# Center of the quantum affine vertex algebra in type $A$

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**Abstract**

We consider the quantum vertex algebra associated with the double Yangian in type $A$ as defined by Etingof and Kazhdan. We show that its center is a commutative associative algebra and construct algebraically independent families of topological generators of the center at the critical level.

# 1 Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and let $\widehat{\mathfrak{g}}$ be the corresponding affine Kac–Moody algebra. The vacuum module $V_\kappa(\mathfrak{g})$ at the level $\kappa \in \mathbb{C}$ over $\widehat{\mathfrak{g}}$ has a vertex algebra structure; see, e.g., books by E. Frenkel and D. Ben-Zvi [12], I. Frenkel, J. Lepowsky and A. Meurman [13] and V. Kac [18]. The center of any vertex algebra is a commutative associative algebra. Unless the level $\kappa$ is critical, the center of the affine vertex algebra $V_\kappa(\mathfrak{g})$ is trivial (coincides with $\mathbb{C}$). By a theorem of B. Feigin and E. Frenkel [7], the center at the critical level $\mathfrak{z}(\widehat{\mathfrak{g}})$ is an algebra of polynomials in infinitely many variables. Moreover, the algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is canonically isomorphic to the algebra of functions on a space of opers; see E. Frenkel [11, Ch. 4] for a detailed exposition.

Explicit formulas for generators of the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ were given in [2] and [3] for type $A$ (see also [24]), in [22] for types $B$, $C$ and $D$; and in [25] for type $G_2$. Due to general results of [8], [9] and [28], these formulas lead to explicit constructions of commutative subalgebras of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ and to explicit higher order Hamiltonians and their eigenvalues on the Bethe vectors in the Gaudin model associated with $\mathfrak{g}$; see also [14], [23].

A general definition of quantum vertex algebra was given by P. Etingof and D. Kazhdan [6]. In accordance with [6], a *quantum affine vertex algebra* can be associated with a rational, trigonometric or elliptic $R$-matrix. In particular, a suitably normalized Yang $R$-matrix gives rise to a quantum vertex algebra structure on the vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$ at the level $c \in \mathbb{C}$ over the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ of type $A$.

In this paper we introduce the *center* $\mathfrak{z}(V)$ of an arbitrary quantum vertex algebra $V$ and describe its general properties. We show that the center is an $\mathcal{S}$-*commutative* associative algebra; see (3.27) for the definition. In general, it need not be commutative. Our main focus will be on the center $\mathfrak{z}\big(\mathcal{V}_c(\mathfrak{gl}_N)\big)$ of the quantum affine vertex algebra $\mathcal{V}_c(\mathfrak{gl}_N)$. The vacuum module is isomorphic to the dual Yangian $\mathrm{Y}^+(\mathfrak{gl}_N)$, as a vector space, and we prove that the center can be identified with a *commutative* subalgebra of $\mathrm{Y}^+(\mathfrak{gl}_N)$. This subalgebra is invariant under a derivation $D$, the *translation operator*, arising from the quantum vertex algebra structure on the vacuum module.

We show that the center at the critical level $c = -N$ possesses large families of algebraically independent topological generators so that a quantum analogue of the Feigin–Frenkel theorem holds. Moreover, unlike the center of the affine vertex algebra $V_{-N}(\mathfrak{gl}_N)$, it turns out to be possible to produce such families parameterized by arbitrary partitions with at most $N$ parts. The construction depends of the *fusion procedure* originated in the work of A. Jucys [17] for the symmetric group providing factorized $R$-matrix formulas for all primitive idempotents. These families thus generalize to the context of quantum vertex algebras the quantum immanants of A. Okounkov [27] which form a basis of the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$.

By taking a classical limit we recover explicit generators of the center of the affine vertex algebra $V_{-N}(\mathfrak{gl}_N)$; cf. [2], [3] and [29]. In principle, this approach is also applicable to construct generators of the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ for an arbitrary simple Lie algebra $\mathfrak{g}$. A required ingredient is a fusion procedure providing $R$-matrix formulas for idempotents in appropriate centralizer algebras. This is already in place for the types $B$, $C$ and $D$ so that the construction of [22] can be reproduced in this way.

We also give a construction of central elements of a completed double Yangian at the critical level prompted by the quantum vertex algebra structure. They are used to show that the center $\mathfrak{z}\big(\mathcal{V}_c(\mathfrak{gl}_N)\big)$ is commutative. If the level is not critical, then the center is trivial in the sense that its generators are elements associated with the center of the Lie algebra $\mathfrak{gl}_N$. They are found as the coefficients of the quantum determinant of the generator matrix of the dual Yangian.

Our arguments are based on explicit constructions of elements of the center of the quantum affine vertex algebra and rely on the $R$-matrix calculations used in [10] to produce explicit Sugawara operators for the quantum affine algebra in type $A$ at the critical level.

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## 2 Vacuum module for the double Yangian

We recall and reproduce some basic properties of the double Yangian for $\mathfrak{gl}_N$. Our definitions follow Etingof and Kazhdan [5], [6] and Iohara [16], where a centrally extended double Yangian over the ring $\mathbb{C}[[h]]$ was considered. To simplify our formulas, we first define this algebra over $\mathbb{C}$ (formally putting $h=-1$ in the notation of [16]), although this will require a certain completion; cf. Nazarov [26]. We will return to the closely related definition of the double Yangian over $\mathbb{C}[[h]]$ to study the associated structure of quantum vertex algebra in Sec. 4.

### 2.1 Yangian and dual Yangian for $\mathfrak{gl}_N$

The *Yangian* $\mathrm{Y}(\mathfrak{gl}_N)$ is the associative algebra with generators $t_{ij}^{(r)}$, where $1 \leqslant i,j \leqslant N$ and $r=1,2,\dots$ and the defining relations

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min\{r,s\}} \Big(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)}\Big), \tag{2.1}$$

where $t_{ij}^{(0)} = \delta_{ij}$. In terms of the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)}\, u^{-r} \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$$

the defining relations can be written as

$$(u-v)\,[t_{ij}(u), t_{kl}(v)] = t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u). \tag{2.2}$$

They admit the following matrix form. Consider the *Yang R-matrix* $R(u)$, which is a rational function in a complex parameter $u$ with values in the tensor product algebra $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N$ defined by

$$R(u) = 1 - P\, u^{-1}, \tag{2.3}$$

where $P$ is the permutation operator in $\mathbb{C}^N \otimes \mathbb{C}^N$. Then (2.1) is equivalent to the *RTT relation*

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v), \tag{2.4}$$

where

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]] \tag{2.5}$$

and the $e_{ij}$ are the matrix units. We use a subscript to indicate a copy of the matrix of the form (2.5) in the multiple tensor product algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \dots \otimes \mathrm{End}\,\mathbb{C}^N}_{m} \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]] \tag{2.6}$$

so that

$$T_a(u) = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes t_{ij}(u). \tag{2.7}$$

We take $m = 2$ for the defining relations (2.4).

This notation for elements of algebras of the form (2.6) will be extended as follows. For an element

$$C = \sum_{i,j,r,s=1}^{N} c_{ijrs}\, e_{ij} \otimes e_{rs} \in \operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N,$$

and any two indices $a, b \in \{1, \dots, m\}$ such that $a \neq b$, we denote by $C_{ab}$ the element of the algebra $(\operatorname{End} \mathbb{C}^N)^{\otimes m}$ with $m \geqslant 2$ given by

$$C_{ab} = \sum_{i,j,r,s=1}^{N} c_{ijrs}\, (e_{ij})_a (e_{rs})_b, \qquad (e_{ij})_a = 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)}. \tag{2.8}$$

We regard the matrix transposition as the linear map

$$t : \operatorname{End} \mathbb{C}^N \to \operatorname{End} \mathbb{C}^N, \qquad e_{ij} \mapsto e_{ji}.$$

For any $a \in \{1, \dots, m\}$ we will denote by $t_a$ the corresponding partial transposition on the algebra (2.6) which acts as $t$ on the $a$-th copy of $\operatorname{End} \mathbb{C}^N$ and as the identity map on all the other tensor factors.

The algebra $\mathrm{Y}(\mathfrak{gl}_N)$ possesses a natural ascending filtration defined by $\deg t_{ij}^{(r)} = r - 1$ for all $r \geqslant 1$. Denote by $\operatorname{gr} \mathrm{Y}(\mathfrak{gl}_N)$ the associated graded algebra. We have the isomorphism $\operatorname{gr} \mathrm{Y}(\mathfrak{gl}_N) \cong \mathrm{U}\big(\mathfrak{gl}_N[t]\big)$. The image $\bar{t}_{ij}^{(r)}$ of the generator $t_{ij}^{(r)}$ in the $(r-1)$-th component of the graded algebra $\operatorname{gr} \mathrm{Y}(\mathfrak{gl}_N)$ corresponds to the element $E_{ij}[r-1]$ of $\mathrm{U}\big(\mathfrak{gl}_N[t]\big)$, where the $E_{ij}$ are the standard basis elements of $\mathfrak{gl}_N$ and we use the notation $X[r] = X\, t^r$ for $X \in \mathfrak{gl}_N$ and any $r \in \mathbb{Z}$.

Let $E = [E_{ij}]$ denote the matrix whose $(i, j)$ entry is the element $E_{ij}$ of $\mathrm{U}(\mathfrak{gl}_N)$. For any $a \in \mathbb{C}$ the mapping

$$\operatorname{ev}_a : T(u) \mapsto 1 + E\,(u - a)^{-1}, \tag{2.9}$$

defines a homomorphism $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$ known as the *evaluation homomorphism*. In terms of generators, $\operatorname{ev}_a : t_{ij}^{(r)} \mapsto E_{ij}\, a^{r-1}$.

For more details on the origins, structure and representations of the Yangian see [21].

The *dual Yangian* $\mathrm{Y}^+(\mathfrak{gl}_N)$ can be defined as the associative algebra with generators $t_{ij}^{(-r)}$, where $1 \leqslant i, j \leqslant N$ and $r = 1, 2, \dots$ subject to the defining relations

$$[t_{ij}^{(-r)}, t_{kl}^{(-s)}] = \delta_{kj}\, t_{il}^{(-r-s)} - \delta_{il}\, t_{kj}^{(-r-s)} + \sum_{a=1}^{\min\{r,s\}} \Big( t_{kj}^{(-r-s+a-1)} t_{il}^{(-a)} - t_{kj}^{(-a)} t_{il}^{(-r-s+a-1)} \Big). \tag{2.10}$$

Combining the generators into the formal power series

$$t^{+}_{ij}(u) = \delta_{ij} - \sum_{r=1}^{\infty} t^{(-r)}_{ij}\, u^{r-1} \in \mathrm{Y}^{+}(\mathfrak{gl}_N)[[u]]$$

we can write the defining relations as

$$(u-v)\,[t^{+}_{ij}(u), t^{+}_{kl}(v)] = t^{+}_{kj}(u)\, t^{+}_{il}(v) - t^{+}_{kj}(v)\, t^{+}_{il}(u) \tag{2.11}$$

which thus take the same form as (2.2). So they are equivalent to

$$R(u-v)\, T^{+}_{1}(u)\, T^{+}_{2}(v) = T^{+}_{2}(v)\, T^{+}_{1}(u)\, R(u-v) \tag{2.12}$$

as in (2.4), where we use the Yang $R$-matrix (2.3) and

$$T^{+}(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t^{+}_{ij}(u) \in \operatorname{End} \mathbb{C}^{N} \otimes \mathrm{Y}^{+}(\mathfrak{gl}_N)[[u]]. \tag{2.13}$$

Consider the ascending filtration on the dual Yangian $\mathrm{Y}^{+}(\mathfrak{gl}_N)$ defined by $\deg t^{(-r)}_{ij} = -r$ for all $r \geqslant 1$. We have the isomorphism for the associated graded algebra

$$\operatorname{gr} \mathrm{Y}^{+}(\mathfrak{gl}_N) \cong \mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big). \tag{2.14}$$

The image $\bar{t}^{(-r)}_{ij}$ of the generator $t^{(-r)}_{ij}$ in the $(-r)$-th component of the graded algebra $\operatorname{gr} \mathrm{Y}^{+}(\mathfrak{gl}_N)$ corresponds to the element $E_{ij}[-r]$ of $\mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)$. The isomorphism relies on the Poincaré–Birkhoff–Witt theorem for $\mathrm{Y}^{+}(\mathfrak{gl}_N)$ which can be proved in a way similar to the Yangian; cf. [21, Ch. 1] and references therein. We will give a more general proof below in the context of the double Yangian which would imply (2.14); see Corollary 2.3. For any nonzero $a \in \mathbb{C}$ the mapping

$$\mathrm{ev}_a : T^{+}(u) \mapsto 1 + E\,(u-a)^{-1}, \tag{2.15}$$

defines the *evaluation homomorphism* $\mathrm{Y}^{+}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$. We assume an expansion into a power series in $u$ so that in terms of generators it takes the form $\mathrm{ev}_a : t^{(-r)}_{ij} \mapsto E_{ij}\, a^{-r}$.

## 2.2 Double Yangian for $\mathfrak{gl}_N$

The *double Yangian* $\mathrm{DY}(\mathfrak{gl}_N)$ for $\mathfrak{gl}_N$ is defined as the associative algebra generated by the central element $C$ and elements $t^{(r)}_{ij}$ and $t^{(-r)}_{ij}$, where $1 \leqslant i,j \leqslant N$ and $r = 1, 2, \dots,$ subject to the defining relations written in terms of the generator matrices (2.5) and (2.13) as follows; see [5], [6] and [16]. They are given by (2.4), (2.12) together with the relation

$$\overline{R}\big(u-v+C/2\big)\, T_1(u)\, T^{+}_{2}(v) = T^{+}_{2}(v)\, T_1(u)\, \overline{R}\big(u-v-C/2\big), \tag{2.16}$$

where

$$\overline{R}(u) = g(u)\,R(u) = g(u)\,\big(1 - P\,u^{-1}\big) \tag{2.17}$$

and

$$g(u) = 1 + \sum_{i=1}^{\infty} g_i\,u^{-i}, \qquad g_i \in \mathbb{C}, \tag{2.18}$$

is a formal power series in $u^{-1}$ whose coefficients are uniquely determined by the relation

$$g(u+N) = g(u)\,(1 - u^{-2}). \tag{2.19}$$

Its first few terms are

$$g(u) = 1 + \frac{1}{N}\,u^{-1} + \frac{N^2+1}{2\,N^2}\,u^{-2} + \frac{N^4+4\,N^2+1}{6\,N^3}\,u^{-3} + \dots.$$

The relation (2.19) ensures that the $R$-matrix $\overline{R}(u) = \overline{R}_{12}(u)$ possesses the *crossing symmetry* properties

$$\big(\overline{R}_{12}(u)^{-1}\big)^{t_1}\,\overline{R}_{12}\big(u+N\big)^{t_1} = 1 \qquad \text{and} \qquad \big(\overline{R}_{12}(u)^{-1}\big)^{t_2}\,\overline{R}_{12}\big(u+N\big)^{t_2} = 1. \tag{2.20}$$

Moreover, the following *unitarity property* holds

$$\overline{R}_{12}(u)\overline{R}_{12}(-u) = 1. \tag{2.21}$$

Indeed, replacing $u$ with $-u-N$ in (2.19) we get

$$g(-u) = g(-u-N)\,\big(1 - (u+N)^{-2}\big)$$

and so

$$g(u)\,g(-u)\,(1-u^{-2}) = g(u+N)\,g(-u-N)\,\big(1-(u+N)^{-2}\big).$$

This means that the series on the left hand side is invariant under the shift $u \mapsto u+N$ which is only possible when

$$g(u)\,g(-u)\,(1-u^{-2}) = 1$$

thus implying (2.21). The series $g(u)$ can be defined equivalently as a unique formal power series of the form (2.18) satisfying the relation

$$g(u)\,g(u+1)\dots g(u+N-1) = \big(1-u^{-1}\big)^{-1}. \tag{2.22}$$

To see the equivalence of the definitions, observe that by (2.19), the series $G(u)$ defined by the left hand side of (2.22) satisfies $G(u+1) = G(u)\,(1-u^{-2})$. However, $G(u)$ is uniquely determined by this relation and so coincides with the right hand side of (2.22).

