proposition 2.38. *Let $\mathcal{E} : \mathfrak{M} \to (-\infty,+\infty]$ be a proper functional with strong upper gradient g, and v a p-gradient flow of $\mathcal{E}$. Then, the following hold.*

(1) *$\mathcal{E}$ is a strict Lyapunov function of v.*

(2) ***(Trend to equilibrium in the entropy sense)*** *If for $t_0 \geq 0$, $\mathcal{E}$ restricted on the set $\overline{\mathcal{I}}_{t_0}(v)$ is lower semicontinuous, then for every $\varphi \in \omega(v)$, one has $\varphi \in D(\mathcal{E})$ and*

$$\lim_{t\to\infty} \mathcal{E}(v(t)) = \mathcal{E}(\varphi) = \inf_{\xi \in \overline{\mathcal{I}}_{t_0}(v)} \mathcal{E}(\xi). \tag{2.23}$$

In particular, if $\omega(v)$ is non-empty, then $\mathcal{E}_{|\overline{\mathcal{I}}_{t_0}(v)}$ is bounded from below.

(3) ***(ω-limit points are equilibrium points of $\mathcal{E}$)*** *Suppose for $t_0 \geq 0$, $\mathcal{E}$ restricted on the set $\overline{\mathcal{I}}_{t_0}(v)$ is bounded from below and g restricted on the set $\overline{\mathcal{I}}_{t_0}(v)$ is lower semicontinuous. Then the ω-limit set $\omega(v)$ of v is contained in the set $\mathbb{E}_g$ of equilibrium points of $\mathcal{E}$.*

Proof. By inequality (2.8), the function $\mathcal{E}(v(\cdot))$ is non-increasing along $(0, +\infty)$. Thus, $\mathcal{E}$ is a Lyapunov function. Now, suppose that there are $t_0 \geq 0$ and $C \in \mathbb{R}$ such that $\mathcal{E}(v(t)) \equiv C$ for every $t \geq t_0$. By dissipation energy equality (2.11), the metric derivative $|v'|(t) = 0$ for all $t \geq t_0$ and so by Proposition 2.4, v is constant on $[t_0, +\infty)$, proving (1).

Next, we assume there is a $t_0 \geq 0$ such that $\mathcal{E}$ restricted on the set $\overline{\mathcal{I}}_{t_0}(v)$ is lower semicontinuous, and let $\varphi \in \omega(v)$. Then, there is a sequence $t_n \uparrow +\infty$ such that $v(t_n) \to \varphi$ in $\mathfrak{M}$ and so, the lower semicontinuity of $\mathcal{E}$ yields $\varphi \in D(\mathcal{E})$ and

$$\liminf_{t\to\infty} \mathcal{E}(v(t_n)) \geq \mathcal{E}(\varphi).$$

By monotonicity of $\mathcal{E}(v(\cdot))$ along $(0, +\infty)$ and since $t_n \uparrow +\infty$, for every $t > t_0$ there is an $t_n > t$ satisfying

$$\mathcal{E}(v(t)) \geq \mathcal{E}(v(t_n)) \geq \mathcal{E}(\varphi).$$

Thus, $\mathcal{E}$ is bounded from below on $\overline{\mathcal{I}}_{t_0}(v)$ and by using again the monotonicity of $\mathcal{E}(v(\cdot))$, we see that limit (2.23) holds, establishing statement (2).

To see that statement (3) hold, we note first that since $\mathcal{E}(v(\cdot))$ is bounded from below on $(t_0, +\infty)$ for some $t_0 > 0$ and since v is a p-gradient flow of $\mathcal{E}$ with strong upper gradient g, we can infer from energy dissipation equality (2.9) that the metric derivative $|v'| \in L^p(t_0, \infty)$. Now, let $\varphi \in \omega(v)$. Then there is sequence $(t_n)_{n\geq 1} \subseteq (t_0, +\infty)$ such that $t_n \uparrow +\infty$ and $v(t_n) \to \varphi$ in $\mathfrak{M}$ as $n \to \infty$. Since $v(t_n) \in D(g)$ for every n and g is lower semicontinuous, we have that $\varphi \in D(g)$. Further, for every $s \in [0, 1]$, Hölder's inequality gives

$$\begin{aligned}
\limsup_{n\to+\infty} d(v(t_n + s), \varphi) &\leq \limsup_{n\to+\infty} \int_{t_n}^{t_n+s} |v'|(r)\,\mathrm{d}r + \limsup_{n\to+\infty} d(v(t_n), \varphi) \\
&\leq \limsup_{n\to+\infty} \left(\int_{t_n}^{t_n+s} |v'|^p(r)\,\mathrm{d}r\right)^{1/p} \\
&\leq \limsup_{n\to+\infty} \left(\int_{t_n}^{+\infty} |v'|^p(r)\,\mathrm{d}r\right)^{1/p} = 0,
\end{aligned}$$

showing that the sequence $(v_n)_{n\geq 1}$ of curves $v_n : [0, 1] \to \mathfrak{M}$ given by

$$v_n(s) = v(t_n + s) \qquad \text{for every } s \in [0, 1]$$

satisfies

$$\lim_{n\to+\infty} \sup_{s\in[0,1]} d(v_n(s), \varphi) = 0. \tag{2.24}$$

Since v is a p-gradient flow of $\mathcal{E}$, also v_n is a p-gradient flow of $\mathcal{E}$ with strong upper gradient g. By assumption, g is lower semicontinuous and since v_n is continuous on $[0, 1]$, $g(v_n)$ is lower semicontinuous on $[0, 1]$ and so, measurable. Thus, by Fatou's lemma applied to (2.24) and by (2.10),

$$0 \leq g^{p'}(\varphi) \leq \liminf_{n\to+\infty} \int_0^1 g^{p'}(v_n(s))\,\mathrm{d}s$$

$$\leq \limsup_{n\to+\infty} \int_{t_n}^{t_n+1} |v'|^p(r)\,\mathrm{d}r \leq \limsup_{n\to+\infty} \int_{t_n}^{+\infty} |v'|^p(r)\,\mathrm{d}r = 0.$$

Therefore, $\varphi \in \mathbb{E}_g$. This completes the proof of this proposition. □

Remark 2.39 (**The ω-limit set $\omega(v)$ for λ-geodesically convex $\mathcal{E}$**). If for $\lambda \in \mathbb{R}$, the functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ is proper, lower semicontinuous and λ-geodesically convex on a length space $\mathfrak{M}$, then by Proposition 2.23, the descending slope $|D^-\mathcal{E}|$ of $\mathcal{E}$ is lower semicontinuous. Thus by Proposition 2.38, for every p-gradient flow v of $\mathcal{E}$, the ω-limit set $\omega(v) \subseteq \mathbb{E}_{|D^-\mathcal{E}|}$.

3. Kurdyka-Łojasiewicz-Simon inequalities in metric spaces

Here, we adapt the classical Kurdyka-Łojasiewicz inequality (cf [44, 46, 16, 24]) to a metric space framework. Throughout this section, let $(\mathfrak{M}, d)$ be a complete metric space and $1 < p < +\infty$.

3.1. Preliminaries to KŁ and ŁS-inequalities.

We begin by fixing the following notation.

Notation 3.1. For a proper function $\mathcal{F} : \mathfrak{M} \to (-\infty, +\infty]$, we write $[|\mathcal{F}| > 0]$ and $[\mathcal{F} > 0]$ for the pre-image sets $\{v \in \mathfrak{M} \,|\, |\mathcal{F}| > 0\}$ and $\{v \in \mathfrak{M} \,|\, \mathcal{F} > 0\}$. For R, $r > 0$, we write $[0 < \mathcal{F} < r]$, $[0 < \mathcal{F} \leq R]$, $[\mathcal{F} = r]$, and $[0 < |\mathcal{F}| \leq R]$ to denote the set $\{v \in \mathfrak{M} \,|\, 0 < \mathcal{F} < R\}$, $\{v \in \mathfrak{M} \,|\, \mathcal{F} = r\}$, $\{v \in \mathfrak{M} \,|\, 0 < \mathcal{F} \leq R\}$, and $\{v \in \mathfrak{M} \,|\, 0 < |\mathcal{F}| < r\}$.

The following inequality plays the key role of this paper. Our definition extends the ones in [16, 24] to the metric space framework.

Definition 3.2. A proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g and equilibrium point $\varphi \in \mathbb{E}_g$ is said to satisfy a *Kurdyka-Łojasiewicz inequality on the set* $\mathcal{U} \subseteq [g > 0] \cap [\theta'(\mathcal{E}(\cdot|\varphi)) > 0]$ if there is a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ and

$$\theta'(\mathcal{E}(v|\varphi))\, g(v) \geq 1 \qquad \text{for all } v \in \mathcal{U}. \tag{3.1}$$

Remark 3.3. Later, one sees that the equilibrium point $\varphi \in \overline{\mathcal{U}}$ if $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on the set $\mathcal{U} \subseteq [g > 0] \cap [\theta'(\mathcal{E}(\cdot|\varphi)) > 0]$ (see Sections 3.3 and 3.4).

In the particular case that for an exponent $\alpha \in (0, 1]$ and $c > 0$,

$$\theta(s) := \tfrac{c}{\alpha}\, |s|^{\alpha-1} s \qquad \text{for every } s \in \mathbb{R}, \tag{3.2}$$

Kurdyka-Łojasiewicz inequality (3.1) reduces to the gradient inequality due to Łojasiewicz [46, 47] in $\mathbb{R}^N$ and Simon [54] in infinite dimensional Hilbert spaces.

Definition 3.4. A proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g and equilibrium point $\varphi \in \mathbb{E}_g$ is said to satisfy a *Łojasiewicz-Simon inequality with exponent* $\alpha \in (0, 1]$ *near* φ if there are $c > 0$ and a set $\mathcal{U} \subseteq D(\mathcal{E})$ with $\varphi \in \mathcal{U}$ such that

$$|\mathcal{E}(v|\varphi)|^{1-\alpha} \leq c\, g(v) \qquad \text{for every } v \in \mathcal{U}. \tag{3.3}$$

3.2. **p-Gradient flows of finite length.** In this part, we show that the validity of a Kurdyka-Łojasiewicz inequality (3.1) on a set $\mathcal{U}$ yields finite length of p-gradient flows in $\mathcal{U}$. Our next theorem generalizes the result [16, $(i) \Rightarrow (ii)$ of Theorem 18]) for proper, lower semicontinuous and semi-convex functionals $\mathcal{E}$ on Hilbert space.

Theorem 3.5 (**Finite length of p-gradient flows**). *Let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper functional with strong upper gradient g, and for $\varphi \in \mathbb{E}_g$, suppose $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on the set $\mathcal{U} \subseteq [g > 0] \cap [\theta'(\mathcal{E}(\cdot|\varphi)) > 0]$. Then, the following statements hold.*

(1) Every piecewise p-gradient flow $v : [0,T) \to \mathfrak{M}$ of $\mathcal{E}$ with $0 < T < +\infty$ such that for some $0 \le t_0 < T$,

$$v(t) \in \mathcal{U} \qquad \text{for almost every } t \in [t_0, T), \tag{3.4}$$

has finite length $\gamma(v)$.

(2) Every piecewise p-gradient flow $v : [0,+\infty) \to \mathfrak{M}$ of $\mathcal{E}$ satisfying (3.4) for some $0 \le t_0 < T = +\infty$ and for which $\mathcal{E} \circ v_{|[t_0,+\infty)}$ is bounded from below, has finite length $\gamma(v)$.

In particular, there is a continuous, strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ such that for every piecewise p-gradient flow $v : [0,T) \to \mathfrak{M}$ of $\mathcal{E}$ satisfying (3.4) for some $0 \le t_0 < T \le +\infty$, one has

$$d(v(s), v(t)) \le \int_s^t |v'|(r)\,\mathrm{d}r \le \theta(\mathcal{E}(v(s)|\varphi)) - \theta(\mathcal{E}(v(t)|\varphi)) \tag{3.5}$$

for every $t_0 \le s < t < T$.

Remark 3.6 (*Trend to equilibrium in the metric sense*). The main task in proving trend to equilibrium in the metric sense of p-gradient flows v is to show that the curve v *once* entered the $\mathcal{U}$ for which (3.1) holds, can not escape from $\mathcal{U}$ any more. In other words, v satisfies (3.4) for some minimal $t_0 \ge 0$ such that $v(t_0) \in \mathcal{U}$ and hence v has finite length (see Section 3.5).

Proof of Theorem 3.5. Let $v : [0,T) \to \mathfrak{M}$ be a piecewise p-gradient flow of $\mathcal{E}$ satisfying (3.4) for some $0 \le t_0 < T < +\infty$. Then, there is a sequence $(t_n)_{n\ge 0}$ such that $\mathcal{P} := \{0 = t_0, t_1, t_2, \dots\}$ is a countable partition of $[0,T)$ and for every $n \ge 1$, the curve $v_{|(t_{n-1},t_n)}$ is a p-gradient flow of $\mathcal{E}$ on (t_{n-1}, t_n).

Let $n \ge 1$. Since $\mathcal{E} \circ v$ is non-increasing on (t_{n-1}, t_n) and locally absolutely continuous, and since $\theta \in W^{1,1}_{loc}(\mathbb{R})$ and strictly increasing, the function

$$\mathcal{H}(t) := \theta(\mathcal{E}(v(t)|\varphi)) \tag{3.6}$$

for every $t \in (t_{n-1}, t_n)$, is non-increasing, differentiable almost everywhere on (t_{n-1}, t_n), and the "chain rule"

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) = \theta'(\mathcal{E}(v(t)|\varphi))\,\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(v(t)|\varphi) \tag{3.7}$$

holds for almost every $t \in (t_{n-1}, t_n)$ (cf [45, Corollary 3.50]). Since $\mathcal{P}$ is a countable partition of $[0,T)$, $\mathcal{H}$ is differentiable almost everywhere on $(0,T)$ and (3.7) holds for almost every $t \in (0,T)$. By condition (3.4) and since $\mathcal{U} \subseteq$

$[g>0]\cap[\theta'(\mathcal{E}(\cdot|\varphi))>0]$, we can apply Kurdyka-Łojasiewicz inequality (3.1) to $v=v(t)$ for almost every $t\in(t_0,T)$. Then by (2.6),

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t)\ge\theta'(\mathcal{E}(u(t)|\varphi))\,g(v(t))\,|v'|(t)\ge|v'|(t) \tag{3.8}$$

for a.e. $t\in(t_0,T)$. Now, let $N\ge1$ such that $t_0\in[t_{N-1},t_N)$. With out loss of generalization, we can assume that $T_{N-1}=t_0$. Then, integrating (3.8) over (t_{N-1},t_N) gives

$$\int_{t_{n-1}}^{t_n}|v'|(s)\,\mathrm{d}s\le\theta(\mathcal{E}(v(t_{n-1}+)|\varphi))-\theta(\mathcal{E}(v(t_n-)|\varphi))$$

for every $n\ge N$. By Definition 2.14, one has that

$$\theta(\mathcal{E}(v(t_n-)|\varphi))\ge\theta(\mathcal{E}(v(t_n+)|\varphi))\qquad\text{for every } n\ge N.$$

Thus, for every integer $M>N$,

$$\begin{aligned}\int_{t_{N-1}=t_0}^{t_M}|v'|(s)\,\mathrm{d}s&=\sum_{n=N}^{M}\int_{t_{n-1}}^{t_n}|v'|(s)\,\mathrm{d}s\\&\le\theta(\mathcal{E}(v(t_0+)|\varphi))+\sum_{n=N}^{M}(-\theta(\mathcal{E}(v(t_n-)|\varphi))+\theta(\mathcal{E}(v(t_n+)|\varphi))\\&\le\theta(\mathcal{E}(v(t_0+).\end{aligned}$$

Now, sending $M\to\infty$ in these estimates yields

$$\int_{t_0}^{T}|v'|(r)\,\mathrm{d}r\le\theta(\mathcal{E}(v(t_0)|\varphi)).$$

Combining this with the fact that $|v'|_{[t_{n-1},t_n)}\in L^p(t_{n-1},t_n)$ and using Hölder's inequality leads to

$$\begin{aligned}\int_0^t|v'|(r)\,\mathrm{d}r&=\int_0^{t_0}|v'|(r)\,\mathrm{d}r+\int_{t_0}^{T}|v'|(r)\,\mathrm{d}r\\&\le\left(\int_0^{t_0}|v'|^p(r)\,\mathrm{d}r\right)^{1/p}t_0^{1/p'}+\theta(\mathcal{E}(v(t_0)|\varphi)).\end{aligned}$$

Thus, the piecewise p-gradient flow v has finite length provided $T<\infty$. The case $T=+\infty$ is shown similarly since by hypothesis, the map $t\mapsto\mathcal{E}(v(t)|\varphi)$ is bounded on $[t_0,+\infty)$ and so, v has finite length. This completes the proof this theorem. □

3.3. **Talweg implies KŁ-inequality.** We start this section by introducing the following assumption on the functional $\mathcal{E}$ and the strong upper gradient g of $\mathcal{E}$ (cf [16] in the Hilbert space framework).

Assumption 3.1 (*Sard-type condition*)**.** For the proper functional $\mathcal{E}:\mathfrak{M}\to(-\infty,+\infty]$ with strong upper gradient g and equilibrium point $\varphi\in\mathbb{E}_g$, let

(H1) the set $\mathcal{U}\subseteq\mathfrak{M}$ satisfy $\quad\mathcal{U}\cap[\mathcal{E}(\cdot|\varphi)>0]\subseteq[g>0]$.

Remark 3.7. The following statements are worth mentioning.

(1) If for $\varphi \in \mathbb{E}_g$ and $\mathcal{U} \subseteq [\theta'(\mathcal{E}(\cdot|\varphi)) > 0]$, the functional $\mathcal{E}$ satisfies the following *stronger form* of Kurdyka-Łojasiewicz inequality:

$$g(v) \geq 1/\theta'(\mathcal{E}(v|\varphi)) \qquad \text{for all } v \in \mathcal{U}, \tag{3.9}$$

where $\theta \in W^{1,1}_{loc}(\mathbb{R})$ is a strictly increasing function satisfying $\theta(0) = 0$, then for the set $\mathcal{U}$ hypothesis (H1) holds and $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on $\mathcal{U}$.

(2) Examples to (1): if for $\varphi \in \mathbb{E}_g$, $\mathcal{E}$ satisfies a Łojasiewicz-Simon inequality (3.3) with exponent $\alpha \in (0, 1]$ on the set $\mathcal{U} \subseteq D(\mathcal{E})$ then $\mathcal{E}$ satisfies (H1) on $\mathcal{U}$.

(3) If the functional $\mathcal{E}$ and $\varphi \in \mathbb{E}_g$, the set $\mathcal{U} \subseteq [\mathcal{E}(\cdot|\varphi) \geq 0]$ satisfies hypothesis (H1), then for every $\tilde{\varphi} \in \mathcal{U} \cap [g = 0]$, one has $\mathcal{E}(\tilde{\varphi}|\varphi) = 0$. In other words, (H1) is a *Sard-type* condition.

We come now to the definition of a *talweg curve* in $\mathfrak{M}$ introduced by Kurdyka [44] in $\mathbb{R}^N$ (see also [16] for the Hilbert space framework).

Definition 3.8. Let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper functional with strong upper gradient g. For $\varphi \in \mathbb{E}_g$ and $R > 0$, suppose the set $\mathcal{U} \subseteq [\mathcal{E}(\cdot|\varphi) \leq R]$ satisfies hypothesis (H1) and $\varphi \in \overline{\mathcal{U}}$. We set

$$s_{\mathcal{U}}(r) = \inf_{\hat{v} \in \mathcal{U} \cap [\mathcal{E} = r + \mathcal{E}(\varphi)]} g(\hat{v}) \qquad \text{for every } r \in (0, R].$$

Then, for given $C > 1$ and $\delta > 0$, we call a piecewise continuous curve $x : (0, \delta] \to \mathfrak{M}$ a *talweg curve* through the C-valley

$$\mathcal{V}_{C,\mathcal{U}}(\varphi) := \Big\{ v \in \mathcal{U} \cap [\mathcal{E}(\cdot|\varphi) > 0] \;\Big|\; g(v) \leq C\, s_{\mathcal{U}}(\mathcal{E}(v|\varphi)) \Big\}$$

if x satisfies

$$x(t) \in \mathcal{V}_{C,\mathcal{U}}(\varphi) \quad \text{for every } t \in (0, \delta] \quad \text{and} \quad \lim_{t \to 0+} \mathcal{E}(x(t)|\varphi) = 0. \tag{3.10}$$

Furthermore, we need that a talweg has the following *regularity*.

Definition 3.9. For $\delta > 0$, a curve $x : (0, \delta] \to \mathfrak{M}$ is called *piecewise AC* if there is a countable partition $\{I_n\}_{n \geq 1}$ of $(0, \delta]$ into nontrivial intervals $I_n = (a_n, b_n] \subseteq (0, \delta]$ such that $x \in AC(I_n, \mathfrak{M})$ for all $n \geq 1$.

Remark 3.10 (*Existence of a talweg curve*). If for $\varphi \in \mathbb{E}_g$ and $R > 0$, a functional $\mathcal{E}$ satisfies the stronger inequality (3.9) on a set

$$\mathcal{U} \subseteq [\mathcal{E}(\cdot|\varphi) \leq R] \cap [\theta'(\mathcal{E}(\cdot|\varphi)) > 0],$$

then one has that

$$s_{\mathcal{U}}(r) \geq 1/\theta'(r) \qquad \text{for every } r \in (0, R] \cap [0 < \theta'(r) < \infty].$$

We emphasize that the existence of a talweg curve depends on the controllability of $s_{\mathcal{U}}(r)$ from below with respect to r.

Our first theorem of this subsection provides *optimal* conditions on the talweg curve implying that the functional $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1). The optimality on the talweg is shown in Example (4.2) in Section 4.1 by investigating the *smooth case* (see, for instance [39] or the examples in [25]), for which it is known that $\mathcal{E}$ satisfies a Łojasiewicz-Simon inequality with exponent $\alpha = 1/2$.

Theorem 3.11 (**Talweg implies KŁ-inequality**). *Let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper functional with strong upper gradient g, and for $\varphi \in \mathbb{E}_g$ and $R > 0$, suppose that the set $\mathcal{U} \subseteq [0 \le \mathcal{E}(\cdot|\varphi) \le R]$ satisfies hypothesis (H1). If there are $C > 1$, $\delta > 0$, and a piecewise AC talweg $x : (0, \delta] \to \mathfrak{M}$ through the C-valley $\mathcal{V}_{C,\mathcal{U}}(\varphi)$ such that*

$$h(t) := \mathcal{E}(x(t)|\varphi) \quad \text{for every } t \in (0, \delta], \quad h(0) := \lim_{t\to 0+} \mathcal{E}(x(t)|\varphi), \tag{3.11}$$

is a strictly increasing function $h : [0, \delta] \to [0, R]$ satisfying $h \in W^{1,1}_{loc}(0, \delta)$, $h(0) = 0$ and $|\{h' = 0\}| = 0$, then there is a subset $\tilde{\mathcal{U}} \subseteq \mathcal{U} \cap [0 < \mathcal{E}(\cdot|\varphi) \le R]$ such that for every $v \in \tilde{\mathcal{U}}$, one has

$$(h^{-1})'(\mathcal{E}(v|\varphi))g(v) \ge \frac{1}{C\,|x'|(t)} \qquad \text{with } t = h^{-1}(\mathcal{E}(v|\varphi)). \tag{3.12}$$

In addition, if $|x'| \in L^\infty(0, \delta)$, then $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on $\tilde{\mathcal{U}}$.

