# Center of the quantum affine vertex algebra associated with trigonometric $R$-matrix

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**Abstract**

We consider the quantum vertex algebra associated with the trigonometric $R$-matrix in type $A$ as defined by Etingof and Kazhdan. We show that its center is a commutative associative algebra and construct an algebraically independent family of topological generators of the center at the critical level.

## 1 Introduction

A general definition of *quantum vertex algebra* was given by P. Etingof and D. Kazhdan [6]. In particular, a *quantum affine vertex algebra* can be associated with a rational, trigonometric or elliptic $R$-matrix. A suitably normalized Yang $R$-matrix gives rise to a quantum vertex algebra structure on the vacuum module $\mathcal{V}_c(\mathfrak{gl}_n)$ over the double Yangian $\mathrm{DY}(\mathfrak{gl}_n)$. In our previous paper [12] coauthored with N. Jing and F. Yang, we introduced the *center* of an arbitrary quantum vertex algebra and described the center $\mathfrak{z}\big(\mathcal{V}_c(\mathfrak{gl}_n)\big)$ of the quantum affine vertex algebra $\mathcal{V}_c(\mathfrak{gl}_n)$. We showed that the center at the critical level $c=-n$ possesses large families of algebraically independent topological generators in a complete analogy with the affine vertex algebra [7]; see also [9].

Our goal in this paper is to give a similar description of the center of the quantized universal enveloping algebra $\mathrm{U}(R)$, associated with a normalized trigonometric $R$-matrix $R$, as a quantum vertex algebra. We show that the center of the $h$-adically completed quantum affine vertex algebra $\mathrm{U}_c(R)$ at the level $c\in\mathbb{C}$ is a commutative algebra. Moreover, we produce a family of algebraically independent topological generators of the center in an explicit form. We show that taking their 'classical limits' reproduces the corresponding generators of the center of the quantum affine vertex algebra $\mathcal{V}_c(\mathfrak{gl}_n)$. In particular, as with the rational case, the center of the $h$-adic completion of $\mathrm{U}_c(R)$ is 'large' at the critical level $c=-n$, and trivial otherwise.

Despite an apparent analogy between the rational and trigonometric cases, there are significant differences in the constructions. In the rational case, the quantum vertex algebra structure is essentially determined by that of the double Yangian $\mathrm{DY}(\mathfrak{gl}_n)$. One could expect that the role of the double Yangian in the trigonometric case to be played by the quantum affine algebra $\mathrm{U}_q(\mathfrak{gl}_n)$. In fact, as explained in [6], a more subtle structure is to be used instead. Nonetheless, the technical part is quite similar to that of the paper [8], where explicit constructions of elements of the center of the completed quantum affine algebra were given, and which stems from the pioneering work of N. Reshetikhin and M. Semenov-Tian-Shansky [15]; see also J. Ding and P. Etingof [3] and E. Frenkel and N. Reshetikhin [10].

# 2 Quantized universal enveloping algebra

In accordance to [6], a normalized $R$-matrix is needed to define an appropriate version of the quantized universal enveloping algebra $\mathrm{U}(R)$. Namely, the $R$-matrix should satisfy the *unitarity* and *crossing symmetry* properties. Its existence is established in [5, Proposition 1.2]. The normalizing factor does not admit a simple closed expression. A description of this factor in the rational case is also given in [12, Section 2.2]. Here we give a similar description in the trigonometric case which also implies the existence of the normalized $R$-matrix. We start by recalling some standard tensor notation.

We let $e_{ij} \in \operatorname{End}\mathbb{C}^n$ denote the standard matrix units. For an element

$$C = \sum_{i,j,r,s=1}^{n} c_{ijrs}\, e_{ij} \otimes e_{rs} \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n,$$

and any two indices $a, b \in \{1, \dots, m\}$ such that $a \neq b$, we denote by $C_{ab}$ the element of the algebra $(\operatorname{End}\mathbb{C}^n)^{\otimes m}$ with $m \geqslant 2$ given by

$$C_{ab} = \sum_{i,j,r,s=1}^{n} c_{ijrs}\, (e_{ij})_a (e_{rs})_b, \qquad (e_{ij})_a = 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)}. \tag{2.1}$$

We regard the matrix transposition as the linear map

$$t : \operatorname{End}\mathbb{C}^n \to \operatorname{End}\mathbb{C}^n, \qquad e_{ij} \mapsto e_{ji}.$$

For any $a \in \{1, \dots, m\}$ we will denote by $t_a$ the corresponding partial transposition on the algebra $(\operatorname{End}\mathbb{C}^n)^{\otimes m}$ which acts as $t$ on the $a$-th copy of $\operatorname{End}\mathbb{C}^n$ and as the identity map on all the other tensor factors.

Introduce the two-parameter $R$-matrix $\overline{R}(u,v) \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n[[u,v,h]]$ as a formal power series in the variables $u,v,h$ by

$$\begin{aligned}\overline{R}(u,v) =& \big(e^{u-h/2} - e^{v+h/2}\big) \sum_i e_{ii} \otimes e_{ii} + \big(e^u - e^v\big) \sum_{i \neq j} e_{ii} \otimes e_{jj} \\ &+ \big(e^{-h/2} - e^{h/2}\big) e^u \sum_{i>j} e_{ij} \otimes e_{ji} + \big(e^{-h/2} - e^{h/2}\big) e^v \sum_{i<j} e_{ij} \otimes e_{ji},\end{aligned} \tag{2.2}$$

where the summation indices run over the set $\{1,\dots,n\}$. We will also use the one-parameter $R$-matrix $\overline{R}(u) \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n\,((u))[[h]]$ defined by

$$\begin{aligned}\overline{R}(u) = \frac{\overline{R}(u,0)}{e^{u-h/2} - e^{h/2}} =& \sum_i e_{ii} \otimes e_{ii} + e^{-h/2} \frac{1-e^u}{1-e^{u-h}} \sum_{i \neq j} e_{ii} \otimes e_{jj} \\ &+ \frac{\big(1-e^{-h}\big)\, e^u}{1-e^{u-h}} \sum_{i>j} e_{ij} \otimes e_{ji} + \frac{1-e^{-h}}{1-e^{u-h}} \sum_{i<j} e_{ij} \otimes e_{ji}.\end{aligned} \tag{2.3}$$

Here and below expressions of the form $\big(1-e^{u+a\,h}\big)^{-1}$ with $a \in \mathbb{C}$ should be understood as elements of the algebra $\mathbb{C}((u))[[h]]$,

$$\big(1-e^{u+a\,h}\big)^{-1} = -u^{-1}\left(\sum_{l \geqslant 1} \frac{(u+a\,h)^{l-1}}{l!}\right)^{-1} \big(1+ah/u\big)^{-1} \in \mathbb{C}((u))[[h]].$$

Denote by $D$ the diagonal $n \times n$ matrix

$$D = \operatorname{diag}\big[e^{\frac{(n-1)\,h}{2}}, e^{\frac{(n-3)\,h}{2}}, \dots, e^{-\frac{(n-1)\,h}{2}}\big] \tag{2.4}$$

with entries in $\mathbb{C}[[h]]$. The following proposition goes back to [11, Proposition 4.7] and is a version of [5, Proposition 1.2]. We use the notation (2.1).

**Proposition 2.1.** *There exists a unique series $g(u) \in 1 + h\mathbb{C}((u))[[h]]$ (depending on $n$) such that the $R$-matrix $R(u) = g(u)\overline{R}(u)$ possesses the unitarity property*

$$R_{12}(u)R_{21}(-u) = 1 \tag{2.5}$$

*and the crossing symmetry properties*

$$\big(R_{12}(u)^{-1}\big)^{t_2} D_2 R_{12}(u+nh)^{t_2} = D_2 \quad \text{and} \quad R_{12}(u+nh)^{t_1} D_1 \big(R_{12}(u)^{-1}\big)^{t_1} = D_1. \tag{2.6}$$

*Proof.* Due to the well-known properties of the $R$-matrix (2.3) (see [11]), identities (2.5) and (2.6) will hold for $R(u) = g(u)\overline{R}(u)$ if and only if $g(u)$ satisfies the relations

$$g(u)\,g(-u) = 1 \tag{2.7}$$

and

$$g(u+nh)=g(u)\,\frac{\big(1-e^{u+h}\big)\big(1-e^{u+(n-1)h}\big)}{\big(1-e^{u}\big)\big(1-e^{u+nh}\big)}. \tag{2.8}$$

It is well known by [11] that there exists a unique formal power series $f(x)\in\mathbb{C}(q)[[x]]$ of the form

$$f(x)=1+\sum_{k=1}^{\infty}f_k x^k,\qquad f_k=f_k(q), \tag{2.9}$$

whose coefficients $f_k$ are determined by the relation

$$f(xq^{2n})=f(x)\,\frac{(1-xq^{2})(1-xq^{2n-2})}{(1-x)(1-xq^{2n})}. \tag{2.10}$$

Equivalently, the series (2.9) is a unique solution of the equation

$$f(x)f(xq^2)\dots f(xq^{2n-2})=\frac{1-x}{1-x\,q^{2n-2}}. \tag{2.11}$$

Observe that $f(x)$ admits the presentation

$$f(x)=1+\sum_{k=1}^{\infty}a_k\Big(\frac{x}{1-x}\Big)^{k}, \tag{2.12}$$

where all rational functions $a_k/(q-1)^k\in\mathbb{C}(q)$ are regular at $q=1$. Indeed, this claim means that all coefficients of the formal series

$$b(z)=1+\sum_{k=1}^{\infty}\frac{a_k}{(q-1)^k}\,z^k$$

are regular at $q=1$. However, rewriting the equation (2.11) in terms of $b(z)$ we get

$$b(z)\,b\Big(\frac{zq^2}{1-z\,\frac{1-q^2}{1-q}}\Big)\dots b\Big(\frac{zq^{2n-2}}{1-z\,\frac{1-q^{2n-2}}{1-q}}\Big)=\frac{1}{1-z\,\frac{1-q^{2n-2}}{1-q}}.$$

This implies a system of recurrence relations for the coefficients of the series $b(z)$ so that an easy induction argument shows that each of the coefficients is regular at $q=1$. Thus, making the substitution

$$x=e^{u}\qquad\text{and}\qquad q=e^{h/2} \tag{2.13}$$

in (2.12) we obtain a well-defined element $\widetilde{g}(u)\in 1+h\mathbb{C}((u))[[h]]$ satisfying (2.8).