Given any $c \in \mathbb{C}$ we will introduce the *double Yangian at the level* $c$ as the quotient $\mathrm{DY}_c(\mathfrak{gl}_N)$ of $\mathrm{DY}(\mathfrak{gl}_N)$ by the ideal generated by $C - c$. In particular, we have the natural epimorphism

$$\varphi : \mathrm{DY}(\mathfrak{gl}_N) \to \mathrm{DY}_0(\mathfrak{gl}_N), \qquad C \mapsto 0, \quad t_{ij}^{(r)} \mapsto t_{ij}^{(r)}. \tag{2.23}$$

Equip the Lie algebra $\mathfrak{gl}_N$ with the invariant symmetric bilinear form given by

$$\langle X, Y \rangle = \mathrm{tr}\,(XY) - \frac{1}{N}\,\mathrm{tr}\,X\,\mathrm{tr}\,Y, \qquad X, Y \in \mathfrak{gl}_N.$$

Consider the corresponding affine Kac–Moody algebra $\widehat{\mathfrak{gl}}_N = \mathfrak{gl}_N[t, t^{-1}] \oplus \mathbb{C}K$ defined by the commutation relations

$$\big[E_{ij}[r], E_{kl}[s]\big] = \delta_{kj}\, E_{il}[r+s] - \delta_{il}\, E_{kj}[r+s] + r\,\delta_{r,-s}\, K\Big(\delta_{kj}\,\delta_{il} - \frac{\delta_{ij}\,\delta_{kl}}{N}\Big), \tag{2.24}$$

and the element $K$ is central.

Introduce the ascending filtration on the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ by

$$\deg t_{ij}^{(r)} = r - 1 \qquad \text{and} \qquad \deg t_{ij}^{(-r)} = -r \tag{2.25}$$

for all $r \geqslant 1$; the degree of the central element $C$ is defined to be equal to zero. Denote by $\mathrm{gr}\,\mathrm{DY}(\mathfrak{gl}_N)$ the corresponding graded algebra. We will use the notation $\bar{t}_{ij}^{(r)}$ and $\bar{t}_{ij}^{(-r)}$ for the images of the generators in the respective components of the graded algebra and let $\overline{C}$ be the image of $C$ in the zeroth component.

**Proposition 2.1.** *The assignments*

$$E_{ij}[r-1] \mapsto \bar{t}_{ij}^{(r)}, \qquad E_{ij}[-r] \mapsto \bar{t}_{ij}^{(-r)} \qquad \textit{and} \qquad K \mapsto \overline{C} \tag{2.26}$$

*with* $r \geqslant 1$ *define a homomorphism*

$$\mathrm{U}(\widehat{\mathfrak{gl}}_N) \to \mathrm{gr}\,\mathrm{DY}(\mathfrak{gl}_N). \tag{2.27}$$

*Proof.* As we pointed out in the previous sections, there are homomorphisms

$$\mathrm{U}\big(\mathfrak{gl}_N[t]\big) \to \mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_N) \qquad \text{and} \qquad \mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big) \to \mathrm{gr}\,\mathrm{Y}^+(\mathfrak{gl}_N)$$

which are defined by the assignments (2.26). We will now use the defining relations (2.16) to verify that the generators $\bar{t}_{ij}^{(r)}$ and $\bar{t}_{kl}^{(-s)}$ with $r, s \geqslant 1$ of the graded algebra satisfy the desired relations in $\mathrm{U}(\widehat{\mathfrak{gl}}_N)$. Introduce re-scaled generators of $\mathrm{DY}(\mathfrak{gl}_N)$ by setting

$$\widetilde{t}_{ij}^{(r)} = h^{r-1}\, t_{ij}^{(r)} \qquad \text{and} \qquad \widetilde{t}_{ij}^{(-r)} = h^{-r}\, t_{ij}^{(-r)}$$

for $r \geqslant 1$, where $h$ is a complex-valued parameter. The relations satisfied by $\bar{t}_{ij}^{(r)}$ and $\bar{t}_{kl}^{(-s)}$ in the graded algebra $\operatorname{gr}\mathrm{DY}(\mathfrak{gl}_N)$ will be recovered by calculating the relations between $\widetilde{t}_{ij}^{(r)}$ and $\widetilde{t}_{kl}^{(-s)}$ and then taking the limit as $h \to 0$. Set

$$\widetilde{t}_{ij}(u) = \sum_{r=1}^{\infty} \widetilde{t}_{ij}^{(r)}\, u^{-r} = \frac{1}{h}\Big(t_{ij}\Big(\frac{u}{h}\Big) - \delta_{ij}\Big)$$

and

$$\widetilde{t}_{kl}^{+}(v) = \sum_{s=1}^{\infty} \widetilde{t}_{kl}^{(-s)}\, v^{s-1} = \frac{1}{h}\Big(\delta_{kl} - t_{kl}^{+}\Big(\frac{v}{h}\Big)\Big).$$

Write (2.16) in terms of the generating series:

$$g\big(u-v+C/2\big)\,\Big(t_{ij}(u)\,t_{kl}^{+}(v) - \frac{1}{u-v+C/2}\,t_{kj}(u)\,t_{il}^{+}(v)\Big)$$
$$= g\big(u-v-C/2\big)\,\Big(t_{kl}^{+}(v)\,t_{ij}(u) - \frac{1}{u-v-C/2}\,t_{kj}^{+}(v)\,t_{il}(u)\Big). \qquad (2.28)$$

Note the expansion into a power series in $(u-v)^{-1}$:

$$\frac{g\big(u-v-C/2\big)}{g\big(u-v+C/2\big)} = 1 + \frac{C}{N\,(u-v)^2} + \dots. \qquad (2.29)$$

Now replace $u$ by $u/h$ and $v$ by $v/h$ in (2.28) to get the corresponding relations between the series $\widetilde{t}_{ij}(u)$ and $\widetilde{t}_{kl}^{+}(v)$. We have

$$\big(\delta_{ij} + h\widetilde{t}_{ij}(u)\big)\big(\delta_{kl} - h\widetilde{t}_{kl}^{+}(v)\big) - \frac{h}{u-v+hC/2}\,\big(\delta_{kj} + h\widetilde{t}_{kj}(u)\big)\big(\delta_{il} - h\widetilde{t}_{il}^{+}(v)\big)$$
$$- \Big(\big(\delta_{kl} - h\widetilde{t}_{kl}^{+}(v)\big)\big(\delta_{ij} + h\widetilde{t}_{ij}(u)\big) - \frac{h}{u-v-hC/2}\,\big(\delta_{kj} - h\widetilde{t}_{kj}^{+}(v)\big)\big(\delta_{il} + h\widetilde{t}_{il}(u)\big)\Big)$$
$$\times\Big(1 + \frac{h^2 C}{N\,(u-v)^2} + \dots\Big) = 0.$$

As a power series in $h$, the left hand side is divisible by $h^2$. Hence, dividing by $h^2$ we get the relation modulo $h$,

$$\big[\widetilde{t}_{ij}(u), \widetilde{t}_{kl}^{+}(v)\big] \equiv \frac{1}{u-v}\Big(\delta_{kj}\,\big(\widetilde{t}_{il}(u) + \widetilde{t}_{il}^{+}(v)\big) - \delta_{il}\,\big(\widetilde{t}_{kj}(u) + \widetilde{t}_{kj}^{+}(v)\big)\Big)$$
$$+ \frac{C}{N\,(u-v)^2}\Big(N\,\delta_{kj}\,\delta_{il} - \delta_{ij}\,\delta_{kl}\Big).$$

Thus, taking the coefficients of $u^{-r}v^{s-1}$ with $r, s \geqslant 1$ on both sides, in the limit $h \to 0$ in the graded algebra we get

$$\big[\bar{t}_{ij}^{(r)}, \bar{t}_{kl}^{(-s)}\big] = \begin{cases} \delta_{kj}\, \bar{t}_{il}^{(r-s)} - \delta_{il}\, \bar{t}_{kj}^{(r-s)} + (r-1)\, \delta_{r,s+1}\, \overline{C}\Big(\delta_{kj}\, \delta_{il} - \dfrac{\delta_{ij}\, \delta_{kl}}{N}\Big) & \text{if} \quad r > s, \\ \delta_{kj}\, \bar{t}_{il}^{(r-s-1)} - \delta_{il}\, \bar{t}_{kj}^{(r-s-1)} + (r-1)\, \delta_{r,s+1}\, \overline{C}\Big(\delta_{kj}\, \delta_{il} - \dfrac{\delta_{ij}\, \delta_{kl}}{N}\Big) & \text{if} \quad r \leqslant s. \end{cases}$$

Comparing with (2.24), we may conclude that the assignments (2.26) define a homomorphism (2.27). $\square$

To equip the double Yangian with a Hopf algebra structure, we will need to use shifts $u \mapsto u + a$ of the variable $u$. They are well-defined for the generator series $t_{ij}(u)$ but not for $t_{ij}^{+}(u)$. So we will consider the completion $\widehat{\mathrm{Y}}^{+}(\mathfrak{gl}_N)$ of the dual Yangian with respect to the *descending* filtration defined by setting the degree of $t_{ij}^{(-r)}$ with $r \geqslant 1$ to be equal to $r$. By the defining relations, the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ is spanned over $\mathbb{C}[C]$ by the products $xy$ with $x \in \mathrm{Y}^{+}(\mathfrak{gl}_N)$ and $y \in \mathrm{Y}(\mathfrak{gl}_N)$. This follows by an easy induction based on the relation obtained by swapping the indices $i$ and $k$ in (2.28) and solving the system of equations for $t_{ij}(u)\, t_{kl}^{+}(v)$ and $t_{kj}(u)\, t_{il}^{+}(v)$. The *extended double Yangian* $\mathrm{DY}^{\circ}(\mathfrak{gl}_N)$ can now be defined as the space of finite $\mathbb{C}[C]$-linear combinations of all products of the form $xy$ with $x \in \widehat{\mathrm{Y}}^{+}(\mathfrak{gl}_N)$ and $y \in \mathrm{Y}(\mathfrak{gl}_N)$ with the multiplication extended by continuity from the double Yangian.

The Hopf algebra structure on $\mathrm{DY}^{\circ}(\mathfrak{gl}_N)$ is defined by the coproduct

$$\begin{aligned} &\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{N} t_{ik}\big(u + C_2/4\big) \otimes t_{kj}\big(u - C_1/4\big), \\ &\Delta : t_{ij}^{+}(u) \mapsto \sum_{k=1}^{N} t_{ik}^{+}\big(u - C_2/4\big) \otimes t_{kj}^{+}\big(u + C_1/4\big), \\ &\Delta : C \mapsto C \otimes 1 + 1 \otimes C, \end{aligned}$$

where $C_1 = C \otimes 1$ and $C_2 = 1 \otimes C$; the antipode

$$S : T(u) \mapsto T(u)^{-1}, \qquad S : T^{+}(u) \mapsto T^{+}(u)^{-1}, \qquad S : C \mapsto -C;$$

and the counit

$$\varepsilon : T(u) \mapsto 1, \qquad \varepsilon : T^{+}(u) \mapsto 1, \qquad \varepsilon : C \mapsto 0.$$

We are now in a position to prove the Poincaré–Birkhoff–Witt theorem for the double Yangian. We will do this for a particular ordering $\prec$ on the set of generators. Observe that due to the defining relations,

$$\big[t_{ij}^{(r)}, t_{ij}^{(s)}\big] = 0 \qquad \text{and} \qquad \big[t_{ij}^{(-r)}, t_{ij}^{(-s)}\big] = 0$$

for all $r, s \geqslant 1$. Hence, a total ordering $\prec$ on the series $t_{ij}(u)$ and $t^+_{ij}(u)$ will induce a well-defined total ordering on the generators (with the central element $C$ included in the ordering in an arbitrary way). We set $t^+_{ij}(u) \prec t_{kl}(u)$ for all $i, j, k, l$. Furthermore, set $t^+_{ij}(u) \prec t^+_{kl}(u)$ and $t_{ij}(u) \prec t_{kl}(u)$ if and only if $(i, j) \prec (k, l)$ in the lexicographical order.

**Theorem 2.2.** *Any element of the algebra* $\mathrm{DY}(\mathfrak{gl}_N)$ *can be written uniquely as a linear combination of ordered monomials in the generators.*

*Proof.* It follows by an easy induction from the defining relations (2.2), (2.11) and (2.28) that the ordered monomials span the algebra $\mathrm{DY}(\mathfrak{gl}_N)$.

The next step is to demonstrate that the ordered monomials are linearly independent. We consider the level zero algebra $\mathrm{DY}_0(\mathfrak{gl}_N)$ first and follow an idea used by Etingof and Kazhdan [4, Proposition 3.15] and by Nazarov [26, Proposition 2.2]. It is based on the existence of the evaluation modules for $\mathrm{DY}_0(\mathfrak{gl}_N)$: for each nonzero $a \in \mathbb{C}$ we have the representation defined by

$$\pi_a : \mathrm{DY}_0(\mathfrak{gl}_N) \to \mathrm{End}\,\mathbb{C}^N, \qquad t^{(r)}_{ij} \mapsto a^{r-1} e_{ij}, \quad t^{(-r)}_{ij} \mapsto a^{-r} e_{ij}$$

for all $r \geqslant 1$. If there is a nontrivial linear combination of ordered monomials equal to zero, we employ the coproduct on $\mathrm{DY}_0(\mathfrak{gl}_N)$ to conclude that the image of this linear combination is zero under any representation $\pi_{a_1} \otimes \ldots \otimes \pi_{a_l}$ with nonzero parameters $a_i$. This leads to a contradiction exactly as in [26] by considering the top degree components of all monomials with respect to the filtration defined by (2.25) and by employing associated evaluation modules over $\mathrm{U}\big(\mathfrak{gl}_N[t, t^{-1}]\big)$ as implied by Proposition 2.1.

To show that ordered monomials are linearly independent in $\mathrm{DY}(\mathfrak{gl}_N)$, observe that $C \neq 0$ due to the existence of the level 1 representations. Here we rely on the work by Iohara [16] providing such representations in terms of the Drinfeld presentation of $\mathrm{DY}(\mathfrak{gl}_N)$ to be written in terms of the $RTT$ presentation via the Ding–Frenkel isomorphism. Now prove by the induction on $k \geqslant 1$ that the powers $1, C, \ldots, C^k$ are linearly independent. Suppose that

$$d_k C^k + \cdots + d_1 C + d_0 = 0, \qquad d_i \in \mathbb{C}, \quad d_k \neq 0.$$

By applying the homomorphism $\varphi$ defined in (2.23) we find that $d_0 = 0$. If $k = 1$ then this makes a contradiction since $C \neq 0$. Now suppose that $k \geqslant 2$ and apply the coproduct map $\Delta$ to get

$$d_k (C \otimes 1 + 1 \otimes C)^k + \cdots + d_1 (C \otimes 1 + 1 \otimes C) = 0.$$

This simplifies to

$$d_k (k C^{k-1} \otimes C + \cdots + k C \otimes C^{k-1}) + \cdots + 2 d_2 C \otimes C = 0.$$

However, this is impossible since the powers $1, C, \ldots, C^{k-1}$ are linearly independent by the induction hypothesis. Now suppose that a linear combination of ordered monomials is zero,

$$A(C)+\sum A^{r_1\dots r_p}_{i_1j_1\dots i_pj_p}(C)\,t^{(r_1)}_{i_1j_1}\dots t^{(r_p)}_{i_pj_p}=0, \tag{2.30}$$

where the summation is over a finite nonempty set of indices, $A(C)$ is a polynomial in $C$, and the coefficients $A^{r_1\dots r_p}_{i_1j_1\dots i_pj_p}(C)$ are nonzero polynomials in $C$. Regarding $\mathrm{DY}(\mathfrak{gl}_N)$ as a subalgebra of $\mathrm{DY}^\circ(\mathfrak{gl}_N)$, apply the homomorphism $\psi=(\mathrm{id}\otimes\varphi)\,\Delta$ to its elements. The action on polynomials in $C$ is given by

$$\psi: B(C)\mapsto B(C)\otimes 1,$$

whereas the images of the generators $t^{(r)}_{ij}$ under $\psi$ are found from the expansions

$$t_{ij}(u)\mapsto\sum_{k=1}^N t_{ik}(u)\otimes t_{kj}\big(u-C_1/4\big)\qquad\text{and}\qquad t^+_{ij}(u)\mapsto\sum_{k=1}^N t^+_{ik}(u)\otimes t^+_{kj}\big(u+C_1/4\big).$$