Remark 3.12. We note that in Theorem 3.11, the hypotheses imply that h is a homeomorphism. In the smooth case (see, for instance, Example 4.2 in Section 4.1), one can show that $h : (0, \delta) \to (0, R)$ is, in fact, at least a C^1-diffeomorphism and the talweg $x \in C^1((0, \delta); \mathfrak{M}) \cap C([0, \delta]; \mathfrak{M})$.

Proof of Theorem 3.11. We begin by defining the strictly increasing homeomorphism $\theta : \mathbb{R} \to \mathbb{R}$ by setting

$$\theta(r) = \begin{cases} h^{-1}(R)(r - R) + h^{-1}(R) & \text{if } r > R, \\ h^{-1}(r) & \text{if } r \in [0, R], \\ -h^{-1}(-r) & \text{if } r \in [-R, 0], \\ -h^{-1}(R)(r + R) - h^{-1}(R) & \text{if } r < -R. \end{cases}$$

By hypothesis, $|\{h' = 0\}| = 0$. Thus, $\theta \in W^{1,1}_{loc}(\mathbb{R})$ and

$$t = (\theta \circ h)(t) = \theta(\mathcal{E}(x(t)|\varphi)) \qquad \text{for every } t \in [0, \delta].$$

Since h is strictly increasing and since $\theta \in W^{1,1}_{loc}(\mathbb{R})$, the chain rule (cf [45, Corollary 3.50]) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(\mathcal{E}(x(t)|\varphi)) = \theta'(\mathcal{E}(x(t)|\varphi))\,\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(x(t)|\varphi)$$

for almost every $t \in (0, \delta)$. Since $\theta'(\mathcal{E}(x(\cdot)|\varphi))$ is positive on $(0, \delta)$, and since g is a strong upper gradient of $\mathcal{E}$, inequality (2.6) yields that

$$1 = \frac{\mathrm{d}}{\mathrm{d}t}(\theta \circ h)(t) = \theta'(\mathcal{E}(x(t)|\varphi))\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(x(t)) \le \theta'(\mathcal{E}(x(t)|\varphi))g(x(t))\,|x'|(t)$$

for almost every $t \in (0, \delta)$. Since $x(t) \in \mathcal{V}_{C,\mathcal{U}}(\varphi)$ for every $t \in (0, \delta]$ and by hypothesis (H1), $g(x(t)) > 0$ for all $t \in (0, \delta]$. Thus, the previous inequality yields that

$$\theta'(\mathcal{E}(x(t)|\varphi))g(x(t)) \ge \frac{1}{|x'|(t)} > 0 \qquad \text{for almost every } t \in (0, \delta). \tag{3.13}$$

Next, we set

$$\mathcal{P} = \Big\{ t \in [0, \delta] \,\Big|\, 0 < h'(t) < +\infty,\ 0 < |x'|(t) < +\infty \Big\}.$$

Since $|\{h' = 0\}| = |\{|x'| = 0\}| = 0$, the sets $\mathcal{Z} := [0,\delta] \setminus \mathcal{P}$ satisfies $|\mathcal{Z}| = 0$ and since $h \in W^{1,1}_{loc}(0,\delta)$, the set $\mathcal{L} := h(\mathcal{Z})$ satisfies $|\mathcal{L}| = 0$. Further, one has that the interval $[0,R] = h(\mathcal{P}) \cup \mathcal{L}$ and $0 < (h^{-1})' < \infty$ on $h(\mathcal{P})$.

Now, let $\tilde{U} := [0 < \mathcal{E}(\cdot|\varphi), \mathcal{E}(\cdot|\varphi) \in h(\mathcal{P})] \cap \mathcal{U}$ and take $v \in \tilde{\mathcal{U}}$. Then $r := \mathcal{E}(v|\varphi) \in (0,R]$ and $r \in h(\mathcal{P})$. Thus, there is a unique $t \in \mathcal{P}$ such that $h(t) = r$, $0 < |x'|(t) < +\infty$ exits and

$$\mathcal{E}(x(t)|\varphi) = h(t) = r = \mathcal{E}(v|\varphi). \tag{3.14}$$

Since $x(t) \in \mathcal{V}_{C,\mathcal{U}}(\varphi)$ and by (3.14), we see that

$$g(x(t)) \le C\, s_{\mathcal{U}}(\mathcal{E}(x(t)|\varphi)) = C \inf_{\hat{v}\in\mathcal{U}\cap[\mathcal{E}=(\mathcal{E}(x(t)|\varphi))+\mathcal{E}(\varphi)]} g(\hat{v}) \le C\, g(v).$$

Since $t \in \mathcal{P}$, $x(t)$ also satisfies (3.13). Thus, and by using again (3.14),

$$\frac{1}{|x'|(t)} \le \theta'(\mathcal{E}(x(t)|\varphi))\, g(x(t)) \le \theta'(\mathcal{E}(x(t)|\varphi)) C\, g(v) = \theta'(\mathcal{E}(v|\varphi))\, C\, g(v),$$

which is inequality (3.12). If, in addition, $|x'| \in L^\infty(0,\delta)$, then $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on the set $\tilde{U}$ for the strictly increasing function $\tilde{\theta} := \||x'|\|_{L^\infty(0,\delta)}\, C\,\theta$. This completes the proof of this theorem. □

In our next result we replace the condition $|x'| \in L^\infty(0,T)$ by $|x'| \in L^1(0,T)$.

Theorem 3.13 (**Talweg implies KŁ-inequality, Version 2**). *Let $\mathcal{E} : \mathfrak{M} \to (-\infty,+\infty]$ be a proper functional with strong upper gradient g, and for $\varphi \in \mathbb{E}_g$ and $R > 0$, suppose the set $\mathcal{U} \subseteq [0 \le \mathcal{E}(\cdot|\varphi) \le R]$ satisfies hypothesis (H1). Further, suppose the set $\mathcal{D} \subseteq \mathcal{U}$ is nonempty and the function*

$$r \mapsto s_{\mathcal{D}}(r) := \inf_{\hat{v}\in\mathcal{D}\cap[\mathcal{E}=r+\mathcal{E}(\varphi)]} g(\hat{v}) \quad \textit{is lower semicontinuous on } (0,R]. \tag{3.15}$$

If for $C > 1$, there is a piecewise AC talweg $x : (0,R] \to \mathcal{V}_{C,\mathcal{D}}(\varphi)$ satisfying

$$\mathcal{E}(x(r)|\varphi) = r \qquad \textit{for every } r \in (0,R], \tag{3.16}$$

and $|x'| \in L^1(0,R)$ (that is, x has finite length $\gamma(x)$), then there is a set $\tilde{\mathcal{U}} \subseteq \mathcal{U} \cap [0 < \mathcal{E}(\cdot|\varphi) \le R]$ such that $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on $\tilde{\mathcal{U}}$.

Our proof adapts an idea from [16] to the metric space framework.

Proof of Theorem 3.13. Let $x : (0,R] \to \mathfrak{M}$ be a piecewise AC talweg through the C-valley $\mathcal{V}_{C,\mathcal{D}}$ around φ satisfying (3.16) with $|x'| \in L^1(0,R)$. If $\mathcal{D} \subseteq \mathcal{U}$ is a nonempty set such that (3.15) holds, then by hypothesis (H1), the function

$$u(r) := \frac{1}{\inf_{\hat{v}\in\mathcal{D}\cap[\mathcal{E}=r+\mathcal{E}(\varphi)]} g(\hat{v})} \qquad \text{for every } r \in (0,R] \tag{3.17}$$

is measurable, strictly positive and upper semicontinuous on $(0,R]$. By (3.16), (2.6), and since $x(r) \in \mathcal{V}_{C,\mathcal{D}}(\varphi)$ for every $r \in (0,\delta]$, we see that

$$1 = \frac{\mathrm{d}}{\mathrm{d}r}\mathcal{E}(x(r)|\varphi) \le g(x(r))\, |x'|(r) \le C \inf_{\hat{v}\in\mathcal{U}\cap[\mathcal{E}=r+\mathcal{E}(\varphi)]} g(y)\, |x'|(r)$$

for almost every $r \in (0,R)$. Since $|x'| \in L^1(0,R)$, the function $u \in L^1(0,R)$. Now, due to [16, Lemma 45], for the measurable function u given by (3.17), there is a continuous function $\tilde{u} : (0, R] \to (0, +\infty)$ satisfying $\tilde{u} \geq u$ and $\tilde{u} \in L^1(0, R)$. Now, let $\theta : \mathbb{R} \to \mathbb{R}$ be defined by

$$\theta(s) = \begin{cases} \displaystyle\int_0^R \tilde{u}(r)\,\mathrm{d}r\,(s - R + 1) & \text{if } s > R, \\ \displaystyle\int_0^s \tilde{u}(r)\,\mathrm{d}r & \text{if } s \in [0, R], \\ \displaystyle -\int_0^{-s} \tilde{u}(r)\,\mathrm{d}r & \text{if } s \in [-R, 0), \\ \displaystyle -\int_0^R \tilde{u}(r)\,\mathrm{d}r\,(s + R + 1) & \text{if } s < -R \end{cases} \tag{3.18}$$

for every $s \in \mathbb{R}$. Then, $\theta \in \theta \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}) \cap W^{1,1}_{loc}(\mathbb{R})$ is strictly increasing, satisfies $\theta(0) = 0$. In particular, for every $v \in \mathcal{U} \cap [0 < \mathcal{E}(\cdot|\varphi)]$, setting $r = \mathcal{E}(v|\varphi)$ yields

$$\theta'(\mathcal{E}(v|\varphi))\, g(v) = u(r)\, g(v) \geq 1,$$

proving that $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) near φ. $\square$

3.4. **Existence of a talweg curve.** Here, our aim is to provide sufficient conditions implying the existence of a talweg curve through a C-valley $\mathcal{V}_{C,\mathcal{U}}(\varphi)$ of a given functional $\mathcal{E}$ near an equilibrium point $\varphi \in \mathbb{E}_g$.

For our main result, we need that the functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ and the strong upper gradient g of $\mathcal{E}$ satisfy the following two conditions.

Assumption 3.2. Suppose, for the proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g the following hold.

(i) For every $v_0 \in D(\mathcal{E})$, there is a unique p-gradient flow v of $\mathcal{E}$ with initial value $v(0+) = v_0$;

(ii) The mapping $S : [0, +\infty) \times D(\mathcal{E}) \to D(\mathcal{E})$ defined by

$$S_t v_0 = v(t) \qquad \text{for all } t \geq 0,\ v_0 \in D(\mathcal{E}), \tag{3.19}$$

is continuous, where v is the p-gradient flow of $\mathcal{E}$ with $v(0+) = v_0$;

(iii) For every $v_0 \in D(\mathcal{E})$ and p-gradient flow v of $\mathcal{E}$ with initial value $v(0+) = v_0$, the map $t \mapsto g(v(t))$ is right-continuous on $[0, +\infty)$.

Remark 3.14 (*Gradient flows of evolution variational inequalities*). It is well-known (cf [32, Theorem 2.6], [31, Proposition 3.1]) that for functionals $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ on a length space $(\mathfrak{M}, d)$ which are λ-geodesically convex for some $\lambda \in \mathbb{R}$ and satisfies coercivity condition (2.21) for 2-gradient flows v, the mapping $S : [0, +\infty) \times D(\mathcal{E}) \to D(\mathcal{E})$ defined by (3.19) satisfies all three conditions in Assumption 3.2. In this case, the descending slope $|D^-\mathcal{E}|$ is the upper gradient of $\mathcal{E}$ and $\mathcal{E}$ generates a so-called *evolution variational inequality* (see [32, Definition 2.4] and [7, Theorem 4.0.4]).

Assumption 3.3. Suppose, for the proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g and equilibrium point $\varphi \in \mathbb{E}_g$, there are $\varepsilon > 0$ and $R > 0$ such that the set

$$\mathcal{U}_{\varepsilon,R} := B(\varphi, \varepsilon) \cap [0 \leq \mathcal{E}(\cdot|\varphi) < R] \tag{3.20}$$

is relatively compact in $\mathfrak{M}$.

The next theorem provides sufficient condition for the existence of a talweg curve in the metric space framework (cf [16, Theorem 18, $(ii) \Rightarrow (iv)$] in the Hilbert space setting).

Theorem 3.15 (**Existence of a talweg curve).** *Let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper lower semicontinuous functional and $g : \mathfrak{M} \to [0, +\infty]$ a proper lower semicontinuous strong upper gradient of $\mathcal{E}$ satisfying Assumption 3.2. Suppose, for $\varphi \in \mathbb{E}_g$, there are $\varepsilon > 0$ and $R > 0$ such that for the set $\mathcal{U}_{\varepsilon,R}$ given by (3.20) the Assumption 3.3 and hypothesis (H1) hold. Further, suppose, there are $0 < \varepsilon_0 < \varepsilon$ and $0 < R_0 < R$ such that the set $\mathcal{D} \subseteq \mathcal{U}_{\varepsilon_0,R_0}$ given by*

$$(3.21)\quad \mathcal{D} = \left\{ v \in \mathcal{U}_{\varepsilon_0,R_0} \;\middle|\; \begin{array}{l} \text{there are } v^0 \in \overline{\mathcal{U}}_{\frac{\varepsilon_0}{3},R_0},\ \text{a } p\text{-gradient flows } \hat{v} \text{ of } \mathcal{E} \\ \text{with } \hat{v}(0+) = v^0 \text{ and } t_0 \geq 0 \text{ s.t. } \hat{v}(t_0) = v \end{array} \right\}$$

is non-empty such that (3.15) holds and for every $v_0 \in \overline{\mathcal{U}}_{\frac{\varepsilon_0}{3},R_0}$ and p-gradient flow v of $\mathcal{E}$ with initial value $v(0+) = v_0$, one has that

$$(3.22)\qquad v([0,t)) \subseteq \mathcal{U}_{\varepsilon_0,R_0} \text{ for } t > 0, \text{ implies } \qquad \int_0^t |v'|(s)\,\mathrm{d}s \leq \frac{\varepsilon_0}{3}.$$

Then, there are $C > 1$, $0 < R_1 \leq R_0$, and a piecewise AC talweg $x : (0, R_1] \to \mathfrak{M}$ of finite length $\gamma(x)$ through the C-valley $\mathcal{V}_{C,\mathcal{D}}(\varphi)$ of $\mathcal{E}$ satisfying (3.16) with R replaced by R_1.

Proof. We begin by showing that due to hypothesis (3.22), the set $\mathcal{D}$ given by (3.21) admits the following *stability property*:

$$(3.23)\qquad \begin{cases} \text{for every } v \in \mathcal{D},\ p\text{-gradient flow } \hat{v} \text{ of } \mathcal{E} \text{ with } \hat{v}(0+) = v \\ \text{and every } t \geq 0 \text{ satisfying } \mathcal{E}(\hat{v}(t)) > 0, \text{ one has } \hat{v}(t) \in \mathcal{D}. \end{cases}$$

To see this, let $v \in \mathcal{D}$, $\hat{v}$ be a p-gradient flow of $\mathcal{E}$ with initial value $\hat{v}(0+) = v$, and $t \geq 0$ such that $\mathcal{E}(\hat{v}(t)|\varphi) > 0$. Then, we show that $\hat{v}(t) \in \mathcal{D}$. For this, it is sufficient to prove that $\hat{v}(t) \in \mathcal{U}_{\varepsilon_0,R_0}$ and there is a $w_0 \in \overline{\mathcal{U}}_{\frac{\varepsilon_0}{3},R_0}$, and p-gradient flow $\hat{w}$ of $\mathcal{E}$ satisfying $\hat{w}(0+) = w_0$ and there is a $\hat{t} \geq 0$ such that $\hat{w}(\hat{t}) = \hat{v}(t)$. Since $v \in \mathcal{D}$, there are $w_0 \in \overline{\mathcal{U}}_{\frac{\varepsilon_0}{3},R_0}$ and a p-gradient flow $\hat{w}$ of $\mathcal{E}$ satisfying $\hat{w}(0+) = w_0$, and there is a $t_0 \geq 0$ such that $\hat{w}(t_0) = v$. By $\mathcal{E}(\hat{v}(t)|\varphi) > 0$ and $\hat{v}(0+) = v = \hat{w}(t_0)$, and since $\mathcal{E}(w_0|\varphi) \leq R_0$, the monotonicity of $\mathcal{E}$ yields that $0 < \mathcal{E}(\hat{w}(s)|\varphi) \leq R_0$ for every $s \in [0, t_0]$. Since $w_0 \in \overline{\mathcal{U}}_{\frac{\varepsilon_0}{3},R_0}$ and $\hat{w}$ is continuous, there is a $0 < \delta \leq t_0$ such that $w(s) \in \mathcal{U}_{\varepsilon_0,R_0}$ for every $s \in [0, \delta]$. Thus, by (3.22),

$$(3.24)\qquad d(\hat{w}(s), \varphi) \leq d(\hat{w}(s), w_0) + d(w_0, \varphi) \leq \int_0^s |\hat{w}'|(r)\,\mathrm{d}r + \frac{\varepsilon_0}{3} \leq 2\,\frac{\varepsilon_0}{3}$$

(firstly) for all $s \in [0, \delta]$, showing that $w(s) \in \overline{B}(\varphi, 2\,\frac{\varepsilon_0}{3}) \cap [0 < \mathcal{E}(\cdot|\varphi) \leq R_0]$ for every $s \in [0, \delta]$. Since the right-hand side of (3.24) is independent of δ, and since $0 < \mathcal{E}(\hat{w}(s)|\varphi) \leq R_0$ for every $s \in [0, t_0]$, we can conclude that $\hat{w}(s) \in \overline{B}(\varphi, 2\,\frac{\varepsilon_0}{3}) \cap [0 < \mathcal{E}(\cdot|\varphi) \leq R_0]$ for all $s \in [0, t_0]$. Now, by $\hat{v}(0+) = v = \hat{w}(t_0)$, Assumption 3.2 implies that $\hat{v}(s) = \hat{w}(s + t_0)$ for every $s \in [0, t]$. Thus, assumption $\mathcal{E}(\hat{v}(t)|\varphi) > 0$ yields that $0 < \mathcal{E}(\hat{w}(s)|\varphi) \leq R_0$ for every $s \in [0, t]$ and so, $\hat{w}(s) \in \overline{B}(\varphi, 2\,\frac{\varepsilon_0}{3}) \cap [0 < \mathcal{E}(\cdot|\varphi) \leq R_0]$ for all $s \in [0, t_0 + t]$, implying that $v(t) \in\in \mathcal{U}_{\varepsilon_0,R_0}$. Moreover, we have shown that there is a $w_0 \in \overline{\mathcal{U}}_{\frac{\varepsilon_0}{3},R_0}$, and p-gradient flow $\hat{w}$ of $\mathcal{E}$ satisfying $\hat{w}(0+) = w_0$ and there is a $\hat{t} := t_0 + t \geq 0$ such that $\hat{w}(\hat{t}) = \hat{v}(t)$.

Next, we show that $\mathcal{D}$ is closed in $[0 < \mathcal{E}(\cdot|\varphi) \leq R_0]$. To see this, let $(v_n)_{n\geq 1} \subseteq \mathcal{D}$ and $v \in [0 < \mathcal{E}(\cdot|\varphi) \leq R_0]$ such that $v_n \to v$ in $\mathfrak{M}$. By definition of $\mathcal{D}$, there are sequences $(t_n^0)_{n\geq 1} \subseteq [0, +\infty)$, $(\hat{v}_n^0)_{n\geq 1} \subseteq \overline{\mathcal{U}}_{\frac{\varepsilon_0}{3}, R_0}$ and a sequence $(\hat{v}_n)_{n\geq 1}$ of p-gradient flows of $\mathcal{E}$ with initial value $\hat{v}_n(0+) = \hat{v}_n^0$ satisfying $\hat{v}_n(t_n^0) = v_n$ for every $n \geq 1$. By the lower semicontinuity of $\mathcal{E}$, there are $N \geq 1$ and $\hat{\varepsilon} > 0$ such that $\mathcal{E}(v_n|\varphi) \geq \hat{\varepsilon}$ for all $n \geq N$. Since $\mathcal{E} \circ \hat{v}_n$ is non-increasing on $[0, +\infty)$ and $\hat{v}_n(t_n^0) = v_n$,

$$\mathcal{E}(v_n^0|\varphi) \geq \hat{\varepsilon} \qquad \text{for all } n \geq N \tag{3.25}$$

In addition,

$$\mathcal{I} := \inf_{v \in \overline{\mathcal{U}}_{\varepsilon_0, R_0} \cap [\mathcal{E}(\cdot|\varphi) \geq \hat{\varepsilon}]} g(v) > 0, \tag{3.26}$$

otherwise, there is a sequence $(\tilde{v}_{n\geq 1})$ in $\overline{\mathcal{U}}_{\varepsilon_0, R_0} \cap [\mathcal{E}(\cdot|\varphi) \geq \hat{\varepsilon}]$ such that $g(\tilde{v}_n) \to 0$ as $n \to \infty$. By Assumption 3.3, there is a $\tilde{v} \in \overline{\mathcal{U}}_{\varepsilon_0, R_0} \cap [\mathcal{E}(\cdot|\varphi) \geq \hat{\varepsilon}]$ such that up to a subsequence, $\tilde{v}_n \to \tilde{v}$ in $\mathfrak{M}$. Now, by the lower semicontinuity of g, $g(\tilde{v}) = 0$. On the other hand, $\mathcal{E}(\tilde{v}) \geq \hat{\varepsilon}$ and so, Assumption 3.1 implies that $g(\tilde{v}) > 0$, showing that we arrived to a contradiction. Thus, (3.26) holds. Now, since each $\hat{v}_n$ is a p-gradient flow of $\mathcal{E}$ with initial value $\hat{v}_n(0+) = \hat{v}_n^0 \in \overline{\mathcal{U}}_{\frac{\varepsilon_0}{3}, R_0}$, *energy dissipation equality* (2.11) gives that

$$0 < t_n^0\, \mathcal{I}^{p'} \leq \int_0^{t_n^0} g^{p'}(\hat{v}_n(s))\, \mathrm{d}s = \mathcal{E}(v_n^0|\varphi) - \mathcal{E}(\hat{v}_n(t_n)|\varphi) \leq \mathcal{E}(v_n^0|\varphi) \leq R_0,$$

showing that the sequence $(t_n^0)_{n\geq 1}$ is bounded. Thus, there is a $t_0 \geq 0$ such that after possibly passing to a subsequence, $t_n^0 \to t_0$ as $n \to +\infty$. Moreover, by (3.25), $(\hat{v}_n^0)_{n\geq N} \subseteq \overline{\mathcal{U}}_{\frac{\varepsilon_0}{3}, R_0} \cap [\mathcal{E}(\cdot|\varphi) \geq \hat{\varepsilon}]$. Hence, Assumption 3.3 implies that there is an $\hat{v}_0 \in \overline{\mathcal{U}}_{\frac{\varepsilon_0}{3}, R_0} \cap [\mathcal{E}(\cdot|\varphi) \geq \hat{\varepsilon}]$ such that after possibly passing to another subsequence, $\hat{v}_n^0 \to \hat{v}_0$ in $\mathfrak{M}$. By Assumption 3.2, there is a p-gradient flow $\hat{v}$ of $\mathcal{E}$ with initial value $\hat{v}(0+) = \hat{v}_0$. Now the continuity of the mapping $S : [0, +\infty) \times D(\mathcal{E}) \to D(\mathcal{E})$ given by (3.19), we have $\hat{v}(t_0) = v$ and so, $\hat{v}(t_0) \in [0 < \mathcal{E}(\cdot|\varphi) \leq R_0]$. Thus and by stability property (3.23), $v \in \mathcal{D}$.