Now set

$$\varphi(u)=\widetilde{g}(u)\widetilde{g}(-u)\in 1+h\mathbb{C}((u))[[h]].$$

Since $\varphi(u)=\varphi(-u)$, the Laurent series

$$\varphi(u)=\sum_{s\in\mathbb{Z}}\varphi_s u^s,\qquad \varphi_s\in\mathbb{C}[[h]],$$

contains only even powers of $u$. Moreover, (2.8) implies the relation

$$\varphi(u) = \varphi(u + nh). \tag{2.14}$$

By comparing the coefficients of the negative powers of $u$ on both sides, we conclude that $\varphi_s = 0$ for all $s < 0$. Similarly, by considering nonnegative powers of $u$ in (2.14) we get $\varphi_s = 0$ for all $s > 0$ so that $\varphi(u) = \varphi_0$ is an element of $1 + h\mathbb{C}[[h]]$. Let $\psi \in 1 + h\mathbb{C}[[h]]$ be such that $\psi^2\varphi_0 = 1$. Then the series $g(u) = \psi\,\widetilde{g}(u)$ satisfies both (2.7) and (2.8).

A direct argument with formal series shows that the conditions (2.7) and (2.8) uniquely determine $g(u)$. The details are given in Appendix A which also contains a direct proof of the existence of $g(u)$. $\square$

The first few terms of the series $g(u)$ are found by

$$g(u) = 1 + \frac{(n-1)\big(1+e^u\big)}{2n\big(1-e^u\big)}h + \frac{(n-1)^2\big(1+e^u\big)^2}{8n^2\big(1-e^u\big)^2}h^2 + \dots .$$

**Corollary 2.2.** *The series $g(u)$ satisfies the relation*

$$g(u)\,g(u+h)\dots g\big(u+(n-1)h\big) = e^{(n-1)h/2}\,\frac{1-e^u}{1-e^{u+(n-1)h}}. \tag{2.15}$$

*Proof.* Denote by $G(u)$ the series on the left hand side of (2.15). This is an element of $1 + h\mathbb{C}((u))[[h]]$ which satisfies

$$G(u)G\big(-u-(n-1)h\big) = 1$$

and

$$G(u+h) = G(u)\,\frac{(1-e^{u+h})\big(1-e^{u+(n-1)h}\big)}{(1-e^u)(1-e^{u+nh})}$$

by (2.7) and (2.8), respectively. The same argument as in Appendix A shows that any series $G(u) \in 1 + h\mathbb{C}((u))[[h]]$ satisfying these two properties is determined uniquely. The series on the right hand side of (2.15) also satisfies these two properties and so the claim follows. $\square$

*Remark* 2.3. The $R$-matrices given in (2.3) and in [8, eq. (2.2)] are related by the change of parameters (2.13). Note, however, that although the relations (2.8) and (2.10) correspond to each other under this change, the difference equation (2.10) determines $f(x)$ uniquely, as an element of $1 + x\mathbb{C}(q)[[x]]$, whereas $g(u)$ is regarded as an element of a different algebra of power series, namely, $1 + h\mathbb{C}((u))[[h]]$. Moreover, as shown in Appendix A, the property (2.7) is necessary to guarantee that $g(u)$ is determined uniquely. $\square$

To make a connection with the quantum affine vertex algebra associated with the rational $R$-matrix (see Proposition 2.4 below), define the $\mathbb{Z}$-gradation on the algebra $\operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n[u^{\pm 1}, h]$ by setting

$$\deg u^k h^l = -k - l \tag{2.16}$$

and assigning the zero degree to elements of $\operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n$. Extend the degree function (2.16) to the algebra of formal series $\operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n((u))[[h]]$ by allowing it to take the infinite value. Elements of finite degree will then form a subalgebra and we denote it by $\operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n((u))[[h]]_{\mathrm{fin}}$.

The $R$-matrix $\overline{R}(u)$ defined in (2.3) belongs to $\operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n((u))[[h]]_{\mathrm{fin}}$ and its degree is zero. Denote by $\overline{R}^{\,\mathrm{rat}}(u)$ its component of degree zero. It is easy to check that $\overline{R}^{\,\mathrm{rat}}(u)$ coincides with the Yang $R$-matrix, up to a scalar factor:

$$\overline{R}^{\,\mathrm{rat}}(u) = \frac{u}{u-h}\Big(1 - \frac{h}{u}P\Big) \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n[[h/u]],$$

where $P \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n$ is the permutation operator given by

$$P = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}. \tag{2.17}$$

Similarly, denote the highest degree component of the $R$-matrix $R(u)$, defined in Proposition 2.1, by $R^{\mathrm{rat}}(u)$. The unitarity property (2.5) and the crossing symmetry properties (2.6) for the $R$-matrix $R(u)$ imply

$$R_{12}^{\mathrm{rat}}(u) R_{21}^{\mathrm{rat}}(-u) = 1 \tag{2.18}$$

and

$$\big(R_{12}^{\mathrm{rat}}(u)^{-1}\big)^{t_2} R_{12}^{\mathrm{rat}}(u+nh)^{t_2} = 1 \qquad \text{and} \qquad R_{12}^{\mathrm{rat}}(u+nh)^{t_1} \big(R_{12}^{\mathrm{rat}}(u)^{-1}\big)^{t_1} = 1.$$

Furthermore, let $g^{\mathrm{rat}}(u)$ be the highest degree component of the series $g(u)$, defined in Proposition 2.1. Then $R^{\mathrm{rat}}(u) = g^{\mathrm{rat}}(u)\overline{R}^{\,\mathrm{rat}}(u)$. It is clear from the proof of Proposition 2.1 that $g^{\mathrm{rat}}(u) \in 1 + (h/u)\mathbb{C}[[h/u]]$. By taking the highest degree components in (2.8) we get

$$g^{\mathrm{rat}}(u+nh) = g^{\mathrm{rat}}(u)\,\frac{(u+h)\big(u+(n-1)h\big)}{u(u+nh)}. \tag{2.19}$$

Since $1 - h/u$ is invertible in $\mathbb{C}[[h/u]]$, by replacing $g^{\mathrm{rat}}(u)$ with $(1-h/u)\overline{g}(u)$ in (2.19) we obtain the equivalent equation

$$\overline{g}(u+nh) = \Big(1 - \frac{h^2}{u^2}\Big)\,\overline{g}(u). \tag{2.20}$$

Equation (2.20) has a unique solution for the series $\overline{g}(u)$ in $1+(h/u)\mathbb{C}[[h/u]]$; see e.g. [12, Section 2.2]. Therefore,

$$R^{\,\mathrm{rat}}(u) = g^{\,\mathrm{rat}}(u)\overline{R}^{\,\mathrm{rat}}(u) = \overline{g}(u)\Big(1-\frac{h}{u}P\Big).$$

The *quantized universal enveloping algebra* $\mathrm{U}(R)$ is the associative algebra over $\mathbb{C}[[h]]$ generated by elements $l_{ij}^{(-r)}$, where $1 \leqslant i,j \leqslant n$ and $r=1,2,\ldots$, subject to the defining relations

$$R(u-v)\,L_1^+(u)\,L_2^+(v) = L_2^+(v)\,L_1^+(u)\,R(u-v), \tag{2.21}$$

where the matrix $L^+(u)$ is given by

$$L^+(u) = \sum_{i,j=1}^{n} e_{ij} \otimes l_{ij}^+(u) \,\in\, \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{U}(R)[[u]] \tag{2.22}$$

and

$$l_{ij}^+(u) = \delta_{ij} - h\sum_{r=1}^{\infty} l_{ij}^{(-r)} u^{r-1} \,\in\, \mathrm{U}(R)[[u]].$$

Here we extend the notation (2.1) to matrices of the form (2.22). A subscript indicates its copy in the multiple tensor product algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^n \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^n}_{m} \otimes \mathrm{U}(R)[[u]],$$

so that

$$L_a^+(u) = \sum_{i,j=1}^{n} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes l_{ij}^+(u).$$

We take $m=2$ for the defining relations (2.21). Note that the $R$-matrix $R(u)$ in (2.21) can be replaced with $\overline{R}(u)$ to define the same algebra $\mathrm{U}(R)$.