Let $p_0$ be the maximum length of the monomials occurring in the linear combination in (2.30) and let $r_0$ be the maximum degree among the monomials of length $p_0$. Now apply the homomorphism $\psi$ to the left hand side of (2.30) and use the defining relations of the double Yangian to write the image as a (possibly infinite) linear combination of products of the form $x\otimes y$, where $x$ and $y$ are ordered monomials in the generators. The defining relations and coproduct formulas imply that the part of this linear combination containing the monomials $y$ of length $p_0$ and degree $r_0$ has the form

$$\sum A^{r_1\dots r_p}_{i_1j_1\dots i_pj_p}(C)\otimes t^{(r_1)}_{i_1j_1}\dots t^{(r_p)}_{i_pj_p}, \tag{2.31}$$

where $p=p_0$ and the sum of the degrees of the generators is equal to $r_0$. On the other hand, the ordered monomials $t^{(r_1)}_{i_1j_1}\dots t^{(r_p)}_{i_pj_p}$ are linearly independent in $\mathrm{DY}(\mathfrak{gl}_N)$ over $\mathbb{C}$, as follows from the Poincaré–Birkhoff–Witt theorem for the level zero algebra $\mathrm{DY}_0(\mathfrak{gl}_N)$ by the application of the homomorphism (2.23). This implies that the coefficients $A^{r_1\dots r_p}_{i_1j_1\dots i_pj_p}(C)$ in (2.31) must be zero, thus making a contradiction with the assumptions in (2.30). Therefore, all ordered monomials are linearly independent. □

**Corollary 2.3.** *The homomorphism* (2.27) *is injective and so it defines an isomorphism*

$$\mathrm{U}(\widehat{\mathfrak{gl}}_N)\cong \mathrm{gr}\,\mathrm{DY}(\mathfrak{gl}_N). \tag{2.32}$$

*Proof.* This is immediate from Theorem 2.2 and the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{U}(\widehat{\mathfrak{gl}}_N)$. □

## 2.3 Invariants of the extended vacuum module

Theorem 2.2 implies the vector space decomposition for the extended double Yangian as a $\mathbb{C}[C]$-module,

$$\mathrm{DY}^\circ(\mathfrak{gl}_N)\cong \widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)\otimes \mathrm{Y}(\mathfrak{gl}_N). \tag{2.33}$$

Introduce the *extended vacuum module* $\widehat{\mathcal{V}}_c(\mathfrak{gl}_N)$ *at the level* $c$ as the quotient of the algebra $\mathrm{DY}^\circ(\mathfrak{gl}_N)$ by the left ideal generated by $C-c$ and all elements $t_{ij}^{(r)}$ with $r \geqslant 1$. We let $\mathbf{1}$ denote the image of 1 in the quotient. As a vector space, $\widehat{\mathcal{V}}_c(\mathfrak{gl}_N)$ is isomorphic to the completed dual Yangian $\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)$ due to the decomposition (2.33).

Now assume that the level is critical, $c=-N$, and set $\widehat{\mathcal{V}}_{\text{cri}} = \widehat{\mathcal{V}}_{-N}(\mathfrak{gl}_N)$. Introduce the subspace of $\mathrm{Y}(\mathfrak{gl}_N)$-invariants by

$$\mathfrak{z}(\widehat{\mathcal{V}}_{\text{cri}}) = \{v \in \widehat{\mathcal{V}}_{\text{cri}} \mid t_{ij}(u)v = \delta_{ij}v\}, \tag{2.34}$$

so that any element of $\mathfrak{z}(\widehat{\mathcal{V}}_{\text{cri}})$ is annihilated by all operators $t_{ij}^{(r)}$ with $r \geqslant 1$. We will discuss the structure of the space $\mathfrak{z}(\widehat{\mathcal{V}}_{\text{cri}})$ below in Sec. 4.4 in the context of quantum vertex algebra structure on $\mathcal{V}_{-N}(\mathfrak{gl}_N)$. In particular, we will see that $\mathfrak{z}(\widehat{\mathcal{V}}_{\text{cri}})$ is a commutative associative algebra which can be identified with a subalgebra of the completed dual Yangian $\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)$. Our goal in this section is to construct some families of elements of $\mathfrak{z}(\widehat{\mathcal{V}}_{\text{cri}})$.

We will work with the tensor product algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^N}_{m} \otimes \widehat{\mathrm{Y}}^+(\mathfrak{gl}_N) \tag{2.35}$$

and introduce the rational function in variables $u_1, \ldots, u_m$ with values in (2.35) (with the identity component in $\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)$) by

$$R(u_1, \ldots, u_m) = \prod_{1 \leqslant a < b \leqslant m} R_{ab}(u_a - u_b), \tag{2.36}$$

where the product is taken in the lexicographical order on the set of pairs $(a, b)$. We point out the identity

$$R(u_1, \ldots, u_m) T_1^+(u_1) \ldots T_m^+(u_m) = T_m^+(u_m) \ldots T_1^+(u_1) R(u_1, \ldots, u_m), \tag{2.37}$$

implied by a repeated application of (2.12).

Suppose that $\mu$ is a Young diagram with $m$ boxes whose length does not exceed $N$. For a standard $\mu$-tableau $\mathcal{U}$ with entries in $\{1, \ldots, m\}$ introduce the contents $c_a = c_a(\mathcal{U})$ for $a = 1, \ldots, m$ so that $c_a = j - i$ if $a$ occupies the box $(i, j)$ in $\mathcal{U}$. Let $e_{\mathcal{U}} \in \mathbb{C}[\mathfrak{S}_m]$ be the primitive idempotent associated with $\mathcal{U}$ through the use of the orthonormal Young bases in the irreducible representations of $\mathfrak{S}_m$. The symmetric group $\mathfrak{S}_m$ acts by permuting the tensor factors in $(\mathbb{C}^N)^{\otimes m}$. Denote by $\mathcal{E}_{\mathcal{U}}$ the image of $e_{\mathcal{U}}$ under this action. We will need an expression for $\mathcal{E}_{\mathcal{U}}$ provided by the fusion procedure originated in [17]; see also [21, Sec. 6.4] for more details and references. By a version of the procedure, the consecutive evaluations of the function $R(u_1, \ldots, u_m)$ are well-defined and the result is proportional to $\mathcal{E}_{\mathcal{U}}$,

$$R(u_1, \ldots, u_m)\big|_{u_1=c_1}\big|_{u_2=c_2} \ldots \big|_{u_m=c_m} = h(\mu)\, \mathcal{E}_{\mathcal{U}}, \tag{2.38}$$

where $h(\mu)$ is the product of all hook lengths of the boxes of $\mu$.

Using the tensor product algebra (2.35), set

$$\mathbb{T}^+_\mu(u) = \mathrm{tr}_{1,\dots,m}\, \mathcal{E}_{\mathcal{U}}\, T^+_1(u+c_1)\dots T^+_m(u+c_m). \tag{2.39}$$

This is a power series in $u$ whose coefficients are elements of the completed dual Yangian $\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)$. The series (2.39) can be regarded as a Yangian extension of the quantum immanants of [27]. In particular, by the argument of [27, Sec. 3.4], this series does not depend on the standard tableau $\mathcal{U}$ of shape $\mu$ thus justifying the notation.

**Theorem 2.4.** *All coefficients of the series $\mathbb{T}^+_\mu(u)\mathbf{1}$ belong to the subspace of invariants $\mathfrak{z}(\widehat{\mathcal{V}}_{cri})$ of the extended vacuum module.*

*Proof.* Consider the tensor product algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^N\otimes\dots\otimes\mathrm{End}\,\mathbb{C}^N}_{m+1}\otimes\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N) \tag{2.40}$$

with the $m+1$ copies of $\mathrm{End}\,\mathbb{C}^N$ labeled by $0,1,\dots,m$. We need to verify the identity

$$T_0(z)\,\mathbb{T}^+_\mu(u)\mathbf{1} = \mathbb{T}^+_\mu(u)\mathbf{1}, \tag{2.41}$$

where we identify the vector spaces $\widehat{\mathcal{V}}_{\mathrm{cri}}\cong\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)$. By the defining relations (2.16), for all $a=1,\dots,m$ we can write

$$T_0(z)\,T^+_a(u+c_a) = \overline{R}_{0a}(z-u-c_a-N/2)^{-1}\,T^+_a(u+c_a)\,T_0(z)\,\overline{R}_{0a}(z-u-c_a+N/2).$$

Hence, suppressing the arguments of the $R$-matrices we get

$$\begin{aligned}
&T_0(z)\,\mathrm{tr}_{1,\dots,m}\,\mathcal{E}_{\mathcal{U}}\,T^+_1(u+c_1)\dots T^+_m(u+c_m)\mathbf{1}\\
&\qquad= \mathrm{tr}_{1,\dots,m}\,\mathcal{E}_{\mathcal{U}}\,\overline{R}_{01}^{\,-1}\dots\overline{R}_{0m}^{\,-1}\,T^+_1(u+c_1)\dots T^+_m(u+c_m)\,T_0(z)\,\overline{R}_{0m}\dots\overline{R}_{01}\mathbf{1}\\
&\qquad= \mathrm{tr}_{1,\dots,m}\,\mathcal{E}_{\mathcal{U}}\,\overline{R}_{01}^{\,-1}\dots\overline{R}_{0m}^{\,-1}\,T^+_1(u+c_1)\dots T^+_m(u+c_m)\,\overline{R}_{0m}\dots\overline{R}_{01}\mathbf{1},
\end{aligned}$$

where the last equality holds since $T_0(z)$ acts as the identity operator on the subspace $\mathrm{End}\,(\mathbb{C}^N)^{\otimes(m+1)}\otimes\mathbf{1}$. Using the notation (2.36) we get

$$\begin{aligned}
&R(u_1,\dots,u_m)\,R_{0m}(u_0-u_m)\dots R_{01}(u_0-u_1)\\
&\qquad= R_{01}(u_0-u_1)\dots R_{0m}(u_0-u_m)\,R(u_1,\dots,u_m),
\end{aligned} \tag{2.42}$$

where $u_0$ is another variable. This follows by a repeated application of the Yang–Baxter equation

$$R_{12}(u)\,R_{13}(u+v)\,R_{23}(v) = R_{23}(v)\,R_{13}(u+v)\,R_{12}(u) \tag{2.43}$$

satisfied by the Yang $R$-matrix. Relation (2.42) will remain valid if each factor $R_{0a}(u_0-u_a)$ is replaced with $\overline{R}_{0a}(u_0-u_a)$. Hence, by the fusion procedure (2.38), the consecutive evaluations $u_a=c_a$ for $a=1,\dots,m$ imply

$$\mathcal{E}_{\mathcal{U}}\,\overline{R}_{0m}(u_0-c_m)\dots\overline{R}_{01}(u_0-c_1)=\overline{R}_{01}(u_0-c_1)\dots\overline{R}_{0m}(u_0-c_m)\,\mathcal{E}_{\mathcal{U}}.$$

By inverting the $R$-matrices we also get

$$\mathcal{E}_{\mathcal{U}}\,\overline{R}_{01}(u_0-c_1)^{-1}\dots\overline{R}_{0m}(u_0-c_m)^{-1}=\overline{R}_{0m}(u_0-c_m)^{-1}\dots\overline{R}_{01}(u_0-c_1)^{-1}\,\mathcal{E}_{\mathcal{U}}.$$

Returning now to the calculation of $T_0(z)\,\mathbb{T}^+_\mu(u)\,\mathbf{1}$, recall that $\mathcal{E}_{\mathcal{U}}$ is an idempotent and use the cyclic property of trace together with (2.37) and (2.38) to write

$$\begin{aligned}\mathrm{tr}_{1,\dots,m}\,\mathcal{E}_{\mathcal{U}}\,X\,Y&=\mathrm{tr}_{1,\dots,m}\,X^\circ\,\mathcal{E}_{\mathcal{U}}\,Y=\mathrm{tr}_{1,\dots,m}\,X^\circ\,\mathcal{E}_{\mathcal{U}}^2\,Y\\&=\mathrm{tr}_{1,\dots,m}\,\mathcal{E}_{\mathcal{U}}\,X\,Y^\circ\,\mathcal{E}_{\mathcal{U}}=\mathrm{tr}_{1,\dots,m}\,X\,Y^\circ\,\mathcal{E}_{\mathcal{U}}^2=\mathrm{tr}_{1,\dots,m}\,X\,Y^\circ\,\mathcal{E}_{\mathcal{U}}=\mathrm{tr}_{1,\dots,m}\,X\,\mathcal{E}_{\mathcal{U}}\,Y,\end{aligned}$$

where we set

$$X=\overline{R}{}^{-1}_{01}\dots\overline{R}{}^{-1}_{0m},\qquad Y=T^+_1(u+c_1)\dots T^+_m(u+c_m)\,\overline{R}_{0m}\dots\overline{R}_{01}$$

and used the notation $X^\circ$ and $Y^\circ$ for the same products written in the opposite order. Thus, we can write

$$T_0(z)\,\mathbb{T}^+_\mu(u)\,\mathbf{1}=\mathrm{tr}_{1,\dots,m}\,X\,\mathcal{E}_{\mathcal{U}}\,Y\,\mathbf{1}=\mathrm{tr}_{1,\dots,m}\,X^{t_1\dots t_m}\big(\mathcal{E}_{\mathcal{U}}\,Y\big)^{t_1\dots t_m}\,\mathbf{1}.$$

We have

$$\big(\mathcal{E}_{\mathcal{U}}\,Y\big)^{t_1\dots t_m}=\overline{R}{}^{t_m}_{0m}\dots\overline{R}{}^{t_1}_{01}\big(\mathcal{E}_{\mathcal{U}}\,T^+_1(u+c_1)\dots T^+_m(u+c_m)\big)^{t_1\dots t_m}$$

and

$$X^{t_1\dots t_m}=\Big(\overline{R}{}^{-1}_{01}\Big)^{t_1}\dots\Big(\overline{R}{}^{-1}_{0m}\Big)^{t_m}.$$

By the crossing symmetry (2.20), we have

$$\Big(\overline{R}{}^{-1}_{0a}\Big)^{t_a}\,\overline{R}{}^{t_a}_{0a}=1$$

for all $a=1,\dots,m$ and so

$$T_0(z)\,\mathbb{T}^+_\mu(u)\,\mathbf{1}=\mathrm{tr}_{1,\dots,m}\big(\mathcal{E}_{\mathcal{U}}\,T^+_1(u+c_1)\dots T^+_m(u+c_m)\big)^{t_1\dots t_m}\,\mathbf{1}=\mathbb{T}^+_\mu(u)\,\mathbf{1}$$

as required. $\square$

Note two important particular cases of Theorem 2.4 where $\mu$ is a row or column diagram. In each case there is a unique standard tableau $\mathcal{U}$, and the corresponding idempotents $\mathcal{E}_{\mathcal{U}}$ coincide with the respective images $H^{(m)}$ and $A^{(m)}$ of the symmetrizer and anti-symmetrizer

$$h^{(m)}=\frac{1}{m!}\sum_{s\in\mathfrak{S}_m}s\qquad\text{and}\qquad a^{(m)}=\frac{1}{m!}\sum_{s\in\mathfrak{S}_m}\mathrm{sgn}\,s\cdot s$$

under the action of $\mathfrak{S}_m$ on $(\mathbb{C}^N)^{\otimes m}$.

**Corollary 2.5.** *All coefficients of the series*

$$\mathrm{tr}_{1,\dots,m}\, H^{(m)}\, T_1^+(u)\dots T_m^+(u+m-1) \qquad \text{and} \qquad \mathrm{tr}_{1,\dots,m}\, A^{(m)}\, T_1^+(u)\dots T_m^+(u-m+1)$$

*belong to the subspace of invariants* $\mathfrak{z}(\widehat{\mathcal{V}}_{cri})$ *of the extended vacuum module.* □

In the particular case $m = N$ the second series coincides with the *quantum determinant* $\mathrm{qdet}\, T^+(u)$ of the matrix $T^+(u)$; see also Proposition 2.8 below.