With these preliminaries, we begin now to construct a talweg curve x. First, we defined the function $s_{\mathcal{D}} : (0, R_0] \to (0, \infty)$ by (3.15) on the interval $(0, R_0]$ and for $C > 1$, let

$$\mathcal{V}_C(r) := \Big\{ v \in \mathcal{D} \cap [\mathcal{E} = r + \mathcal{E}(\varphi)] \,\Big|\, g(v) \leq C\, s_D(r) \Big\} \subseteq V_{C,\mathcal{D}}(\varphi). \tag{3.27}$$

By hypothesis, the set $\mathcal{D} \subseteq \mathcal{U}_{\varepsilon_0, R_0}$ is non-empty. Thus, there are $0 < R_1 \leq R_0$ and $v_0 \in \overline{\mathcal{U}}_{\frac{\varepsilon_0}{3}, R_0}$ such that $\mathcal{E}(v_0|\varphi) = R_1$. By Assumption 3.2, there is a p-gradient flow v of $\mathcal{E}$ with $v(0+) = v_0$, and by stability property (3.23),

$$v(t) \in \mathcal{D} \qquad \text{for all } t \in [0, T) \text{ with } \mathcal{E}(v(t)|\varphi) > 0. \tag{3.28}$$

Let $T := \inf\{t > 0 \,|\, \mathcal{E}(v(t)|\varphi) = 0\}$. Then, since $\mathcal{E}(v_0|\varphi) = R_1 > 0$, the continuity of v yields that $0 < T \leq +\infty$ and

$$\lim_{t \to T-} \mathcal{E}(v(t)|\varphi) = 0.$$

By Proposition 2.16 and hypothesis (H1), $\mathcal{E}\circ v$ is strictly decreasing on $[0,T)$ and by Proposition 2.11, $\mathcal{E}\circ v$ is locally absolutely continuous on $[0,T)$. Thus, by the intermediate value theorem, the mapping $\mathcal{E}\circ v : [0,T) \to (0,R_1]$ is a homeomorphism. Moreover, by (3.28), the set

$$\mathcal{V}_C(r) \neq \emptyset \qquad \text{for all } r \in (0,R_1] \text{ and all } C > 1.$$

To see this, assume that the contrary holds. Then, by (3.28), there are $r_1 \in (0,R_1]$ and $C_1 > 1$ such that $g(v) > C_1\, s_D(r_1)$ for all $v \in \mathcal{D} \cap [\mathcal{E} = r + \mathcal{E}(\varphi)]$, implying that $1 > C_1$, which obviously is a contradiction.

Now, for $1 < C_1 < \frac{C_1+C}{2} < C$, let $v_0 \in \mathcal{V}_{C_1}(R_1)$ and v be the p-gradient flow of $\mathcal{E}$ with initial value $v(0+) = v_0$. By Assumption 3.2, the map $t \mapsto g(v(t))$ is right-continuous. Thus, there is a $T_0 > 0$ such that

$$g(v(t)) < \frac{C_1 + C}{2}\, S_{\mathcal{D}}(R_1) \qquad \text{for all } t \in [0,T_0).$$

By hypothesis, the function $t \mapsto S_{\mathcal{D}}(\mathcal{E}(v(t)))$ is lower semicontinuous. Hence, there is a $T_1 \in (0,T_0)$ such that

$$C\, S_{\mathcal{D}}(\mathcal{E}(v(t)|\varphi)) > \frac{C_1 + C}{2}\, s_{\mathcal{D}}(R_1) \qquad \text{for all } t \in [0,T_1).$$

Combining these two inequalities on the interval $[0,T_1)$ yields that

$$g(v(t)) < \frac{C_1 + C}{2}\, S_{\mathcal{D}}(r) < C\, S_{\mathcal{D}}(\mathcal{E}(v(t)|\varphi)) \qquad \text{for all } t \in [0,T_1). \tag{3.29}$$

Since there might be some $t \in (0,T_1)$ such that $v(t) \neq \mathcal{D}$, we need to apply the continuity of the p-gradient flow v and stability property (3.23) to conclude that there is a $0 < T_* \leq T_1$ such that $v(t) \in \mathcal{D}$ and (3.29) holds for all $t \in [0,T)$ and so,

$$v(t) \in \mathcal{V}_{C,\mathcal{D}}(\mathcal{E}(v(t)|\varphi)) \qquad \text{for all } t \in [0,T_*). \tag{3.30}$$

Recall, $\mathcal{E}\circ v$ is strictly decreasing and continuous on $[0,T_*]$. Next, we consider two cases:

1. Case : Suppose

$$\lim_{t \to T_*-} \mathcal{E}(v(t)|\varphi) = 0. \tag{3.31}$$

Then, $\mathcal{E}(v(\cdot)|\varphi) : [0,T_*] \to [0,R_1]$ is a strictly decreasing homeomorphism. By Proposition 2.16 and due to hypothesis (H1), $\mathcal{E}(v(\cdot)|\varphi) \in W^{1,1}(0,T_*)$ satisfying $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(v(t)|\varphi) < 0$ for a.e. $t \in (0,T_*)$. Let $\hat{h} : [0,T_*] \to [0,R_1]$ be defined by $\hat{h}(t) := \mathcal{E}(v(T_* - t)|\varphi)$ for every $t \in [0,T_*]$. Then, $\hat{h}$ also belongs to $W^{1,1}(0,T_*) \cap C([0,T_*])$, $\hat{h}$ is strictly increasing, $|\{\hat{h}'(t) = 0\}| = 0$ and $\hat{h}(0) = 0$. Now, we defined the curve $x : [0,R_1] \to \mathfrak{M}$ by

$$x(r) = v(\hat{h}^{-1}(r)) \qquad \text{for every } r \in [0,R_1]. \tag{3.32}$$

Then, by (3.30) and by construction of $\hat{h}^{-1}$, x is a talweg through the C-valley $\mathcal{V}_{C,\mathcal{D}}(\varphi)$ satisfying

$$x(r) \in \mathcal{V}_{C,\mathcal{D}}(\mathcal{E}(x(r)|\varphi)) \qquad \text{for all } r \in (0,R_1]. \tag{3.33}$$

and (3.16) with R replaced by R_1. Since $\hat{h} \in W^{1,1,}(0,\delta)$ satisfying $\hat{h}' > 0$ a.e. on $[0,\delta]$, the inverse $\hat{h}^{-1}$ satisfies Lusin's condition (N) on $[0,R_1]$ and hence $\hat{h}^{-1} \in W^{1,1}(0,R_1)$, implying $x = v \circ \hat{h}^{-1} \in AC_{loc}(0,R_1;\mathfrak{M})$. Furthermore, by (3.32), x is p-gradient curve of $\mathcal{E}$ and by (3.33), $x((0,R_1])\subseteq\mathcal{D}$. Since $\mathcal{D}\subseteq\mathcal{U}_{\varepsilon_0,R_0}$, hypothesis (3.22) yields that x has finite length $\gamma(x)$.

2. Case : Suppose

$$\lim_{t\to T_*-}\mathcal{E}(v(t)|\varphi)=\mathcal{E}(v(T_*)|\varphi)>0. \tag{3.34}$$

Then, by stability property (3.23), $v(T_*)\in\mathcal{V}_C(\mathcal{E}(v(T_*)|\varphi))$. Now, the above argument shows that iteratively, for fixed $1<C_1<\frac{C_1+C}{2}<C$, setting $r_0=R_1$, $v_0^{(0)}=v_0$, $t_0=T_*$, $r_1=\mathcal{E}(v(t_0)|\varphi)$, $v_1^{(0)}=v(t_0)$, and $v_1=v$, and for every integer $n\geq 1$, there are $r_n\in(0,r_{n-1})$, $v_n^{(0)}\in\mathcal{V}_{C_1}(r_{n-1})$, $t_n>0$, and a p-gradient flow v_n of $\mathcal{E}$ satisfying $v_n(0+)=v_n^{(0)}$ and if $\mathcal{E}(v(t_n)|\varphi)>0$ then by stability property (3.23),

$$v_n(t)\in\mathcal{V}_C(\mathcal{E}(v_n(t)|\varphi))\qquad\text{for all } t\in[0,t_n]. \tag{3.35}$$

Moreover, there is a strictly increasing homeomorphism $\hat{h}_n^{-1}:[r_n,r_{n-1}]\to[0,t_n]$ satisfying $\hat{h}_n^{-1}(r_{n-1})=0$, $\hat{h}_n^{-1}(r_n)=t_n$, $\hat{h}_n^{-1}\in W^{1,1}(r_n,r_{n-1})$ with $|\{(\hat{h}_n^{-1})'=0\}|=0$. Thus, $x_n:=v_n\circ\hat{h}^{-1}\in AC(r_n,r_{n-1};\mathfrak{M})$. In addition, by construction of $\hat{h}_n^{-1}$, for every $r\in[r_n,r_{n-1}]$, there is a unique $t\in[0,t_n]$ such that $\hat{h}_n^{-1}(r)=t$ and $\mathcal{E}(v_n(t)|\varphi)=r$.

Now, if there is an $N>1$ such that the curve v_N is the first among $\{v_n\}_{n=1}^{N-1}$ satisfying the limit (3.31) where T_* is replaced by t_N, then $r_N=0$ and there is a *finite* partition

$$\mathcal{P}\quad:\quad 0=r_N<r_{N-1}<\cdots<r_1<r_0=R_1$$

of the interval $[0,R_1]$ and a curve $x:[0,R_1]\to\mathfrak{M}$ defined by

$$x(t)=\sum_{i=0}^{N}v_n(\hat{h}_n^{-1}(t))\,\mathbb{1}_{(r_n,r_{n-1}]}(t)\qquad\text{for every } t\in[0,R_1] \tag{3.36}$$

which is a *piecewise AC* talweg through the C-valley $\mathcal{V}_{C,\mathcal{D}}(\varphi)$ satisfying (3.16) for R replaced by R_1 and $x_{|(r_n,r_{n-1}]}=x_n\in AC(r_n,r_{n-1};\mathfrak{M})$. Furthermore, by (3.36), x is piecewise p-gradient curve of $\mathcal{E}$ and by (3.35), $x((0,R_1])\subseteq\mathcal{D}$. Since $\mathcal{D}\subseteq\mathcal{U}_{\varepsilon_0,R_0}$, hypothesis (3.22) yields that x has finite length $\gamma(x)$.

If for every integer $n\geq 1$, v_n satisfies (3.34) with T_* replaced by t_n, then stability property (3.23) yields that each v_n satisfies (3.35) with $t=\delta_n$. There is an $\alpha\in[0,R_1)$ such that the family $\{(r_n,r_{n-1}]\}_{n\geq1}$ defines a countable partition of the interval $(\alpha,R_1]$ and the function $x_\alpha:(\alpha,R_1]\to\mathcal{V}_{C,\mathcal{D}}(\varphi)$ defined by

$$x_\alpha(t)=\sum_{i=0}^{\infty}v_n(\hat{h}_n^{-1}(t))\,\mathbb{1}_{(r_n,r_{n-1}]}(t) \tag{3.37}$$

for every $t\in(\alpha,R_1]$ has the properties that $x_\alpha\in AC((r_n,r_{n-1}],\mathfrak{M})$ for all $n\geq1$, and

$$x_\alpha(t)\in\mathcal{V}_{C,\mathcal{D}}(\mathcal{E}(x_n(t)|\varphi))\qquad\text{for all } t\in(\alpha,R_1] \tag{3.38}$$

and

$$\mathcal{E}(x_\alpha(r)|\varphi)=r\qquad\text{for every } r\in(\alpha,R_1]. \tag{3.39}$$

Moreover, by (3.37), x is piecewise p-gradient curve of $\mathcal{E}$ and by (3.38), $x((0,R_1])\subseteq\mathcal{D}$. Since $\mathcal{D}\subseteq\mathcal{U}_{\varepsilon_0,R_0}$, hypothesis (3.22) yields that x has finite length $\gamma(x)$.

Let $\mathcal{T}$ be the set of all pairs (α,x_α) for every $\alpha\in[0,R_1)$ and curves $x_\alpha:(\alpha,R_1]\to\mathcal{V}_{C,\mathcal{D}}(\varphi)$ of finite length $\gamma(x_\alpha)$ satisfying (3.38), (3.39), and there is countable partition $\{I_n\}_{n\geq 1}$ of $(\alpha,R_1]$ of nontrivial intervals $I_n\subseteq(0,R_1]$, for which $x_\alpha\in AC(I_n,\mathfrak{M})$ for all $n\geq 1$. Then, due to the function x_α constructed in (3.37), the set $\mathcal{T}$ is non-empty. We can define a *partial ordering* "$\leq$" on $\mathcal{T}$ by setting that for all (α,x_α), $(\hat\alpha,x_{\hat\alpha})\in\mathcal{T}$, one has

$$(\alpha,x_\alpha)\leq(\hat\alpha,x_{\hat\alpha})\qquad\text{if } \hat\alpha\leq\alpha \text{ and } x_{\hat\alpha|(\alpha,R_1]}=x_\alpha.$$

Then, by Zorn's Lemma, there is a maximal element $(\alpha_0,x_{\alpha_0})\in\mathcal{T}$. If we assume that $\alpha_0>0$, then by stability property (3.23),

$$x(\alpha_0)\in\mathcal{V}_{C,\mathcal{D}}(\mathcal{E}(x(\alpha_0)|\varphi))$$

and so by using the same arguments as given at the beginning of *Case 2.*, we construct an element $(\hat\alpha,x_{\hat\alpha})\in\mathcal{T}$ satisfying $(\alpha_0,x_{\alpha_0})\leq(\hat\alpha,x_{\hat\alpha})$, which contradicts the fact that (α_0,x_{α_0}) is the maximal element of $\mathcal{T}$. Therefore, $\varepsilon_0=0$ and thereby we have shown that there is a *piecewise AC* talweg $x_{\alpha_0}:(0,R_1]\to\mathcal{V}_{C,\mathcal{D}}(\varphi)$ through the C-valley $\mathcal{V}_{C,\mathcal{D}}(\varphi)$ of finite length $\gamma(x_{\alpha_0})$ satisfying (3.16). This complete the proof of this theorem. □

Due to Theorem 3.15, we can characterize the validity of the Kurdyka-Łojasiewicz inequality (3.1) for functionals $\mathcal{E}$ defined on a metric space.

Theorem 3.16 (Characterization of KŁ inequality). *Let $\mathcal{E}:\mathfrak{M}\to(-\infty,+\infty]$ be a proper lower semicontinuous functional and $g:\mathfrak{M}\to[0,+\infty]$ a proper lower semicontinuous strong upper gradient of $\mathcal{E}$ satisfying Assumption 3.2. Suppose, for $\varphi\in\mathbb{E}_g$, there are $\varepsilon>0$ and $R>0$ such that the set $\mathcal{U}_{\varepsilon,R}$ given by (3.20) satisfies Assumption 3.3 and hypothesis (H1) holds. Further, suppose, there are $0<\varepsilon_0<\varepsilon$ and $0<R_0<R$ such that the set $\mathcal{D}\subseteq\mathcal{U}_{\varepsilon_0,R_0}$ given by (3.21) is non-empty and (3.15) holds.*

Then the following statements are equivalent.

(1) *(**$\mathcal{E}$ satisfies a KŁ inequality**) There is an $0<R_1\leq R_0$ such that $\mathcal{E}$ satisfies Kurdyka-Lojasiewicz inequality (3.1) on*

$$\tilde U:=\mathcal{U}_{\varepsilon,R}\cap[0<\mathcal{E}(\cdot|\varphi)\leq R_1].$$

(2) *(**p-gradient flows of finite length**) There is an $0<R_1\leq R_0$ such that for $0<T\leq\infty$, every piecewise p-gradient flow $v:[0,T)\to\mathfrak{M}$ of $\mathcal{E}$ satisfying (3.4) for some $0\leq t_0<T$ and with $\mathcal{U}$ replaced by $\tilde{\mathcal{U}}$, has finite length $\gamma(v)$ given by (2.3). In particular, there is a continuous, strictly increasing function $\theta\in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0)=0$ such that for every p-gradient flow $v:[0,T)\to\mathfrak{M}$ of $\mathcal{E}$ satisfying (3.4), one has that (3.5) holds.*

(3) *(**Existence of a piecewise AC-talweg**) There are $C>1$, $0<R_1\leq R_0$ and a piecewise AC talweg $x:(0,R_1]\to\mathfrak{M}$ of finite length $\gamma(x)$ through the C-valley $\mathcal{V}_{C,\mathcal{D}}(\varphi)$ satisfying (3.16) with R replaced by R_1.*

Proof. We only need to note that the implication (1) $\Rightarrow$ (2) is a consequence of Theorem 3.5, (2) $\Rightarrow$ (3) holds by Theorem 3.15, and (3) $\Rightarrow$ (1) follows from Theorem 3.13. $\square$

3.5. **Trend to equilibrium of p-gradient flows in the metric sense.** This subsection is dedicated to establishing the trend to equilibrium in the metric sense of p-gradient flows in metric spaces. The following theorem is the main result.

Theorem 3.17 (**Trend to equilibrium in the metric sense**)**.** *Let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper functional with strong upper gradient g, and v be a p-gradient flow of $\mathcal{E}$ with non-empty ω-limit set $\omega(v)$. Suppose, $\mathcal{E}$ is lower semicontinuous on $\overline{\mathcal{I}}_{\bar{t}}(v)$ for some $\bar{t} \geq 0$ and for $\varphi \in \omega(v) \cap \mathbb{E}_g$, there is an $\varepsilon > 0$ such that the set $B(\varphi, \varepsilon)$ satisfies hypothesis (H1).*

If there is a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ and $|[\theta > 0, \theta' = 0]| = 0$ such that $\mathcal{E}$ satisfies the Kurdyka-Łojasiewicz inequality (3.1) on

$$\mathcal{U}_\varepsilon = B(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) > 0] \cap [\theta'(\mathcal{E}(\cdot|\varphi)) > 0], \tag{3.40}$$

then v has finite length and

$$\lim_{t \to \infty} v(t) = \varphi \qquad \text{in } \mathfrak{M}. \tag{3.41}$$

Remark 3.18 (*ω-limit point and points of equilibrium*). Note, due to statement (3) of Proposition 2.38, if the strong upper gradient g of $\mathcal{E}$ is lower semicontinuous on $\mathfrak{M}$, then one has that $\omega(v) \subseteq \mathbb{E}_g$.

Remark 3.19 (*Consistency with the Łojasiewicz-Simon inequality*). The function θ given by (3.2) satisfies the condition $|[\theta > 0, \theta' = 0]| = 0$ in Theorem 3.17.

Proof of Theorem 3.17. Let $\varphi \in \omega(v) \cap \mathbb{E}_g$. Then, there is a sequence $(t_n)_{n \geq 1}$ such that $t_n \uparrow \infty$ and

$$\lim_{n \to \infty} v(t_n) = \varphi \qquad \text{in } \mathfrak{M}. \tag{3.42}$$

Since $\mathcal{E}$ is lower semicontinuous on $\overline{\mathcal{I}}_{\bar{t}}(v)$ for some $\bar{t} \geq 0$, Proposition 2.38 implies that limit (2.23) holds. Thus and since $\mathcal{E}$ is a strict Lyapunov function of v, to show that the limit (3.41) holds, it is sufficient to consider the case $\mathcal{E}(v(t)|\varphi) > 0$ on $[0, +\infty)$.

By hypothesis, for $\varphi \in \omega(v) \cap \mathbb{E}_g$, there is $\varepsilon > 0$ such that the set $B(\varphi, \varepsilon)$ satisfies hypothesis (H1) and $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality on the set $\mathcal{U}_\varepsilon$ given by (3.40). Thus, we intend to apply Theorem 3.5 to v. For this, we need to show that v satisfies condition (3.4) with $\mathcal{U}$ replaced by $\mathcal{U}_\varepsilon$ and for some $0 \leq t_0 < T = +\infty$.

By (3.42), there is a $n_0 \geq 1$ such that

$$v(t_n) \in B(\varphi, \varepsilon) \qquad \text{for all } n \geq n_0. \tag{3.43}$$

The, for every $n \geq n_0$, we define the *first exit time with respect to t_n* by

$$t_n^{(1)} := \inf\left\{ t \geq t_n \,\middle|\, d(v(t), \varphi) = \varepsilon \right\}. \tag{3.44}$$

Since v is continuous on $(0, \infty)$ with values in $\mathfrak{M}$, it follows that $t_n^{(1)} > t_n$ for every $n \geq n_0$. To see that v satisfies (3.4) with $\mathcal{U}$ replaced by $\mathcal{U}_\varepsilon$ and $T = +\infty$, we need to show that there is an $n_1 \geq n_0$ such that $t_{n_1}^{(1)} = +\infty$.

To prove this claim, we assume that the contrary is true and we shall arrive at a contradiction. Then,

$$0 < t_n < t_n^{(1)} < \infty \qquad \text{for all } n \geq n_0$$

and by the continuity of v, we see that

$$d(v(t_n^{(1)}), \varphi) = \varepsilon \qquad \text{for all } n \geq n_0. \tag{3.45}$$

By hypothesis, there is a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ and $|[\theta > 0, \theta' = 0]| = 0$. We use the auxiliary function $\mathcal{H}$ from (3.6) on the interval $(0, +\infty)$. Since $\theta \in AC_{loc}(\mathbb{R})$ and $t \mapsto \mathcal{E}(v(t)|\varphi)$ is decreasing, $\mathcal{H}$ is differentiable a.e. on $(0, +\infty)$ and the chain rule (3.7) holds for a.e. $t \in (0, +\infty)$. Since $\mathcal{E}(v(t)|\varphi) > 0$ on $(0, +\infty)$ and $|[\theta > 0, \theta' = 0]| = 0$, we have that $\theta'(\mathcal{E}(v(t))|\varphi) > 0$ for a.e. $t \in (0, +\infty)$. Thus, for every $n \geq n_0$, $v(t) \in \mathcal{U}_\varepsilon$ for a.e. $t \in (t_n, t_n^1)$ and so by Kurdyka-Łojasiewicz inequality (3.1), we can conclude that inequality (3.8) holds for a.e. $t \in (t_n, t_n^1)$. Integrating inequality (3.8) over (t_n, t) for $t \in (t_n, t_n^{(1)}]$ and using Proposition 2.4 together with the fact that $\mathcal{H}(t) > 0$ for every $t > 0$, we get

$$\begin{aligned} d(v(t), \varphi) &\leq d(v(t), v(t_n)) + d(v(t_n), \varphi) \\ &\leq \int_{t_n}^{t} |v'|(r)\,\mathrm{d}r + d(v(t_n), \varphi) \leq \mathcal{H}(t_n) + d(v(t_n), \varphi) \end{aligned} \tag{3.46}$$

for every $t \in (t_n, t_n^{(1)}]$ and $n \geq n_0$. In particular,

$$d(v(t_n^{(1)}), \varphi) \leq \mathcal{H}(t_n) + d(v(t_n), \varphi) \qquad \text{for all } n \geq n_0$$

By limit (2.23), the continuity of θ, and since $\theta(0) = 0$, we have that

$$\lim_{t \to \infty} \mathcal{H}(t) = 0.$$

Thus and by (3.42), we can conclude that

$$\lim_{n \to \infty} d(v(t_n^{(1)}), \varphi) = 0,$$

which contradicts (3.45). Therefore, our assumption is false and our claim that there is a $n_1 \geq n_0$ satisfying $t_{n_1}^1 = +\infty$ holds, proving condition (3.4) for some $0 \leq t_0 < T = +\infty$ where $\mathcal{U}$ is replaced by $\mathcal{U}_\varepsilon$. Thus, Theorem 3.5 yields that the p-gradient flow v has finite length. In other words, the metric derivative $|v'|$ of v belongs to $L^1(0, \infty)$ and by (2.1) with $m = |v'|$, we can apply the Cauchy criterion to conclude that $\lim_{t \to +\infty} v(t)$ exists in $\mathfrak{M}$. By (3.42) this limit needs to coincide with limit (3.41) and therefore the statement of this theorem holds. ∎

3.6. **Decay rates and finite time of extinction.** In contrast to the general Kurdyka-Łojasiewicz inequality (3.1), the Łojasiewicz-Simon inequality (3.3) has the advantage to derive *decay estimates* of the trend to equilibrium in the metric sense and to provide upper bounds on the *extinction time*. We emphasize that p-gradient flows trend with *polynomial rate* to an equilibrium in the metric sense if the Łojasiewicz-exponent $0 < \alpha < 1/p$, with *exponential rate* if $\alpha = 1/p$ and p-gradient flows *extinct in finite time* if $1/p < \alpha \le 1$.