Recall that the *dual Yangian* $\mathrm{Y}^+(\mathfrak{gl}_n)$ for $\mathfrak{gl}_n$ is the associative algebra over $\mathbb{C}[[h]]$ generated by elements $t_{ij}^{(-r)}$, where $1 \leqslant i,j \leqslant n$ and $r=1,2,\ldots$, subject to the defining relations

$$R^{\,\mathrm{rat}}(u-v)T_1^+(u)T_2^+(v) = T_2^+(v)T_1^+(u)R^{\,\mathrm{rat}}(u-v), \tag{2.23}$$

where the matrix $T^+(u)$ is given by

$$T^+(u) = \sum_{i,j=1}^{n} e_{ij} \otimes t_{ij}^+(u) \,\in\, \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{Y}^+(\mathfrak{gl}_n)[[u]]$$

and

$$t_{ij}^+(u) = \delta_{ij} - h\sum_{r=1}^{\infty} t_{ij}^{(-r)} u^{r-1} \,\in\, \mathrm{Y}^+(\mathfrak{gl}_n)[[u]].$$

Introduce the descending filtration

$$\cdots \supset \mathrm{U}^{(2)} \supset \mathrm{U}^{(1)} \supset \mathrm{U}^{(0)} \supset \mathrm{U}^{(-1)} \supset \mathrm{U}^{(-2)} \supset \cdots$$

on $\mathrm{U}(R)$ by setting

$$\deg h = -1 \qquad \text{and} \qquad \deg l_{ij}^{(-r)} = r \tag{2.24}$$

so that for any $r \in \mathbb{Z}$ the subspace $\mathrm{U}^{(r)}$ is the linear span of the elements of $\mathrm{U}(R)$ whose degrees do not exceed $r$. Let

$$\mathrm{gr}\, \mathrm{U}(R) = \bigoplus_{r\in\mathbb{Z}} \mathrm{U}^{(r)}/\mathrm{U}^{(r-1)}$$

be the associated graded algebra. It inherits a $\mathbb{C}[h]$-module structure from $\mathrm{U}(R)$. Let $\bar{l}_{ij}^{(-r)}$ denote the image of $l_{ij}^{(-r)}$ in the $r$-th component of $\mathrm{gr}\, \mathrm{U}(R)$. We will also write

$$\overline{L}^{+}(u) = \sum_{i,j=1}^{n} e_{ij} \otimes \bar{l}_{ij}^{+}(u) \;\in\; \mathrm{End}\, \mathbb{C}^n \otimes \mathrm{gr}\, \mathrm{U}(R)[[u]],$$

where

$$\bar{l}_{ij}^{+}(u) = \delta_{ij} - h \sum_{r=1}^{\infty} \bar{l}_{ij}^{(-r)} u^{r-1}.$$

**Proposition 2.4.** *We have an isomorphism*

$$\mathrm{gr}\, \mathrm{U}(R) \otimes_{\mathbb{C}[h]} \mathbb{C}[[h]] \cong \mathrm{Y}^{+}(\mathfrak{gl}_n)$$

*defined on the generators by*

$$\bar{l}_{ij}^{(-r)} \mapsto t_{ij}^{(-r)} \tag{2.25}$$

*for all* $1 \leqslant i, j \leqslant n$ *and* $r = 1, 2, \dots$.

*Proof.* Observe that both sides of (2.21) are elements of finite degrees of the algebra

$$\mathrm{End}\, \mathbb{C}^n \otimes \mathrm{End}\, \mathbb{C}^n \otimes \mathrm{U}(R)((u))((v))[[h]],$$

with respect to the degree function defined by (2.16) and (2.24) together with $\deg v = -1$. Moreover, by taking the highest degree components (which are of the zero degree) we get the defining relations (2.23) for the dual Yangian $\mathrm{Y}^{+}(\mathfrak{gl}_n)$:

$$R^{\mathrm{rat}}(u-v)\,\overline{L}_1^{+}(u)\,\overline{L}_2^{+}(v) = \overline{L}_2^{+}(v)\,\overline{L}_1^{+}(u)\,R^{\mathrm{rat}}(u-v).$$

Note that the coefficients of any monomial $u^a v^b$ on both sides of this relation coincide with the highest degree components of the respective coefficients of the monomial $u^a v^b$ on both sides of (2.21). This implies that the mapping $t_{ij}^{(-r)} \mapsto \bar{l}_{ij}^{(-r)}$ defines a homomorphism from the Yangian to the extended graded algebra. This homomorphism is clearly surjective, while its injectivity follows from well-known versions of the Poincaré–Birkhoff–Witt theorem for the algebras $\mathrm{Y}^{+}(\mathfrak{gl}_n)$ and $\mathrm{U}(R)$; see [4, Section 3.4]. $\square$

# 3 Quantum affine vertex algebra

We follow [6] to introduce a quantum vertex algebra structure on the $h$-adic completion $\widetilde{\mathrm{U}}(R)$ of the quantized universal enveloping algebra $\mathrm{U}(R)$. The corresponding structure associated with the rational $R$-matrix was studied in [12] where a description of the center of the quantum vertex algebra was given. Our goal is to obtain an analogous description of the center of $\widetilde{\mathrm{U}}(R)$ at the critical level.

## 3.1 The center of a quantum vertex algebra

We shall say that the $\mathbb{C}[[h]]$-module $V$ is topologically free if $V = V_0[[h]]$ for some complex vector space $V_0$. Denote by $V_h((z))$ the space of all Laurent series

$$v(z) = \sum_{r\in\mathbb{Z}} v_r z^{-r-1} \in V[[z^{\pm 1}]]$$

satisfying $v_r \to 0$ as $r \to \infty$, in the $h$-adic topology.

**Definition 3.1.** Let $V = V_0[[h]]$ be a topologically free $\mathbb{C}[[h]]$-module. A *quantum vertex algebra* $V$ over $\mathbb{C}[[h]]$ is the following data.

(a) A $\mathbb{C}[[h]]$-module map (the *vertex operators*)

$$Y:\ V\otimes V \ \to\ V_h((z)), \qquad v\otimes w \mapsto Y(z)(v\otimes w). \tag{3.1}$$

Setting $Y(v,z)w = Y(z)(v\otimes w)$ defines the map $Y(v,z) : V \to V_h((z))$ which satisfies the *weak associativity property*: for any $u, v, w \in V$ and $n \in \mathbb{Z}_{\geqslant 0}$ there exists $\ell \in \mathbb{Z}_{\geqslant 0}$ such that

$$(z_0+z_2)^{\ell}\, Y(v, z_0+z_2)Y(w,z_2)u - (z_0+z_2)^{\ell}\, Y\big(Y(v,z_0)w, z_2\big)\, u \in h^n V[[z_0^{\pm 1}, z_2^{\pm 1}]].$$

(b) A vector $\mathbf{1} \in V$ (the *vacuum vector*) which satisfies $Y(\mathbf{1},z)v = v$ for all $v \in V$, and for any $v \in V$ the series $Y(v,z)\mathbf{1}$ is a Taylor series in $z$ with the property

$$Y(v,z)\mathbf{1}\big|_{z=0} = v. \tag{3.2}$$

(c) A $\mathbb{C}[[h]]$-module map $D\colon V \to V$ (the *translation operator*) which satisfies

$$D\mathbf{1} = 0 \qquad \text{and} \qquad \frac{d}{dz}Y(v,z) = [D, Y(v,z)] \quad \text{for all } v \in V.$$

(d) A $\mathbb{C}[[h]]$-module map $\mathcal{S}\colon V\otimes V \to V\otimes V\otimes \mathbb{C}((z))$ which satisfies

$$\mathcal{S}(z)(v\otimes w) - v\otimes w\otimes 1 \in h\,V\otimes V\otimes\mathbb{C}((z)) \quad \text{for } v,w\in V,$$
$$[D\otimes 1,\mathcal{S}(z)] = -\frac{d}{dz}\mathcal{S}(z),$$

the *Yang–Baxter equation*

$$\mathcal{S}_{12}(z_1)\mathcal{S}_{13}(z_1+z_2)\mathcal{S}_{23}(z_2) = \mathcal{S}_{23}(z_2)\mathcal{S}_{13}(z_1+z_2)\mathcal{S}_{12}(z_1),$$

the *unitarity condition* $\mathcal{S}_{21}(z) = \mathcal{S}^{-1}(-z)$, and the $\mathcal{S}$-*locality*: for any $v,w\in V$ and $n\in\mathbb{Z}_{\geqslant 0}$ there exists $\ell\in\mathbb{Z}_{\geqslant 0}$ such that for any $u\in V$

$$(z_1-z_2)^{\ell}\,Y(z_1)\big(1\otimes Y(z_2)\big)\big(\mathcal{S}(z_1-z_2)(v\otimes w)\otimes u\big) \\ -(z_1-z_2)^{\ell}\,Y(z_2)\big(1\otimes Y(z_1)\big)(w\otimes v\otimes u) \in h^n V[[z_1^{\pm 1},z_2^{\pm 1}]].$$

□

The tensor products in Definition 3.1 are understood as $h$-adically completed. In particular, $V\otimes V$ denotes the space $(V_0\otimes V_0)[[h]]$ and $V\otimes V\otimes\mathbb{C}((z))$ denotes the space $\big(V_0\otimes V_0\otimes\mathbb{C}((z))\big)[[h]]$.