One more family of elements of $\mathfrak{z}(\widehat{\mathcal{V}}_{\mathrm{cri}})$ can be constructed by making use of the well-known fact that the matrix $M = T^+(u)e^{-\partial_u}$ with entries in the extended algebra $\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)[[u,\partial_u]]$ is a *Manin matrix*; see [1]. Namely, by the Newton identity [1, Theorem 4], we have

$$\partial_z\, \mathrm{cdet}(1+z\,M) = \mathrm{cdet}(1+z\,M)\sum_{m=0}^{\infty}(-z)^m\, \mathrm{tr}\, M^{m+1}, \tag{2.44}$$

where

$$\mathrm{cdet}(1+z\,M) = \sum_{m=0}^{N} z^m\, \mathrm{tr}_{1,\dots,m}\, A^{(m)} M_1\dots M_m.$$

**Corollary 2.6.** *All coefficients of the series*

$$\mathrm{tr}\, T^+(u)\dots T^+(u-m+1), \qquad m \geqslant 1,$$

*belong to the subspace of invariants* $\mathfrak{z}(\widehat{\mathcal{V}}_{cri})$ *of the extended vacuum module.*

*Proof.* Note that

$$M^m = T^+(u)\dots T^+(u-m+1)\, e^{-m\,\partial_u}$$

so that the claim follows from (2.44). □

*Remark* 2.7. The MacMahon Master Theorem for Manin matrices [15] implies a relationship between the two families of Corollary 2.5:

$$\big[\mathrm{cdet}(1-z\,M)\big]^{-1} = \sum_{m=0}^{\infty} z^m\, \mathrm{tr}_{1,\dots,m}\, H^{(m)} M_1\dots M_m \tag{2.45}$$

for $M = T^+(u)e^{-\partial_u}$; see also [24] for another proof and a super-extension. □

We have the following well-known properties of quantum determinants.

**Proposition 2.8.** *The coefficients of the quantum determinants*

$$\mathrm{qdet}\, T(u) = \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}\,\sigma\cdot t_{\sigma(1)\,1}(u)\dots t_{\sigma(N)\,N}(u-N+1), \tag{2.46}$$

$$\mathrm{qdet}\, T^+(u) = \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}\,\sigma\cdot t^+_{\sigma(1)\,1}(u)\dots t^+_{\sigma(N)\,N}(u-N+1), \tag{2.47}$$

*belong to the center of the extended double Yangian* $\mathrm{DY}^\circ(\mathfrak{gl}_N)$.[1]

*Proof.* The respective coefficients of $\operatorname{qdet} T(u)$ and $\operatorname{qdet} T^+(u)$ are central in the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ and completed dual Yangian $\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)$; see e.g. [21, Ch. 1]. Furthermore, in the algebra (2.35) with $m = N$ we have

$$A^{(N)} T_1^+(u) \dots T_N^+(u - N + 1) = A^{(N)} \operatorname{qdet} T^+(u). \tag{2.48}$$

Arguing as in the beginning of the proof of Theorem 2.4, and keeping the notation, we find

$$\begin{aligned} &T_0(z)\, A^{(N)} T_1^+(u) \dots T_N^+(u - N + 1) \\ &\qquad = A^{(N)}\, \overline{R}_{01}^{\,-1} \dots \overline{R}_{0N}^{\,-1}\, T_1^+(u) \dots T_N^+(u - N + 1)\, T_0(z)\, \overline{R}_{0N} \dots \overline{R}_{01}. \end{aligned}$$

Now use the identity

$$A^{(N)}\, \overline{R}_{0N}(v + N - 1) \dots \overline{R}_{01}(v) = A^{(N)}.$$

It is implied by (2.22) and the following property of the Yang $R$-matrix (2.3)

$$A^{(N)}\, R_{0N}(v + N - 1) \dots R_{01}(v) = A^{(N)}\, (1 - v^{-1});$$

see e.g. [21, Ch. 1]. This proves that $T_0(z)$ commutes with $\operatorname{qdet} T^+(u)$. By the same calculation, $T_0^+(z)$ commutes with $\operatorname{qdet} T(u)$. $\square$

Recall that the *vacuum module at the critical level* $V_{\mathrm{cri}} = V_{-N}(\mathfrak{gl}_N)$ over the affine Kac–Moody algebra $\widehat{\mathfrak{gl}}_N$ is defined as the quotient of $\mathrm{U}(\widehat{\mathfrak{gl}}_N)$ by the left ideal generated by $\mathfrak{gl}_N[t]$ and $K + N$. The *Feigin–Frenkel center* is the subspace $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$ of invariants

$$\mathfrak{z}(\widehat{\mathfrak{gl}}_N) = \{ v \in V_{\mathrm{cri}} \mid \mathfrak{gl}_N[t]\, v = 0 \}. \tag{2.49}$$

This subspace is a commutative associative algebra which can be identified with a subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)$. By a theorem of Feigin and Frenkel [7], $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$ is an algebra of polynomials in infinitely many variables; see [11] for a detailed exposition of these results.

Our goal now is to use a classical limit to reproduce a construction of elements of $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$; cf. [29]. By Theorem 2.2, we can regard elements of the completed dual Yangian $\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)$ as infinite linear combinations

$$\sum A^{r_1 \dots r_p}_{i_1 j_1 \dots i_p j_p}\, t^{(r_1)}_{i_1 j_1} \dots t^{(r_p)}_{i_p j_p}$$

[1]The corresponding result for the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ stated in [10, Lemma 4.3] holds for an arbitrary level as well (not just for the critical level). This follows from the property $f(x)f(xq^2)\dots f(xq^{2n-2}) = (1-x)/(1-xq^{2n-2})$ of the series $f(x)$ used in the proof of that lemma.

of ordered monomials over $\mathbb{C}$, where all $r_i$ are negative integers. This shows that the isomorphism (2.14) will hold true in the same form, if we replace $\mathrm{Y}^+(\mathfrak{gl}_N)$ with the completed dual Yangian $\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)$ equipped with the inherited ascending filtration defined by $\deg t_{ij}^{(-r)} = -r$; see Corollary 2.3. Moreover, for any element $S \in \mathfrak{z}(\widehat{\mathcal{V}}_{\mathrm{cri}})$ its image $\overline{S}$ in the graded algebra, regarded as an element of $\mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)$, belongs to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$. We will use Corollary 2.5 to construct appropriate linear combinations of elements of $\mathfrak{z}(\widehat{\mathcal{V}}_{\mathrm{cri}})$ whose graded images will be generators of $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$.

Extend the ascending filtration on the completed dual Yangian to the algebra of formal series $\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)[[u, \partial_u]]$ by setting $\deg u = 1$ and $\deg \partial_u = -1$ so that the associated graded algebra is isomorphic to $\mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)[[u, \partial_u]]$. Then the element

$$\mathrm{tr}_{1,\dots,m} A^{(m)} \big(1 - T_1^+(u)\, e^{-\partial_u}\big) \dots \big(1 - T_m^+(u)\, e^{-\partial_u}\big) \tag{2.50}$$

has degree $-m$ and its image in the graded algebra coincides with

$$\mathrm{tr}_{1,\dots,m} A^{(m)} \big(\partial_u + E_+(u)_1\big) \dots \big(\partial_u + E_+(u)_m\big), \tag{2.51}$$

where

$$E_+(u) = \sum_{r=1}^{\infty} E[-r]\, u^{r-1}. \tag{2.52}$$

On the other hand, the element (2.50) equals

$$\mathrm{tr}_{1,\dots,m} A^{(m)} \sum_{k=0}^{m} \sum_{1 \leqslant i_1 < \dots < i_k \leqslant m} (-1)^k\, T_{i_1}^+(u) \dots T_{i_k}^+(u - k + 1)\, e^{-k\,\partial_u}.$$

Transform this expression by applying conjugations by suitable elements of $\mathfrak{S}_m$ and using the cyclic property of trace to bring it to the form

$$\mathrm{tr}_{1,\dots,m} A^{(m)} \sum_{k=0}^{m} (-1)^k \binom{m}{k}\, T_1^+(u) \dots T_k^+(u - k + 1)\, e^{-k\,\partial_u}.$$

Calculating partial traces of the anti-symmetrizer, we can write this as

$$\sum_{k=0}^{m} (-1)^k \binom{N-k}{m-k} \mathrm{tr}_{1,\dots,k} A^{(k)}\, T_1^+(u) \dots T_k^+(u - k + 1)\, e^{-k\,\partial_u}. \tag{2.53}$$

By Corollary 2.5, we can conclude that all coefficients of (2.51) belong to $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$.

Together with a similar argument for the other two families of invariants in Corollaries 2.5 and 2.6, we thus reproduce the following result on generators of $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$ from [2], [3] and [24]; see Corollary 2.9 below. Alternatively, for those two families it can also be derived with the use of the observation that $M = \partial_u + E_+(u)$ is a Manin matrix and applying (2.44) and (2.45). Introduce the power series $\phi_{m\,a}(u)$, $\psi_{m\,a}(u)$ and $\theta_{m\,a}(u)$ by the expansions:

$$\mathrm{tr}_{1,\dots,m}\, A^{(m)}\big(\partial_u + E_+(u)_1\big)\dots\big(\partial_u + E_+(u)_m\big) = \phi_{m\,0}(u)\,\partial_u^m + \dots + \phi_{m\,m}(u), \tag{2.54}$$

$$\mathrm{tr}_{1,\dots,m}\, H^{(m)}\big(\partial_u + E_+(u)_1\big)\dots\big(\partial_u + E_+(u)_m\big) = \psi_{m\,0}(u)\,\partial_u^m + \dots + \psi_{m\,m}(u), \tag{2.55}$$

and

$$\mathrm{tr}\,\big(\partial_u + E_+(u)\big)^m = \theta_{m\,0}(u)\,\partial_u^m + \theta_{m\,1}(u)\,\partial_u^{m-1} + \dots + \theta_{m\,m}(u). \tag{2.56}$$

Define their coefficients by

$$\phi_{m\,m}(u) = \sum_{r=0}^{\infty} \phi_{m\,m}^{(r)}\, u^r, \qquad \psi_{m\,m}(u) = \sum_{r=0}^{\infty} \psi_{m\,m}^{(r)}\, u^r \qquad \text{and} \qquad \theta_{m\,m}(u) = \sum_{r=0}^{\infty} \theta_{m\,m}^{(r)}\, u^r.$$

**Corollary 2.9.** *Each family $\phi_{m\,m}^{(r)}$, $\psi_{m\,m}^{(r)}$ and $\theta_{m\,m}^{(r)}$ with $m = 1, \dots, N$ and $r = 0, 1, \dots$ is algebraically independent and generates the algebra $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$.*

*Proof.* The algebraic independence follows by considering the symbols of the elements in the symmetric algebra as in [11, Ch. 3]. □

# 3 Quantum vertex algebras

We will follow [6] to introduce quantum vertex algebras. We will be most concerned with the center of a quantum vertex algebra which we introduce by analogy with vertex algebra theory. Our goal is to use the constructions of invariants of the extended vacuum module given in Sec. 2.3 to describe the structure of the center; see Sec. 4 below.

## 3.1 Definition and basic properties

Let $h$ be a formal parameter, $V_0$ a complex vector space and $V = V_0[[h]]$ a topologically free $\mathbb{C}[[h]]$-module. Denote by $V_h((z))$ the space of all Laurent series

$$v(z) = \sum_{r\in\mathbb{Z}} v_r z^{-r-1} \in V[[z^{\pm 1}]]$$

satisfying $v_r \to 0$ as $r \to \infty$, in the $h$-adic topology. More precisely, $V_h((z))$ consists of all Laurent series $v(z)$ satisfying the following condition: for every $n \in \mathbb{Z}_{\geqslant 0}$ there exists $s \in \mathbb{Z}$ such that $r \geqslant s$ implies $v_r \in h^n V$. Note that the space $V_h((z))$ can be identified with $V_0((z))[[h]]$.

**Definition 3.1.** Let $V$ be a topologically free $\mathbb{C}[[h]]$-module. A *quantum vertex algebra* $V$ over $\mathbb{C}[[h]]$ is the following data.

(a) A $\mathbb{C}[[h]]$-module map (the *vertex operators*)

$$Y: V \otimes V \to V_h((z)), \qquad v \otimes w \mapsto Y(z)(v \otimes w). \tag{3.1}$$

For any $v \in V$ the map $Y(v,z): V \to V_h((z))$ is then defined by

$$Y(v,z)w = Y(z)(v \otimes w)$$

and which satisfies the *weak associativity property*: for any $u, v, w \in V$ and $n \in \mathbb{Z}_{\geqslant 0}$ there exists $\ell \in \mathbb{Z}_{\geqslant 0}$ such that

$$(z_0+z_2)^\ell\, Y(v, z_0+z_2)Y(w,z_2)\,u - (z_0+z_2)^\ell\, Y\big(Y(v,z_0)w, z_2\big)\,u \in h^n V[[z_0^{\pm 1}, z_2^{\pm 1}]]. \tag{3.2}$$

(b) A vector $\mathbf{1} \in V$ (the *vacuum vector*) which satisfies

$$Y(\mathbf{1}, z)v = v \quad \text{for all } v \in V, \tag{3.3}$$

and for any $v \in V$ the series $Y(v,z)\mathbf{1}$ is a Taylor series in $z$ with the property

$$Y(v,z)\mathbf{1}\big|_{z=0} = v. \tag{3.4}$$

(c) A $\mathbb{C}[[h]]$-module map $D\colon V \to V$ (the *translation operator*) which satisfies

$$D\mathbf{1} = 0; \tag{3.5}$$

$$\frac{d}{dz}Y(v,z) = [D, Y(v,z)] \quad \text{for all } v \in V. \tag{3.6}$$

(d) A $\mathbb{C}[[h]]$-module map $\mathcal{S}\colon V \otimes V \to V \otimes V \otimes \mathbb{C}((z))$ which satisfies

$$\mathcal{S}(z)(v \otimes w) - v \otimes w \otimes 1 \,\in\, h\, V \otimes V \otimes \mathbb{C}((z)) \quad \text{for } v, w \in V, \tag{3.7}$$

$$[D \otimes 1, \mathcal{S}(z)] = -\frac{d}{dz}\mathcal{S}(z), \tag{3.8}$$

the *Yang–Baxter equation*

$$\mathcal{S}_{12}(z_1)\mathcal{S}_{13}(z_1+z_2)\mathcal{S}_{23}(z_2) = \mathcal{S}_{23}(z_2)\mathcal{S}_{13}(z_1+z_2)\mathcal{S}_{12}(z_1), \tag{3.9}$$

the *unitarity condition*

$$\mathcal{S}_{21}(z) = \mathcal{S}^{-1}(-z), \tag{3.10}$$

and the $\mathcal{S}$-*locality*: for any $v, w \in V$ and $n \in \mathbb{Z}_{\geqslant 0}$ there exists $\ell \in \mathbb{Z}_{\geqslant 0}$ such that for any $u \in V$

$$\begin{aligned}(z_1 - z_2)^\ell\, Y(z_1)\big(1 \otimes Y(z_2)\big)\big(\mathcal{S}(z_1-z_2)(v \otimes w) \otimes u\big)\\ - (z_1-z_2)^\ell\, Y(z_2)\big(1 \otimes Y(z_1)\big)(w \otimes v \otimes u) \in h^n V[[z_1^{\pm 1}, z_2^{\pm 1}]].\end{aligned} \tag{3.11}$$

In the relations used in Definition 3.1 we applied a common expansion convention: for $\ell < 0$ an expression of the form $(z \pm w)^{\ell}$ should be expanded into a Taylor series of the variable appearing on the right. For example,

$$(z-w)^{-1} = \sum_{r \geqslant 0} \frac{w^r}{z^{r+1}} \in \mathbb{C}((z))[[w]] \qquad \text{and} \qquad (w-z)^{-1} = \sum_{r \geqslant 0} \frac{z^r}{w^{r+1}} \in \mathbb{C}((w))[[z]].$$

We will apply this convention throughout the paper, unless stated otherwise. Also, the tensor products are understood as $h$-adically completed. In particular, $V \otimes V$ denotes the space $(V_0 \otimes V_0)[[h]]$ and $V \otimes V \otimes \mathbb{C}((z))$ denotes the space $\big(V_0 \otimes V_0 \otimes \mathbb{C}((z))\big)[[h]]$.

For any $r \in \mathbb{Z}$ the *$r$-product* $v_r w$ of elements $v$ and $w$ of $V$ is defined as the Laurent coefficient of the series

$$Y(v,z)w = Y(z)(v \otimes w) = \sum_{r \in \mathbb{Z}} (v_r w)\, z^{-r-1}.$$

It is clear from Definition 3.1 that the quotient $\overline{V} = V/hV$ of a quantum vertex algebra is a vertex algebra, as defined, e.g. in [12], [13] and [18].