Our next result generalizes (partially) the ones in [40], [27, Theorem 2.7 & Remark 2.8] in the Hilbert space setting, and [15] in the 2-Wasserstein setting.

Theorem 3.20 (**Decay estimates and finite time of extinction).** *Let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper functional with strong upper gradient g, and v be a p-gradient flow of $\mathcal{E}$ with non-empty ω-limit set $\omega(v)$. Suppose, $\mathcal{E}$ is lower semicontinuous on $\overline{\mathcal{I}}_{\bar{t}}(v)$ for some $\bar{t} > 0$, and for $\varphi \in \omega(v) \cap \mathbb{E}_g$ there are $\varepsilon > 0$, $c > 0$, and $\alpha \in (0, 1]$ such that $\mathcal{E}$ satisfies a Łojasiewicz-Simon inequality (3.3) with exponent α on $B(\varphi, \varepsilon) \cap D(\mathcal{E})$. Then*

$$
\begin{aligned}
&d(v(t), \varphi) \le \tfrac{c}{\alpha} \left(\mathcal{E}(v(t)|\varphi)\right)^{\alpha} = \mathcal{O}\left(t^{-\frac{\alpha(p-1)}{1-p\alpha}}\right) && \text{if } 0 < \alpha < \tfrac{1}{p}\\
&d(v(t), \varphi) \le c\,p \left(\mathcal{E}(v(t)|\varphi)\right)^{\frac{1}{p}} \le c\,p\,\left(\mathcal{E}(v(t_0)|\varphi)\right)^{\frac{1}{p}}\, e^{-\frac{t}{pc^{p'}}} && \text{if } \alpha = \tfrac{1}{p}\\
&d(v(t), \varphi) \le \begin{cases} \tilde{c}\,(\hat{t} - t)^{\frac{\alpha(p-1)}{p\alpha-1}} & \text{if } \ t_0 \le t \le \hat{t},\\ 0 & \text{if } t > \hat{t}, \end{cases} && \text{if } \tfrac{1}{p} < \alpha \le 1,
\end{aligned}
$$

where,

$$
\begin{aligned}
\tilde{c} &:= \left[\left[\tfrac{1}{\alpha^{\alpha-1}c}\right]^{\frac{p'-1}{\alpha}}\ \tfrac{p\alpha-1}{\alpha(p-1)} \right]^{\frac{\alpha(p-1)}{p\alpha-1}},\\
\hat{t} &:= t_0 + \alpha^{\frac{\alpha-1}{\alpha(p-1)}}\, c^{\frac{1}{\alpha(p-1)}}\, \tfrac{\alpha(p-1)}{p\alpha-1}\, \left(\mathcal{E}(v(t_0)|\varphi)\right)^{\frac{p\alpha-1}{\alpha(p-1)}},
\end{aligned}
$$

and $t_0 \ge 0$ can be chosen to be the "first entry time", that is, $t_0 \ge 0$ is the smallest time $\hat{t}_0 \in [0, +\infty)$ such that $v([\hat{t}_0, +\infty)) \subseteq B(\varphi, \varepsilon)$.

Proof. As in the previous proof, it remains to consider the situation, when $\mathcal{E}(v(t)) > \mathcal{E}(\varphi)$ for all $t \ge 0$. In addition, we assume that $\mathcal{E}$ satisfies a Łojasiewicz-Simon inequality for exponent $\alpha \in (0, 1]$ on $B(\varphi, \varepsilon) \cap D(g)$. In this case, the function θ is given by (3.2) and so, $\mathcal{H}$ defined in (3.6) reduces to

$$
\mathcal{H}(t) = \frac{c}{\alpha} \left(\mathcal{E}(v(t)|\varphi)\right)^{\alpha} \tag{3.47}
$$

for every $t \ge 0$. Let $t_0 \ge 0$ be the *first entry time* of v in $B(\varphi, \varepsilon)$. By Theorem 3.17 and since $v(t) \in D(g)$ for a.e. $t \in (0, +\infty)$, v satisfies condition (3.4) for $0 \le t_0 < T = +\infty$ where $\mathcal{U}$ is replaced by $B(\varphi, \varepsilon) \cap D(g)$ and so, we can apply the Łojasiewicz-Simon inequality (3.3) to $v = v(t)$ for a.e. $t \in [t_0, +\infty)$. Moreover, the function $\mathcal{H}$ is differentiable a.e. on $(0, +\infty)$ and chain rule (3.7) holds for a.e. $t \in (0, +\infty)$. Thus and since v is a p-gradient flow of $\mathcal{E}$ with respect to strong upper gradient g, we can conclude by (2.10) that

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) &= c\left(\mathcal{E}(v(t)|\varphi)\right)^{\alpha-1} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(v(t))\right)\\
&= c\left(\mathcal{E}(v(t)|\varphi)\right)^{\alpha-1} g(v(t))^{p'}\\
&\ge \left[\tfrac{1}{c}\right]^{p'-1} \left(\mathcal{E}(v(t)|\varphi)\right)^{\frac{1-\alpha}{p-1}}\\
&= \left[\tfrac{1}{\alpha^{\alpha-1}c}\right]^{\frac{p'-1}{\alpha}}\ \mathcal{H}^{\frac{1-\alpha}{\alpha(p-1)}}(t)
\end{aligned}
$$

for a.e. $t \in (t_0, +\infty)$. Therefore, for a.e. $t \in (t_0, +\infty)$, one has

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{H}^{-\frac{1-p\alpha}{\alpha(p-1)}}(t) \geq \left[\tfrac{1}{\alpha^{\alpha-1}c}\right]^{\frac{p'-1}{\alpha}} \frac{1-p\alpha}{\alpha(p-1)} && \text{if} \quad 0<\alpha<\tfrac{1}{p}\\
&\frac{\mathrm{d}}{\mathrm{dt}}\log \mathcal{H}(t) \leq -\left[\tfrac{1}{\alpha^{\alpha-1}c}\right]^{\frac{p'-1}{\alpha}} = -\frac{1}{pc^{p'}} && \text{if} \quad \alpha=\tfrac{1}{p}\\
&\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{H}^{\frac{p\alpha-1}{\alpha(p-1)}}(t) \leq -\left[\tfrac{1}{\alpha^{\alpha-1}c}\right]^{\frac{p'-1}{\alpha}} \frac{p\alpha-1}{\alpha(p-1)} && \text{if} \quad \tfrac{1}{p}<\alpha\leq 1.
\end{aligned}$$

Integrating these inequalities over (t_0, t) for any $t > t_0$ and rearranging the resulting inequalities yields

$$\begin{aligned}
&\mathcal{H}(t) \leq \left[\left[\tfrac{1}{\alpha^{\alpha-1}c}\right]^{\frac{p'-1}{\alpha}} \frac{1-p\alpha}{\alpha(p-1)}(t-t_0) + \mathcal{H}^{-\frac{1-p\alpha}{\alpha(p-1)}}(t_0)\right]^{-\frac{\alpha(p-1)}{1-p\alpha}} && \text{if } 0<\alpha<\tfrac{1}{p},\\
&\mathcal{H}(t) \leq \mathcal{H}(t_0)e^{-\frac{t}{pc^{p'}}} && \text{if } \alpha=\tfrac{1}{p},
\end{aligned}$$

and in the case $\frac{1}{p} < \alpha \leq 1$,

$$\mathcal{H}^{\frac{p\alpha-1}{\alpha(p-1)}}(t) \leq \left[\tfrac{1}{\alpha^{\alpha-1}c}\right]^{\frac{p'-1}{\alpha}} \frac{p\alpha-1}{\alpha(p-1)}(t_0-t) + \mathcal{H}^{\frac{p\alpha-1}{\alpha(p-1)}}(t_0)$$

for every $t > t_0$. Now, for

$$\hat{t} = t_0 + \frac{\alpha(p-1)}{\alpha p-1}\,\alpha^{\frac{\alpha-1}{\alpha(p-1)}}\,c^{\frac{1}{\alpha(p-1)}}\,(\mathcal{E}(v(t_0)|\varphi))^{\frac{\alpha p-1}{p-1}},$$

if $t = \hat{t}$, then the right hand side in the latter inequality becomes 0 and hence $\mathcal{H}(\hat{t}) = 0$. By (3.47) and since $\mathcal{E}$ is a strict Lyapunov function of v, this implies that $v(t) \equiv \varphi$ for all $t \geq \hat{t}$. Finally, by (2.5), (3.8) and Theorem 3.17, we have

$$d(v(t), \varphi) \leq \int_t^\infty |v'|(s)\,\mathrm{d}s \leq \mathcal{H}(t)$$

for every $t \geq t_0$. Therefore, the previous three inequalities yield the claim of this theorem. □

As an immediate consequence of Theorem 3.20, we obtain the well-known exponential convergence result (cf [7, Theorem 2.4.14]) of 2-gradient flows of functionals $\mathcal{E}$ defined on a length space $\mathfrak{M}$ and which are λ-geodesically convex for some $\lambda > 0$. Corollary 3.21 shows that the approach using the Łojasiewicz-Simon inequality provides the same rate of convergence and so, this theory is consistent with the classical one.

Corollary 3.21. *Let $(\mathfrak{M}, d)$ be a length space and $\mathcal{E} : \mathfrak{M} \to \mathbb{R}\cup\{+\infty\}$ a proper, lower semicontinuous functional that is λ-geodesically convex for $\lambda > 0$ and admit a global minimizer $\varphi \in D(\mathcal{E})$. Then, every gradient flow v of $\mathcal{E}$ satisfies*

$$d(v(t), \varphi) = \mathcal{O}\left(e^{-\lambda t}\right) \qquad \text{as } t \to \infty.$$

Remark 3.22. We note that a similar statement of Corollary 3.21 can not hold for *geodesically convex* functionals $\mathcal{E}$ since the class of *convex*, proper and lower semicontinuous functionals on Hilbert spaces belong to this case where the counter-example [13] by Baillon is known.

3.7. **Lyapunov stable equilibrium points.** In this subsection, our aim is to show that if a functional $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) in a neighborhood of an equilibrium point $\varphi \in \mathbb{E}_g$ of $\mathcal{E}$, then the *Lyapunov stability* of φ can be characterized with the property that φ is a local minimum of $\mathcal{E}$. This result generalizes the main theorem in [1] for functionals $\mathcal{E}$ defined on the Euclidean space $\mathbb{R}^N$ and satisfying a Łojasiewicz inequality (1.7).

We begin by recalling the notion of Lyapunov stable points of equilibrium.

Definition 3.23. For a given proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g a point of equilibrium $\varphi \in \mathbb{E}_g$ of $\mathcal{E}$ is called *Lyapunov stable* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that for every $v_0 \in B(\varphi, \delta) \cap D(\mathcal{E})$ and every p-gradient flow v of $\mathcal{E}$ with initial value $v(0+) = v_0$, one has

$$v(t) \in B(\varphi, \varepsilon) \qquad \text{for all } t \geq 0. \tag{3.48}$$

The property that an equilibrium point φ of $\mathcal{E}$ is Lyapunov stable is a *local* property. To characterize such point, we need the following assumption on the existence of p-gradient curves with initial values in a neighborhood of φ.

Assumption 3.4 (*Existence of p-gradient flows for small initial values*)**.** Suppose, for the proper energy functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g and given $\varepsilon > 0$ and $\varphi \in D(\mathcal{E})$ the following holds:
for all $v_0 \in D(\mathcal{E}) \cap B(\varphi, \varepsilon)$, there is a p-gradient flow v of $\mathcal{E}$ with $v(0+) = v_0$.

The next theorem is the main result of this section.

Theorem 3.24 (**Lyapunov stability and local Minima**)**.** *Let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper, lower semicontinuous functional and g be a with proper strong upper gradient of $\mathcal{E}$. Then the following statements hold.*

(1) *Suppose g is lower semicontinuous and for $\varphi \in \mathbb{E}_g$, there is a $\varepsilon > 0$ such that $\mathcal{E}$ is bounded from below on $B(\varphi, \varepsilon)$, $\mathcal{E}$ and g satisfy Assumption 3.4, the set*

$$\overline{B}(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) \neq 0] \quad \textit{is contained in} \quad [g > 0], \tag{H1*}$$

and there is a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ and $|[\theta \neq 0, \theta' = 0]| = 0$, for which $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on

$$\mathcal{U}_\varepsilon := B(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) \neq 0] \cap [\theta'(\mathcal{E}(\cdot|\varphi)) > 0]. \tag{3.49}$$

Then, if φ is Lyapunov stable, φ is a local minimum of $\mathcal{E}$.

(2) *Suppose for $\varphi \in \mathbb{E}_g$, there is an $\varepsilon > 0$ such that*

$$\begin{cases} \text{for every } \eta > 0, \text{ there is a } 0 < \delta \leq \varepsilon \text{ such that} \\ \quad \mathcal{E}(v|\varphi) < \eta \text{ for all } v \in B(\varphi, \delta) \cap D(\mathcal{E}), \end{cases} \tag{3.50}$$

the set $\overline{B}(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) \neq 0]$ satisfies (H1), and there is a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ and $|[\theta \neq 0, \theta' = 0]| = 0$, for which $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on $\mathcal{U}_\varepsilon$. Then, if φ is a local minimum of $\mathcal{E}$, φ is Lyapunov stable.*

Remark 3.25. Concerning Theorem 3.24, we note the following.

(1) The hypothesis that $\mathcal{E}$ is bounded from below on $B(\varphi,\varepsilon)$ is a necessary condition for φ being a local minimum of $\mathcal{E}$.

(2) If the functional $\mathcal{E}$ is continuous at the equilibrium point $\varphi\in\mathbb{E}_g$, then $\mathcal{E}$ is necessarily locally bounded and satisfies condition (3.50).

Proof of Theorem 3.24. We begin by showing that statement (1) holds. To do this, we prove the contrapositive. Thus, suppose φ is not a local minimum of $\mathcal{E}$. Then we shall show that there is $\varepsilon>0$ such that for every $\delta>0$ there is a $v_\delta^0\in D(\mathcal{E})\cap B(\varphi,\delta)$ and a p-gradient flow v_δ of $\mathcal{E}$ with initial value $v_\delta(0+)=v_\delta^0$ satisfying

$$v([0,+\infty))\not\subseteq B(\varphi,\varepsilon). \tag{3.51}$$

If φ is not a local minimum of $\mathcal{E}$ then for every $\delta>0$ there is a $v_\delta^0\in D(\mathcal{E})\cap B(\varphi,\delta)$ satisfying

$$\mathcal{E}(v_\delta^0|\varphi)<\mathcal{E}(\varphi|\varphi)=0. \tag{3.52}$$

Now, by hypothesis, there is an $\varepsilon>0$ and a strictly increasing function $\theta\in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0)=0$ and $|[\theta\neq 0,\theta'=0]|=0$, for which $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on $\mathcal{U}_\varepsilon$. For every $0<\delta<\varepsilon$, Assumption 3.4 ensures that there is a p-gradient flow v_δ of $\mathcal{E}$ with initial value $v_\delta(0+)=v_\delta^0\in D(\mathcal{E})\cap B(\varphi,\delta)$ satisfying (3.52). Since $\mathcal{E}$ is a Lyapunov function of v_δ (cf Proposition 2.38), inequality (3.52) implies

$$\mathcal{E}(v_\delta(t)|\varphi)\le\mathcal{E}(v_\delta(0+)|\varphi)<\mathcal{E}(\varphi|\varphi)=0\qquad\text{for all } t\ge 0. \tag{3.53}$$

If we assume that v_δ satisfies (3.48), then by (3.53) and since $|[\theta\neq 0,\theta'=0]|=0$, the trajectory v_δ satisfies condition (3.4) in Theorem 3.5 with $0=t_0<T=+\infty$, where $\mathcal{U}$ is replaced by $\mathcal{U}_\varepsilon$. Since $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on $\mathcal{U}_\varepsilon$ and since $\mathcal{E}$ is bounded from below on $\mathcal{U}_\varepsilon$, we can conclude that v_δ has finite length. Since $\mathfrak{M}$ is complete, there is an element $\hat{\varphi}\in\overline{B}(\varphi,\varepsilon)$ such that $v_\delta(t)\to\hat{\varphi}$ in $\mathfrak{M}$ as $t\to\infty$. By hypothesis, $\mathcal{E}$ and g are lower semicontinuous, and $\mathcal{E}$ is bounded from below on $B(\varphi,\varepsilon)$. Thus, Proposition 2.38 yields that $g(\hat{\varphi})=0$. But on the other hand, by (3.53), $|\mathcal{E}(\hat{\varphi}|\varphi)|>0$ and so, by hypothesis (H1*), $g(\hat{\varphi})>0$ which is a contradiction to $g(\hat{\varphi})=0$. Thus, our assumption is false, and therefore we have shown the existence of a p-gradient flow v_δ satisfying (3.51). Since this holds for all $0<\delta<\frac{\varepsilon}{7}$, we have thereby proved that φ is not Lyapunov stable.

Next, we prove that statement (2) is true. Thus, suppose φ is a local minimum of $\mathcal{E}$. Then, there is an $r>0$ such that

$$\mathcal{E}(v|\varphi)\ge 0\qquad\text{for all } v\in B(\varphi,r)\cap D(\mathcal{E}) \tag{3.54}$$

By hypothesis, there are $\varepsilon_0>0$ and a continuous strictly increasing function $\theta\in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0)=0$ and $|[\theta\neq 0,\theta'=0]|=0$, for which $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on the set $\mathcal{U}_{\varepsilon_0}$. Now, let $0<\varepsilon\le\varepsilon_0$. Then by (3.50), there is a $0<\delta<\min\{\frac{\varepsilon}{2},r\}$ such that

$$\mathcal{E}(v|\varphi)<\theta^{-1}(\tfrac{\varepsilon}{2})\qquad\text{for every } v\in B(\varphi,\delta)\cap D(\mathcal{E}). \tag{3.55}$$

Now, let $v_0\in B(\varphi,\delta)\cap D(\mathcal{E})$ and $v:[0,+\infty)\to\mathfrak{M}$ be a p-gradient flow of $\mathcal{E}$ with initial value $v(0+)=v_0$. Then by the continuity of v, there is a

$0 < T \le +\infty$ such that

$$v(t) \in B(\varphi, \varepsilon) \qquad \text{for all } 0 \le t < T. \tag{3.56}$$

Hence, by (3.55), since $|[\theta \neq 0, \theta' = 0]| = 0$, and since δ and $\varepsilon \le \varepsilon_0$, we have that $v(t) \in \mathcal{U}_{\varepsilon_0}$ for a.e. $0 \le t < T$. Since $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on $\mathcal{U}_\varepsilon$, we can apply Theorem 3.5 to conclude that the restriction $v_T := v_{|[0,T)}$ of v on $[0, T)$ has finite arc-length $\gamma(v_T)$. Let $T = sup\{T > 0 \,|\, v([0, T)) \subseteq B(\varphi, \varepsilon)\}$. Then, to complete this proof, it remains to show that $T = +\infty$. Thus, assume that T is finite and then we shall arrive to a contradiction. By Lemma 2.6, we can parametrize the curve v_T by its arc-length $\gamma(v_T)$ on $(0, T)$. Let $\hat{v}_T : [0, \gamma(v_T)] \to \mathfrak{M}$ be this reparametrization of v_T by its arc-length. Then, $\hat{v}_T$ is a p-gradient flow of $\mathcal{E}$ with metric derivative $|\hat{v}'| = 1$ a.e. on $(0, \gamma(v_T))$ satisfying satisfies (2.10) and $\hat{v}_T(t) \in \mathcal{U}_\varepsilon$ for all $t \in [0, \gamma(v_T))$. Thus and since $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1), we can conclude that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\theta(\mathcal{E}(\hat{v}(s)|\varphi)) &= \theta'(\mathcal{E}(\hat{v}(s)|\varphi))\, \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\hat{v}(s)|\varphi) \\
&= \theta'(\mathcal{E}(\hat{v}(s)|\varphi))\, (-g(\hat{v}(s)))\, |v'|(s) \\
&= -\theta'(\mathcal{E}(\hat{v}(s)|\varphi))\, g(\hat{v}(s)) \\
&\le -1
\end{aligned}$$

for a.e. $t \in (0, \gamma(v_T))$. Integrating this inequality from $(0, \gamma(v_T))$ leads to

$$(\theta(\mathcal{E}(\hat{v}(\gamma_T)|\varphi)) - \theta(\mathcal{E}(v_0|\varphi))) \le -\gamma(v_T).$$

Rearranging this inequality, then applying (3.54) and (3.55) and using the monotonicity of θ shows that the length $\gamma(v_T)$ of v on $[0, T)$ satisfies

$$\gamma(v_T) \le \theta(\mathcal{E}(v_0|\varphi)) < \frac{\varepsilon}{2}.$$

Therefore,

$$d(v(t), \varphi) \le d(v(t), v_0) + d(v_0, \varphi) \le \gamma(v_T) + \frac{\varepsilon}{2} < \varepsilon.$$

Thus, by the continuity of v, and since we have assumed that T is finite, there is a $T' > T$ such that v satisfies (3.56) for T replaced by T'. But this is a contradiction to the fact that T is maximal such that (3.56) holds, implying that our assumption is false. Therefore, v satisfies (3.56) with $T = +\infty$. Since $0 < \varepsilon \le \varepsilon_0$ and the p-gradient flow v with initial value $v(0+) = v_0 \in B(\varphi, \delta) \cap D(\mathcal{E})$ where arbitrary, we have thereby shown that the local minimizer φ of $\mathcal{E}$ is Lyapunov stable. □

3.8. **Entropy-transportation inequality and KŁ inequality.** In this last part of Section 3, we show that a generalized *entropy-transportation inequality* (1.5) is equivalent to Kurdyka-Łojasiewicz inequality (3.1). We investigate this in two cases; when $\mathcal{E}$ satisfies these inequalities *locally* and *globally*.