Let $V$ be a quantum vertex algebra. As in [12], we define the *center* of $V$ as the $\mathbb{C}[[h]]$-submodule

$$\mathfrak{z}(V) = \big\{v\in V \mid w_r v = 0 \text{ for all } w\in V \text{ and all } r\geqslant 0\big\}.$$

It was proved in [12] that the center of a quantum vertex algebra is a unital associative algebra with the product $\mathfrak{z}(V)\otimes\mathfrak{z}(V)\to\mathfrak{z}(V)$ given by

$$v\cdot w = v_{-1}w \quad \text{for all } v,w\in V.$$

The algebra $\mathfrak{z}(V)$ need not be commutative; see [12, Proposition 4.3]. Instead, it possesses the $\mathcal{S}$-*commutativity* property as demonstrated in [12, Proposition 3.7]. The next proposition shows that this property (as given in (3.3) below) is characteristic for elements of the center.

**Proposition 3.2.** *Let $V$ be a quantum vertex algebra. Vector $v\in V$ belongs to $\mathfrak{z}(V)$ if and only if*

$$Y(z_1)\left(1\otimes Y(z_2)\right)\left(\mathcal{S}(z_1-z_2)(v\otimes w)\otimes u\right) = Y(w,z_2)Y(v,z_1)u \tag{3.3}$$

*for all $w\in V$ and $u\in\mathfrak{z}(V)$.*

*Proof.* Let $v \in V$ satisfy (3.3) for all $w \in V$ and $u \in \mathfrak{z}(V)$. Recall the $\mathcal{S}$-Jacobi identity:

$$
\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right) Y(w, z_2)Y(v, z_1)u \\
&\quad - z_0^{-1}\delta\left(\frac{z_1 - z_2}{-z_0}\right) Y(z_1)\big(1 \otimes Y(z_2)\big)\big(\mathcal{S}(-z_0)(v \otimes w) \otimes u\big) \\
&\qquad = z_1^{-1}\delta\left(\frac{z_2 - z_0}{z_1}\right) Y\big(Y(w, z_0)v, z_1\big)\, u,
\end{aligned}
$$

which holds in any quantum vertex algebra; see [13]. Taking the residue $\operatorname{Res}_{z_0}$ on both sides, we obtain the $\mathcal{S}$-commutator formula

$$
\begin{aligned}
Y(w, z_2)Y(v, z_1)u - Y(z_1)\left(1 \otimes Y(z_2)\right)&\left(\mathcal{S}(z_1 - z_2)(v \otimes w) \otimes u\right) \\
&= \operatorname*{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2 - z_0}{z_1}\right) Y\big(Y(w, z_0)v, z_1\big)\, u.
\end{aligned}
$$

The left hand side is equal to zero by (3.3), so that

$$
\operatorname*{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2 - z_0}{z_1}\right) Y\big(Y(w, z_0)v, z_1\big)\, u = 0.
$$

This implies

$$
\sum_{r \geqslant 0} (-1)^r \binom{b + r}{r} (w_r v)_{-a-b-r-2} u = 0 \qquad \text{for all} \quad a, b \in \mathbb{Z}.
$$

Let $m > 0$. By (3.1) there exists $r_0 > 0$ such that $w_r v \in h^m V$ for all $r > r_0$. Therefore,

$$
\sum_{r=0}^{r_0} (-1)^r \binom{b + r}{r} (w_r v)_{-a-b-r-2} u = 0 \mod h^m V \quad \text{for all} \quad a, b \in \mathbb{Z}.
$$

By evaluating $(a, b) = (-i - 1 + c, i)$ for all $i = 0, \ldots, r_0$ with a fixed integer $c$, we obtain a system of $r_0 + 1$ homogeneous linear equations in the variables $(w_r v)_{-r-c-1} u$ with $r = 0, \ldots, r_0$. It is easily verified that its matrix is invertible, so there is a unique solution,

$$
(w_r v)_{-r-c-1} u = 0 \mod h^m V \qquad \text{for all} \quad r = 0, \ldots, r_0.
$$

By taking here $c = -r$, $u = \mathbf{1}$ and using (3.2) we get $w_r v \in h^m V$ for all $r = 0, \ldots, r_0$. We may conclude that $w_r v \in h^m V$ for all $r, m \geqslant 0$. Since $V$ is a topologically free $\mathbb{C}[[h]]$-module, $V$ is separated, so that $\cap_{m \geqslant 1} h^m V = 0$. This implies that $w_r v = 0$ for all $r \geqslant 0$ which means that the vector $v$ belongs to the center of $V$.

The "only if" part holds due to [12, Proposition 3.7]. $\square$

## 3.2 The center of the quantum affine vertex algebra

Here we recall some results of [6] describing the quantum vertex algebra structure on $\mathrm{U}(R)$. Let $\widetilde{\mathrm{U}}(R) = \mathrm{U}(R)[[h]]$ be the $h$-adic completion of $\mathrm{U}(R)$. The following property will play a central role; see [6, Lemma 2.1]. For any nonnegative integer $m$, consider the tensor product space $(\operatorname{End}\mathbb{C}^n)^{\otimes(m+1)} \otimes \widetilde{\mathrm{U}}(R)$ with the copies of the endomorphism algebra labelled by $0, 1, \dots, m$. Let $v_1, \dots, v_m$ be variables.

**Lemma 3.3.** *For any $c \in \mathbb{C}$ there exists a unique series $L(u) \in \operatorname{End}\mathbb{C}^n \otimes \big(\operatorname{End}\widetilde{\mathrm{U}}(R)\big)_h((u))$ such that for all $m \geqslant 0$ we have*

$$L_0(u)L_1^+(v_1)\dots L_m^+(v_m)1 = R_{01}(u - v_1 + hc/2)^{-1}\dots R_{0m}(u - v_m + hc/2)^{-1} \\ \times L_1^+(v_1)\dots L_m^+(v_m)R_{0m}(u - v_m - hc/2)\dots R_{01}(u - v_1 - hc/2)1.$$

Fix an arbitrary complex number $c$. As the action of the operator $L(u)$ on $\widetilde{\mathrm{U}}(R)$ depends on the choice of $c$, we will indicate this dependence by denoting the completed quantized universal enveloping algebra $\widetilde{\mathrm{U}}(R)$ by $\widetilde{\mathrm{U}}_c(R)$. The complex number $c$ will be called the *level* of $\widetilde{\mathrm{U}}_c(R)$. This terminology is motivated by the fact that the classical limit of the quantum vertex algebra $\widetilde{\mathrm{U}}_c(R)$ coincides with the affine vertex algebra for $\mathfrak{gl}_n$ at the level $c$; see [6]. The following relations hold for operators on $\operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n \otimes \widetilde{\mathrm{U}}_c(R)$:

$$R(u - v)L_1(u)L_2(v) = L_2(v)L_1(u)R(u - v), \tag{3.4}$$

$$R(u - v + hc/2)L_1(u)L_2^+(v) = L_2^+(v)L_1(u)R(u - v - hc/2). \tag{3.5}$$

Given a variable $z$ and a family of variables $u = (u_1, \dots, u_m)$, set

$$L_{[m]}(u|z) = L_1(z + u_1)\dots L_m(z + u_m), \qquad L_{[m]}^+(u|z) = L_1^+(z + u_1)\dots L_m^+(z + u_m).$$

The respective components of the matrices $L^+(u)$ and $L(u)$ are understood as operators on $\widetilde{\mathrm{U}}_c(R)$. The series $L_i(z + u_i)$ should be expanded in nonnegative powers of $u_i$.

By the results of Etingof and Kazhdan [6], for any $c \in \mathbb{C}$ there exists a unique well-defined structure of quantum vertex algebra on the quantized universal enveloping algebra $\widetilde{\mathrm{U}}_c(R)$. In particular, the vacuum vector is $\mathbf{1} = 1 \in \widetilde{\mathrm{U}}_c(R)$, the vertex operators are defined by

$$Y\big(L_{[m]}^+(u|0)\mathbf{1}, z\big) = L_{[m]}^+(u|z)\, L_{[m]}(u|z + hc/2)^{-1}$$

and the translation operator $D$ is given by

$$e^{zD}\, L^+(u_1)\dots L^+(u_m)\mathbf{1} = L^+(z + u_1)\dots L^+(z + u_m)\mathbf{1}.$$

We will not reproduce the definition of the map $\mathcal{S}$ as it requires some additional notation and it will not be used below. The map is defined in the same way as for the rational $R$-matrix; see also [12, Section 4.2].