*Remark* 3.2. In the original definition of the quantum VOA in [6], the *hexagon identity*

$$\mathcal{S}(z_1)\big(Y(z_2) \otimes 1\big) = \big(Y(z_2) \otimes 1\big)\mathcal{S}_{23}(z_1)\mathcal{S}_{13}(z_2 + z_1) \tag{3.12}$$

was considered instead of the weak associativity property (3.2). It was proved therein that (3.12) implies (3.2). Furthermore, the authors introduced the notion of *nondegenerate vertex algebra* and proved that if the other axioms hold, then the hexagon identity is equivalent to the weak associativity property when $V/hV$ is a nondegenerate vertex algebra.

*Remark* 3.3. A more general notion of the *$\hbar$-adic (weak) quantum vertex algebra* was studied in [20]. The author proved that weak associativity (3.2) and a certain weaker form of the $\mathcal{S}$-locality (3.11) imply the *Jacobi identity*

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right) Y(w,z_2)Y(v,z_1)u \\ &\quad - z_0^{-1}\delta\left(\frac{z_1 - z_2}{-z_0}\right) Y(z_1)\big(1 \otimes Y(z_2)\big)\big(\mathcal{S}(-z_0)(v \otimes w) \otimes u\big) \\ &\quad = z_1^{-1}\delta\left(\frac{z_2 - z_0}{z_1}\right) Y\big(Y(w,z_0)v, z_1\big)\, u \end{aligned} \tag{3.13}$$

so it holds for any elements $u, v, w$ of a quantum vertex algebra $V$. On the other hand, the $\mathcal{S}$-locality and weak associativity can be recovered from (3.13) by using properties of the formal $\delta$-function defined by

$$\delta(z) = \sum_{r \in \mathbb{Z}} z^r.$$

In particular, since $Y(w,z_0)v \in V_h((z_0))$, for every $n \geqslant 0$ there exists $\ell \geqslant 0$ such that $z_0^\ell Y(w,z_0)v$ is a Taylor series in $z_0$ modulo $h^n V[[z_0^{\pm 1}]]$. By taking the residue $\operatorname{Res}_{z_0} z_0^\ell$ in (3.13) we recover the $\mathcal{S}$-locality (3.11).

As with the vertex algebra theory, the translation operator $D$ is determined by the vertex operators. Namely, suppose that $v$ and $w$ are elements of a quantum vertex algebra $V$. By applying (3.6) to the vector $w$ and considering the coefficient of $z^{-r-1}$ we get

$$-\,r\,v_{r-1}w = D\,v_r w - v_r\,Dw \qquad \text{for all } r \in \mathbb{Z}. \tag{3.14}$$

Now taking $w = \mathbf{1}$, $r = -1$ in (3.14) and using (3.4) and (3.5) we obtain

$$Dv = v_{-2}\mathbf{1} \quad \text{for all } v \in V. \tag{3.15}$$

## 3.2 The center of a quantum vertex algebra

Let $\overline{V}$ be a vertex algebra. Recall that the *center* of $\overline{V}$ is defined by

$$\mathfrak{z}(\overline{V}) = \big\{ v \in \overline{V} \mid w_r v = 0 \;\text{ for all } w \in \overline{V} \text{ and all } r \geqslant 0 \big\}\,; \tag{3.16}$$

see, e.g., [11] and [12]. Equivalently, in terms of the vertex operators we have

$$\mathfrak{z}(\overline{V}) = \big\{ v \in \overline{V} \mid [Y(v,z_1), Y(w,z_2)] = 0 \;\text{ for all } w \in \overline{V} \big\}\,.$$

Consequently, the center of a vertex algebra has a structure of a unital commutative associative algebra equipped with a derivation. The multiplication is defined by the $(-1)$-product $v \cdot w = v_{-1}w$ for $v, w \in \overline{V}$.

We will now introduce a quantum version of $\mathfrak{z}(\overline{V})$. Let $V$ be a quantum vertex algebra. Define the *center* of $V$ as the subspace

$$\mathfrak{z}(V) = \{ v \in V \mid w_r v = 0 \;\text{ for all } w \in V \text{ and all } r \geqslant 0 \}\,. \tag{3.17}$$

**Proposition 3.4.** *Let $V$ be a quantum vertex algebra. For any element $v \in V$ and for any $w, u \in \mathfrak{z}(V)$ we have*

$$Y(v, z_0 + z_2)\,Y(w, z_2)\,u = Y\big(Y(v,z_0)w, z_2\big)\,u. \tag{3.18}$$

*Proof.* By (3.2) for every $n \geqslant 0$ there exists $\ell \geqslant 0$ such that

$$(z_0+z_2)^\ell\, Y(v,z_0+z_2)Y(w,z_2)u - (z_0+z_2)^\ell\, Y\big(Y(v,z_0)w,z_2\big)\,u \in h^n V[[z_0^{\pm 1}, z_2^{\pm 1}]]. \tag{3.19}$$

Set

$$A(z_0,z_2) = Y(v,z_0+z_2)Y(w,z_2)\,u \qquad \text{and} \qquad B(z_0,z_2) = Y\big(Y(v,z_0)w,z_2\big)\,u.$$

The definition of the center (3.17), together with the assumptions $w, u \in \mathfrak{z}(V)$, imply $B(z_0, z_2) \in V[[z_0, z_2]]$. Similarly, we have $Y(w, z_2)u \in V[[z_2]]$ because $u \in \mathfrak{z}(V)$ and so $A(z_0, z_2) \in V((z_0))[[z_2]]$. However, by (3.1) we have $Y(a, z)b \in V_h((z))$ for all $a, b \in V$, and hence $A(z_0, z_2) \in V_0((z_0))[[h]][[z_2]]$. Furthermore, observe that $(z_0 + z_2)^{\pm\ell} \in \mathbb{C}((z_0))((z_2))$.

Since $V_0$ is a vector space over $\mathbb{C}$, we may regard $V_F = V_0((z_0))((h))((z_2))$ as a vector space over the field $F = \mathbb{C}((z_0))((h))((z_2))$. By the above argument,

$$A(z_0, z_2), B(z_0, z_2) \in V_F \qquad \text{and} \qquad (z_0 + z_2)^{\pm\ell} \in F.$$

Therefore, multiplying (3.19) by $(z_0 + z_2)^{-\ell} \in \mathbb{C}((z_0))((z_2)) \subset F$ we find

$$Y(v, z_0 + z_2)Y(w, z_2)u - Y\big(Y(v, z_0)w, z_2\big)\, u \in h^n V[[z_0^{\pm 1}, z_2^{\pm 1}]]. \tag{3.20}$$

Relation (3.20) holds for all $n \geqslant 0$ which implies

$$Y(v, z_0 + z_2)Y(w, z_2)u - Y\big(Y(v, z_0)w, z_2\big)\, u = 0,$$

as required. $\square$

We point out some consequences of Proposition 3.4. Observe that the right hand side of (3.18) is a Taylor series in the variables $z_0, z_2$:

$$Y\big(Y(v, z_0)w, z_2\big)\, u = \sum_{m,n<0} (v_m w)_n u \, z_0^{-m-1} z_2^{-n-1}. \tag{3.21}$$

The left hand side of (3.18) can be written as

$$\begin{aligned} Y(v, z_0 + z_2)Y(w, z_2)u &= \sum_{\substack{r\in\mathbb{Z} \\ s<0}} v_r w_s u \,(z_0 + z_2)^{-r-1} z_2^{-s-1} \\ &= \sum_{r,s<0} v_r w_s u \,(z_0 + z_2)^{-r-1} z_2^{-s-1} + \sum_{\substack{r\geqslant 0 \\ s<0}} v_r w_s u \,(z_0 + z_2)^{-r-1} z_2^{-s-1}. \end{aligned} \tag{3.22}$$

Since the expressions in (3.21) and (3.22) are equal by (3.18), we get

$$\sum_{\substack{r\geqslant 0 \\ s<0}} v_r w_s u (z_0 + z_2)^{-r-1} z_2^{-s-1} = 0 \qquad \text{and} \tag{3.23}$$

$$\sum_{r,s<0} v_r w_s u (z_0 + z_2)^{-r-1} z_2^{-s-1} = \sum_{m,n<0} (v_m w)_n u z_0^{-m-1} z_2^{-n-1}. \tag{3.24}$$

**Proposition 3.5.** *The center of a quantum vertex algebra is closed under all $s$-products with $s \in \mathbb{Z}$.*

*Proof.* Let $v \in V$ be an element of a quantum vertex algebra $V$ and let $w, u \in \mathfrak{z}(V)$. We can write (3.23) as

$$0 = \sum_{\substack{r \geqslant 0 \\ s < 0}} v_r w_s u (z_0 + z_2)^{-r-1} z_2^{-s-1} = \sum_{\substack{r \geqslant 0 \\ s < 0}} \sum_{\ell \geqslant 0} \binom{-r-1}{\ell} v_r w_s u z_0^{-r-\ell-1} z_2^{\ell-s-1}. \tag{3.25}$$

The coefficient of $z_0^{-1}$ in (3.25) is zero and so $v_0 w_s u = 0$ for all $s \in \mathbb{Z}$ (note that $w_s u = 0$ for $s \geqslant 0$). By taking the coefficient of $z_0^{-2}$ we find that $v_1 w_s u = 0$ for all $s \in \mathbb{Z}$. Continuing by the induction on the power of $z_0$ in (3.25) we conclude that $v_r w_s u = 0$ for all $s \in \mathbb{Z}$ and $r \geqslant 0$. Hence, $w_s u \in \mathfrak{z}(V)$ for all $s \in \mathbb{Z}$ so the proposition follows. $\square$

Define the product $\mathfrak{z}(V) \otimes \mathfrak{z}(V) \to \mathfrak{z}(V)$ on the center of a quantum vertex algebra $V$ by setting

$$w \cdot u = w_{-1} u \qquad \text{for any } w, u \in \mathfrak{z}(V). \tag{3.26}$$

By Proposition 3.5 the product is well-defined.

**Proposition 3.6.** *The product (3.26) defines the structure of a unital associative algebra on $\mathfrak{z}(V)$. Moreover, this algebra is equipped with a derivation defined as the restriction of the translation operator $D$.*

*Proof.* Let $v, w, u$ be arbitrary elements of $\mathfrak{z}(V)$. By taking the constant terms in (3.24) we get $(v \cdot w) \cdot u = v \cdot (w \cdot u)$. Furthermore, (3.3) implies $\mathbf{1} \cdot v = v$, while (3.4) implies $v \cdot \mathbf{1} = v$ and $\mathbf{1} \in \mathfrak{z}(V)$, so $\mathbf{1}$ is the identity in the associative algebra $\mathfrak{z}(V)$.

Taking $r \geqslant 0$ in (3.14), we find that $Dw \in \mathfrak{z}(V)$ for any $w \in \mathfrak{z}(V)$ so that the restriction of $D$ is a well-defined operator on $\mathfrak{z}(V)$. By setting $w = \mathbf{1}$ and considering the coefficient of $z_0$ in (3.24) we get $v_{-2}\mathbf{1}_{-1} u = (v_{-2}\mathbf{1})_{-1} u$ for all $v, u \in \mathfrak{z}(V)$. Therefore, using (3.15) and vacuum axiom (3.3) we calculate

$$(Dv) \cdot u = (Dv)_{-1} u = (v_{-2}\mathbf{1})_{-1} u = v_{-2}\mathbf{1}_{-1} u = v_{-2} u.$$

Using (3.14) with $r = -1$ we can write this as $Dv_{-1}u - v_{-1}Du = D(v \cdot u) - v \cdot D(u)$. Since $D\mathbf{1} = 0$ by (3.5), we conclude that $D \colon \mathfrak{z}(V) \to \mathfrak{z}(V)$ is a derivation. $\square$

The final result of this section will demonstrate that the center of a quantum vertex algebra is $\mathcal{S}$-*commutative*, as stated in the next proposition. This replaces the commutativity property of the center of a vertex algebra in the quantum case. In general, the center of a quantum vertex algebra need not be commutative, as demonstrated by Proposition 4.2 below.

**Proposition 3.7.** *Let $V$ be a quantum vertex algebra. For any $w \in V$ and any $v, u \in \mathfrak{z}(V)$ we have*

$$Y(z_1)\big(1 \otimes Y(z_2)\big)\big(\mathcal{S}(z_1 - z_2)(v \otimes w) \otimes u\big) = Y(z_2)\big(1 \otimes Y(z_1)\big)(w \otimes v \otimes u). \tag{3.27}$$

*Proof.* By (3.11) for any $w \in V$, $v \in \mathfrak{z}(V)$ and $n \geqslant 0$ there exists $\ell \geqslant 0$ such that for any $u \in \mathfrak{z}(V)$ we have

$$(z_1 - z_2)^{\ell}\, Y(z_1)\big(1 \otimes Y(z_2)\big)\big(\mathcal{S}(z_1 - z_2)(v \otimes w) \otimes u\big) \tag{3.28}$$

$$- (z_1 - z_2)^{\ell}\, Y(z_2)\big(1 \otimes Y(z_1)\big)(w \otimes v \otimes u) \in h^n V[[z_1^{\pm 1}, z_2^{\pm 1}]]. \tag{3.29}$$

Since $v$ and $u$ lie in the center of $V$ and the center is closed under all $s$-products by Proposition 3.5, the expression $Y(z_1)(v \otimes u)$ occurring in (3.29) is a Taylor series in $z_1$ with coefficients in $\mathfrak{z}(V)$. Therefore,

$$Y(z_2)\big(1 \otimes Y(z_1)\big)(w \otimes v \otimes u) \in V[[z_1, z_2]].$$

Now consider (3.28). Recall that $\mathcal{S}(z)(a \otimes b) \in V \otimes V \otimes \mathbb{C}((z))$ for any $a, b \in V$. Since $u \in \mathfrak{z}(V)$, the expression $\big(1 \otimes Y(z_2)\big)\big(\mathcal{S}(z_1 - z_2)(v \otimes w) \otimes u\big)$ lies in $(V_0 \otimes V_0)((z_1))[[h]][[z_2]]$ and can be written as

$$\sum_{k \geqslant 0} \Big( \sum_{\text{fin}} v_k^{(1)} \otimes v_k^{(2)} \otimes a_k(z_1 - z_2) \Big) h^k, \tag{3.30}$$

where the internal sum is finite and denotes an element of $V_0 \otimes V_0 \otimes \mathbb{C}((z_1))[[z_2]]$. Applying the operator $Y(z_1)$ to (3.30) we get

$$\sum_{k \geqslant 0} \Big( \sum_{\text{fin}} Y(v_k^{(1)}, z_1) v_k^{(2)} \otimes a_k(z_1 - z_2) \Big) h^k. \tag{3.31}$$

For every $m \geqslant 0$ the coefficient of $z_2^m$ in $a_k(z_1 - z_2)$ lies in $\mathbb{C}[z_1^{-1}]$, so the internal finite sum is an element of $V_h((z_1))[[z_2]] \equiv V_0((z_1))[[h]][[z_2]]$ for every $k \geqslant 0$. Hence, we conclude that (3.31) lies in $V_0((z_1))[[h]][[z_2]]$. The proof is now completed as for Proposition 3.4 by multiplying the expression which occurs in (3.28) and (3.29) by $(z_1 - z_2)^{-\ell}$. $\square$

*Remark* 3.8. By the definition of $\mathfrak{z}(V)$, the center $\mathfrak{z}(\overline{V})$ of the vertex algebra $\overline{V} = V/hV$ coincides with $\mathfrak{z}(V)/h\,\mathfrak{z}(V)$. Hence, due to the property (3.7) of the map $\mathcal{S}$, we recover from Proposition 3.7 that the product on $\mathfrak{z}(\overline{V})$ is commutative.

# 4 Quantum affine vertex algebra

Following [6] we introduce the quantum vertex algebra associated with the double Yangian for $\mathfrak{gl}_N$. In accordance with the general definitions of Sec. 3, we will consider this algebra as a module over $\mathbb{C}[[h]]$. So we will start by restating definitions of Sec. 2.2 in this context and then verify the axioms for the quantum vertex algebra on the vacuum module.