We begin by introducing the notion of local and global *entropy-transportation inequality* (cf [61]).

Definition 3.26. A proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g is said to satisfy *locally a generalized entropy-transportation (ET-)*

inequality at a point of equilibrium $\varphi \in \mathbb{E}_g$ if there are $\varepsilon > 0$ and a strictly increasing function $\Psi \in C(\mathbb{R})$ satisfying $\Psi(0) = 0$ and

$$\inf_{\hat{\varphi}\in\mathbb{E}_g\cap B(\varphi,\varepsilon)} d(v,\hat{\varphi}) \le \Psi(\mathcal{E}(v|\varphi)) \tag{3.57}$$

for every $v \in B(\varphi,\varepsilon) \cap D(\mathcal{E})$. Further, a functional $\mathcal{E}$ is said to satisfy *globally a generalized entropy-transportation inequality at* $\varphi \in \mathbb{E}_g$ if $\mathcal{E}$ satisfies

$$\inf_{\hat{\varphi}\in\mathbb{E}_g} d(v,\hat{\varphi}) \le \Psi(\mathcal{E}(v|\varphi)) \quad \text{for every } v \in D(\mathcal{E}). \tag{3.58}$$

Remark 3.27 (*isolated equilibrium points*). If a functional $\mathcal{E}$ with strong upper gradient g admits a point of equilibrium $\varphi \in \mathbb{E}_g$ satisfying

$$\mathbb{E}_g \cap B(\varphi,\varepsilon) = \{\varphi\}$$

for some $\varepsilon > 0$, then the generalized entropy-transportation inequality (3.57) reduces to inequality (1.5). This is, for instance, the case when $\mathcal{E}$ is λ-geodesically convex with $\lambda > 0$ (see Proposition 2.28).

The following theorem is our first main result of this subsection.

Theorem 3.28 (Equivalent between Local KŁ- and ET-inequality). *Let $\mathcal{E} : \mathfrak{M} \to (-\infty,+\infty]$ be a proper, lower semicontinuous functional with strong upper gradient g. Then, the following statements hold.*

*(1) **(KŁ-inequality implies ET-inequality)** Let g be lower semicontinuous and the equilibrium point $\varphi \in \mathbb{E}_g$ be Lyapunov stable. Suppose, there is an $\varepsilon > 0$ such that $\mathcal{E}$ is bounded from below on $B(\varphi,\varepsilon)$, $\mathcal{E}$ and g satisfy Assumption 3.4, and $\overline{B}(\varphi,\varepsilon) \cap [\mathcal{E}(\cdot|\varphi) \neq 0]$ satisfies (H1*).*

If there is a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ and $|[\theta \neq 0, \theta' = 0]| = 0$ and such that $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on the set $\mathcal{U}_\varepsilon$ given by (3.49), then $\mathcal{E}$ satisfies locally a generalized entropy-transportation inequality (3.57) at φ.

*(2) **(ET-inequality implies KŁ-inequality)** Suppose $\varphi \in \mathbb{E}_g$ is a local minimum of $\mathcal{E}$ and there is an $\varepsilon > 0$ such that $\mathcal{E}$ and g satisfy*

$$\mathcal{E}(v|\varphi) \le g(v)\, d(v,\varphi) \qquad \textit{for all } v \in B(\varphi,\varepsilon) \cap D(\mathcal{E}). \tag{3.59}$$

If there is a strictly increasing function $\Psi \in C(\mathbb{R})$ satisfying $\Psi(0) = 0$ and such that $s \mapsto \Psi(s)/s$ belongs to $L^1_{loc}(\mathbb{R})$, and $\mathcal{E}$ satisfies a generalized entropy-transportation inequality (3.57) on $B(\varphi,\varepsilon)\cap D(\mathcal{E})$, then $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on $\mathcal{U} \cap [\mathcal{E}(\cdot|\varphi) > 0]$.

Remark 3.29. It is worth noting that every proper, λ-geodesically convex functional $\mathcal{E}$ with $\lambda \ge 0$ satisfies condition (3.59) with $g = |D^-\mathcal{E}|$ the descending slope of $\mathcal{E}$ (see Proposition 2.23).

Remark 3.30 (*The role of the set $D(\mathcal{E})$ in Theorem 3.28 and "invariant sets"*). We note that Assumption 3.4 in Theorem 3.28 is only needed to establish statement (1). Further, the proof of statement (1) in Theorem 3.28 shows, that the set $D(\mathcal{E})$ in the two inequalities (3.1) and (3.57) could be replaced by every subset $\mathcal{D} \subseteq D(\mathcal{E})$ which the flow map $S : [0, +\infty) \times D(\mathcal{E}) \to 2^{D(\mathcal{E})}$ defined by (3.19) leaves *invariant*, that is,

$$\begin{cases} \text{for every } v_0 \in \mathcal{D} \text{ and every } p\text{-gradient flow } v \text{ of } \mathcal{E} \text{ with} \\ \text{initial value } v(0+) = v_0, \text{ one has } v(t) \in \mathcal{D} \text{ for all } t \geq 0. \end{cases} \tag{3.60}$$

After these remarks, we turn now to the proof.

Proof of Theorem 3.28. We begin by showing that statement (1) holds. By hypothesis, for $\varphi \in \mathbb{E}_g$, there are $\varepsilon > 0$ and a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ such that $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on the set $\mathcal{U}_\varepsilon$ given by (3.49). Now, let $v \in D(\mathcal{E}) \cap B(\varphi, \varepsilon)$. By Assumption 3.4 and since φ is Lyapunov stable, there is a p-gradient flow $\hat{v} : [0, +\infty) \to \mathfrak{M}$ of $\mathcal{E}$ with initial value $\hat{v}(0+) = v$ satisfying (3.48) and by Theorem 3.24, φ is a local minimum of $\mathcal{E}$. Thus, $\mathcal{E}(\hat{v}(t)|\varphi) \geq 0$ for all $t \geq 0$.

We set $T := \sup\{t \geq 0 \,|\, \mathcal{E}(\hat{v}(t)|\varphi) > 0\}$. Then, we need to consider three cases. First, suppose $T = +\infty$. Then, $\mathcal{E}(\hat{v}(t)|\varphi) > 0$ for all $t \geq 0$. Since $|[\theta \neq 0, \theta' = 0]| = 0$ and the set $\overline{B}(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) > 0]$ satisfies (H1*), it follows that $\hat{v}(t) \in \mathcal{U}_\varepsilon$ for almost every $t \geq 0$. Since $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on the set $\mathcal{U}_\varepsilon$ and since $\mathcal{E}$ is bounded from below on $B(\varphi, \varepsilon)$, Theorem 3.5 yields that $\hat{v}$ has finite length (2.3). By the completeness of $\mathfrak{M}$ and by statement (3) of Proposition 2.38, there is a $\hat{\varphi} \in \mathbb{E}_g \cap \overline{B}(\varphi, \varepsilon)$ such that $\hat{v}(t) \to \hat{\varphi}$ in $\mathfrak{M}$ as $t \to +\infty$. Moreover, the function θ and $\hat{v}$ satisfy inequality (3.5) in Theorem 3.5. Thus,

$$d(v, \hat{v}(t)) \leq \int_0^t |\hat{v}'|(s)\,\mathrm{d}s \leq \theta(\mathcal{E}(v|\varphi)) - \theta(\mathcal{E}(\hat{v}(t)|\varphi)) \tag{3.61}$$

for all $t \geq 0$. Since $\hat{\varphi} \in \mathbb{E}_g \cap \overline{B}(\varphi, \varepsilon)$, hypothesis (H1*) implies that $\mathcal{E}(\hat{\varphi}|\varphi) = 0$. Thus, sending $t \to +\infty$ in (3.61) yields

$$d(v, \hat{\varphi}) \leq \theta(\mathcal{E}(v|\varphi)) \tag{3.62}$$

and by taking the infimum of $d(v, \cdot)$ over all equilibrium points $\tilde{\varphi} \in \mathcal{E}_g \cap B(\varphi, \varepsilon)$ of $\mathcal{E}$ on the left-hand side of (3.62) gives

$$\inf_{\tilde{\varphi} \in \mathcal{E}_g \cap B(\varphi, \varepsilon)} d(v, \tilde{\varphi}) \leq d(v, \hat{\varphi}) \leq \theta(\mathcal{E}(v|\varphi)),$$

which for $\Psi = \theta$, is an entropy-transportation inequality (3.57) at φ.

Next, suppose $0 < T < \infty$. Since $\mathcal{E}$ is a Lyapunov function of $\hat{v}$ (see Proposition 2.38), $\hat{v}(t) = \hat{v}(T)$ for all $t \geq T$ and since g is lower semicontinuous on $\mathfrak{M}$, statement (3) of Proposition 2.38 implies that $\hat{v}(T) \in \mathbb{E}_g \cap B(\varphi, \varepsilon)$ with $\mathcal{E}(\hat{v}(T)) = \mathcal{E}(\varphi)$. Since $\hat{v}(t) \in \mathcal{U}_\varepsilon$ for almost every $t \in (0, T)$ and since $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on the set $\mathcal{U}_\varepsilon$, it follows from Theorem 3.5 that $\hat{v}$ satisfies inequality (3.61) for $t = T$ and so,

$$\inf_{\tilde{\varphi} \in \mathcal{E}_g \cap B(\varphi, \varepsilon)} d(v, \tilde{\varphi}) \leq d(v, \hat{v}(T)) \leq \theta(\mathcal{E}(v|\varphi)).$$

In the case $T = 0$, the initial value $v \in \mathbb{E}_g \cap B(\varphi, \varepsilon)$ satisfying $\mathcal{E}(v|\varphi) = 0$. Since $\theta(0) = 0$ and $d(v, v) = 0$, a generalized entropy-transportation inequality (3.57) at φ with $\Psi := \theta$, trivially holds. Therefore and since $v \in D(\mathcal{E}) \cap B(\varphi, \varepsilon)$ was arbitrary, the proof of statement (1) is complete.

Next, we intend to show that statement (2) holds. To see this, we note that by condition (3.59) and since φ is a local minimum,

$$\mathcal{E}(\tilde{\varphi}) = \mathcal{E}(\varphi) \qquad \text{for all } \tilde{\varphi} \in \mathbb{E}_g \cap B(\varphi, \varepsilon),$$

or equivalently, for the set $B(\varphi, \varepsilon)$, $\mathcal{E}$ and g satisfy condition (H1) in Section 3.3. By using again that φ is a local minimum, one sees that the two sets $B(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) \neq 0]$ and $B(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) > 0]$ coincide. Thus, and by (3.59),

$$\frac{\mathcal{E}(v|\varphi)}{g(v)} \leq \inf_{\tilde{\varphi} \in \mathbb{E}_g \cap B(\varphi,\varepsilon)} d(v, \tilde{\varphi})$$

for every $v \in B(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) > 0]$, or equivalently,

$$\mathcal{E}(v|\varphi) \leq g(v) \inf_{\tilde{\varphi} \in \mathbb{E}_g \cap B(\varphi,\varepsilon)} d(v, \tilde{\varphi}). \tag{3.63}$$

By hypothesis, there is a strictly increasing function $\Psi \in C(\mathbb{R})$ satisfying $\Psi(0) = 0$ such that $\mathcal{E}$ satisfies a generalized entropy-transportation inequality (3.57) on $B(\varphi, \varepsilon) \cap D(\mathcal{E})$. Combining (3.57) with (3.63), one finds that

$$\mathcal{E}(v|\varphi) \leq g(v) \inf_{\tilde{\varphi} \in \mathbb{E}_g \cap B(\varphi,\varepsilon)} d(v, \tilde{\varphi}) \leq g(v)\Psi(\mathcal{E}(v|\varphi)) \tag{3.64}$$

for every $v \in B(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) > 0]$. By hypothesis, Ψ is a continuous, strictly increasing function on $\mathbb{R}$ satisfying $\Psi(0) = 0$, and $s \mapsto \Psi(s)/s$ belongs to $L^1_{loc}(\mathbb{R})$. Thus, if $\theta : \mathbb{R} \to (0, +\infty)$ is defined by

$$\theta(s) = \int_0^s \frac{\Psi(r)}{r}\, \mathrm{d}r \qquad \text{for every } s \in \mathbb{R},$$

then $\theta \in W^{1,1}_{loc}(\mathbb{R})$, θ is strictly increasing and satisfies $\theta(0) = 0$ and $\theta'(s) = \frac{\Psi(s)}{s} > 0$ for all $s \in \mathbb{R} \setminus \{0\}$. Note, that this implies that $|\{\theta \neq 0, \theta' = 0\}| = 0$. Moreover, in terms of the function θ, inequality (3.64) can be rewritten as

$$1 \leq g(v)\, \frac{\Psi(\mathcal{E}(v|\varphi))}{\mathcal{E}(v|\varphi)} = g(v)\, \theta'(\mathcal{E}(v|\varphi))$$

for every $v \in B(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) > 0]$, proving that $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on $\mathcal{U} = B(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) > 0]$. This completes the proof of this theorem. □

To show that a *global* Kurdyka-Łojasiewicz inequality (3.1) implies a *global* entropy-transportation inequality (1.5), we need the following assumption instead of Assumption 3.4.

Assumption 3.5 (*Existence of p-gradient flows*)**.** Suppose, for the proper energy functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g holds:

for all $v_0 \in D(\mathcal{E})$*, there is a p-gradient flow* v *of* $\mathcal{E}$ *with* $v(0+) = v_0$*.*

We recall that by Fermat's rule (Proposition 2.28), for λ-geodesically convex functionals $\mathcal{E}$ with $\lambda \geq 0$, every equilibrium point $\varphi \in \mathbb{E}_{|D^-\mathcal{E}|}$ of $\mathcal{E}$ is a *global* minimum of $\mathcal{E}$. Thus, by following the idea of the proof of Theorem 3.28 and using statement (3) of Proposition 2.23 together with Remark 3.29, one sees that the following result on the equivalence of *global* Kurdyka-Łojasiewicz inequality (3.1) and *global* entropy-transportation inequality (1.5) holds. We omit the proof of Theorem 3.31 since it is repetitive to the previous one.

Theorem 3.31 (**Global KŁ- and ET-inequality**). *For $\lambda \ge 0$, let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper, lower semicontinuous, λ-geodesically convex functional on a length space $(\mathfrak{M}, d)$. Suppose, $\mathcal{E}$ and the descending slope $|D^{-}\mathcal{E}|$ satisfy Assumption 3.5 and for $\varphi \in \mathbb{E}_{|D^{-}\mathcal{E}|}$, the set $[\mathcal{E}(\cdot|\varphi) > 0]$ satisfies hypothesis (H1*). Then, the following statements are equivalent.*

*(1) (**KŁ-inequality**) There is a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ and $|[\theta \neq 0, \theta' = 0]| = 0$, and $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality (3.1) on $\mathcal{U} := [\mathcal{E}(\cdot|\varphi) > 0] \cap [\theta'(\mathcal{E}(\cdot|\varphi)) > 0]$.*

*(2) (**ET-inequality**) There is a strictly increasing function $\Psi \in C(\mathbb{R})$ satisfying $\Psi(0) = 0$ and $s \mapsto \Psi(s)/s$ belongs to $L^1_{loc}(\mathbb{R})$ such that $\mathcal{E}$ satisfies the generalized entropy-transportation inequality*

$$\inf_{\tilde{\varphi} \in \mathrm{argmin}(\mathcal{E})} d(v, \tilde{\varphi}) \le \Psi(\mathcal{E}(v|\varphi)) \qquad \text{for all } v \in D(\mathcal{E}). \tag{3.65}$$

Under the hypotheses of Theorem 3.31, if $\mathcal{E}$ satisfies a global Łojasiewicz-Simon inequality (3.3) with exponent $\alpha \in (0, 1]$ at $\varphi \in \mathbb{E}_{|D^{-}\mathcal{E}|}$, then the proof of statement (1) of Theorem 3.28 shows that $\mathcal{E}$ satisfies the generalized entropy-transportation inequality (3.65) for the function $\psi = \theta$ given by (3.2). Conversely, if $\mathcal{E}$ satisfies the the generalized entropy-transportation inequality (3.65) for $\psi(s) = \frac{c}{\alpha}|s|^{\alpha-1}s$, then the proof of statement (2) of Theorem 3.28 yields that $\mathcal{E}$ satisfies the global Łojasiewicz-Simon inequality (3.3) at φ for the function $\theta(s) = \frac{C}{\alpha^2}|s|^{\alpha-1}s$. Summarizing, we state the following result.

Corollary 3.32 (**Global ŁS- and ET-inequality**). *For $\lambda \ge 0$, let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper, lower semicontinuous, λ-geodesically convex functional on a length space $(\mathfrak{M}, d)$. Suppose, $\mathcal{E}$ and the descending slope $|D^{-}\mathcal{E}|$ satisfy Assumption 3.5 and for $\varphi \in \mathbb{E}_{|D^{-}\mathcal{E}|}$, the set $[\mathcal{E}(\cdot|\varphi) > 0]$ satisfies hypothesis (H1*). Then, for $\alpha \in (0, 1]$, the following statements hold.*

*(1) (**ŁS-inequality implies ET-inequality**) If there is a $c > 0$ such that $\mathcal{E}$ satisfies*

$$(\mathcal{E}(v|\varphi))^{1-\alpha} \le c|D^{-}\mathcal{E}|(v) \qquad \text{for all } v \in D(\mathcal{E}) \tag{3.66}$$

then $\mathcal{E}$ satisfies

$$\inf_{\tilde{\varphi} \in \mathrm{argmin}(\mathcal{E})} d(v, \tilde{\varphi}) \le \frac{c}{\alpha}(\mathcal{E}(v|\varphi))^{\alpha} \qquad \text{for all } v \in D(\mathcal{E}). \tag{3.67}$$

*(2) (**ET-inequality implies ŁS-inequality**) If there is a $c > 0$ such that $\mathcal{E}$ satisfies (3.67), then $\mathcal{E}$ satisfies*

$$(\mathcal{E}(v|\varphi))^{1-\alpha} \le \frac{c}{\alpha}|D^{-}\mathcal{E}|(v) \qquad \text{for all } v \in D(\mathcal{E}). \tag{3.68}$$

Remark 3.33. For Theorem 3.31 and Corollary 3.32, the same comments hold as stated in Remark 3.30.

Remark 3.34 (*The classical case $\lambda > 0$*). For $\lambda > 0$, let $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ be a proper, lower semicontinuous, λ-geodesically convex functional on a length space $(\mathfrak{M}, d)$. Then by Proposition 2.29, if $\mathcal{E}$ admits a (unique) minimizer $\varphi \in D(\mathcal{E})$ of $\mathcal{E}$, then (2.22) shows that the Łojasiewicz-Simon inequality (3.3) with exponent $\alpha = \frac{1}{2}$ holds and immediately induces that $\mathcal{E}$ satisfies an entropy-transportation inequality (1.5) with $\Psi(s) = \frac{2}{\lambda}|s|^{-1/2}s$. On the other hand, Corollary 3.32 yields for $\lambda > 0$ that entropy-transportation inequality (1.5) for $\Psi(s) = \frac{2}{\lambda}|s|^{-1/2}s$ implies that $\mathcal{E}$ satisfies a Łojasiewicz-Simon inequality (3.3) with exponent $\alpha = \frac{1}{2}$.

4. Applications

4.1. **The classical Banach and Hilbert space case.** We begin by considering the metric derivative of Banach space-valued curves.

Suppose that $\mathfrak{M} = X$ is a reflexive Banach space. Then, a curve $v : (0, +\infty) \to \mathfrak{M}$ belongs to the class $AC^p_{loc}(0, +\infty; X)$ if and only if v is differentiable at a.e. $t \in (0, +\infty)$, $v' \in L^p_{loc}(0, +\infty; X)$ and

$$v(t) - v(s) = \int_s^t v'(r)\,dr \qquad \text{for all } 0 < s \leq t < +\infty.$$

In particular, the *metric derivative* $|v'|$ of v is given by

$$|v'|(t) = \|v'(t)\|_X \qquad \text{for a.e. } t \in (0, +\infty).$$

Remark 4.1 (*λ-convex functions on H*). Due to the identity

$$t\,\|v_1\|_H^2 + (1-t)\|v_0\|_H^2 - t\,(1-t)\|v_1 - v_0\|_H^2 = \|tv_1 + (1-t)v_0\|_H^2$$

holding for every element v_0, v_1 of an Hilbert space H and $t \in [0, 1]$, a functional $\mathcal{E} : H \to (-\infty, \infty]$ is λ-convex for some $\lambda \in \mathbb{R}$ along every *line segment* $\gamma = \overline{v_0 v_1} \subseteq H$ if and only if $v \mapsto \mathcal{E}_{-\lambda}(v) := \mathcal{E}(v) - \frac{\lambda}{2}\|v\|_H^2$ is convex on H, or equivalently (following the notion in [28]), $\mathcal{E}$ is *semi-convex* on H for $\omega = -\lambda$.

It is important to see that the theory in the *non-smooth* framework is consistent with the *smooth* one. We begin with the smooth setting. For this, we recall that a functional $\mathcal{E} : \mathcal{U} \to \mathbb{R}$ is Fréchet differentiable on an open set $\mathcal{U} \subseteq X$ if for every $v \in \mathcal{U}$, there is a (unique) element T of the dual space X' of X such that

$$\mathcal{E}(v + h) = \mathcal{E}(v) + T(h) + o(h) \qquad \text{for } h \to 0 \text{ in } X.$$

Then, one sets $\mathcal{E}'(v) = T$ and calls the mapping $\mathcal{E}' : \mathcal{U} \to X'$ the Fréchet differential of $\mathcal{E}$. It is outlined in the book [7] that a mapping $g : X \to [0, +\infty]$ is an *upper gradient of* $\mathcal{E}$ if and only if

$$g(v) \geq \|\mathcal{E}'(v)\|_{X'} \qquad \text{for all } v \in \mathcal{U},$$

where $\|\cdot\|_{X'}$ denotes the norm of the dual space X'.

Next, we revisit an interesting example of a smooth energy functional $\mathcal{E}$ given in [40, Proposition 1.1] (see also [25, Corollary 3.13], [26]) to demonstrate that the hypotheses on the talweg curve x in Theorem 3.11 provides an *optimal* function θ in Kurdyka-Łojasiewicz inequality (3.1). To measure the optimality on θ, we focus on the class of functions θ by (3.2) for some exponent $\alpha \in (0, 1]$.

Example 4.2. Let $\mathcal{E} : \mathcal{U} \to \mathbb{R}$ be a twice continuously differentiable function on an open neighborhood $\mathcal{U} \subseteq X$. Suppose $\varphi \in \mathcal{U}$ is a *local minimum* of $\mathcal{E}$ and $\mathcal{E}''(\varphi) : X \to X'$ is *invertible*. Then, by [40, Proposition 1.1], there is an $R_L > 0$ such that $\mathcal{E}$ satisfies Łojasiewicz-Simon inequality (3.3) on $B(\varphi, R_L)$.