Consider the $h$-permutation operator $P^h \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n[[h]]$ defined by

$$P^h = \sum_i e_{ii} \otimes e_{ii} + e^{h/2} \sum_{i>j} e_{ij} \otimes e_{ji} + e^{-h/2} \sum_{i<j} e_{ij} \otimes e_{ji}. \tag{3.6}$$

The symmetric group $\mathfrak{S}_k$ acts on the space $(\mathbb{C}^n)^{\otimes k}$ by $s_a \mapsto P^h_{s_a} = P^h_{a\,a+1}$ for $a = 1, \dots, k-1$, where $s_a$ denotes the transposition $(a, a+1)$. For a reduced decomposition $\sigma = s_{a_1} \dots s_{a_l}$ of an element $\sigma \in \mathfrak{S}_k$ set $P^h_\sigma = P^h_{s_{a_1}} \dots P^h_{s_{a_l}}$. Denote by $A^{(k)}$ the image of the normalized symmetrizer under this action, so that

$$A^{(k)} = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}\sigma \cdot P^h_\sigma.$$

Using the two-parameter $R$-matrix (2.2), for arbitrary variables $u_1, \dots, u_k$ set

$$\overline{R}(u_1, \dots, u_k) = \prod_{1 \leqslant a < b \leqslant k} \overline{R}_{ab}(u_a, u_b),$$

where the product is taken in the lexicographical order on the pairs $(a, b)$. Note that (2.21) implies the relation

$$\overline{R}(u_1, \dots, u_k) L^+_1(u_1) \dots L^+_k(u_k) = L^+_k(u_k) \dots L^+_1(u_1) \overline{R}(u_1, \dots, u_k). \tag{3.7}$$

We will need the following well-known case of the fusion procedure for the $R$-matrix (2.2) going back to [1].

**Lemma 3.4.** *Set $u_a = u - (a-1)h$ for $a = 1, \dots, k$. We have*

$$\overline{R}(u_1, \dots, u_k) = k!\, e^{uk(k-1)/2} \prod_{0 \leqslant a < b \leqslant k-1} (e^{-ah} - e^{-bh}) A^{(k)}.$$

□

Combining relation (3.7) and Lemma 3.4 we get for $u_a = u - (a-1)h$

$$A^{(k)} L^+_1(u_1) \dots L^+_k(u_k) = L^+_k(u_k) \dots L^+_1(u_1) A^{(k)}. \tag{3.8}$$

Also, since

$$\overline{R}(u, v) D_1 D_2 = D_2 D_1 \overline{R}(u, v),$$

we have

$$A^{(k)} D_1 \dots D_k = D_k \dots D_1 A^{(k)} \tag{3.9}$$

for the diagonal matrix (2.4).

Now consider the $h$-adically completed quantum vertex algebra $\widetilde{\mathrm{U}}_c(R)$ at the critical level $c = -n$ and introduce its elements as the coefficients of the power series in $u$ given by

$$\phi_k(u) = \operatorname{tr}_{1,\dots,k} A^{(k)} L^+_1(u) \dots L^+_k(u - (k-1)h) D_1 \dots D_k \tag{3.10}$$

for $k = 1, \dots, n$.

**Proposition 3.5.** *The coefficients of all series $\phi_k(u)$ belong to $\mathfrak{z}(\widetilde{\mathrm{U}}_{-n}(R))$.*

*Proof.* The argument is essentially a version of the proofs of [8, Theorem 3.2] and [12, Theorem 2.4] which we adjust for the current context. It is sufficient to show that

$$L_0(v)\,\phi_k(u) = \phi_k(u)$$

for all $k = 1, \dots, n$. Relation (3.5) implies

$$\begin{aligned} L_0(v)L_1^+(u_1)\dots L_k^+(u_k) &= R_{01}(v-u_1-hn/2)^{-1}\dots R_{0k}(v-u_k-hn/2)^{-1} \\ &\times\ L_1^+(u_1)\dots L_k^+(u_k)L_0(v)R_{0k}(v-u_k+hn/2)\dots R_{01}(v-u_1+hn/2). \end{aligned} \tag{3.11}$$

Since $L_0(v)$ commutes with $A^{(k)}$ and $D_a$ for $a = 1, \dots, k$, using (3.11) and $L_0(v)\mathbf{1} = \mathbf{1}$ we get

$$\begin{aligned} L_0(v)\,\phi_k(u) &= \mathrm{tr}_{1,\dots,k}\, A^{(k)}R_{01}(v-u_1-hn/2)^{-1}\dots R_{0k}(v-u_k-hn/2)^{-1} \\ &\times\ L_1^+(u_1)\dots L_k^+(u_k)\, R_{0k}(v-u_k+hn/2)\dots R_{01}(v-u_1+hn/2)D_1\dots D_k. \end{aligned} \tag{3.12}$$

Hence, we only have to prove that the right hand side in (3.12) equals $\phi_k(u)$. Set

$$\begin{aligned} X &= R_{01}(v-u_1-hn/2)^{-1}\dots R_{0k}(v-u_k-hn/2)^{-1} \\ Y &= L_1^+(u_1)\dots L_k^+(u_k)R_{0k}(v-u_k+hn/2)\dots R_{01}(v-u_1+hn/2)D_1\dots D_k \end{aligned}$$

and denote by $X^{\mathrm{op}}$ and $Y^{\mathrm{op}}$ the respective expressions obtained from these products by reversing the order of some factors:

$$\begin{aligned} X^{\mathrm{op}} &= R_{0k}(v-u_k-hn/2)^{-1}\dots R_{01}(v-u_1-hn/2)^{-1} \\ Y^{\mathrm{op}} &= L_k^+(u_k)\dots L_1^+(u_1)R_{01}(v-u_1+hn/2)\dots R_{0k}(v-u_k+hn/2)D_k\dots D_1. \end{aligned}$$

The Yang-Baxter equation for the $R$-matrix (2.2) and Lemma 3.4 imply

$$\begin{aligned} A^{(k)}R_{01}(v-u_1-hn/2)^{-1}&\dots R_{0k}(v-u_k-hn/2)^{-1} \\ &= R_{0k}(v-u_k-hn/2)^{-1}\dots R_{01}(v-u_1-hn/2)^{-1}A^{(k)}, \end{aligned}$$

that is, $A^{(k)}X = X^{\mathrm{op}}A^{(k)}$, and also

$$\begin{aligned} A^{(k)}R_{0k}(v-u_k+hn/2)&\dots R_{01}(v-u_1+hn/2) \\ &= R_{01}(v-u_1+hn/2)\dots R_{0k}(v-u_k+hn/2)A^{(k)}. \end{aligned} \tag{3.13}$$

Combining (3.8), (3.9) and (3.13) we get $A^{(k)}Y = Y^{\mathrm{op}}A^{(k)}$. Since $A^{(k)}$ is an idempotent, we proceed as follows:

$$\begin{aligned}\mathrm{tr}_{1,\dots,k}A^{(k)}XY &= \mathrm{tr}_{1,\dots,k}X^{\mathrm{op}}A^{(k)}Y = \mathrm{tr}_{1,\dots,k}X^{\mathrm{op}}\left(A^{(k)}\right)^2 Y\\ &= \mathrm{tr}_{1,\dots,k}X^{\mathrm{op}}A^{(k)}A^{(k)}Y = \mathrm{tr}_{1,\dots,k}A^{(k)}XY^{\mathrm{op}}A^{(k)}.\end{aligned}$$

By the cyclic property of trace, this equals

$$\mathrm{tr}_{1,\dots,k}A^{(k)}XY^{\mathrm{op}}A^{(k)} = \mathrm{tr}_{1,\dots,k}XY^{\mathrm{op}}\left(A^{(k)}\right)^2 = \mathrm{tr}_{1,\dots,k}XY^{\mathrm{op}}A^{(k)} = \mathrm{tr}_{1,\dots,k}XA^{(k)}Y,$$

and so $\mathrm{tr}_{1,\dots,k}A^{(k)}XY = \mathrm{tr}_{1,\dots,k}XA^{(k)}Y$. As a final step, we use the property

$$\mathrm{tr}_{1,\dots,k}XA^{(k)}Y = \mathrm{tr}_{1,\dots,k}X^{t_1\dots t_k}\left(A^{(k)}Y\right)^{t_1\dots t_k}.$$

Write

$$\begin{aligned}X^{t_1\dots t_k}\left(A^{(k)}Y\right)^{t_1\dots t_k} &= \left(R_{01}(v-u_1-hn/2)^{-1}\right)^{t_1}\dots\left(R_{0k}(v-u_k-hn/2)^{-1}\right)^{t_k}\\ &\times D_1\dots D_k R_{0k}(v-u_k+hn/2)^{t_k}\dots R_{01}(v-u_1+hn/2)^{t_1}\left(A^{(k)}L^+_1(u_1)\dots L^+_k(u_k)\right)^{t_1\dots t_k}\end{aligned}$$

and apply the first crossing symmetry property in (2.6) to get

$$\begin{aligned}X^{t_1\dots t_k}\left(A^{(k)}Y\right)^{t_1\dots t_k} &= D_1\dots D_k\left(A^{(k)}L^+_1(u_1)\dots L^+_k(u_k)\right)^{t_1\dots t_k}\\ &= \left(A^{(k)}L^+_1(u_1)\dots L^+_k(u_k)D_1\dots D_k\right)^{t_1\dots t_k}.\end{aligned}$$

This implies

$$\begin{aligned}\mathrm{tr}_{1,\dots,k}XA^{(k)}Y &= \mathrm{tr}_{1,\dots,k}\left(A^{(k)}L^+_1(u_1)\dots L^+_k(u_k)D_1\dots D_k\right)^{t_1\dots t_k}\\ &= \mathrm{tr}_{1,\dots,k}A^{(k)}L^+_1(u_1)\dots L^+_k(u_k)D_1\dots D_k = \phi_k(u),\end{aligned}$$

thus completing the proof. □

Consider the $h$-permutation operator $P^h_{(k,k-1,\dots,1)}$ which is associated with the $k$-cycle $(k,k-1,\dots,1) = s_{k-1}\dots s_1$ so that

$$P^h_{(k,k-1,\dots,1)} = P^h_{k-1\,k}\dots P^h_{12}.$$

Introduce another family of elements of $\widetilde{\mathrm{U}}_{-n}(R)$ as the coefficients of the power series in $u$ defined by

$$\theta_k(u) = \mathrm{tr}_{1,\dots,k}P^h_{(k,k-1,\dots,1)}L^+_1(u)\dots L^+_k(u-(k-1)h)\,D_1\dots D_k \tag{3.14}$$

for all $k \geqslant 1$ and $\theta_0(u) = n$.