## 4.1 Double Yangian over $\mathbb{C}[[h]]$

From now on we will work with algebras and modules over $\mathbb{C}[[h]]$ and keep the same notation for the objects associated with the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ as in Sec. 2.2. The definitions of the algebras are readily translated into the $\mathbb{C}[[h]]$-module context by the formal re-scaling $u \mapsto u/h$ of the 'spectral parameter' and generators

$$t_{ij}^{(r)} \mapsto h^{r-1}\, t_{ij}^{(r)}, \qquad t_{ij}^{(-r)} \mapsto h^{-r}\, t_{ij}^{(-r)}, \qquad C \mapsto C,$$

for $r \geqslant 1$. Conversely, the formal evaluation $h = 1$ can be used to recover some of the definitions and formulas of Sec. 2.2. The Yang $R$-matrix (2.3) now takes the form

$$R(u) = 1 - P\, h u^{-1}, \tag{4.1}$$

while for the normalized $R$-matrix (2.17) we have

$$\overline{R}(u) = g(u/h)\,\big(1 - P\, h u^{-1}\big). \tag{4.2}$$

The double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ is now defined as the associative algebra over $\mathbb{C}[[h]]$ generated by the central element $C$ and elements $t_{ij}^{(r)}$ and $t_{ij}^{(-r)}$, where $1 \leqslant i, j \leqslant N$ and $r = 1, 2, \dots$, subject to the defining relations

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v), \tag{4.3}$$

$$R(u-v)\, T_1^+(u)\, T_2^+(v) = T_2^+(v)\, T_1^+(u)\, R(u-v), \tag{4.4}$$

$$\overline{R}\big(u - v + hC/2\big)\, T_1(u)\, T_2^+(v) = T_2^+(v)\, T_1(u)\, \overline{R}\big(u - v - hC/2\big), \tag{4.5}$$

where the matrices $T(u)$ and $T^+(u)$ are given by

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \qquad \text{and} \qquad T^+(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}^+(u) \tag{4.6}$$

with

$$t_{ij}(u) = \delta_{ij} + h \sum_{r=1}^{\infty} t_{ij}^{(r)}\, u^{-r} \qquad \text{and} \qquad t_{ij}^+(u) = \delta_{ij} - h \sum_{r=1}^{\infty} t_{ij}^{(-r)}\, u^{r-1}.$$

The coproduct now takes the form

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{N} t_{ik}\big(u + hC_2/4\big) \otimes t_{kj}\big(u - hC_1/4\big),$$

$$\Delta : t_{ij}^+(u) \mapsto \sum_{k=1}^{N} t_{ik}^+\big(u - hC_2/4\big) \otimes t_{kj}^+\big(u + hC_1/4\big),$$

$$\Delta : C \mapsto C \otimes 1 + 1 \otimes C,$$

where the tensor products are understood as $h$-adically completed. The antipode and counit are defined by the same formulas as for the extended double Yangian $\mathrm{DY}^{\circ}(\mathfrak{gl}_N)$; see Sec. 2.2.

The Poincaré–Birkhoff–Witt theorem for the double Yangian extends to the algebra $\mathrm{DY}(\mathfrak{gl}_N)$ over $\mathbb{C}[[h]]$; see Theorem 2.2. Therefore the subalgebra of $\mathrm{DY}(\mathfrak{gl}_N)$ generated by the elements $t_{ij}^{(r)}$ with $1 \leqslant i, j \leqslant N$ and $r \geqslant 1$ can be identified with the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ defined by the relations (4.3). Similarly, the subalgebra generated by the elements $t_{ij}^{(-r)}$ with $1 \leqslant i, j \leqslant N$ and $r \geqslant 1$ can be identified with the dual Yangian $\mathrm{Y}^+(\mathfrak{gl}_N)$ defined by the relations (4.4).

## 4.2 Vacuum module as a quantum vertex algebra

The double Yangian at the level $c \in \mathbb{C}$ is the quotient $\mathrm{DY}_c(\mathfrak{gl}_N)$ of $\mathrm{DY}(\mathfrak{gl}_N)$ by the ideal generated by $C - c$. Similar to Sec. 2.3, the *vacuum module* $\mathcal{V}_c(\mathfrak{gl}_N)$ *at the level* $c$ over the double Yangian is the quotient

$$\mathcal{V}_c(\mathfrak{gl}_N) = \mathrm{DY}_c(\mathfrak{gl}_N)/\mathrm{DY}_c(\mathfrak{gl}_N)\langle t_{ij}^{(r)} \,|\, r \geqslant 1\rangle. \tag{4.7}$$

By the Poincaré–Birkhoff–Witt theorem (Theorem 2.2), we can identify this quotient with the dual Yangian $\mathrm{Y}^+(\mathfrak{gl}_N)$ as a $\mathbb{C}[[h]]$-module.

As demonstrated in [6], the $h$-adically completed vacuum module possesses a quantum vertex algebra structure. In the classical limit $h \to 0$ it turns into the affine vertex algebra $V_c(\mathfrak{gl}_N)$. Accordingly, (4.7) is called the *quantum affine vertex algebra*. To introduce the structure, we need some notation. For a positive integer $n$, consider the tensor product space

$$(\mathrm{End}\,\mathbb{C}^N)^{\otimes n} \otimes \mathcal{V}_c(\mathfrak{gl}_N). \tag{4.8}$$

Given a variable $z$ and a family of variables $u = (u_1, \dots, u_n)$, set

$$T_n(u) = T_{1\,n+1}(u_1) \dots T_{n\,n+1}(u_n),$$

$$T_n^+(u) = T_{1\,n+1}^+(u_1) \dots T_{n\,n+1}^+(u_n),$$

$$T_n(u|z) = T_{1\,n+1}(z + u_1) \dots T_{n\,n+1}(z + u_n),$$

$$T_n^+(u|z) = T_{1\,n+1}^+(z + u_1) \dots T_{n\,n+1}^+(z + u_n).$$

Here we extend the notation (2.8) to include the vacuum module as a component of tensor products so that the subscript $n+1$ corresponds to $\mathcal{V}_c(\mathfrak{gl}_N)$. For $n = 0$ these products will be considered as being equal to the identity. The respective components of the matrices (4.6) are understood as operators on $\mathcal{V}_c(\mathfrak{gl}_N)$. The series $T_{i\,n+1}(z + u_i)$ and $T_{i\,n+1}^+(z + u_i)$ should be expanded in nonpositive and nonnegative powers of $z$, respectively.

For nonnegative integers $m$ and $n$ introduce functions depending on a variable $z$ and the families of variables $u=(u_1,\dots,u_n)$ and $v=(v_1,\dots,v_m)$ with values in the space

$$(\operatorname{End}\mathbb{C}^N)^{\otimes n}\otimes(\operatorname{End}\mathbb{C}^N)^{\otimes m} \tag{4.9}$$

by

$$R^{12}_{nm}(u|v|z)=\prod_{j=1,\dots,n}^{\longrightarrow}\ \prod_{i=n+1,\dots,n+m}^{\longleftarrow} R_{ji}(z+u_j-v_i) \tag{4.10}$$

with the arrows indicating the order of the factors, where we use the Yang $R$-matrix (4.1) and adopt the matrix notation as in (2.8). As above, empty products will be understood as being equal to the identity. We also define $\overline{R}^{12}_{nm}(u|v|z)$ by the same formula (4.10), where the $R$-matrix (4.2) is used instead of $R(u)$. The superscripts 1 and 2 are meant to indicate the tensor factors in (4.9). We also adopt the superscript notation for multiple tensor products of the form

$$(\operatorname{End}\mathbb{C}^N)^{\otimes n}\otimes(\operatorname{End}\mathbb{C}^N)^{\otimes m}\otimes(\operatorname{End}\mathbb{C}^N)^{\otimes k}\otimes\mathcal{V}_c(\mathfrak{gl}_N)\otimes\mathcal{V}_c(\mathfrak{gl}_N)\otimes\mathcal{V}_c(\mathfrak{gl}_N). \tag{4.11}$$

Expressions like $T^{14}_n(u)$ or $T^{35}_k(u)$ will be understood as the respective operators $T_n(u)$ or $T_k(u)$, whose non-identity components belong to the corresponding tensor factors. In particular, the non-identity components of $T^{35}_k(u)$ belong to the factors

$$n+m+1,\ n+m+2,\ \dots,\ n+m+k \quad \text{and} \quad n+m+k+2.$$

Employing this notation, we point out some immediate consequences of the defining relations (4.3)–(4.5) for operators on

$$(\operatorname{End}\mathbb{C}^N)^{\otimes n}\otimes(\operatorname{End}\mathbb{C}^N)^{\otimes m}\otimes\mathcal{V}_c(\mathfrak{gl}_N).$$

They follow by a straightforward induction and take the form

$$R^{12}_{nm}(u|v|z-w)T^{+13}_n(u|z)T^{+23}_m(v|w)=T^{+23}_m(v|w)T^{+13}_n(u|z)R^{12}_{nm}(u|v|z-w), \tag{4.12}$$

$$R^{12}_{nm}(u|v|z-w)T^{13}_n(u|z)T^{23}_m(v|w)=T^{23}_m(v|w)T^{13}_n(u|z)R^{12}_{nm}(u|v|z-w), \tag{4.13}$$

$$\overline{R}^{12}_{nm}(u|v|z-w+hc/2)T^{13}_n(u|z)T^{+23}_m(v|w)=T^{+23}_m(v|w)T^{13}_n(u|z)\overline{R}^{12}_{nm}(u|v|z-w-hc/2). \tag{4.14}$$

It will also be convenient to use an ordered product notation for elements of the tensor product of two associative algebras $\mathcal{A}\otimes\mathcal{B}$. Suppose that $A_1,A_2\in\mathcal{A}$ and $B_1,B_2\in\mathcal{B}$. Let $F=A_1\otimes B_1$ and define the following products

$$\begin{aligned} {}^{ll}F(A_2\otimes B_2)&=A_1A_2\otimes B_1B_2, &\qquad {}^{lr}F(A_2\otimes B_2)&=A_1A_2\otimes B_2B_1,\\ {}^{rl}F(A_2\otimes B_2)&=A_2A_1\otimes B_1B_2, &\qquad {}^{rr}F(A_2\otimes B_2)&=A_2A_1\otimes B_2B_1, \end{aligned} \tag{4.15}$$

indicating the left and right multiplication of the components. For $\alpha, \beta \in \{l, r\}$ we will denote by $({}^{\alpha\beta}F)^{-1}$ the operator $G$ such that $({}^{\alpha\beta}G)\,F = 1$. Note that $({}^{\alpha\beta}F)^{-1}$ and ${}^{\alpha\beta}\big(F^{-1}\big)$ need not be equal.

This notation will often be applied to products of $R$-matrices $F = R^{12}_{nm}(u|v|z)$, where the roles of $\mathcal{A}$ and $\mathcal{B}$ will be played by the first and second components in (4.9). We point out the formulas for the inverse operators associated with the $R$-matrix (4.1):

$$\begin{aligned}\big({}^{lr}R(u)\big)^{-1} &= \big({}^{rl}R(u)\big)^{-1} = \big(1 - hNu^{-1}\big)^{-1}\big(R(-u) - hNu^{-1}\big),\\ \big({}^{ll}R(u)\big)^{-1} &= \big({}^{rr}R(u)\big)^{-1} = R(u)^{-1} = \big(1 - h^2u^{-2}\big)^{-1}R(-u),\end{aligned}$$

which can be used to calculate the inverse operators corresponding to $F = R^{12}_{nm}(u|v|z)$.

We will now use the general definition of quantum vertex algebra reproduced in Sec. 3; see Definition 3.1. The following theorem is due to Etingof and Kazhdan [6].

**Theorem 4.1.** *There exists a unique well-defined structure of quantum vertex algebra on the vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$ with the following data.*

*(a) The vacuum vector is*

$$\mathbf{1} = 1 \in \mathcal{V}_c(\mathfrak{gl}_N). \tag{4.16}$$

*(b) The vertex operators are defined by*

$$Y\big(T^+_n(u)\mathbf{1}, z\big) = T^+_n(u|z)\, T_n(u|z + hc/2)^{-1}. \tag{4.17}$$

*(c) The translation operator $D$ is defined by*

$$e^{zD}\, T^+(u_1)\dots T^+(u_n)\mathbf{1} = T^+(z+u_1)\dots T^+(z+u_n)\mathbf{1}. \tag{4.18}$$

*(d) The map $\mathcal{S}$ is defined by the relation*

$$\begin{aligned}\mathcal{S}_{34}(z)\Big(\overline{R}^{\,12}_{nm}(u|v|z)^{-1}\, T^{+24}_m(v)\, \overline{R}^{\,12}_{nm}(u|v|z-hc)\, T^{+13}_n(u)(\mathbf{1}\otimes\mathbf{1})\Big)&\\ = T^{+13}_n(u)\, \overline{R}^{\,12}_{nm}(u|v|z+hc)^{-1}\, T^{+24}_m(v)\, \overline{R}^{\,12}_{nm}(u|v|z)(\mathbf{1}\otimes\mathbf{1})&\end{aligned} \tag{4.19}$$

*for operators on*

$$(\operatorname{End}\mathbb{C}^N)^{\otimes n} \otimes (\operatorname{End}\mathbb{C}^N)^{\otimes m} \otimes \mathcal{V}_c(\mathfrak{gl}_N) \otimes \mathcal{V}_c(\mathfrak{gl}_N). \tag{4.20}$$

*Proof.* We add some details as compared to [6], to take care of the variations of the definition of the quantum vertex algebra. Let $V = \mathcal{V}_c(\mathfrak{gl}_N)$. We start by pointing out that $Y$ is a well-defined operator as in (3.1). Indeed, since the coefficients of the series $T^+_n(u)\mathbf{1}$ span

$\mathcal{V}_c(\mathfrak{gl}_N)$, it suffices to verify that $Y$ preserves the ideal of relations of the dual Yangian. This follows by employing (4.3) and (4.4) as in the proof of [6, Lemma 2.1]. As a next step, we will verify the weak associativity property (3.2). Let $m$, $n$ and $k$ be nonnegative integers and let $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_m)$ and $w = (w_1, \dots, w_k)$ be families of variables. Note the following relation which is a consequence of (4.14):

$$T_n^{14}(u|z_0 + hc/2)^{-1}\, \overline{R}_{nm}^{12}(u|v|z_0 + hc)^{-1}\, T_m^{+24}(v) \\ = T_m^{+24}(v)\, \overline{R}_{nm}^{12}(u|v|z_0)^{-1}\, T_n^{14}(u|z_0 + hc/2)^{-1}. \tag{4.21}$$

Here and below we use the additional variables $z_0$ and $z_2$ as in (3.2). Using the definition of the vacuum module together with (4.17) and (4.21) we get

$$\begin{aligned} Y\big(T_n^{+14}(u)\mathbf{1}, z_0\big)\, \overline{R}_{nm}^{12}(u|v|z_0 + hc)^{-1}\, T_m^{+24}(v)\mathbf{1} & \\ = T_n^{+14}(u|z_0)\, T_n^{14}(u|z_0 + hc/2)^{-1}\, \overline{R}_{nm}^{12}(u|v|z_0 + hc)^{-1}\, T_m^{+24}(v)\mathbf{1} & \\ = T_n^{+14}(u|z_0)\, T_m^{+24}(v)\, \overline{R}_{nm}^{12}(u|v|z_0)^{-1}\, T_n^{14}(u|z_0 + hc/2)^{-1}\mathbf{1} & \\ = T_n^{+14}(u|z_0)\, T_m^{+24}(v)\, \overline{R}_{nm}^{12}(u|v|z_0)^{-1}\mathbf{1}. & \end{aligned}$$