Now, our aim is to construct a talweg $x : (0, \delta] \to X$ through the C-valley $\mathcal{V}_{C,\mathcal{D}}(\varphi)$ (for some $C > 1$, $\delta > 0$ and $\mathcal{D} \subseteq B(\varphi, R_L)$) which satisfies the hypotheses of Theorem 3.11 such that $\theta := h^{-1}$ for $h(\cdot) := \mathcal{E}(x(\cdot)|\varphi)$ does not grow faster or slower as $s \mapsto |s|^{1-\alpha}s$ with exponent $\alpha = \frac{1}{2}$.

By assumption, there is an $r_0 > 0$ such that

$$\mathcal{E}(v|\varphi) \geq 0 \qquad \text{for all } v \in B(\varphi, r_0). \tag{4.1}$$

Applying Taylor's theorem gives that

$$\mathcal{E}'(v) = \mathcal{E}''(\varphi)(v - \varphi) + o(\|v - \varphi\|_X) \tag{4.2}$$

for every $v \in B(\varphi, r_1)$, for some $0 < r_1 \leq r_0$. By possibly choosing $r_1 \in (0, r_0]$ a bit smaller, we can conclude from (4.2) that

$$\|\mathcal{E}'(v)\|_{X'} \leq C_1 \|v - \varphi\|_X \tag{4.3}$$

for every $v \in B(\varphi, r_1)$, where $C_1 = (\|\mathcal{E}''(\varphi)\|_{X''} + 1)$. Since $\mathcal{E}''(\varphi)$ is invertible, there is a $C_2 > 0$ such that

$$\|v - \varphi\|_X \leq C_2 \|\mathcal{E}''(\varphi)(v - \varphi)\|_{X'}$$

for all $v \in X$. Combining this inequality with (4.2), and by possibly choosing $r_1 \in (0, r_0]$ again smaller, we get

$$\|\mathcal{E}'(v)\|_{X'} \geq \frac{1}{2C_2}\|v - \varphi\|_X \tag{4.4}$$

for all $v \in B(\varphi, r_1)$. Thus, functional $\mathcal{E}$ satisfies Assumption 3.1 on $B(\varphi, r_1)$. Further, Taylor's expansion gives

$$\mathcal{E}(v|\varphi) = \tfrac{1}{2}\langle \mathcal{E}''(\varphi)(v - \varphi), v - \varphi\rangle_{X',X} + o(\|v - \varphi\|_X^2) \tag{4.5}$$

for every $v \to \varphi$. Thus, by possibly replacing r_1 by a smaller $\hat{r}_1 > 0$, we see that

$$|\mathcal{E}(v|\varphi)|^{1/2} \leq C_3 \|v - \varphi\|_X \tag{4.6}$$

for every $v \in B(\varphi, r_1)$. Now, fix an element $v_0 \in \partial B(\varphi, r_1)$ and let $x : [0, 1] \to X$ be the *straight line* from φ to v_0 given by

$$x(r) = \varphi + r(v_0 - \varphi) \qquad \text{for every } r \in [0, 1]$$

Then, by using again (4.5), there is an $0 < r_2 < \min\{1, r_1\}$ such that

$$|\mathcal{E}(x(r)|\varphi)|^{1/2} \geq C_4 \|x(r) - \varphi\|_X \tag{4.7}$$

for every $r \in [0, r_2]$, where $C_4 = \frac{1}{2}\frac{|\langle \mathcal{E}''(\varphi)(v_0-\varphi), v_0-\varphi\rangle_{X',X}|^{1/2}}{\|v_0-\varphi\|_X^{1/2}} > 0$. In fact, inserting $v = x(r)$ into (4.5) gives

$$\mathcal{E}(x(r)|\varphi) = \tfrac{r^2}{2}\langle \mathcal{E}''(\varphi)(v_0 - \varphi), v_0 - \varphi\rangle_{X',X} + o(r^2) \quad \text{as } r \to 0+.$$

Since $\frac{o(r^2)}{r^2} \to 0$ as $r \to 0+$ and $|\langle \mathcal{E}''(\varphi)(v_0 - \varphi), v_0 - \varphi\rangle_{X',X}| > 0$, there is $0 < \delta \leq r_2 < 1$ such that

$$\frac{o(r^2)}{r^2} < \frac{1}{4}|\langle \mathcal{E}''(\varphi)(v_0 - \varphi), v_0 - \varphi\rangle_{X',X}|$$

for every $0 < r \le \delta$. From this and by using the triangle inequality, we see that

$$\begin{aligned}|\mathcal{E}(x(r)|\varphi)| &= |\tfrac{r^2}{2}\langle \mathcal{E}''(\varphi)(v_0-\varphi), v_0-\varphi\rangle_{X',X} + o(r^2)|\\ &\ge |\tfrac{r^2}{4}\langle \mathcal{E}''(\varphi)(v_0-\varphi), v_0-\varphi\rangle_{X',X}|\\ &= C_4^2\,\|x(r)-\varphi\|_X^2\end{aligned}$$

for every $r \in [0,\delta]$. Now, set $\mathcal{D} = \overline{B}(\varphi,\delta)$ and let $\hat{v} \in \overline{B}(\varphi,\delta) \cap [\mathcal{E} = \mathcal{E}(x(r))]$ for $r \in (0,\delta]$. Note, $\delta > 0$ can always be chosen smaller such that $\mathcal{D} \subseteq B(\varphi, R_L)$ for the $R_L > 0$ given by [40]. Then, by (4.3), (4.7), and by (4.6) combined with (4.4),

$$\begin{aligned}\|\mathcal{E}'(x(r))\|_{X'} &\le C_1\,\|x(r)-\varphi\|_X\\ &\le \frac{C_1}{C_4}|\mathcal{E}(x(r)|\varphi)|^{1/2}\\ &= \frac{C_1}{C_4}|\mathcal{E}(\hat{v}|\varphi)|^{1/2} \le \frac{C_1}{C_4}\frac{C_3}{2C_2}\|\mathcal{E}'(\hat{v})\|_{X'}.\end{aligned}\tag{4.8}$$

Taking the infimum over all $\hat{v} \in \mathcal{D} \cap [\mathcal{E} = \mathcal{E}(x(r)|\varphi) + \mathcal{E}(\varphi)]$, gives

$$\|\mathcal{E}'(x(r))\|_{X'} \le \frac{C_1}{C_4}\frac{C_3}{2C_2}\, s_{B(\varphi,R)}(\mathcal{E}(x(r)|\varphi)).$$

Since $r \in (0,\delta]$ were arbitrary, we have thereby shown that the line curve x is a *talweg curve* through the C-valley $\mathcal{V}_{C,\mathcal{D}}(\varphi)$ of $\mathcal{E}$ for $C = \frac{C_1}{C_4}\frac{C_3}{2C_2} > 1$. Moreover, $x'(r) = (v_0 - \varphi) \neq 0$ for every $r \in [0,\delta]$. Hence, $x \in AC^\infty(0,\delta;X)$. Further, the function

$$h(r) = \mathcal{E}(x(r)|\varphi) \qquad \text{for every } r \in [0,\delta],$$

satisfies $h(0) = 0$ and since $\mathcal{E} \in C^2$ and $\mathcal{E}''(\varphi) \neq 0$,

$$\begin{aligned}&h'(r) = \langle \mathcal{E}'(x(r)), v_0-\varphi\rangle_{X',X} \neq 0 &&\text{for all } t \in (0,\delta), \text{ and}\\ &h''(r) = \langle \mathcal{E}''(x(r))(v_0-\varphi), v_0-\varphi\rangle_{X',X} \neq 0 &&\text{for all } r \in (0,\delta).\end{aligned}$$

Thus, $h(r)$ is a homeomorphism from $[0,\delta] \to [0,R]$ for $R := \mathcal{E}(x(\delta)|\varphi)$ and a diffeomorphism from $(0,\delta) \to (0,R)$. This shows that $\mathcal{E}$ satisfies the hypotheses of Theorem 3.11 and so $\mathcal{E}$ satisfies Kurdyka-Łojasiewicz inequality (3.1) near φ. Moreover, by the inequalities (4.6) and (4.7),

$$\frac{1}{C_3\,\|v_0-\varphi\|_X}|h(r)|^{1/2} \le r \le \frac{1}{C_4\,\|v_0-\varphi\|_X}|h(r)|^{1/2}$$

for every $r \in [0,\delta]$ and so, the function $\theta = h^{-1}$ satisfies

$$\frac{1}{C_3\,\|v_0-\varphi\|_X}|s|^{1/2} \le \theta(s) \le \frac{1}{C_4\,\|v_0-\varphi\|_X}|s|^{1/2}$$

for every $s \in [0,R]$. In particular, by inequality (4.8),

$$1 \le C\,\|\mathcal{E}'(v)\|_{X'}|\mathcal{E}(v|\varphi)|^{-1/2}\tag{4.9}$$

for all $v \in B(\varphi,\delta)$, where $C = \frac{C_3}{2C_2} > 0$. This shows that $\mathcal{E}$ satisfies Kurdyka-Łojasiewicz inequality (3.1) near φ with $\theta(s) = 2C|s|^{\frac{1}{2}-1}s$, or, Łojasiewicz-Simon inequality (3.3) near φ with exponent $\alpha = \frac{1}{2}$, showing the optimality of θ for this example of an energy $\mathcal{E}$.

Next, we consider the non-smooth case of energy functionals $\mathcal{E} : X \to (-\infty, +\infty]$. Here, one assumes that $\mathcal{E}$ can be decomposed as $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$, where $\mathcal{E}_1$ is a proper, lower semicontinuous, convex functional and $\mathcal{E}_2$ a continuously differentiable functional on X. Then the *sub-differential* $\partial\mathcal{E}$ of $\mathcal{E}$ is given by

$$\partial\mathcal{E} = \left\{(v, x') \in X \times X' \,\middle|\, \liminf_{t\downarrow 0} \tfrac{\mathcal{E}(v+th)-\mathcal{E}(v)}{t} \geq \langle x', h\rangle_{X',X} \text{ for all } h \in X\right\}.$$

By [7, Corollary 1.4.5], for $v \in D(|D^-\mathcal{E}|)$, the *descending slope*

$$|D^-\mathcal{E}|(v) = \min\left\{\|x'\|_{X'} \,\middle|\, x' \in \partial\mathcal{E}(v)\right\} \tag{4.10}$$

and $|D^-\mathcal{E}|$ is a strong upper gradient of $\mathcal{E}$. Since $\partial\mathcal{E}^\circ(v)$ denotes the set of all *sub-gradients* $x' \in \partial\mathcal{E}(v)$ with minimal (dual) norm, the relation (4.10) can be rewritten as $|D^-\mathcal{E}|(v) = \|\partial\mathcal{E}^\circ(v)\|_{X'}$.

In the case $X = (H, (.,.)_H)$ is a real Hilbert space and $\mathcal{E} : H \to (-\infty, +\infty]$ a proper, lower semicontinuous and semi-convex functional. Then, the following well-known *generation theorem* [20, Proposition 3.12] (see also [28]) holds for solutions to gradient system (1.2).

For every $v_0 \in \overline{D(\mathcal{E})}$, there is a unique solution v of

$$\begin{cases} v'(t) + \partial\mathcal{E}(v(t)) \ni 0, & t \in (0, \infty), \\ v(0) = v_0. \end{cases} \tag{1.2}$$

Since semi-convexity coincides with the notion of $(-\lambda)$-geodesically convexity for some $\lambda \in \mathbb{R}$ (see Remark 4.1) and by (4.10), solution v of (1.2) are the (2-)*gradient flows* of $\mathcal{E}$ in the metric space $\mathfrak{M} := \overline{D(\mathcal{E})}$ equipped with the induced metric of H. In particular, $\mathcal{E}$ generates an *evolution variational inequality* (see Remark 3.14 and also [32] and [7]).

As a consequence of Theorem 3.17, we obtain the following stability result for the Hilbert space framework which generalize [27, Theorem 2.6] (also, compare with [26, Theorem 12.2], and [16, Theorem 18, $(i) \Rightarrow (ii)$]).

Corollary 4.3. *Let $\mathcal{E} : H \to (-\infty, +\infty]$ be a proper, lower semicontinuous and semi-convex functional on a Hilbert space H. Suppose v is a gradient flow of $\mathcal{E}$ and there are $\varepsilon > 0$ and an equilibrium point $\varphi \in \omega(v)$ such that the set $B(\varphi, \varepsilon)$ satisfies hypothesis (H1). If there is a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ and $|[\theta > 0, \theta' = 0]| = 0$, for which $\mathcal{E}$ satisfies a Kurdyka-Lojasiewicz inequality on the set $\mathcal{U}_\varepsilon$ given by (3.40), then the following statements hold.*

(1) The gradient flow v trends to the equilibrium point φ of $\mathcal{E}$ in the metric sense of H.

(2) If there is a Banach space V which is continuously embedded into H and the gradient flow $v \in C((0, \infty); V)$ and has relatively compact image in V, then v trends to the equilibrium point φ of $\mathcal{E}$ in the metric sense of V.

Further, from Theorem 3.20, we can conclude the following result of decay estimates in the Hilbert space framework.

Corollary 4.4. *Let $\mathcal{E} : H \to (-\infty, +\infty]$ be a proper, lower semicontinuous and semi-convex functional on a Hilbert space H. Suppose v is a gradient flow of $\mathcal{E}$ and there are c, $\varepsilon > 0$ and an equilibrium point $\varphi \in \omega(v)$ such that $\mathcal{E}$ satisfies a Łojasiewicz-Simon inequality (3.3) with exponent $\alpha \in (0, 1]$ on $B(\varphi, \varepsilon)$. Then,*

$$\|v(t) - \varphi\|_H \le \tfrac{c}{\alpha}\,(\mathcal{E}(v(t)|\varphi))^{\alpha} = \mathcal{O}\left(t^{-\frac{\alpha}{1-2\alpha}}\right) \qquad \text{if } 0 < \alpha < \tfrac{1}{2}$$

$$\|v(t) - \varphi\|_H \le c\,2\,(\mathcal{E}(v(t)|\varphi))^{\frac{1}{2}} \le c\,2\,(\mathcal{E}(v(t_0)|\varphi))^{\frac{1}{2}}\, e^{-\frac{t}{2c^2}} \qquad \text{if } \alpha = \tfrac{1}{2}$$

$$\|v(t) - \varphi\|_H \le \begin{cases} \tilde{c}\,(\hat{t} - t)^{\frac{\alpha}{2\alpha-1}} & \text{if } \ t_0 \le t \le \hat{t}, \\ 0 & \text{if } t > \hat{t}, \end{cases} \qquad \text{if } \tfrac{1}{2} < \alpha \le 1,$$

where,

$$\tilde{c} := \left[\left[\tfrac{1}{\alpha^{\alpha-1}c}\right]^{\frac{1}{\alpha}}\, \tfrac{2\alpha-1}{\alpha}\right]^{\frac{\alpha}{2\alpha-1}}, \quad \hat{t} := t_0 + \alpha^{\frac{\alpha-1}{\alpha}}\, c^{\frac{1}{\alpha}}\, \tfrac{\alpha}{2\alpha-1}\,(\mathcal{E}(v(t_0)|\varphi))^{\frac{2\alpha-1}{\alpha}},$$

and $t_0 \ge 0$ can be chosen to be the "first entry time", that is, $t_0 \ge 0$ is the smallest time $\hat{t}_0 \in [0, +\infty)$ such that $v([\hat{t}_0, +\infty)) \subseteq B(\varphi, \varepsilon)$.

4.1.1. ***Strategy to derive global ET- and ŁS-inequalities.*** In the next two examples, we want to illustrate how an abstract *Poincaré-Sobolev inequality* (see (4.11) below) leads to an *entropy-transportation inequality* (3.58) and from this via the equivalence relation to Łojasiewicz-Simon inequality (3.3) (Corollary 3.32) to *decay estimates* of the trend to equilibrium and *finite time of extinction* (via Corollary 4.4). In particular, we provide a *simple* method to establish upper bounds on the *arrival time to equilibrium* (also called *extinction time*)

$$T^*(v_0) := \inf\left\{t > 0 \,\middle|\, v(s) = \varphi \quad \text{for all } s \ge t\right\}$$

of solutions v of parabolic boundary-value problems on a domain $\Omega \subseteq \mathbb{R}^N$, ($N \ge 1$) with initial value $v(0) = v_0$ and given equilibrium point φ. To do this, we revisit two classical examples on the total variational flow from [9] and [11] (see also [37] and [17]). But we stress that this method can easily be applied to any other nonlinear parabolic boundary-value problem that can be realized as an abstract gradient system (1.2) for a functional $\mathcal{E}$ defined on a Hilbert space H and for which a *polynomial* entropy-transportation inequality

$$\|v - \varphi\|_H \le C\,(\mathcal{E}(v|\varphi))^{\beta} \qquad (v \in D(\mathcal{E})),$$

holds for a global minimizer φ of $\mathcal{E}$ and given C, $\beta > 0$ (see also Remark 4.9 below). Taking βth root on both sides of the last inequality, then we call

$$\|v - \varphi\|_H^{1/\beta} \le \tilde{C}\,\mathcal{E}(v|\varphi) \qquad (v \in D(\mathcal{E})), \tag{4.11}$$

an (abstract) *Poincaré-Sobolev inequality* since (4.11) is either obtained by only a known Sobolev inequality (see, for instance, [6, Theorem 3.47] or [45, Chapter 11, 13.5, 14.6]) or by a Sobolev inequality combined with a Poincaré inequality (see, for instance, [6, Theorem 3.44]).

We note that the idea to employ Łojasiewicz-Simon inequality (3.3) for deriving *decay estimates* and *finite time of extinction* of solution of semi-linear and quasilinear parabolic problems is well-known (see, for instance, [40, 27]). But, our approach to exploit the equivalence relation between entropy-transportation inequality (3.58) and Łojasiewicz-Simon inequality (3.3) (Corollary 3.32) seems to be new. On the other hand, we also need to mention that Corollary 4.4 does not lead to optimal bounds on the extinction time $T^*(v_0)$.

4.1.2. ***Finite Extinction time of the Dirichlet-Total Variational Flow.*** In [9] (see also [11]), the following parabolic initial boundary-value problem

$$\begin{cases} v_t = \mathrm{div}\left(\frac{Dv}{|Dv|}\right) & \text{in } \Omega\times(0,+\infty),\\ v = 0 & \text{on } \partial\Omega\times(0,+\infty),\\ v(0) = v_0 & \text{on } \Omega, \end{cases} \tag{4.12}$$

related to the *total variational flow with homogeneous Dirichlet boundary conditions* was studied on a bounded connected extension domain $\Omega\subseteq\mathbb{R}^N$. In particular, it was shown that problem (4.12) can be rewritten as an abstract initial value problem (1.2) in the Hilbert space $H = L^2(\Omega)$ for the energy functional $\mathcal{E} : L^2(\Omega)\to(-\infty,+\infty]$ given by

$$\mathcal{E}(v) := \begin{cases} \displaystyle\int_\Omega |Dv| + \int_{\partial\Omega}|v| & \text{if } v\in BV(\Omega)\cap L^2(\Omega),\\ +\infty & \text{if otherwise.} \end{cases} \tag{4.13}$$

In fact (cf [9, Theorem 3], see also [11, Theorem 5.14]), $\mathcal{E}$ is a proper, convex and lower semicontinuous functional on $L^2(\Omega)$ with dense domain. Thus, by [19, Proposition 3.12], for every $v_0\in L^2(\Omega)$, there is a unique solution of problem (4.12).

Now, due to Corollary 4.4, we can show that for $N\le 2$, every solution v of problem (4.12) with initial value $v_0\in L^2(\Omega)$ has finite *extinction time*

$$T^*(v_0) := \inf\left\{t>0\,\middle|\, v(s)=0\quad\text{for all } s\ge t\right\}.$$

Theorem 4.5. *Suppose $N\le 2$ and for $v_0\in L^2(\Omega)$, let v be the unique strong solution of problem* (4.12)*. Then,*

$$T^*(v_0)\le\begin{cases} s + S_1\,|\Omega|^{1/2}\,\mathcal{E}(v(s)) & \text{if } N=1,\\ s+S_2\,\mathcal{E}(v(s)) & \text{if } N=2,\end{cases}$$

for arbitrarily small $s>0$, and

$$\|v(t)\|_{L^2(\Omega)}\le\begin{cases} \tilde{c}\,(T^*(v_0)-t) & \text{if } 0\le t\le T^*(v_0),\\ 0 & \text{if } t>T^*(v_0),\end{cases} \tag{4.14}$$

where S_N is the best constant in Sobolev inequality (4.15)*, and $\tilde{c}>0$.*

Proof. By the Sobolev inequality for BV-functions (see [6, Theorem 3.47]),

$$\|u\|_{L^{1^*}(\mathbb{R}^N)}\le S_N\int_{\mathbb{R}^N}|Du|\qquad\text{for every } u\in BV(\mathbb{R}^N), \tag{4.15}$$

where $1^*=\infty$ if $n=1$, $1^*=2$ if $n=2$. Since Ω is an extension domain, for every $v\in BV(\Omega)$, the extension $\hat{v}$ of v given by

$$\hat{v}(x) := \begin{cases} v(x) & \text{if } x\in\Omega\\ 0 & \text{if } x\in\mathbb{R}^N\setminus\overline{\Omega},\end{cases}$$

belongs to $BV(\mathbb{R}^N)$ and satisfies

$$\int_{\mathbb{R}^N} |D\hat{v}| = \int_{\Omega} |Dv| + \int_{\partial\Omega} |v|.$$

(cf [6, Theorem 3.89]) and so by inequality (4.15), we find that

$$\|v\|_{L^{1^*}(\Omega)} \le S_N \left(\int_{\Omega} |Dv| + \int_{\partial\Omega} |v| \right) \tag{4.16}$$

for all $v \in BV(\Omega)$. Note that for $v \in BV(\Omega) \cap L^2(\Omega)$, the right-hand side in (4.16) is $\mathcal{E}(v)$. Moreover, $\varphi \equiv 0$ is the (unique) global minimizer of $\mathcal{E}$. Thus and since for $N \le 2$, the Sobolev exponent $1^* \ge 2$, we can apply Hölder's inequality to conclude from (4.16) that

$$\|v\|_{L^2(\Omega)} \le C\, \mathcal{E}(v|\varphi) \qquad \text{for all } v \in D(\mathcal{E}), \tag{4.17}$$

where the constant $C = S_1\,|\Omega|^{1/2}$ if $N = 1$ and $C = S_2$ if $N = 1$. Due to the (abstract) Sobolev-inequality (4.17), $\mathcal{E}$ satisfies, in fact, an *entropy-transportation inequality* (1.5) for $\Psi(s) = C\,s$, $(s \in \mathbb{R})$, which by Corollary 3.32, is equivalent to the Łojasiewicz-Simon inequality

$$1 \le C\, \|\partial^{\circ}\mathcal{E}(v)\|_H = C\,|D^-\mathcal{E}|(v), \qquad (v \in [\mathcal{E} > 0]).$$

Thereby, we have shown that the energy functional $\mathcal{E}$ given by (4.13) satisfies a global Łojasiewicz-Simon inequality (3.3) with exponent $\alpha = 1$ at the equilibrium point $\varphi \equiv 0$. Now, Corollary 4.4 implies that the statement of this theorem holds, where the entry time $t_0 > 0$ can be chosen arbitrarily small since $v(t) \in D(\mathcal{E})$ for every $t > 0$. □