**Corollary 3.6.** *The coefficients of all series $\theta_k(u)$ belong to $\mathfrak{z}(\widetilde{\mathrm{U}}_{-n}(R))$.*

*Proof.* Relation (2.21) implies that $M = L^+(u)\,D\,e^{-h\partial_u}$ is a *q-Manin matrix* with entries in $\widetilde{\mathrm{U}}_{-n}(R)[[u,\partial_u]]$ as follows from [2, Lemma 5.1], with the notation (2.13). Hence, applying the Newton identity for $q$-Manin matrices [2, Theorem 5.7], we can express the coefficients of each series $\theta_k(u)$ as polynomials in the coefficients of the series $\phi_k(u)$. So the corollary is a consequence of Proposition 3.5. $\square$

Let $\widetilde{\mathrm{U}}(R)^{\mathrm{ext}}$ denote the extension of $\widetilde{\mathrm{U}}(R)$ to an algebra over the field $\mathbb{C}((h))$. For all $m \geqslant 0$ introduce elements $\Theta_m^{(r)}$ of this algebra as the coefficients of the series

$$\Theta_m(u) = \sum_{r=0}^{\infty} \Theta_m^{(r)} u^r \in \widetilde{\mathrm{U}}(R)^{\mathrm{ext}}[[u]],$$

where we use the series (3.14) and set

$$\Theta_m(u) = h^{-m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \theta_k(u). \tag{3.15}$$

**Proposition 3.7.** *All the elements $\Theta_m^{(r)}$ with $r, m \geqslant 0$ pairwise commute.*

*Proof.* This follows from the corresponding well-known property of the coefficients of the series (3.10); see e.g. [2, Proposition 6.5] for a proof, which is quite similar to the rational case; cf. [14, Section 1.14]. The property extends to the coefficients of the series (3.14) due to the Newton identity; see [2, Theorem 6.6]. $\square$

**Theorem 3.8.** *All coefficients of the series $\Theta_m(u)$ belong to the $\mathbb{C}[[h]]$-module $\mathfrak{z}(\widetilde{\mathrm{U}}_{-n}(R))$. Moreover, the family $\Theta_m^{(r)}$ with $m = 1, \dots, n$ and $r = 0, 1, \dots$ is algebraically independent.*

*Proof.* We will need the usual permutation $P$ given in (2.17) along with the $h$-permutation operator $P^h$ defined in (3.6). Let $M = L^+(u)\,D\,e^{-h\partial_u}$ as before, and for each $m \geqslant 1$ consider the expression

$$\mathcal{M}_m = h^{-m}\big(1 - (M_m)^{\rightarrow}\big)\Big(P_{m-1\,m} - P^h_{m-1\,m}(M_{m-1})^{\rightarrow}\Big) \\ \times \cdots \times \Big(P_{23} - P^h_{23}(M_2)^{\rightarrow}\Big)\Big(P_{12} - P^h_{12} M_1\Big), \tag{3.16}$$

where the arrow in the superscript indicates that the corresponding factor appears on the right:

$$\Big(P_{a\,a+1} - P^h_{a\,a+1}\big(M_a\big)^{\rightarrow}\Big)\, X := P_{a\,a+1}\, X - P^h_{a\,a+1}\, X\, M_a. \tag{3.17}$$

We verify first that the expression (3.16), as a Laurent series in $h$, does not contain negative powers of $h$. Indeed, write

$$P_{a\,a+1} - P^h_{a\,a+1}\big(M_a\big)^{\rightarrow} = P_{a\,a+1}\Big(1 - P_{a\,a+1}P^h_{a\,a+1}\big(L^+_a(u)D_a e^{-h\partial_u}\big)^{\rightarrow}\Big)$$

and observe that

$$P_{a\,a+1}P^h_{a\,a+1}\big|_{h=0}=1.$$

Therefore, the expression in (3.17) vanishes at $h=0$ for any element $X$ of the $\mathbb{C}[[h]]$-module $(\operatorname{End}\mathbb{C}^n)^{\otimes m}\otimes\widetilde{\mathrm{U}}_{-n}(R)$. Hence each of the $m$ factors in (3.16) is divisible by $h$.

As a next step, expand the product in (3.16) to get the expression

$$\mathcal{M}_m=h^{-m}\sum_{k=0}^{m}\sum_{1\leqslant a_1<\dots<a_k\leqslant m}(-1)^k\,\Pi_{a_1,\dots,a_k}\,M_{a_1}\dots M_{a_k},$$

where

$$\Pi_{a_1,\dots,a_k}=P_{(m,m-1,\dots,a_k+1)}P^h_{a_k\,a_k+1}P_{(a_k,\dots,a_{k-1}+1)}P^h_{a_{k-1}\,a_{k-1}+1}\dots P_{(a_2,\dots,a_1+1)}P^h_{a_1\,a_1+1}P_{(a_1,\dots,1)}.$$

Now consider the trace $\operatorname{tr}_{1,\dots,m}\mathcal{M}_m$. Let us verify that for the partial trace we have

$$\operatorname{tr}_{\{1,\dots,m\}\setminus\{a_1,\dots,a_k\}}\Pi_{a_1,\dots,a_k}=P^h_{a_{k-1}\,a_k}P^h_{a_{k-2}\,a_{k-1}}\dots P^h_{a_1\,a_2}. \tag{3.18}$$

For any permutation $\sigma\in\mathfrak{S}_m$ we have $P_\sigma P^h_{ab}=P^h_{\sigma(a)\,\sigma(b)}P_\sigma$. Moreover, $\operatorname{tr}_a P_{ab}=\operatorname{tr}_a P^h_{ab}=1$ for any $a\neq b$. Therefore,

$$\operatorname{tr}_{a_r+1,\dots,a_{r+1}-1}P_{(a_{r+1},\dots,a_r+1)}\,P^h_{a_r\,a_r+1}=\operatorname{tr}_{a_r+1,\dots,a_{r+1}-1}P^h_{a_r\,a_{r+1}}\,P_{(a_{r+1},\dots,a_r+1)}=P^h_{a_r\,a_{r+1}}$$

for $r=1,\dots,k$ with $a_{k+1}:=m$. This implies (3.18). Hence we obtain

$$\begin{aligned}\operatorname{tr}_{1,\dots,m}\mathcal{M}_m&=h^{-m}\sum_{k=0}^{m}\sum_{1\leqslant a_1<\dots<a_k\leqslant m}(-1)^k\operatorname{tr}_{a_1,\dots,a_k}P^h_{a_{k-1}\,a_k}P^h_{a_{k-2}\,a_{k-1}}\dots P^h_{a_1\,a_2}M_{a_1}\dots M_{a_k}\\&=h^{-m}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\operatorname{tr}_{1,\dots,k}P^h_{k-1\,k}P^h_{k-2\,k-1}\dots P^h_{1\,2}M_1\dots M_k.\end{aligned}$$

Since $P^h_{k-1\,k}P^h_{k-2\,k-1}\dots P^h_{1\,2}=P^h_{(k,k-1,\dots,1)}$ and

$$M_1\dots M_k=L^+_1(u)\dots L^+_k(u-(k-1)h)\,D_1\dots D_k\,e^{-kh\partial_u},$$

using (3.14) we can write

$$\operatorname{tr}_{1,\dots,m}\mathcal{M}_m=h^{-m}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\theta_k(u)\,e^{-kh\partial_u}.$$

Regarding this expression as a power series in $\partial_u$, we conclude that its constant term coincides with $\Theta_m(u)$ as defined in (3.15). The first part of the theorem now follows from Corollary 3.6.