For fixed positive integers $M$ and $p$ and operators $A$ and $B$ on (4.11) of this form, we will say that $A$ and $B$ are *equivalent*, if the coefficients of all monomials

$$u_1^{r_1} \dots u_n^{r_n} v_1^{s_1} \dots v_m^{s_m} w_1^{t_1} \dots w_k^{t_k} \qquad \text{with} \quad 0 \leqslant r_1, \dots, r_n, s_1, \dots, s_m, t_1, \dots, t_k \leqslant M \tag{4.22}$$

in $A - B$ belong to the subspace $h^p V[[z_0^{\pm 1}, z_2^{\pm 1}]]$. Let $\ell$ be a nonnegative integer such that the coefficients of the monomials (4.22) in the operator

$$z^\ell\, T_n^{14}(u|z + hc/2)^{-1}\, T_k^{+34}(w)\mathbf{1}$$

have only nonnegative powers of $z$ modulo $h^p$. By the above calculation, the operator

$$(z_0 + z_2)^\ell\, Y\Big(Y\big(T_n^{+14}(u)\mathbf{1}, z_0\big)\, \overline{R}_{nm}^{12}(u|v|z_0 + hc)^{-1}\, T_m^{+24}(v)\mathbf{1}, z_2\Big)\, T_k^{+34}(w)\mathbf{1} \tag{4.23}$$

equals

$$(z_0 + z_2)^\ell\, Y\Big(T_n^{+14}(u|z_0)\, T_m^{+24}(v)\, \overline{R}_{nm}^{12}(u|v|z_0)^{-1}\mathbf{1}, z_2\Big)\, T_k^{+34}(w)\mathbf{1}$$

which by (4.17) coincides with

$$(z_0 + z_2)^\ell\, T_n^{+14}(u|z_2 + z_0)\, T_m^{+24}(v|z_2)\, T_m^{24}(v|z_2 + hc/2)^{-1} \\ \times T_n^{14}(u|z_2 + z_0 + hc/2)^{-1}\, \overline{R}_{nm}^{12}(u|v|z_0)^{-1}\, T_k^{+34}(w)\mathbf{1}.$$

By our assumption on $\ell$, only nonnegative powers of $z_0+z_2$ will occur in the expansion of this operator modulo $h^p$, so that we may swap $z_0$ and $z_2$ to get an equivalent operator

$$(z_0+z_2)^\ell\, T_n^{+14}(u|z_0+z_2)\, T_m^{+24}(v|z_2)\, T_m^{24}(v|z_2+hc/2)^{-1} \\ \times T_n^{14}(u|z_0+z_2+hc/2)^{-1}\, \overline{R}_{nm}^{12}(u|v|z_0)^{-1}\, T_k^{+34}(w)\mathbf{1}. \quad (4.24)$$

On the other hand, by (4.17) the operator

$$Y\big(T_n^{+14}(u)\mathbf{1}, z_0+z_2\big)\, Y\big(\overline{R}_{nm}^{12}(u|v|z_0+hc)^{-1} T_m^{+24}(v)\mathbf{1}, z_2\big)\, T_k^{+34}(w)\mathbf{1} \quad (4.25)$$

equals

$$T_n^{+14}(u|z_0+z_2)\, T_n^{14}(u|z_0+z_2+hc/2)^{-1} \\ \times \overline{R}_{nm}^{12}(u|v|z_0+hc)^{-1}\, T_m^{+24}(v|z_2)\, T_m^{24}(v|z_2+hc/2)^{-1}\, T_k^{+34}(w)\mathbf{1}.$$

Applying (4.21) and then (4.13) we can write this as

$$T_n^{+14}(u|z_0+z_2)\, T_m^{+24}(v|z_2) \\ \times T_m^{24}(v|z_2+hc/2)^{-1}\, T_n^{14}(u|z_0+z_2+hc/2)^{-1}\, \overline{R}_{nm}^{12}(u|v|z_0)^{-1}\, T_k^{+34}(w)\mathbf{1}.$$

Observe that after multiplication by $(z_0+z_2)^\ell$ this coincides with (4.24). Therefore, when the operator (4.25) is multiplied by $(z_0+z_2)^\ell$, it will be equal to (4.23) modulo $h^p$. By applying ${}^{rl}\big(({}^{rl}\overline{R}_{nm}^{12}(u|v|z_0+hc)^{-1})^{-1}\big)$ to both sides of this equality we get (3.2), as required.

The vacuum axioms (3.3) and (3.4) are immediate from the definitions of the vacuum vector and vertex operators.

Now we verify the translation operator $D$ is well-defined by (4.18) and satisfies the axioms (3.5) and (3.6). We need to check that $D$ preserves the defining relations (4.4) of the dual Yangian. This is a straightforward calculation; cf. [6, Lemma 2.1]. Furthermore, $e^{zD}\mathbf{1}=\mathbf{1}$ so that (3.5) holds. Now suppose that $m$ and $n$ are nonnegative integers. Taking the coefficient of $z$ in (4.18) we get

$$D\, T^+(u_1)\dots T^+(u_n)\mathbf{1} = \Big(\sum_{l=1}^{n} \frac{\partial}{\partial u_l}\Big)\, T^+(u_1)\dots T^+(u_n)\mathbf{1}. \quad (4.26)$$

Therefore, using (4.17) we obtain

$$\frac{\partial}{\partial z}\, Y\big(T_n^{+13}(u)\mathbf{1}, z\big)\, T_m^{+23}(v)\mathbf{1} = \frac{\partial}{\partial z}\, T_n^{+13}(u|z)\, T_n^{13}(u|z+hc/2)^{-1}\, T_m^{+23}(v)\mathbf{1}$$

which can be written as

$$\Big(\sum_{l=1}^{n} \frac{\partial}{\partial u_l}\Big)\, T_n^{+13}(u|z)\, T_n^{13}(u|z+hc/2)^{-1}\, T_m^{+23}(v)\mathbf{1}.$$

This coincides with

$$D\,Y\big(T_n^{+13}(u)\mathbf{1},z\big)\,T_m^{+23}(v)\mathbf{1}-Y\big(T_n^{+13}(u)\mathbf{1},z\big)\,D\,T_m^{+23}(v)\mathbf{1},$$

since

$$\begin{aligned}&D\,Y\big(T_n^{+13}(u)\mathbf{1},z\big)\,T_m^{+23}(v)\mathbf{1}\\&\qquad=\Big(\sum_{l=1}^{n}\frac{\partial}{\partial u_l}+\sum_{k=1}^{m}\frac{\partial}{\partial v_k}\Big)\,T_n^{+13}(u|z)\,T_n^{13}(u|z+hc/2)^{-1}\,T_m^{+23}(v)\mathbf{1}\end{aligned}$$

and

$$Y\big(T_n^{+13}(u)\mathbf{1},z\big)\,D\,T_m^{+23}(v)\mathbf{1}=T_n^{+13}(u|z)\,T_n^{13}(u|z+hc/2)^{-1}\Big(\sum_{k=1}^{m}\frac{\partial}{\partial v_k}\Big)\,T_m^{+23}(v)\mathbf{1},$$

thus verifying (3.6). Now turn to the axioms concerning the map $\mathcal{S}$. Using the notation (4.15), we can write the operators appearing in (4.19) in the form

$$\begin{aligned}&\overline{R}_{nm}^{12}(u|v|z)^{-1}\,T_m^{+24}(v)\,\overline{R}_{nm}^{12}(u|v|z-hc)\,T_n^{+13}(u)\\&\qquad={}^{ll}\overline{R}_{nm}^{12}(u|v|z)^{-1}\,{}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc)\,T_n^{+13}(u)T_m^{+24}(v)\end{aligned}$$

and

$$\begin{aligned}&T_n^{+13}(u)\,\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}\,T_m^{+24}(v)\,\overline{R}_{nm}^{12}(u|v|z)\\&\qquad={}^{rr}\overline{R}_{nm}^{12}(u|v|z)\,{}^{rl}\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}\,T_n^{+13}(u)\,T_m^{+24}(v).\end{aligned}$$

Hence (4.19) can be written as

$$\begin{aligned}&\mathcal{S}_{34}(z)\Big({}^{ll}\overline{R}_{nm}^{12}(u|v|z)^{-1}\,{}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc)\,T_n^{+13}(u)T_m^{+24}(v)(\mathbf{1}\otimes\mathbf{1})\Big)\\&\qquad={}^{rr}\overline{R}_{nm}^{12}(u|v|z)\,{}^{rl}\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}\,T_n^{+13}(u)\,T_m^{+24}(v)(\mathbf{1}\otimes\mathbf{1})\end{aligned}$$

which is equivalent to

$$\begin{aligned}\mathcal{S}_{34}(z)\Big(T_n^{+13}(u)\,T_m^{+24}(v)(\mathbf{1}\otimes\mathbf{1})\Big)&={}^{lr}\big(({}^{rl}\overline{R}_{nm}^{12}(u|v|z-hc))^{-1}\big)\ {}^{ll}\overline{R}_{nm}^{12}(u|v|z)\\&\quad\times{}^{rr}\overline{R}_{nm}^{12}(u|v|z)\ {}^{rl}\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}\,T_n^{+13}(u)\,T_m^{+24}(v)(\mathbf{1}\otimes\mathbf{1}).\end{aligned}$$

This form of $\mathcal{S}$ is convenient for checking that the map is well-defined; cf. [6, Lemma 2.1]. Furthermore, (3.7) clearly holds since the value of the $R$-matrix (4.2) at $h=0$ is the identity operator. Property (3.8) is checked in the same way as (3.6) with the use of (4.26). The Yang–Baxter equation (3.9), the unitarity condition (3.10) and the $\mathcal{S}$-locality property (3.11) are verified by straightforward calculations which rely on the Yang–Baxter equation (2.43) satisfied by the $R$-matrix (4.2) and the unitarity property (2.21). $\square$

We now give an example based on the structure of the dual Yangian to demonstrate that the center of a quantum vertex algebra need not be commutative, in general. We use the same notation for products of generators matrices as in the beginning of this section.

**Proposition 4.2.** *There exists a unique well-defined structure of quantum vertex algebra on the $\mathbb{C}[[h]]$-module $V = \mathrm{Y}^+(\mathfrak{gl}_N)$ with the following data.*

*(a) The vacuum vector is*

$$\mathbf{1} = 1 \in \mathrm{Y}^+(\mathfrak{gl}_N). \tag{4.27}$$

*(b) The vertex operators are defined by*

$$Y\big(T_n^+(u)\mathbf{1}, z\big) = T_n^+(u|z). \tag{4.28}$$

*(c) The translation operator $D$ is defined by*

$$e^{zD}\, T^+(u_1)\dots T^+(u_n)\mathbf{1} = T^+(z+u_1)\dots T^+(z+u_n)\mathbf{1}. \tag{4.29}$$

*(d) The map $\mathcal{S}$ is defined by the relation*

$$\mathcal{S}_{34}(z)\Big(T_n^{+13}(u)T_m^{+24}(v)(\mathbf{1}\otimes\mathbf{1})\Big)$$
$$= \overline{R}_{nm}^{12}(u|v|z)\, T_n^{+13}(u)\, T_m^{+24}(v)\, \overline{R}_{nm}^{12}(u|v|z)^{-1}(\mathbf{1}\otimes\mathbf{1}). \tag{4.30}$$

*Moreover, the center $\mathfrak{z}(V)$ of the quantum vertex algebra $V$ coincides with $V$.*

*Proof.* The last claim follows since the image of the vertex operator map $Y$ is contained in $V[[z]]$. In particular, $\mathfrak{z}(V)$ is not commutative for $N \geqslant 2$.

The maps $Y$, $D$ and $\mathcal{S}$ are well-defined, as follows by the same arguments as for the proof of Theorem 4.1. The quantum vertex algebra axioms are also checked in a similar way with some obvious modifications. We only verify the $\mathcal{S}$-commutativity (3.27) which implies the $\mathcal{S}$-locality property (3.11). Set $z = z_1 - z_2$ and consider the left hand side in (3.27). The application of $\mathcal{S}_{45}(z)\otimes 1$ to

$$T_n^{+14}(u)\, T_m^{+25}(v)\, T_k^{+36}(w)(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1})$$

gives

$$\overline{R}_{nm}^{12}(u|v|z)\, T_n^{+14}(u)\, T_m^{+25}(v)\, \overline{R}_{nm}^{12}(u|v|z)^{-1}\, T_k^{+36}(w)(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}).$$

Further applying $1\otimes Y(z_2)$ we get

$$\overline{R}_{nm}^{12}(u|v|z)\, T_n^{+14}(u)\, T_m^{+25}(v|z_2)\, \overline{R}_{nm}^{12}(u|v|z)^{-1}\, T_k^{+35}(w)(\mathbf{1}\otimes\mathbf{1})$$

which becomes

$$\overline{R}^{12}_{nm}(u|v|z)\, T^{+14}_{n}(u|z_1)\, T^{+24}_{m}(v|z_2)\, \overline{R}^{12}_{nm}(u|v|z)^{-1}\, T^{+34}_{k}(w)\mathbf{1} \tag{4.31}$$

after the application of $Y(z_1)$. For the right hand side we have

$$T^{+24}_{m}(v)\, T^{+15}_{n}(u)\, T^{+36}_{k}(w)(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}) \xmapsto{1\otimes Y(z_1)} T^{+24}_{m}(v)\, T^{+15}_{n}(u|z_1)\, T^{+35}_{k}(w)(\mathbf{1}\otimes\mathbf{1})$$

and the application of $Y(z_2)$ gives

$$T^{+24}_{m}(v|z_2)\, T^{+14}_{n}(u|z_1)\, T^{+34}_{k}(w)\mathbf{1}. \tag{4.32}$$

Now (4.12) implies that (4.31) coincides with (4.32) and so the $\mathcal{S}$-commutativity property (3.27) follows. □

## 4.3 Central elements of the completed double Yangian

As with the affine vertex algebras, the vertex operator formulas (4.17) suggest a construction of central elements of a completed double Yangian; cf. [11, Sec. 4.3.2] and Remark 4.6 below. However, we will not use the quantum vertex algebra structure, but rather give a direct proof as in [10].

Introduce the completion of the double Yangian $\mathrm{DY}_c(\mathfrak{gl}_N)$ at the level $c$ as the inverse limit

$$\widetilde{\mathrm{DY}}_c(\mathfrak{gl}_N) = \lim_{\longleftarrow} \mathrm{DY}_c(\mathfrak{gl}_N)/I_p, \tag{4.33}$$

where $p \geqslant 1$ and $I_p$ denotes the left ideal of $\mathrm{DY}_c(\mathfrak{gl}_N)$, generated by all elements $t^{(r)}_{ij}$ with $r \geqslant p$. Using the idempotents $\mathcal{E}_{\mathcal{U}}$ as in Theorem 2.4, introduce the Laurent series in $u$ with coefficients in the $h$-adically completed algebra of formal power series $\widetilde{\mathrm{DY}}_{-N}(\mathfrak{gl}_N)$ at the critical level $c = -N$ by

$$\begin{aligned}\widetilde{\mathbb{T}}_{\mu}(u) = \mathrm{tr}_{1,\dots,m}\, \mathcal{E}_{\mathcal{U}}\, T^{+}_{1}(u+hc_1)\dots T^{+}_{m}(u+hc_m)\\ \times T_m\big(u+hc_m-hN/2\big)^{-1}\dots T_1\big(u+hc_1-hN/2\big)^{-1},\end{aligned} \tag{4.34}$$

where $c_a = c_a(\mathcal{U})$ is the content of the box occupied by $a \in \{1,\dots,m\}$ in the standard tableau $\mathcal{U}$. By the argument of [27, Sec. 3.4], the series $\widetilde{\mathbb{T}}_{\mu}(u)$ does not depend on the standard tableau $\mathcal{U}$ of shape $\mu$.