Remark 4.6. It was shown in [57] that in dimension $N = 2$, Sobolev inequality (4.15) has the sharp constant $S_2 = \frac{1}{\sqrt{2\pi}}$ with Sobolev exponent $1^* = 2$. Thus, Corollary 4.4 applied to $v_0 \in D(\mathcal{E}) = BV(\Omega)$ with $t_0 = 0$ and $\alpha = 1$ yields that in dimension $N = 2$, the *extinction time*

$$T^*(v_0) \le \frac{1}{\sqrt{2\pi}} \int_{\Omega} |Dv_0|.$$

In particular, if $E \subsetneq \Omega$ is a set of finite perimeter and $v_0 = a\,\mathbb{1}_E$, then

$$T^*(v_0) \le \frac{a}{\sqrt{2\pi}} \mathrm{Per}(E). \tag{4.18}$$

Let us point out that the upper bound of $T^*(v_0)$ given by (4.18) is not optimal. In fact, it was shown in [10] that if $0 \in \Omega$, $a > 0$ and for sufficiently small $R > 0$, then

$$v(t,x) := \frac{2}{R} \left[\frac{a\,R}{2} - t \right]^+ \mathbb{1}_{B(0,R)}(x) \qquad \text{for every } x \in \Omega,\, t > 0,$$

is the unique solution of (4.12) with initial datum $v_0 = a\,\mathbb{1}_{B(0,R)}$. But, v has the extinction time $T^*(v_0) = \frac{a\,R}{2}$, which is smaller than the upper bound

$$\frac{a}{\sqrt{2\pi}} \mathrm{Per}(B(0,R)) = a\,R\sqrt{2\pi} \qquad \text{given by (4.18).}$$

4.1.3. ***Finite extinction time to equilibrium of the Neumann-Total Variational Flow.*** In [11], the following *Neumann problem*

$$
\begin{cases} \quad v_t = \mathrm{div}\left(\frac{Dv}{|Dv|}\right) & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ v(0) = v_0 & \text{on } \Omega, \end{cases} \tag{4.19}
$$

related to the *total variational flow* was studied. Here, $\Omega \subseteq \mathbb{R}^N$ is a bounded connected extension domain and if ν is the outward pointing unit normal vector at $\partial\Omega$ then $\frac{\partial v}{\partial \nu}$ denote the *co-normal derivative* $\frac{Dv}{|Dv|} \cdot \nu$ associated with $\mathcal{A} := -\mathrm{div}\,(Dv/|Dv|)$. It was shown in [11, Theorem 2.3] that for the energy functional $\mathcal{E} : L^2(\Omega) \to (-\infty, +\infty]$ given by

$$
\mathcal{E}(v) := \begin{cases} \displaystyle\int_\Omega |Dv| & \text{if } v \in BV(\Omega) \cap L^2(\Omega), \\ +\infty & \text{if otherwise,} \end{cases} \tag{4.20}
$$

problem (4.19) can be rewritten as an abstract initial value problem (1.2) in the Hilbert space $H = L^2(\Omega)$. Moreover, $\mathcal{E}$ is a proper, convex and lower semicontinuous functional on $L^2(\Omega)$ with dense domain. Thus, by [19, Proposition 3.12], for every $v_0 \in L^2(\Omega)$, there is a unique strong solution of (4.19). Due to [11, Theorem 2.20], for every gradient flow v of $\mathcal{E}$ the ω-limit set

$$
\omega(v) = \Big\{ \overline{v_0} = \tfrac{1}{|\Omega|} \int_\Omega v_0 \, \mathrm{d}x \Big\}.
$$

Now, as an application of Corollary 4.4 and due to the Poincaré-Sobolev inequality ([6, Remark 3.50])

$$
\|v - \overline{v}\|_{1^*} \le C_{PS_N} \int_\Omega |Dv| = C_{PS_N}\, \mathcal{E}(v|\overline{v}), \qquad (v \in BV(\Omega)), \tag{4.21}
$$

every solution v of problem (4.19) arrive *in finite time* to the media $\overline{v_0}$ of its initial condition $v(0) = v_0 \in L^2(\Omega)$. We omit the details of the proof since it follows the same idea as the previous one.

Theorem 4.7. *Suppose $N \le 2$ and for $v_0 \in L^2(\Omega)$, let v be the unique strong solution of problem* (4.19)*. Then,*

$$
T^*(v_0) \le \begin{cases} s + C_{PS_1} \, |\Omega|^{1/2} \, \mathcal{E}(v(s)) & \text{if } N = 1, \\ s + C_{PS_2} \, \mathcal{E}(v(s)) & \text{if } N = 2, \end{cases}
$$

for arbitrarily small $s > 0$, and

$$
\|v(t) - \overline{v_0}\|_2 \le \begin{cases} \tilde{c}\,(T^*(v_0) - t) & \text{if } 0 \le t \le T^*(v_0), \\ 0 & \text{if } t > T^*(v_0), \end{cases} \tag{4.22}
$$

where C_{PS_N} is the best constant in Poincaré-Sobolev inequality (4.21) *below, and $\tilde{c} > 0$.*

Remark 4.8. Similarly to Remark 4.6, Corollary 4.4 yields that for every $v_0 \in D(\mathcal{E})$, the *time to the equilibrium*

$$
T^*(v_0) \le C_{PS_N} \int_\Omega |Dv_0|.
$$

We emphasize that this estimate of $T^*(v_0)$ is not contained in [11]. In particular, if $E \subsetneq \Omega$ is a set of finite perimeter, $a \in \mathbb{R}$ and $v_0 = a\,\mathbb{1}_E$, then for the gradient flow v of $\mathcal{E}$ defined in (4.20) with initial value $v(0) = v_0$, the extinction time

$$T^*(v_0) \le C_{PS_N}\, a\,\mathrm{Per}(E).$$

Remark 4.9 (*Extinction time in fourth order total variational flow problems*). It is important to mention the rigorous study [37] on the extinction time T^* of solutions of the total variational flow equipped with Dirichlet (including $\Omega = \mathbb{R}^N$), Neumann, and periodic boundary conditions. The results in [37, Theorem 2.4 and Theorem 2.5] only depend on the L^N-norm of the initial data. Furthermore, in [37] upper estimates are established on the extinction time T^* of solutions of the *fourth-order total variation flow equation*

$$v_t = -\Delta\left[\mathrm{div}\left(\frac{Dv}{|Dv|}\right)\right]. \tag{4.23}$$

After adding the right choice of boundary conditions, equation (4.23) can be realized as a gradient system (1.2) in $H = H^{-1}(\Omega)$ for the energy

$$\mathcal{E}(v) := \begin{cases} \displaystyle\int_\Omega |Dv| & \text{if } v \in BV(\Omega) \cap H^{-1}(\Omega),\\ +\infty & \text{if otherwise,} \end{cases}$$

for every $v \in H^{-1}(\Omega)$. Due to *entropy-transportation inequality*

$$\|v\|_{H^{-1}(\Omega)} \le C \int_\Omega |Dv| = C\,\mathcal{E}(v),$$

(for instance, cf [37, (29)] in dimension $N = 4$), Corollary 3.32 and Corollary 4.4 yield the existence of upper estimates of the extinction time $T^*(v_0)$ for solution of (4.23).

4.2. **Gradient flows in spaces of probability measures.** In the second part of Section 4, we turn to the stability analysis of p-gradient flows in spaces of probability measures.

Throughout this section, we assume that the reader is familiar with the basics of optimal transport and refer for further reading to the standard literature (see, for instance, [60], [62], [8], or [53]). Here, we only recall the notations and results which are important to us for this article.

Let $(X, \mathcal{B}, d)$ be a *Polish space*, that is, a complete separable metric space, equipped with their Borel σ-algebra $\mathcal{B}$. Then, we denote by $\mathcal{P}(X)$ the space of probability measures on $(X, \mathcal{B})$ and $\mathcal{P}(X \times X)$ the space of probability measures on the product space $(X \times X, \mathcal{B} \otimes \mathcal{B})$. If $(Y, \mathcal{B}_{\mathcal{Y}}, d_Y)$ is a second Polish space, $\mu \in \mathcal{P}(X)$, and $T : X \to Y$ be a μ-measurable map, then the *push-forward of* μ *through* T is defined by

$$T_\# \mu(B) := \mu(T^{-1}(B)) \qquad \text{for every } B \in \mathcal{B}_Y.$$

For $s \in \{0, 1\}$, let $\boldsymbol{p}_s : X \times X \to X$ be defined by $\boldsymbol{p}_s(x, y) := (1-s)x + sy$. Then, $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ are projection maps and for given $\mu \in \mathcal{P}(X \times X)$, $\boldsymbol{p}_{0\#}\mu \in \mathcal{P}(X)$ and $\boldsymbol{p}_{1\#}\mu \in \mathcal{P}(X)$ are the *marginals* of μ. For given measures μ_0, $\mu_1 \in \mathcal{P}(X)$, the set of *transport plans with marginals* μ_0 *and* μ_1 is denoted by

$$\Pi(\mu_0, \mu_1) := \Big\{\pi \in \mathcal{P}(X \times X) \,\Big|\, \boldsymbol{p}_{0\#}\pi = \mu_0,\ \boldsymbol{p}_{1\#}\pi = \mu_1\Big\}.$$

Now, let $c : X \times X \to [0, +\infty]$ be a given Borel-measurable function. Then for given μ_0, $\mu_1 \in \mathcal{P}(X)$, the celebrated *Monge problem* (also-called *optimal transport problem*) consists in establishing the existence of a μ_0-measurable map $T^* : X \to X$ called *optimal transport map* satisfying

$$\int_X c(x, T^*(x)) \,\mathrm{d}\mu_0(x) = \min \left\{ \int_X c(x, \hat{T}(x)) \,\mathrm{d}\mu_0(x) \;\middle|\; \begin{array}{l} \hat{T} : X \to X \ \mu_0\text{-meas-} \\ \text{urable } \& \ \hat{T}_{\#}\mu_0 = \mu_1 \end{array} \right\}.$$

The function c in Monge's minimization problem is called a *cost function*. If μ_0 is a Dirac mass δ_a for some $a \in X$ and μ_1 is not a Dirac mass, then there might be no transport map T^* such that $T^*_{\#}\mu_0 = \mu_1$. Thus, the Monge problem can be ill posed, and so, one rather considers a *weaker version* of it instead; namely, the so-called *Monge-Kantorovich problem*

$$\inf \left\{ \int_{X\times X} c(x, y) \,\mathrm{d}\pi(x, y) \;\middle|\; \pi \in \Pi(\mu_0, \mu_1) \right\}. \tag{4.24}$$

If the cost function c is lower semicontinuous on $X \times X$, then the infimum in (4.24) is attained (see e.g., [60, Theorem 1.3]). Each minimizer $\pi^* \in \Pi(\mu_0, \mu_1)$ of the Monge-Kantorovich problem (4.24) is called *optimal transport plan*.

Next, for $1 \le p < +\infty$, the *p-Wasserstein distance* $W_{p,d}(\mu_1, \mu_2)$ between μ_0 and $\mu_1 \in \mathcal{P}(X)$ is defined by

$$W_{p,d}(\mu_1, \mu_2) = \left(\inf_{\pi\in\Pi(\mu_0,\mu_1)} \int_{X\times X} d(x, y)^p \,\mathrm{d}\pi(x, y) \right)^{\frac{1}{p}}.$$

Note, for given μ_0, $\mu_1 \in \mathcal{P}(X)$, the minimization problem

$$W^p_{p,d}(\mu_1, \mu_2) = \inf_{\pi\in\Pi(\mu_0,\mu_1)} \int_{X\times X} d(x, y)^p \,\mathrm{d}\pi(x, y)$$

is, in fact, the Monge-Kantorovich problem (4.24) for the cost function $c(x, y) = d(x, y)^p$ and if for some arbitrary but fixed $x_0 \in X$, the probability measures μ_0 and μ_1 are elements of the *space of finite p-moment*

$$\mathcal{P}_{p,d}(X) := \left\{ \mu \in \mathcal{P}(X) \;\middle|\; \int_X d(x_0, x)^p \,\mathrm{d}\mu(x) < +\infty \right\},$$

then it is not difficult to see that $W^p_{p,d}(\mu_1, \mu_2)$ is finite (cf [62, Chapter 6, p 107]). Note, the space $\mathcal{P}_{p,d}(X)$ is independent of the choice of $x_0 \in X$ and $W_{p,d}$ defines a finite distance on $\mathcal{P}_{p,d}(X)$ (cf [62, Chapter 6, p 107]). The pair $(\mathcal{P}_{p,d}(X), W_{p,d})$ is called the *p-Wasserstein space*.

In order to understand the topology induced by the p-Wasserstein metric $W_{p,d}$ on $\mathcal{P}_{p,d}(X)$, we briefly recall that a sequence $(\mu_n)_{n\ge1}$ of probability measures $\mu_n \in \mathcal{P}_{p,d}(X)$ *converges weakly in* $\mathcal{P}_{p,d}(X)$ to $\mu \in \mathcal{P}_{p,d}(X)$ if for every continuous function $\varphi : X \to \mathbb{R}$ satisfying $|\varphi(x)| \le C(1 + d^p(x_0, x))$ for every $x \in X$, one has

$$\int_X \varphi \,\mathrm{d}\mu_n \to \int_X \varphi \,\mathrm{d}\mu \qquad \text{as } n \to +\infty.$$

By [62, Theorem 6.9], the p-Wasserstein metric $W_{p,d}$ on $\mathcal{P}_{p,d}(X)$ metricizes the weak convergence on $\mathcal{P}_{p,d}(X)$, $(\mathcal{P}_{p,d}(X), W_{p,d})$ is a complete separable metric space (see [62, Theorem 6.18]), and $(\mathcal{P}_{p,d}(X), W_{p,d})$ is compact if (X, d) is compact (see [62, Remark 6.19]).

4.2.1. ***Doubly-nonlinear equations on*** $\mathbb{R}^N$ ***and*** HWI***-inequalities***. In this part, we focus on the Euclidean case $X = \mathbb{R}^N$ equipped with its natural distance $d(x,y) = |x-y|$. Throughout this subsection, let $1 < p < \infty$ and $p' = \frac{p}{p-1}$. Then, we abbreviate our notation from above and write for the space of finite p-moment

$$\mathcal{P}_p(\mathbb{R}^N) = \left\{\mu \in P(\mathbb{R}^N) \;\Big|\; \int_{\mathbb{R}^N} |x|^p \mathrm{d}\mu(x) < +\infty\right\}$$

and for every two probability measure μ_1, $\mu_2 \in \mathcal{P}_p(\mathbb{R}^N)$, the p-Wasserstein distance

$$W_p(\mu_1,\mu_2) = \left(\inf_{\pi\in\Pi(\mu_0,\mu_1)} \int_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^p \,\mathrm{d}\pi(x,y)\right)^{\frac{1}{p}}.$$

Since the cost function $c_p(x,y) := |x-y|^p$, (x, $y \in \mathbb{R}^N$), is continuous on $\mathbb{R}^N \times \mathbb{R}^N$ and bounded from below, for every pair μ_1 and $\mu_2 \in \mathcal{P}_p(\mathbb{R}^N)$, there is an optimal plan $\pi^* \in \Pi(\mu_1,\mu_2)$ such that

$$W_p^p(\mu_1,\mu_2) = \int_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^p \,\mathrm{d}\pi^*(x,y).$$

For the existence (and uniqueness) of an optimal transport map $T^* : \mathbb{R}^N \to \mathbb{R}^N$ (cf [18] for the case $p = 2$ and [36] for $1 < p < \infty$), one needs that μ_0 belongs to the subclass $\mathcal{P}_p^{ac}(\mathbb{R}^N)$ of measures $\mu \in \mathcal{P}_p(\mathbb{R}^N)$ which are absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N$ on $\mathbb{R}^N$, that is,

$$\mu_1(B) = 0 \qquad \text{for all } B \in \mathcal{B} \text{ with } \mathcal{L}^N(B) = 0.$$

Then, for every $\mu_1 \in \mathcal{P}_p^{ac}(\mathbb{R}^N)$ and $\mu_2 \in \mathcal{P}_p(\mathbb{R}^N)$, the p-Wasserstein distance can be rewritten as

$$W_p^p(\mu_1,\mu_2) = \int_{\mathbb{R}^N} |x-T^*(x)|^p \,\mathrm{d}\mu_1(x). \tag{4.25}$$

for the optimal transport map T^* satisfying $T^*_{\#}\mu_2 = \mu_1$.

Next, consider the *free energy* $\mathcal{E} : \mathcal{P}_p(\mathbb{R}^N) \to (-\infty,+\infty]$ composed by

$$\mathcal{E} = \mathcal{H}_F + \mathcal{H}_V + \mathcal{H}_W \tag{4.26}$$

of the *internal energy*

$$\mathcal{H}_F(\mu) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} F(\rho)\,\mathrm{d}x & \text{if } \mu = \rho\,\mathcal{L}^N,\\ +\infty & \text{if } \mu \in \mathcal{P}_p(\mathbb{R}^N)\setminus\mathcal{P}_p^{ac}(\mathbb{R}^N),\end{cases}$$

the *potential energy*

$$\mathcal{H}_V(\mu) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} V\,\mathrm{d}\mu & \text{if } \mu = \rho\,\mathcal{L}^N,\\ +\infty & \text{if } \mu \in \mathcal{P}_p(\mathbb{R}^N)\setminus\mathcal{P}_p^{ac}(\mathbb{R}^N),\end{cases}$$

and the *interaction energy*

$$\mathcal{H}_W(\mu) := \begin{cases} \displaystyle\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} W(x-y)\,\mathrm{d}(\mu\otimes\mu)(x,y) & \text{if } \mu = \rho\,\mathcal{L}^N,\\ +\infty & \text{if } \mu \in \mathcal{P}_p(\mathbb{R}^N)\setminus\mathcal{P}_p^{ac}(\mathbb{R}^N),\end{cases}$$

Here, we assume that the function

(F) : $F : [0, +\infty) \to \mathbb{R}$ is a convex differential function satisfying

$$F(0) = 0, \quad \liminf_{s\downarrow 0} \frac{F(s)}{s^\alpha} > -\infty \quad \text{for some } \alpha > N/(N+p), \tag{4.27}$$

$$\text{the map } s \mapsto s^N F(s^{-N}) \text{ is convex and non increasing in } (0, +\infty), \tag{4.28}$$

there is a $C_F > 0$ such that

$$F(s+\hat{s}) \le C_F\,(1 + F(s) + F(\hat{s})) \quad \text{for all } s,\ \hat{s} \ge 0, \text{ and} \tag{4.29}$$

$$\lim_{s\to+\infty} \frac{F(s)}{s} = +\infty \qquad \text{(super-linear growth at infinity);} \tag{4.30}$$

(V) : $V : \mathbb{R}^N \to (-\infty, +\infty]$ is a proper, lower semicontinuous, λ-convex for some $\lambda \in \mathbb{R}$, and the effective domain $D(V)$ has a convex, nonempty interior $\Omega := \operatorname{int} D(V) \subseteq \mathbb{R}^N$.

(W) : $W : \mathbb{R}^N \to [0, +\infty)$ is a convex, differentiable, and even function and there is a $C_W > 0$ such that

$$W(x+\hat{x}) \le C_W\,(1 + W(x) + W(\hat{x})) \quad \text{for all } x,\ \hat{x} \in \mathbb{R}^N. \tag{4.31}$$

Remark 4.10. The study of gradient flows generated by the free energy functional $\mathcal{E}$ given by (4.26) on the Wasserstein space $(\mathcal{P}_2(\mathbb{R}^N), W_2)$ was done independently in [23] and [7].

We note that the conditions (4.27) on F ensure that for every $\rho \in L^1(\mathbb{R}^N)$, the *negative part* $F^-(s) = \max\{-F(s), 0\}$ satisfies $F^-(\rho) \in L^1(\mathbb{R}^N)$, that is, (4.27) provides growth bounds on F^- (cf [7, Remark 9.3.7]), due to condition (4.28), the functional $\mathcal{H}_F$ is geodesically convex on $\mathcal{P}_p(\mathbb{R}^N)$, and thanks to condition (4.30), $\mathcal{H}_F$ is lower semicontinuous on $\mathcal{P}_p(\mathbb{R}^N)$ (cf [7, Remark 9.3.8]). The so-called *doubling* condition (4.29) is needed to calculate the directional derivative of $\mathcal{H}_F$ (cf [7, Lemma 10.4.4]).

Concerning the functional $\mathcal{H}_V$, the hypotheses that V is convex, proper and lower semicontinuous imply that $\mathcal{H}_V$ is bounded from below by an affine support function and hence, by Cauchy-Schwarz' and Young's inequality, for every $1 < p < +\infty$, there are A, $B > 0$ such that $V(x) \ge -A - B|x|^p$. From this, we can deduce that $\mathcal{H}_V$ is lower semicontinuous on $\mathcal{P}_p(\mathbb{R}^N)$ (cf [7, Lemma 5.1.7]). By [7, Proposition 9.3.2], $\mathcal{H}_V$ is λ-geodesically convex in $\mathcal{P}_p(\mathbb{R}^N)$ if $p \le 2$ and $\lambda \ge 0$, or if $p \ge 2$ and $\lambda \le 0$.

Similarly to $\mathcal{H}_V$, the convexity of W, the fact that W is proper, and the lower semicontinuity of W, imply that $\mathcal{H}_W$ is lower semicontinuous in $\mathcal{P}_p(\mathbb{R}^N)$ and by [7, Proposition 9.3.5], $\mathcal{H}_W$ is geodesically convex on $\mathcal{P}_p(\mathbb{R}^N)$. The doubling condition (4.31) is sufficient for characterizing the decreasing slope $|D^-\mathcal{E}_W|$ of $\mathcal{H}_W$ (cf [7, Theorem 10.4.11]).

For an open set $\Omega \subseteq \mathbb{R}^N$, we denote by $\mathcal{P}_p(\Omega)$ is the closed subspace of probability measures $\mu \in \mathcal{P}_p(\mathbb{R}^N)$ with support $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$. Moreover, $\mathcal{P}_p^{ac}(\Omega) := \mathcal{P}_p(\Omega) \cap \mathcal{P}_p^{ac}(\mathbb{R}^N)$.

Under the hypotheses **(F)**, **(V)** and **(W)**, Proposition 2.23 implies that the descending slope $|D^-\mathcal{E}|$ of $\mathcal{E}$ is lower semicontinuous. Moreover, by [7, Theorem 10.4.13], $|D^-\mathcal{E}|$ can be characterized as follows.