Note that the level $c$ is irrelevant for the second part of the theorem. Extend the degree function (2.24) to the algebra $\widetilde{\mathrm{U}}(R)$ by allowing it to take the infinite value. Elements of finite degree will then form a subalgebra which we denote by $\widetilde{\mathrm{U}}(R)_{\mathrm{fin}}$. Similarly, by using (2.16) introduce the subalgebra $\widetilde{\mathrm{U}}(R)[[u]]_{\mathrm{fin}}$ of $\widetilde{\mathrm{U}}(R)[[u]]$ formed by elements of finite degree. Observe that the series $\Theta_m(u)$ belongs to $\widetilde{\mathrm{U}}(R)[[u]]_{\mathrm{fin}}$. Take the highest degree component of this series (that is, the associated element of the graded algebra $\mathrm{gr}\,\widetilde{\mathrm{U}}(R)[[u]]_{\mathrm{fin}}$) and identify it with an element of the algebra $\mathrm{Y}^+(\mathfrak{gl}_n)[[h,u]]$ via an extension of the map (2.25). This element coincides with the series

$$\overline{\Theta}_m(u) = h^{-m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathrm{tr}\, T^+(u) \cdots T^+(u-(k-1)h). \tag{3.19}$$

Write

$$\overline{\Theta}_m(u) = \sum_{r=0}^{\infty} \overline{\Theta}_m^{(r)} u^r.$$

By [12, Proposition 4.7], the coefficients $\overline{\Theta}_m^{(r)}$ with $m = 1, \dots, n$ and $r = 0, 1, \dots$ are algebraically independent elements of the $h$-adically completed dual Yangian $\mathrm{Y}^+(\mathfrak{gl}_n)$. Note that the coefficient of any power $u^a$ on the right hand side of (3.19) coincides with the highest degree component of the coefficient of $u^a$ on the right hand side of (3.15). Hence, the corresponding coefficients $\Theta_m^{(r)}$ are also algebraically independent. □

**Theorem 3.9.** *The center at the critical level $\mathfrak{z}(\widetilde{\mathrm{U}}_{-n}(R))$ is a commutative algebra. It is topologically generated by the family $\Theta_m^{(r)}$ with $m = 1, \dots, n$ and $r = 0, 1 \dots$.*

*Proof.* By Proposition 3.7 and Theorem 3.8, the coefficients $\Theta_m^{(r)}$ generate a commutative subalgebra of $\mathfrak{z}(\widetilde{\mathrm{U}}_{-n}(R))$. Let $X$ be an arbitrary element of $\mathfrak{z}(\widetilde{\mathrm{U}}_{-n}(R))$. We will prove by induction that for each $k \geqslant 0$ there exists a polynomial

$$Q \in \mathbb{C}[\Theta_m^{(r)}][h], \qquad m = 1, \dots, n \quad \text{and} \quad r = 0, 1, \dots$$

such that $X - Q \in h^k \widetilde{\mathrm{U}}_{-n}(R)$. The induction base is clear. Suppose that the property holds for some $k \geqslant 0$ so that

$$X - Q \ = \ h^k X_k + h^{k+1} X_{k+1} + \dots \quad \text{with } X_s \in V_0, \tag{3.20}$$

where we write $\widetilde{\mathrm{U}}_{-n}(R) = V_0[[h]]$ for some complex vector space $V_0$. By Lemma 3.3 the operator $L(u)$ can be written as

$$L(u) = \sum_{i,j=1}^{n} e_{ij} \otimes l_{ij}(u) \qquad \text{with} \quad l_{ij}(u) = \delta_{ij} + h \sum_{r \in \mathbb{Z}} \tilde{l}_{ij}^{(r)} u^r \in \big(\mathrm{End}\, \widetilde{\mathrm{U}}_{-n}(R)\big)_h((u)).$$

Since $X-Q$ belongs to $\mathfrak{z}(\widetilde{\mathrm{U}}_{-n}(R))$, we have

$$\tilde{l}_{ij}^{(r)}(X-Q)=0 \qquad \text{for all} \quad i,j=1,\dots,n \quad \text{and} \quad r\in\mathbb{Z}.$$

Thus, (3.20) implies

$$\tilde{l}_{ij}^{(r)}X_k\equiv 0 \mod h, \qquad \text{hence} \quad X_k\in\mathfrak{z}(\widetilde{\mathrm{U}}_{-n}(R)) \mod h. \tag{3.21}$$

Consider the symbol (the highest degree component) of $X_k$ in the graded algebra $\mathrm{gr}\,\mathrm{U}_{-n}(R)$. Its image under the isomorphism of Proposition 2.4 is an element $\overline{X}_k\in\mathrm{Y}^+(\mathfrak{gl}_n)$. We will also regard it as an element of the quantum vertex algebra $\mathcal{V}_{\mathrm{cri}}=\mathrm{Y}^+(\mathfrak{gl}_n)[[h]]$ at the critical level $c=-n$, associated with the double Yangian; see [12, Theorem 4.1] for a precise definition of the quantum vertex algebra structure. As shown in [12, Section 4.4], the center of this quantum vertex algebra coincides with the subspace of invariants

$$\mathfrak{z}(\mathcal{V}_{\mathrm{cri}})=\{U\in\mathcal{V}_{\mathrm{cri}}\mid t_{ij}^{(r)}U=0 \qquad \text{for } r\geqslant 1 \ \text{ and all } \ i,j\}. \tag{3.22}$$

Here the operators $t_{ij}^{(r)}$ are found as the coefficients of the series

$$t_{ij}(u)=\delta_{ij}+h\sum_{r=1}^{\infty}t_{ij}^{(r)}u^{-r}\in\big(\mathrm{End}\,\mathcal{V}_{\mathrm{cri}}\big)_h((u)),$$

which are the entries of the matrix operator $T(u)$

$$T(u)=\sum_{i,j=1}^{n}e_{ij}\otimes t_{ij}(u)$$

uniquely determined by the relations

$$\begin{aligned}T_0(u)T_1^+(v_1)\dots T_m^+(v_m)1&=R_{01}^{\mathrm{rat}}(u-v_1-hn/2)^{-1}\dots R_{0m}^{\mathrm{rat}}(u-v_m-hn/2)^{-1}\\&\times T_1^+(v_1)\dots T_m^+(v_m)R_{0m}^{\mathrm{rat}}(u-v_m+hn/2)\dots R_{01}^{\mathrm{rat}}(u-v_1+hn/2)1\end{aligned} \tag{3.23}$$

with the notation as in Lemma 3.3. Similar to the proof of Theorem 3.8, extend the degree function defined in (2.16) and (2.24) by setting $\deg v_i=-1$ for all $i$. As pointed out in Section 2, the rational $R$-matrix $R^{\mathrm{rat}}(u)$ coincides with the highest degree component of the trigonometric $R$-matrix $R(u)$. Write the expression $\mathcal{L}=L_0(u)\,L_1^+(v_1)\dots L_m^+(v_m)1$ as a series in the monomials $u^a v_1^{b_1}\dots v_m^{b_m}$ with coefficients in $\widetilde{\mathrm{U}}_{-n}(R)$. The symbol (the highest degree component) of $\mathcal{L}$ is an element of the graded algebra $\mathrm{gr}\,\widetilde{\mathrm{U}}_{-n}(R)((u))[[v_1,\dots,v_m,h]]$. Also writing this symbol as a series in the monomials $u^a v_1^{b_1}\dots v_m^{b_m}$, observe that if $a\leqslant 0$ then the coefficient of such a monomial in the symbol coincides with the symbol of the coefficient of the same monomial in the expansion of $\mathcal{L}$. This implies that the image of the component of the symbol of $\mathcal{L}$ corresponding to nonpositive powers of $u$ under an extension of the map (2.25) coincides with the right-hand side in (3.23). Together with (3.21) this shows that for $r \geqslant 1$ and all $i, j \in \{1, \dots, n\}$

$$t_{ij}^{(r)}\, \overline{X}_k \equiv 0 \mod h, \qquad \text{hence} \quad \overline{X}_k \in \mathfrak{z}(\mathcal{V}_{\text{cri}}) \mod h.$$

It was proved in [12, Theorem 4.8] that the elements $\overline{\Theta}_m^{(r)}$ topologically generate $\mathfrak{z}(\mathcal{V}_{\text{cri}})$. Therefore, $\overline{X}_k \equiv \overline{S} \mod h$ for some polynomial

$$\overline{S} \in \mathbb{C}[\overline{\Theta}_m^{(r)}], \qquad m = 1, \dots, n \quad \text{and} \quad r = 0, 1, \dots.$$

Note that $\deg \overline{\Theta}_m^{(r)} = \deg \overline{\Theta}_m^{(r)}\big|_{h=0}$ and so

$$\deg \overline{S} = \deg \overline{S}\big|_{h=0} = \deg \overline{X}_k. \tag{3.24}$$

Replace the variables $\overline{\Theta}_m^{(r)}$ in $\overline{S}$ with the respective elements $\Theta_m^{(r)}$ to get a polynomial

$$S \in \mathbb{C}[\Theta_m^{(r)}].$$

Let us consider the difference $X_k - S$ and take its symbol (the highest degree component) in the graded algebra $\operatorname{gr} \widetilde{\mathrm{U}}(R)_{\text{fin}}$. It belongs to $h \operatorname{gr} \widetilde{\mathrm{U}}(R)_{\text{fin}}$. Therefore, we can conclude that

$$X_k - S \in X_k^{(1)} + h\, \widetilde{\mathrm{U}}_{-n}(R)$$

for some $X_k^{(1)} \in V_0$ whose degree is lower than that of the symbol. Hence, we can write (3.20) in the form

$$X - Q - h^k S \;=\; h^k X_k^{(1)} + h^{k+1} X_{k+1}^{(1)} + \dots \quad \text{with } X_s^{(1)} \in V_0. \tag{3.25}$$

Recall that $\deg l_{ij}^{(-r)} = r$, so the elements of $V_0$ have nonnegative degrees; see (2.24). Due to (3.24) we have $\deg S = \deg X_k$, so that $\deg X_k > \deg X_k^{(1)} \geqslant 0$.