**Theorem 4.3.** *All coefficients of $\widetilde{\mathbb{T}}_{\mu}(u)$ belong to the center of the $h$-adically completed algebra $\widetilde{\mathrm{DY}}_{-N}(\mathfrak{gl}_N)$.*

*Proof.* We need to show that

$$T_0(z)\,\widetilde{\mathbb{T}}_\mu(u) = \widetilde{\mathbb{T}}_\mu(u)\,T_0(z) \qquad \text{and} \qquad T_0^+(z)\,\widetilde{\mathbb{T}}_\mu(u) = \widetilde{\mathbb{T}}_\mu(u)\,T_0^+(z). \tag{4.35}$$

Repeat the corresponding part of the proof of Theorem 2.4 and use the relations

$$T_0(z)\,\overline{R}_{0a}(z-u-h\,c_a+h\,N/2)\,T_a\big(u+h\,c_a-h\,N/2\big)^{-1} \\ = T_a\big(u+h\,c_a-h\,N/2\big)^{-1}\overline{R}_{0a}(z-u-h\,c_a+h\,N/2)\,T_0(z)$$

implied by (2.4) to get

$$T_0(z)\,\widetilde{\mathbb{T}}_\mu(u) = \widetilde{\mathbb{T}}^{\,\prime}_\mu(u)\,T_0(z),$$

where we set

$$\widetilde{\mathbb{T}}^{\,\prime}_\mu(u) = \mathrm{tr}_{1,\dots,m}\,\mathcal{E}_{\mathcal{U}}\,\overline{R}_{01}^{\,-1}\dots\overline{R}_{0m}^{\,-1}\,T_1^+(u+h\,c_1)\dots T_m^+(u+h\,c_m) \\ \times T_m\big(u+h\,c_m-h\,N/2\big)^{-1}\dots T_1\big(u+h\,c_1-h\,N/2\big)^{-1}\overline{R}_{0m}\dots\overline{R}_{01}.$$

The same argument as in the proof of Theorem 2.4 shows that $\widetilde{\mathbb{T}}^{\,\prime}_\mu(u) = \widetilde{\mathbb{T}}_\mu(u)$ thus verifying the first relation in (4.35). A similar calculation verifies the second relation. It relies on the identity

$$T_0^+(z)\,T_a^+(u+h\,c_a) = \overline{R}_{0a}(z-u-h\,c_a)^{-1}\,T_a^+(u+h\,c_a)\,T_0^+(z)\,\overline{R}_{0a}(z-u-h\,c_a)$$

implied by (2.4), and

$$T_0^+(z)\,\overline{R}_{0a}(z-u-h\,c_a)\,T_a\big(u+h\,c_a-h\,N/2\big)^{-1} \\ = T_a\big(u+h\,c_a-h\,N/2\big)^{-1}\,\overline{R}_{0a}(z-u-h\,c_a+h\,N)\,T_0^+(z)$$

which follows from (2.16) with the use of (2.21). $\square$

The following formula for $\widetilde{\mathbb{T}}_\mu(u)$ in the case where $\mu = (1^N)$ is a column diagram is a consequence of (2.48) and its counterpart for the matrix $T(u)$.

**Proposition 4.4.** *At the critical level $c = -N$ we have*

$$\widetilde{\mathbb{T}}_{(1^N)}(u) = \mathrm{qdet}\,T^+(u)\big(\mathrm{qdet}\,T(u-h\,N/2)\big)^{-1}.$$

By applying $\widetilde{\mathbb{T}}_\mu(u)$ to the vacuum vector of the module $\widehat{\mathcal{V}}_{\mathrm{cri}}$ we get

$$\widetilde{\mathbb{T}}_\mu(u)\,\mathbf{1} = \mathbb{T}_\mu^+(u)\,\mathbf{1}, \tag{4.36}$$

where

$$\mathbb{T}_\mu^+(u) = \mathrm{tr}_{1,\dots,m}\,\mathcal{E}_{\mathcal{U}}\,T_1^+(u+h\,c_1)\dots T_m^+(u+h\,c_m), \tag{4.37}$$

in accordance with Sec. 2.3. In particular, Theorem 2.4 follows from Theorem 4.3; cf. [10]. As another application of (4.36), we get the following.

**Corollary 4.5.** *The coefficients of all series $\mathbb{T}^+_\mu(u)$ generate a commutative subalgebra of the $h$-adically completed dual Yangian $\mathrm{Y}^+(\mathfrak{gl}_N)$.*

*Proof.* Let $\mu$ and $\nu$ be partitions having at most $N$ parts. By Theorem 4.3 we have

$$\widetilde{\mathbb{T}}_\mu(u)\widetilde{\mathbb{T}}_\nu(u)\mathbf{1} = \widetilde{\mathbb{T}}_\mu(u)\mathbb{T}^+_\nu(u)\mathbf{1} = \mathbb{T}^+_\nu(u)\widetilde{\mathbb{T}}_\mu(u)\mathbf{1} = \mathbb{T}^+_\nu(u)\mathbb{T}^+_\mu(u)\mathbf{1}.$$

Swapping the operators, we conclude that the coefficients of the series $\mathbb{T}^+_\mu(u)$ and $\mathbb{T}^+_\nu(u)$ pairwise commute in the dual Yangian. $\square$

*Remark* 4.6. Following (2.39), set

$$\mathbb{T}^+_\mu(u) = \mathrm{tr}_{1,\dots,m}\,\mathcal{E}_{\mathcal{U}}\,T^+_1(u+hc_1)\dots T^+_m(u+hc_m) \in \mathcal{V}_{-N}(\mathfrak{gl}_N)[[u]].$$

By the definition (4.17) of the vertex operators, we have $Y\big(\mathbb{T}^+_\mu(0)\mathbf{1},z\big) = \widetilde{\mathbb{T}}_\mu(z)$, where $\widetilde{\mathbb{T}}_\mu(z)$ is given by (4.34), but the coefficients of this series are now understood as operators on the vacuum module; cf. [11, Sec. 3.2.2]. $\square$

## 4.4 Center of the quantum affine vertex algebra

By Proposition 3.6, the center of a quantum vertex algebra is an associative algebra with respect to the product defined in (3.26). Moreover, due to Proposition 3.7, this algebra is $\mathcal{S}$-commutative, i.e. its elements satisfy (3.27). The results of this section will imply that the center of the quantum affine vertex algebra $\mathcal{V}_c(\mathfrak{gl}_N)$ associated with $\mathfrak{gl}_N$ is *commutative*, so it shares the commutativity property of the center of a vertex algebra; cf. [11, Lemma 3.3.2]. It follows from the definition (3.17) that the center coincides with the subspace of invariants

$$\mathfrak{z}\big(\mathcal{V}_c(\mathfrak{gl}_N)\big) = \{v \in \mathcal{V}_c(\mathfrak{gl}_N) \mid t^{(r)}_{ij}\,v = 0 \qquad \text{for } r \geqslant 1 \text{ and all } i,j\} \tag{4.38}$$

of the $h$-adically completed vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$; cf. (2.34). Hence, $\mathfrak{z}\big(\mathcal{V}_c(\mathfrak{gl}_N)\big)$ can be identified with a subspace of the $h$-adically completed dual Yangian $\mathrm{Y}^+(\mathfrak{gl}_N)$. Moreover, it follows from (4.17) that the product (3.26) on the center coincides with the product in the algebra $\mathrm{Y}^+(\mathfrak{gl}_N)$. Therefore, by Proposition 3.6 the center can be regarded as a $D$-invariant associative subalgebra of the dual Yangian.

Now assume that the level is critical, $c=-N$, and set $\mathcal{V}_{\mathrm{cri}} = \mathcal{V}_{-N}(\mathfrak{gl}_N)$. In Corollaries 2.5 and 2.6 we constructed three families of invariants of the extended vacuum module at the critical level. In accordance with the definition (3.17), we can reformulate these results for the current setting by stating that all coefficients of the series

$$\mathrm{tr}_{1,\dots,m}\,H^{(m)}\,T^+_1(u-hm+h)\dots T^+_m(u), \qquad \mathrm{tr}_{1,\dots,m}\,A^{(m)}\,T^+_1(u)\dots T^+_m(u-hm+h)$$

and

$$\operatorname{tr} T^{+}(u) \dots T^{+}(u - hm + h),$$

belong to the center $\mathfrak{z}(\mathcal{V}_{\text{cri}})$ of the quantum affine vertex algebra $\mathcal{V}_{\text{cri}}$ (we have used the shift $u \mapsto u - hm + h$ for the first series). We will use these families to produce generators of $\mathfrak{z}(\mathcal{V}_{\text{cri}})$. Extend $\mathcal{V}_{\text{cri}} \cong \mathrm{Y}^{+}(\mathfrak{gl}_N)$ to a module over the field $\mathbb{C}((h))$ and introduce its elements as coefficients of the series

$$\Phi_m(u) = h^{-m} \sum_{k=0}^{m} (-1)^k \binom{N-k}{m-k} \operatorname{tr}_{1,\dots,k} A^{(k)}\, T_1^{+}(u) \dots T_k^{+}(u - hk + h),$$

$$\Psi_m(u) = h^{-m} \sum_{k=0}^{m} (-1)^k \binom{N+m-1}{m-k} \operatorname{tr}_{1,\dots,k} H^{(k)}\, T_1^{+}(u - hk + h) \dots T_k^{+}(u),$$

and

$$\Theta_m(u) = h^{-m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \operatorname{tr} T^{+}(u) \dots T^{+}(u - hk + h).$$

Define the coefficients by

$$\Phi_m(u) = \sum_{r=0}^{\infty} \Phi_m^{(r)} u^r, \qquad \Psi_m(u) = \sum_{r=0}^{\infty} \Psi_m^{(r)} u^r \qquad \text{and} \qquad \Theta_m(u) = \sum_{r=0}^{\infty} \Theta_m^{(r)} u^r.$$

**Proposition 4.7.** *All coefficients of the series $\Phi_m(u)$, $\Psi_m(u)$ and $\Theta_m(u)$ belong to the $\mathbb{C}[[h]]$-module $\mathfrak{z}(\mathcal{V}_{\text{cri}})$. Moreover, each family $\Phi_m^{(r)}$, $\Psi_m^{(r)}$ and $\Theta_m^{(r)}$ with $m = 1, \dots, N$ and $r = 0, 1, \dots$ is algebraically independent.*

*Proof.* First consider the series $\Phi_m(u)$. As in Sec. 2.3 embed the dual Yangian into the algebra of formal series $\mathrm{Y}^{+}(\mathfrak{gl}_N)[[u, \partial_u]]$ and introduce the element

$$\operatorname{tr}_{1,\dots,m} A^{(m)} \big(1 - T_1^{+}(u) e^{-h\partial_u}\big) \dots \big(1 - T_m^{+}(u) e^{-h\partial_u}\big) \tag{4.39}$$

as in (2.50). By repeating the corresponding argument in Sec. 2.3 we find that the element (4.39) coincides with

$$\sum_{k=0}^{m} (-1)^k \binom{N-k}{m-k} \operatorname{tr}_{1,\dots,k} A^{(k)}\, T_1^{+}(u) \dots T_k^{+}(u - hk + h) e^{-kh\partial_u}. \tag{4.40}$$

Observe that the constant term of (4.40), as a formal power series in $\partial_u$, coincides with $h^m \Phi_m(u)$. On the other hand, each factor in (4.39) takes the form

$$1 - T_i^{+}(u) e^{-h\partial_u} \equiv h\,(\partial_u + T^{(-1)} + T^{(-2)} u + \dots) \mod h^2\, \mathcal{V}_{\text{cri}},$$

where $T^{(-r)} = [t_{ij}^{(-r)}]$ is the matrix of generators. This shows that the series $h^m \Phi_m(u)$ belongs to $h^m \mathrm{Y}^{+}(\mathfrak{gl}_N)[[u]]$ and so all coefficients of $\Phi_m(u)$ belong to the $\mathbb{C}[[h]]$-module

$\mathfrak{z}(\mathcal{V}_{\text{cri}})$. Furthermore, taking the classical limit $h = 0$ we find that the image of the series $T^{(-1)} + T^{(-2)} u + \dots$ in the algebra $\mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)[[u]]$ coincides with $E_+(u)$ as defined in (2.52). By Corollary 2.9, the family of elements $\phi^{(r)}_{m\,m}$, found as constant terms of the polynomials (2.53) in $\partial_u$, is algebraically independent. Hence so is the family of the coefficients $\Phi^{(r)}_m$. Indeed, if there is a polynomial with coefficients in $\mathbb{C}[[h]]$ providing an algebraic dependence of the $\Phi^{(r)}_m$, we may assume that at least one of its coefficients is not zero modulo $h$. Then the evaluation $h = 0$ makes a contradiction.

The arguments for the families $\Psi_m(u)$ and $\Theta_m(u)$ are quite similar. One additional observation for the family $\Psi_m(u)$ is the identity

$$\mathrm{tr}_{1,\dots,m}\, H^{(m)}\, T^+_1(u - hm + h) \dots T^+_m(u) = \mathrm{tr}_{1,\dots,m}\, T^+_1(u) \dots T^+_m(u - hm + h)\, H^{(m)}.$$

It follows by applying the fusion formula (2.38) for $H^{(m)}$, then the defining relations (2.12) and the conjugation by the longest permutation of $\mathfrak{S}_m$. $\square$

We can now prove a quantum analogue of the Feigin–Frenkel theorem [7]; see Sec. 2.3.

**Theorem 4.8.** *The center at the critical level $\mathfrak{z}(\mathcal{V}_{\text{cri}})$ is a commutative algebra. It is topologically generated by each of the families $\Phi^{(r)}_m$, $\Psi^{(r)}_m$ and $\Theta^{(r)}_m$ with $m = 1, \dots, N$ and $r = 0, 1, \dots$*

*Proof.* First we point out that the coefficients of all series $\Phi_m(u)$, $\Psi_m(u)$ and $\Theta_m(u)$ pairwise commute. This is well-known for the Yangian counterparts of the series introduced in Corollaries 2.5 and 2.6 (with the matrix $T^+(u)$ replaced with $T(u)$) in relation with Bethe subalgebras [19]; see also [21, Ch. 1]. The same proof applies for the dual Yangian. Alternatively, this fact is obtained as a consequence of Corollary 4.5.

Now suppose that $w \in \mathfrak{z}(\mathcal{V}_{\text{cri}})$. We will prove by induction that for all $n \geqslant 0$ there exists a polynomial $Q$ in the variables $\Phi^{(r)}_m$ such that $w - Q \in h^n \mathcal{V}_{\text{cri}}$. Assuming that this holds for some $n \geqslant 0$, write

$$w - Q = h^n w_n + h^{n+1} w_{n+1} + \dots \qquad \text{with } w_k \in V_0,$$

where we assume that $\mathcal{V}_{\text{cri}} = V_0[[h]]$. Since $w - Q$ belongs to the center of the vacuum module, we can conclude that $w_n \in \mathfrak{z}(\mathcal{V}_{\text{cri}}) \mod h$. Taking the classical limit $h = 0$ we find that the image $\overline{w}_n$ of $w_n$ in $\mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)$ belongs to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$. Therefore, $\overline{w}_n$ is a polynomial $S$ in the variables $\phi^{(r)}_{m\,m}$; see Corollary 2.9. Replace these variables with the respective elements $\Phi^{(r)}_m$ to get a polynomial $S' \in \mathfrak{z}(\mathcal{V}_{\text{cri}})$. The difference $w_n - S'$ belongs to $h\mathcal{V}_{\text{cri}}$. Therefore,

$$w - Q - h^n S' \in h^{n+1}\mathcal{V}_{\text{cri}},$$

which completes the induction argument.

Thus, any element $w \in \mathfrak{z}(\mathcal{V}_{\mathrm{cri}})$ can be approximated by polynomials in the variables $\Phi_m^{(r)}$ and so they are topological generators of the center. The same argument works for the other two families. In particular, this implies that the algebra $\mathfrak{z}(\mathcal{V}_{\mathrm{cri}})$ is commutative. $\square$

Finally, consider the quantum affine vertex algebra $\mathcal{V}_c(\mathfrak{gl}_N)$ with $c \neq -N$. The center of the affine vertex algebra $V_\kappa(\mathfrak{gl}_N)$ with $\kappa \neq -N$ is known to be generated by the elements

$$E_{11}[-r-1] + \dots + E_{NN}[-r-1], \qquad r = 0, 1, \dots . \tag{4.41}$$

By Proposition 2.8, the coefficients of the quantum determinant

$$\operatorname{qdet} T^+(u) = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn} \sigma \cdot t^+_{\sigma(1)\,1}(u) \dots t^+_{\sigma(N)\,N}(u - hN + h), \tag{4.42}$$

as defined in (2.47), belong to the center $\mathfrak{z}\big(\mathcal{V}_c(\mathfrak{gl}_N)\big)$. Write

$$\operatorname{qdet} T^+(u) = 1 - h\left(d_0 + d_1\, u + d_2\, u^2 + \dots\right).$$

Under the classical limit $h \to 0$, the image of $d_r$ in $\mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)$ coincides with the element (4.41). The same argument as in the proof of Theorem 4.8 yields the following.

**Proposition 4.9.** *The center $\mathfrak{z}\big(\mathcal{V}_c(\mathfrak{gl}_N)\big)$ with $c \neq -N$ is a commutative algebra. It is topologically generated by the family $d_0, d_1, \dots$ of algebraically independent elements.* $\square$

In particular, $\mathfrak{z}\big(\mathcal{V}_c(\mathfrak{gl}_N)\big)$ is a commutative algebra for all values of $c$.

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