Proposition 4.11. *Suppose, the functions F, V and W satisfy the hypotheses* **(F)**, **(V)** *and* **(W)**, *and $\mathcal{E} : \mathcal{P}_p(\mathbb{R}^N) \to (-\infty, +\infty]$ is the functional given* *by (4.26). Then, for $\mu = \rho\mathcal{L}^N \in D(\mathcal{E})$, one has $\mu \in D(|D^-\mathcal{E}|)$ if and only if*

$$P_F(\rho) \in W^{1,1}_{loc}(\Omega), \quad \rho\,\xi_\rho = \nabla P_F(\rho) + \rho\nabla V + \rho(\nabla W) * \rho \tag{4.32}$$

for some $\xi_\rho \in L^{p'}(\mathbb{R}^N, \mathbb{R}^N; \mathrm{d}\mu)$, where $P_F(x) := xF'(x) - F(x)$ is the associated "pressure function" of F. Moreover, the vector field ξ_ρ satisfies

$$|D^-\mathcal{E}|(\mu) = \left(\int_{\mathbb{R}^N} |\xi_\rho(x)|^{p'}\,\mathrm{d}\mu\right)^{\frac{1}{p'}}. \tag{4.33}$$

By following the idea of the proof of [43, Proposition 4.1] and replacing $p = 2$ by general $1 < p < +\infty$, one sees that under the hypotheses **(F)**, **(V)** and **(W)**, the free energy $\mathcal{E} : \mathcal{P}_p(\mathbb{R}^N) \to (-\infty, +\infty]$ given by (4.26) satisfies the hypotheses of Theorem 2.20. Therefore and by the regularity result in [7, Theorem 11.3.4], for every $\mu_0 \in D(\mathcal{E})$, there is a p-gradient flow $\mu : [0, +\infty) \to \mathcal{P}_p(\mathbb{R}^N)$ of $\mathcal{E}$ with initial value $\lim_{t\downarrow 0}\mu(t) = \mu_0$. Moreover, for every $t > 0$, $\mu(t) = \rho(t)\,\mathcal{L}^N$ with $\mathrm{supp}(\rho(t)) \subseteq \overline{\Omega}$, and ρ is a distributional solution of the following quasilinear parabolic-elliptic boundary-value problem

$$\begin{cases} \rho_t + \mathrm{div}(\rho\,\boldsymbol{U}_\rho) = 0 & \text{in } (0, +\infty) \times \Omega, \\ \boldsymbol{U}_\rho = -|\xi_\rho|^{p'-2}\xi_\rho & \text{in } (0, +\infty) \times \Omega, \\ \boldsymbol{U}_\rho \cdot \boldsymbol{n} = 0 & \text{in } (0, +\infty) \times \partial\Omega, \end{cases} \tag{4.34}$$

with $P_F(\rho) \in L^1_{loc}((0, +\infty); W^{1,1}_{loc}(\Omega))$ and

$$\xi_\rho = \frac{\nabla P_F(\rho)}{\rho} + \nabla V + (\nabla W) * \rho \in L^\infty_{loc}((0, +\infty); L^{p'}(\Omega, \mathbb{R}^N; \mathrm{d}\mu(\cdot))),$$

where, $\boldsymbol{n}$ in (4.34) denotes the outward unit normal to the boundary $\partial\Omega$ which in the case $\Omega = \mathbb{R}^N$ needs to be neglected.

If the function $F \in C^2(0, +\infty)$, then one has that

$$-\boldsymbol{U}_\rho = |F''(\rho)\nabla\rho + \nabla V + (\nabla W) * \rho|^{p'-2}\left(F''(\rho)\nabla\rho + \nabla V + (\nabla W) * \rho\right).$$

Thus (cf [4], [23], [22]), problem (4.34) includes the

- *doubly nonlinear diffusion equation*

$$\rho_t - \mathrm{div}(|\nabla\rho^m|^{p'-2}\,\nabla\rho^m) = 0$$

  ($V = W = 0$, $F(s) = \frac{m\,s^q}{q(q-1)}$ for $q = m + 1 - \frac{1}{p'-1}$, $\frac{1}{p'-1} \neq m \geq \frac{N-(p'-1)}{N(p'-1)}$)

- *Fokker-Planck equation with interaction term through porous medium*

$$\rho_t = \Delta\rho^m + \mathrm{div}\left(\rho(\nabla V + (\nabla W) * \rho)\right)$$

  ($p = 2$, $F(s) = \frac{s^m}{(m-1)}$ for $1 \neq m \geq 1 - \frac{1}{N}$).

Due to Proposition 4.11, every equilibrium point $\nu = \rho_\infty\mathcal{L}^N \in \mathbb{E}_{|D^-\mathcal{E}|}$ of $\mathcal{E}$ can be characterized by

$$\begin{cases} P_F(\rho_\infty) \in W^{1,1}_{loc}(\Omega) \qquad \text{with} \\ \xi_{\rho_\infty} = \dfrac{\nabla P_F(\rho_\infty)}{\rho_\infty} + \nabla V + (\nabla W) * \rho_\infty = 0 \quad \text{a.e. on } \Omega. \end{cases} \tag{4.35}$$

Further, for every p-gradient flow μ of $\mathcal{E}$ and equilibrium point $\nu \in \mathbb{E}_{|D^-\mathcal{E}|}$, equation (2.10) in Proposition 2.16 reads as follows

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\mu(t)) = -|D^-\mathcal{E}|^{p'}(\mu(t)) = -\mathcal{I}_{p'}(\mu(t)|\nu), \tag{4.36}$$

where due to Proposition 4.11, the *generalized relative Fischer information of μ with respect to ν* is given by

$$\mathcal{I}_{p'}(\mu|\nu) = \int_\Omega -\boldsymbol{U}_\rho \cdot \xi_\rho \,\mathrm{d}\mu.$$

For our next lemma, we introduce the notion of *uniformly λ-p convex* functions (cf [29, 3]), which for differentiable f generalizes the notion of λ-convexity on the Euclidean space $\mathfrak{M} = \mathbb{R}^N$ (cf Definition 2.21).

Definition 4.12. We call a functional $f : \mathbb{R}^N \to (-\infty, +\infty]$ *uniformly λ-p-convex* for some $\lambda \in \mathbb{R}$ if the interior $\Omega = \mathrm{int}(D(f))$ of f is nonempty, f is differentiable on Ω and for every $x \in \Omega$,

$$f(x) - f(x) \geq \nabla f(x) \cdot (y - x) + \lambda\, |y - x|^p \qquad \text{for all } y \in \mathbb{R}^N.$$

Further, we need the following definition from [7].

Definition 4.13. Let $c : \mathbb{R}^N \times \mathbb{R}^N \to [0, +\infty]$ be a proper and lower semicontinuous function. Then, for $u : \mathbb{R}^N \to [+\infty, +\infty]$, the *$c$-transform* $u^c : \mathbb{R}^N \to [+\infty, +\infty]$ is defined by

$$u^c(y) = \inf_{x \in \mathbb{R}^N} (c(x, y) - u(x)) \qquad \text{for every } y \in \mathbb{R}^N.$$

A function $u : \mathbb{R}^N \to [+\infty, +\infty]$ is called *c-concave* if there is a function $v : \mathbb{R}^N \to [+\infty, +\infty]$ such that $u = v^c$.

With these preliminaries, we can state the following result which generalizes [29, Theorem 2.4, Theorem 4.1], [3, inequality (5)] (see also [2, Theorem 2.1]).

Lemma 4.14. *Suppose, the functions F, V and W satisfy the hypotheses* **(F)**,

- **(V*)** *$V : \mathbb{R}^N \to (-\infty, +\infty]$ is proper, lower semicontinuous function, the effective domain $D(V)$ of V has nonempty interior $\Omega := \mathrm{int}\, D(V) \subseteq \mathbb{R}^N$, and V is uniformly λ_V-p-convex for some $\lambda_V \in \mathbb{R}$;*
- **(W*)** *$W : \mathbb{R}^N \to [0, +\infty)$ is a differentiable, even function satisfying (4.31), and for some $\lambda_W \in \mathbb{R}$, W is uniformly λ_W-p-convex for some $\lambda_W \in \mathbb{R}$.*

Further, let $F \in C^2(0, \infty) \cap C[0, +\infty)$ and $\mathcal{E} : \mathcal{P}_p(\mathbb{R}^N) \to (-\infty, +\infty]$ be the functional given by (4.26). Then, for every probability measures $\mu_1 = \rho_1 \mathcal{L}^N$, $\mu_2 = \rho_2 \mathcal{L}^N \in \mathcal{P}_p^{ac}(\Omega)$ with $\rho_2 \in W^{1,\infty}(\Omega)$ and $\inf \rho_2 > 0$, one has

$$\begin{aligned} \mathcal{E}(\mu_1|\mu_2) \geq& \int_\Omega (T^*(x) - x) \cdot \xi_{\rho_2} \mathrm{d}\mu_2 + (\lambda_V + \tfrac{\lambda_W}{2})\, W_p^p(\mu_1, \mu_2) \\ &+ \tfrac{p\,\lambda_W}{2} \int_{\Omega\times\Omega} \tilde{\nabla}\theta(x) \cdot (T^*(y) - y)\, \rho_2(x)\, \rho_2(y) \mathrm{d}x\mathrm{d}y, \end{aligned} \tag{4.37}$$

where T^* is the optimal transport map satisfying (4.25) with $T^*_\# \mu_2 = \mu_1$ and

$$x - T^*(x) = |\tilde{\nabla}\theta(x)|^{p'-2} \tilde{\nabla}\theta(x)$$

for a c_p-concave function θ with $c_p(x, y) := \frac{1}{p}|x - y|^p$.

Proof. Under the hypotheses of this lemma, [29, Theorem 2.4] yields

$$\mathcal{H}_{F+V}(\mu_1|\mu_2) \geq \int_\Omega (T^*(x)-x)\cdot(F''(\rho)\nabla\rho+\nabla V)\mathrm{d}\mu_2+\lambda_V\, W_p^p(\mu_1,\mu_2) \tag{4.38}$$

where we set $\mathcal{H}_{F+V}=\mathcal{H}_F+\mathcal{H}_V$. Next, we deal with the interaction energy $\mathcal{H}_W$. For this, we follow an idea given in [29]. Since $T^*_\#\mu_2=\mu_1$, we can rewrite

$$\mathcal{H}_W(\mu_1)=\tfrac{1}{2}\int_{\Omega\times\Omega} W(T^*(x)-T^*(y))\,\rho_2(x)\,\rho_2(y)\mathrm{d}x\,\mathrm{d}y$$

Thus, and since W is uniformly λ_W-c_p-convex for some $\lambda_W\in\mathbb{R}$, we have

$$\begin{aligned}\mathcal{H}_W(\mu_1)\geq \mathcal{H}_W(\mu_2)+\tfrac{1}{2}\int_{\Omega\times\Omega}\Big[\nabla W(x-y)\cdot\big((T^*(x)-x)&\\ -(T^*(y)-y)\big)\Big]\,\rho_2(x)\,\rho_2(y)\mathrm{d}x\mathrm{d}y&\\ +\tfrac{\lambda_W}{2}\int_{\Omega\times\Omega}|(T^*(x)-x)-(T^*(y)-y)|^p\,\rho_2(x)\,\rho_2(y)\,\mathrm{d}x\mathrm{d}y&\end{aligned}$$

and since by hypothesis, ∇W is odd,

$$\begin{aligned}\mathcal{H}_W(\mu_1)\geq \mathcal{H}_W(\mu_2)+\int_\Omega \nabla(W*\rho_2)\cdot(T^*(x)-x)\,\mathrm{d}\mu_2(x)&\\ +\tfrac{\lambda_W}{2}\int_{\Omega\times\Omega}|(T^*(x)-x)-(T^*(y)-y)|^p\,\rho_2(x)\,\rho_2(y)\,\mathrm{d}x\mathrm{d}y.&\end{aligned}$$

Due to the elementary inequality $|a-b|^p\geq|a|^p-p|a|^{p-2}a\cdot b$, $(a,\,b\in\mathbb{R}^N)$,

$$\begin{aligned}\mathcal{H}_W(\mu_1)\geq \mathcal{H}_W(\mu_2)+\int_{\mathbb{R}^N}\nabla(W*\rho_2)\cdot(T^*(x)-x)\,\mathrm{d}\mu_2(x)+\tfrac{\lambda_W}{2}W_p^p(\mu_1,\mu_2)&\\ +\tfrac{p\,\lambda_W}{2}\int_{\Omega\times\Omega}|T^*(x)-x|^{p-2}(T^*(x)-x)(T^*(y)-y)\,\rho_2(x)\,\rho_2(y)\mathrm{d}x\mathrm{d}y.&\end{aligned}$$

Combining this inequality with (4.38) yields the desired inequality (4.37). □

We note that if the interaction term W satisfying hypothesis **(W)** then hypothesis **(W*)** holds with $\lambda_W=0$. Thus, we can announce for the free energy $\mathcal{E}=\mathcal{H}_F+\mathcal{H}_V+\mathcal{H}_W$ the following *ET-inequality*, *generalized Log-Sobolev inequality* (cf [3, Proposition 1.1]), and *generalized p-HWI inequality* (cf [3, Theorem 1.2] , [49, Theorem 3] and [22, Theorem 2.1]).

Theorem 4.15. *Suppose that the functions F, V and W satisfy the hypotheses* **(F)**, **(V*)** *with $\lambda_V\in\mathbb{R}$ and* **(W)**. *Further, suppose $F\in C^2(0,\infty)\cap C[0,+\infty)$ and let $\mathcal{E}:\mathcal{P}_p(\mathbb{R}^N)\to(-\infty,+\infty]$ be the functional given by (4.26). Then, the following statements hold.*

(1) **(ET-inequality)** *For an equilibrium point $\nu=\rho_\infty\mathcal{L}^N\in\mathbb{E}_{|D^-\mathcal{E}|}$ of $\mathcal{E}$ with $\rho_\infty\in W^{1,\infty}(\Omega)$ and $\inf\rho_\infty>0$, and every $\mu=\rho\mathcal{L}^N\in D(\mathcal{E})$,*

$$\lambda_V\,W_p^p(\mu,\nu)\leq\mathcal{E}(\mu|\nu). \tag{4.39}$$

(2) **(p-Talagrand transportation inequality)** *If $\lambda_V>0$, then entropy-transportation inequality (4.39) is equivalent to the p-Talagrand inequality*

$$W_p(\mu,\nu)\leq\tfrac{1}{\lambda_V^{1/p}}\sqrt[p]{\mathcal{E}(\mu|\nu)} \tag{4.40}$$

holding for an equilibrium point $\nu = \rho_\infty \mathcal{L}^N \in \mathbb{E}_{|D^-\mathcal{E}|}$ *of* $\mathcal{E}$ *with* $\rho_\infty \in W^{1,\infty}(\Omega)$ *and all* $\mu = \rho\mathcal{L}^N \in D(\mathcal{E})$.

(3) **(generalized ŁS-inequality)** *If* $\hat{\lambda} > 0$ *then for every probability measures* $\mu_1 = \rho_1\mathcal{L}^N$, $\mu_2 = \rho_2\mathcal{L}^N \in \mathcal{P}_p^{ac}(\Omega)$ *with* $\rho_2 \in W^{1,\infty}(\Omega)$ *and* $\inf \rho_2 > 0$, *one has that*

$$\mathcal{E}(\mu_2|\mu_1) + (\lambda_V - \hat{\lambda})\, W_p^p(\mu_1,\mu_2) \le \frac{p-1}{p^{p'}} \frac{1}{\hat{\lambda}^{1/(p-1)}} \, |D^-\mathcal{E}|^{p'}(\mu_2). \tag{4.41}$$

(4) **(generalized Log-Sobolev inequality)** *If* $\lambda_V > 0$, *then for every probability measures* $\mu_1 = \rho_1\mathcal{L}^N$, $\mu_2 = \rho_2\mathcal{L}^N \in \mathcal{P}_p^{ac}(\Omega)$ *with* $\rho_2 \in W^{1,\infty}(\Omega)$ *and* $\inf \rho_2 > 0$ *and* $\nu \in \mathbb{E}_{|D^-\mathcal{E}|}$, *one has that*

$$\mathcal{E}(\mu_2|\mu_1) \le \frac{p-1}{p^{p'}} \frac{1}{\lambda_V^{1/(p-1)}} \mathcal{I}_{p'}(\mu_2|\nu). \tag{4.42}$$

(5) **(p-HWI inequality)** *For every probability measures* $\mu_1 = \rho_1\mathcal{L}^N$, $\mu_2 = \rho_2\mathcal{L}^N \in \mathcal{P}_p^{ac}(\Omega)$ *with* $\rho_2 \in W^{1,\infty}(\Omega)$ *and* $\inf \rho_2 > 0$, *one has that*

$$\mathcal{E}(\mu_2|\mu_1) + \lambda_V\, W_p^p(\mu_1,\mu_2) \le \mathcal{I}_{p'}^{1/p'}(\mu_2|\nu)\, W_p(\mu_1,\mu_2). \tag{4.43}$$

Remark 4.16. We note that for $\lambda_V > 0$, $\mathcal{E}$ satisfies an entropy-transportation inequality (4.39) at an equilibrium point $\nu = \rho_\infty\mathcal{L}^N \in \mathbb{E}_{|D^-\mathcal{E}|}$ of $\mathcal{E}$ with $\rho_\infty \in W^{1,\infty}(\Omega)$, then ν is the unique minimizer of $\mathcal{E}$.

Remark 4.17 (**The case** $V = W = 0$.). It is well-known that also in the case $V = W = 0$ that for $\mathcal{E}$ given by (4.26) a Sobolev inequalities holds (which imply a Log-Sobolev inequalities of the form (4.42)). For further details, we refer the interested reader to [3, 4].

Proof of Theorem 4.15. The idea of our proofs follows the one in [3]. Thus, we only sketch the ideas here. Inequality (4.39) follows directly from (4.37) by taking $\mu_2 = \nu$ and applying the characterization (4.35) for the equilibrium point $\nu \in \mathbb{E}_{|D^-\mathcal{E}|}$. Talagrand inequality (4.40) is equivalent to ET-inequality (4.39) by simply taking p'th root or power. Next, by Young's inequality, we see that

$$(x - T^*(x)) \cdot \xi_{\rho_2} \le \hat{\lambda}\, |x - T^*(x)|^p + \frac{p-1}{p^{p'}} \frac{1}{\hat{\lambda}^{1/(p-1)}} \, |\xi_{\rho_2}|^{p'}.$$

Applying this to (4.37) and using (4.33), we see that

$$\lambda_V\, W_p^p(\mu_1,\mu_2) \le \mathcal{E}(\mu_1|\mu_2) + \hat{\lambda}\, W_p^p(\mu_1,\mu_2) + \tfrac{p-1}{p^{p'}} \frac{1}{\hat{\lambda}^{1/(p-1)}} \, |D^-\mathcal{E}|^{p'}(\mu_2) \tag{4.44}$$

from where inequality (4.41) follows. By taking $\hat{\lambda} = \lambda_V > 0$ in (4.41) and by (4.36), we see that the Log-Sobolev inequality (4.42) holds. Finally, to see that p-HWI inequality (4.43) is true, one shows that the function

$$\hat{\lambda} \mapsto \hat{\lambda}\, W_p^p(\mu_1,\mu_2) + \frac{p-1}{p^{p'}} \frac{1}{\hat{\lambda}^{1/(p-1)}} \, |D^-\mathcal{E}|^{p'}(\mu_2)$$

over $(0,+\infty)$ attains its minimum at

$$\hat{\lambda}_0 = \frac{1}{p} \frac{|D^-\mathcal{E}|(\mu_2)}{W_p^{p-1}(\mu_1,\mu_2)}.$$

Insert $\hat{\lambda}_0$ into (4.44) and using the identity (4.36) yields p-HWI inequality (4.43). $\square$

Our next corollary shows that even if V fails to be uniformly λ_V-p-convex for some $\lambda_V > 0$, *equivalence* between entropy transportation inequality (4.39), Łojasiewicz-Simon inequality inequality (3.3), and the logarithmic Sobolev inequality (4.42) holds for the free energy functional $\mathcal{E}$ given by (4.26) (cf [49, Corollary 3.1], [29, Proposition 3.6]). Our next result, is a special case of Corollary 3.32 adapted to the framework in $\mathcal{P}_p(\mathbb{R}^N)$.

Corollary 4.18 (Equivalence between global ET-, ŁS- and Log-Sobolev inequality). *Suppose that the functions F, V and W satisfy the hypotheses* **(F)**, **(V)** *and* **(W)**. *Further, suppose $F \in C^2(0,\infty) \cap C[0,+\infty)$ and let $\mathcal{E} : \mathcal{P}_p(\mathbb{R}^N) \to (-\infty,+\infty]$ be the functional given by (4.26). Then, the following statements hold.*

(1) If for $\nu \in \mathbb{E}_{|D^-\mathcal{E}|}$, there is some $\hat{\lambda} > 0$ such that $\mathcal{E}$ satisfies entropy transportation inequality

$$W_p(\mu,\nu) \le \hat{\lambda}\,(\mathcal{E}(\mu|\nu))^{\frac{1}{p}} \qquad \text{for all } \mu \in D(\mathcal{E}), \tag{4.45}$$

then $\mathcal{E}$ satisfies the Lojasiewicz-Simon inequality

$$\mathcal{E}(\mu|\mu_\infty)^{1-\frac{1}{p}} \le \hat{\lambda}\,|D^-\mathcal{E}|(\mu) \qquad \text{for all } \mu \in D(|D^-\mathcal{E}|), \tag{4.46}$$

or equivalently, $\mathcal{E}$ satisfies the Log-Sobolev inequality

$$\mathcal{E}(\mu|\mu_\infty)^{1-\frac{1}{p}} \le \hat{\lambda}^{\frac{1}{1-\frac{1}{p}}}\,\mathcal{I}_{p'}(\mu|\nu) \qquad \text{for all } \mu \in D(|D^-\mathcal{E}|), \tag{4.47}$$

(2) If for $\nu \in \mathbb{E}_{|D^-\mathcal{E}|}$, there is some $\hat{\lambda} > 0$ such that $\mathcal{E}$ satisfies Log-Sobolev inequality (4.47), then $\mathcal{E}$ satisfies entropy transportation inequality

$$W_p(\mu,\nu) \le \hat{\lambda} p\,(\mathcal{E}(\mu|\nu))^{\frac{1}{p}} \qquad \text{for all } \mu \in D(\mathcal{E}),$$

From Theorem 4.15 and Theorem 3.20, we can conclude the following exponential decay rates (cf [3, 4, 50] and [2, Corollary 5.1]).

Corollary 4.19 (Trend to equilibrium and exponential decay rates). *Suppose that the functions F, V and W satisfy the hypotheses* **(F)**, **(V*)** *with $\lambda_V > 0$ and* **(W)**. *Further, suppose $F \in C^2(0,\infty) \cap C[0,+\infty)$ and let $\mathcal{E} : \mathcal{P}_p(\mathbb{R}^N) \to (-\infty,+\infty]$ be the functional given by $\mathcal{E} = \mathcal{H}_F + \mathcal{H}_V + \mathcal{H}_W$. Then, there is a unique minimizer $\nu = \rho_\infty \mathcal{L}^N \in \mathbb{E}_{|D^-\mathcal{E}|}$ of $\mathcal{E}$ satisfying (4.35) and for every initial value $\mu_0 \in D(\mathcal{E})$, the p-gradient flow μ of $\mathcal{E}$ trends to ν in $\mathcal{P}_p(\Omega)$ as $t \to +\infty$ and for all $t \ge 0$,*

$$W_p(\mu(t),\nu) \le \frac{(p-1)^{1/p'}}{\lambda_V^{1/p}}\,(\mathcal{E}(\mu(t)|\nu))^{\frac{1}{p}} \le \frac{(p-1)^{1/p'}}{\lambda_V^{1/p}}\,(\mathcal{E}(\mu_0|\nu))^{\frac{1}{p}}\, e^{-\frac{tp^{\frac{1}{p}}}{p-1}\lambda_V^{\frac{1}{p-1}}}. \tag{4.48}$$

Remark 4.20. From (4.48), one can deduce strong convergence in $L^1(\mathbb{R}^N)$ (cf [21, 50]) or even strong convergence in $BV(\mathbb{R}^N)$ (cf [62, Remark 22.12]) by using a Csiszar-Kullback(-Pinsker) inequality.

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