Now we can repeat the same argument as above, but working with (3.25) instead of (3.20) so that the role of $X_k$ is played by $X_k^{(1)}$. An obvious induction on the degree of the element $\deg X_k$ allows us to conclude that there exists a polynomial $P \in \mathbb{C}[\Theta_m^{(r)}][h]$ satisfying $X - P \in h^{k+1}\, \widetilde{\mathrm{U}}_{-n}(R)$ thus completing the induction step and the proof. $\square$

Now consider the quantum vertex algebra $\widetilde{\mathrm{U}}_c(R)$ at a noncritical level $c \neq -n$. Define the quantum determinant of the matrix $L^+(u)$ by

$$\operatorname{qdet} L^+(u) = \sum_{\sigma \in \mathfrak{S}_n} \left(-e^{-h/2}\right)^{l(\sigma)} l^+_{\sigma(1)1}(u) \dots l^+_{\sigma(n)n}(u - (n-1)h), \tag{3.26}$$

where $l(\sigma)$ equals the number of inversions in the sequence $(\sigma(1), \dots, \sigma(n))$. Write

$$\operatorname{qdet} L^+(u) = 1 - h\,(d_0 + d_1 u + d_2 u^2 + \dots).$$

**Proposition 3.10.** *The center $\mathfrak{z}(\widetilde{\mathrm{U}}_c(R))$ at a noncritical level $c \neq -n$ is a commutative algebra. It is topologically generated by the family $d_0, d_1, \dots$ of algebraically independent elements.*

*Proof.* The property that the coefficients $d_0, d_1, \dots$ are pairwise commuting elements of $\mathfrak{z}(\widetilde{\mathrm{U}}_c(R))$ can be verified by repeating the arguments of the proof of [8, Lemma 4.3] with the use of Corollary 2.2 (the assumption that the level is critical is unnecessary for the lemma to hold true; see also [12, Proposition 2.8]). In fact, these arguments demonstrate that the coefficients belong to the center of the algebra $\widetilde{\mathrm{U}}_c(R)$ and so, in particular, they commute pairwise.

Note that the series $\operatorname{qdet} L^+(u)$ belongs to the algebra $\widetilde{\mathrm{U}}(R)[[u]]_{\mathrm{fin}}$ introduced in the proof of Theorem 3.8. The symbol of $\operatorname{qdet} L^+(u)$ (the highest degree component) belongs to the graded algebra $\operatorname{gr} \widetilde{\mathrm{U}}(R)[[u]]_{\mathrm{fin}}$. Identify the symbol with an element of the $\mathrm{Y}^+(\mathfrak{gl}_n)[[u,h]]$ via an extension of the map (2.25). This element equals

$$\operatorname{qdet} T^+(u) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn} \sigma \cdot t^+_{\sigma(1)1}(u) \dots t^+_{\sigma(n)n}(u-(n-1)h) \in \mathrm{Y}^+(\mathfrak{gl}_n)[[u,h]]. \qquad (3.27)$$

Now we will use some results concerning the quantum vertex algebra $\mathcal{V}_{\mathrm{c}} := \mathrm{Y}^+(\mathfrak{gl}_n)[[h]]$ at the level $c \neq -n$; see [12, Theorem 4.1]. It was proved in [12, Proposition 4.5] that the center $\mathfrak{z}(\mathcal{V}_{\mathrm{c}})$ is topologically generated by the algebraically independent family of coefficients $\overline{d}_0, \overline{d}_1, \dots$ of the quantum determinant

$$\operatorname{qdet} T^+(u) = 1 - h\,(\overline{d}_0 + \overline{d}_1 u + \overline{d}_2 u^2 + \dots).$$

This implies that the coefficients $d_0, d_1, \dots$ are algebraically independent. Finally, the property that the family $d_0, d_1, \dots$ topologically generates the center $\mathfrak{z}(\widetilde{\mathrm{U}}_c(R))$ is verified by the same argument as in the proof of Theorem 3.9. $\square$

# A A direct proof of Proposition 2.1

We start by describing solutions of (2.8), regarding it as an equation for $g(u) \in \mathbb{C}((u))[[h]]$ (we suppress the dependence on $n$ and $h$ in the notation of these series). Write

$$g(u) = \sum_{l \geqslant 0} g_l(u) h^l, \qquad g_l(u) \in \mathbb{C}((u)). \qquad (A.1)$$

Using the Taylor expansion formula

$$g(u+nh) = \sum_{l \geqslant 0} \Big( \sum_{k=0}^{l} \frac{n^k}{k!} g^{(k)}_{l-k}(u) \Big) h^l,$$

from (2.8) we get

$$\left((1-e^u)-e^u(1-e^u)\sum_{l\geqslant 0}\frac{n^l}{l!}h^l\right)\cdot\sum_{l\geqslant 0}\Big(\sum_{k=0}^{l}\frac{n^k}{k!}g_{l-k}^{(k)}(u)\Big)h^l$$
$$=\left(1+e^u\sum_{l\geqslant 0}\frac{e^un^l-(n-1)^l-1}{l!}h^l\right)\cdot\sum_{l\geqslant 0}g_l(u)h^l. \tag{A.2}$$

By equating the coefficients of the same powers of $h$ on both sides of (A.2), we get a system of differential equations in $g_k(u)$ with $k\geqslant 0$. For the constant term the equation holds identically, while considering the coefficients of $h$ in (A.2) we get $g_0'(u)=0$ so that $g_0(u)=c_0$ for $c_0\in\mathbb{C}$. Taking the coefficients of $h^l$ in (A.2), for an arbitrary $l\geqslant 1$ we get

$$g_{l-1}'(u)=\frac{e^u}{(1-e^u)^2}\sum_{k=0}^{l-2}\sum_{m=0}^{l-k-1}p_{k,m}(e^u)\,g_k^{(m)}(u)-\sum_{k=0}^{l-2}\frac{n^{l-k-1}}{(l-k)!}g_k^{(l-k)}(u) \tag{A.3}$$

for some polynomials $p_{k,m}(z)\in\mathbb{C}[z]$ of degree not exceeding 1 such that

$$p_{k,m}(1)=0\quad\text{when } k+m=l-1,$$

and $p_{k,0}(z)$ are constants. The right hand side of (A.3) is understood as being equal to zero for $l=1$.

Fix $r\geqslant 1$ and suppose that series $g_0(u),\dots,g_{r-1}(u)\in\mathbb{C}((u))$ satisfy (A.3) for $l=1,\dots,r$. We will prove by induction on $r$ that a solution $g_r(u)\in\mathbb{C}((u))$ of (A.3) with $l=r+1$ exists and, up to an additive constant, it has the form of a linear combination of the series

$$\frac{p(e^u)}{(1-e^u)^t}$$

for some polynomials $p(z)\in\mathbb{C}[z]$ and positive integers $t$ such that $t-\deg p\geqslant 1$ and $t\leqslant r$. Indeed, by the induction hypothesis, all derivatives $g_k^{(m)}(u)$ with $m\geqslant 1$ and $k=0,\dots,r-1$ can be expressed as linear combinations of series of the form

$$\frac{e^u\,q(e^u)}{(1-e^u)^{t+m}}$$

for some polynomials $q(z)\in\mathbb{C}[z]$ and positive integers $t$ such that $t+m-\deg q\geqslant 2$ and $t\leqslant k$. This implies that the right hand side in (A.3) for $l=r+1$ can be written as a linear combination of series of the form

$$\frac{e^u\,q(e^u)}{(1-e^u)^t}$$

for some polynomials $q(z)\in\mathbb{C}[z]$ and positive integers $t$ such that $t-\deg q\geqslant 2$ and $t\leqslant r+1$ so that $g_r(u)\in\mathbb{C}((u))$ does exist and takes the required form.

Thus, by taking $c_0 = 1$ in the above argument, we may conclude that equation (2.8) has a solution $g(u) \in 1 + (h/u)\mathbb{C}[[h/u, u]]$. As shown in the proof of Proposition 2.1 in Section 2, by multiplying $g(u)$ by an appropriate element of $\mathbb{C}[[h]]$ we get a solution of both (2.7) and (2.8). To show that any such solution is determined uniquely, expand $g(u)$ as in (A.1). We find from (2.7) that

$$g_0(u) = 1 \qquad \text{and} \qquad \sum_{k=0}^{l} g_k(u)\, g_{l-k}(-u) = 0 \quad \text{for all} \quad l \geqslant 1. \tag{A.4}$$

Returning to the induction argument in the first part of the above proof, assume that the coefficients $g_0(u), \dots, g_{r-1}(u) \in \mathbb{C}((u))$ are uniquely determined for some $r \geqslant 1$. Relation (A.3) with $l = r + 1$ now determines $g_r(u)$ uniquely, up to an additive constant. However, its value is fixed by the second condition in (A.4) for $l = r$.

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