ABSORPTION OF DIRECT FACTORS WITH RESPECT TO THE MINIMAL FAITHFUL PERMUTATION DEGREE OF A FINITE GROUP

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Abstract. The minimal faithful permutation degree $\mu(G)$ of a finite group G is the least nonnegative integer n such that G embeds in the symmetric group $\mathrm{Sym}(n)$. We prove that if H is a group then $\mu(G) = \mu(G \times H)$ for some group G if and only if H embeds in $A \times Q^k$ for some abelian group of odd order, some generalised quaternion 2-group and some nonnegative integer k. As a consequence, $\mu(G^{n+1}) = \mu(G^n)$ for some nonnegative integer n if and only if G is trivial.

1. Introduction

Throughout this paper all groups are assumed to be finite. The *minimal faithful permutation degree* $\mu(G)$ of a group G is the smallest nonnegative integer n such that G embeds in the symmetric group $\mathrm{Sym}(n)$. Note that $\mu(G) = 0$ if and only if G is trivial. For any groups G and H and subgroups S of G, we always have the inequality

$$\mu(G \times H) \leq \mu(G) + \mu(H). \tag{1}$$

Many sufficient conditions are known for equality to occur in (1), for example, when G and H have coprime order (Johnson [5, Theorem 1]), when G and H are nilpotent (Wright [10]), when G and H are direct products of simple groups (Easdown and Praeger [2]), and when $G \times H$ embeds in $\mathrm{Sym}(9)$ (Easdown and Saunders [3]). The first published example where the inequality in (1) is strict appears in Wright [10], where $G \times H$ is a subgroup of $\mathrm{Sym}(15)$. Saunders [6, 7] describes an infinite class of examples, which includes the example in [10] as a special case, where strict inequality takes place in (1). The smallest example in his class occurs when $G \times H$ embeds in $\mathrm{Sym}(10)$. In all of these examples of strict inequality, the groups G and H have the properties that H is cyclic of prime order and

$$\mu(G \times H) = \mu(G). \tag{2}$$

In this article, when (2) occurs, we say that H is *absorbed by* G. Our main theorem below (Theorem 4.1) is that a group H is absorbed by some group G if and only if H embeds in the direct product of an abelian group of odd order with some power of a generalised quaternion 2-group. A consequence (Corollary 4.2) is that it is never possible for some power of a nontrivial group to absorb a further copy of itself. The forward direction of the proof of Theorem 4.1 invokes a number of classical results about group actions, centralisers of subgroup of symmetric groups and wreath products. The backward direction involves the delicate construction of appropriate examples, and is motivated by the ideas that appear in the work of Saunders [6, 7], the seminal example in Wright's original paper [10] and

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Hendriksen's thesis [4]. These examples are also closely related to examples in the last section of [1], where it is shown that is is possible to absorb an arbitrary large finite direct product H of elementary abelian groups (with possibly mixed primes) using a group G that does not decompose as a nontrivial direct product.

2. Preliminaries

Recall that if G is nontrivial then $\mu(G)$ is the smallest sum of indices for a collection of subgroups $\mathscr{C} = \{H_1, \ldots, H_k\}$ such that $\cap_{i=1}^k H_i$ is core-free. In this case we say that *$\mathscr{C}$ affords a minimal faithful representation of G.* The subgroups $H_1, \ldots, H_k$ become the respective point-stabilisers for the action of G on its orbits and letters in the ith orbit may be identified with cosets of H_i for $i = 1 \ldots, k$. If $k = 1$ then the representation afforded by $\mathscr{C}$ is transitive and H_1 is a core-free subgroup.

Lemma 2.1. *Let $\mathscr{C}$ be a collection of subgroups affording a minimal faithful representation of a group G. Let $\mathscr{D}$ be a nonempty subset of $\mathscr{C}$. Then $\{K/N \mid K \in \mathscr{D}\}$ affords a minimal faithful representation of G/N where*

$$N \;=\; \bigcap_{H\in\mathscr{D}} \operatorname{core}(H)\ .$$

In particular, if $H \in \mathscr{C}$ then $H/\operatorname{core}(H)$ affords a minimal faithful transitive representation of $G/\operatorname{core}(H)$.

Proof. Note that

$$\bigcap_{K\in\mathscr{D}} \operatorname{core}_{G/N}(K/N) \;=\; \Big(\bigcap_{K\in\mathscr{D}} \operatorname{core}_G(K)\Big)/N \;=\; N/N\ ,$$

so that $\{K/N \mid K \in \mathscr{D}\}$ affords a faithful representation of G/N. If this representation is not minimal, then we could replace $\mathscr{D}$ by $\mathscr{E}$ with a smaller index sum and the same core intersection, and then $(\mathscr{C}\backslash\mathscr{D}) \cup \mathscr{E}$ would afford a faithful representation of G of smaller degree than that afforded by $\mathscr{C}$, which is a contradiction. $\square$

Remark 2.2. The previous lemma is a reformulation of part (i) of Lemma 2.7 of [3].

Lemma 2.3. *Let G be a transitive subgroup of $\mathrm{Sym}(X)$ and H the stabiliser of a point in X. Then $C_{\mathrm{Sym}(X)}(G) \cong N_G(H)/H$. Any subgroup K of $C_{\mathrm{Sym}(X)}(G)$ acts semiregularly on X and the orbits of K form a block system for G.*

Proof. Put $C = C_{\mathrm{Sym}(X)}(G)$. The isomorphism and semiregular action of C are well-known (see [11, Theorem 5]). Let K be a subgroup of C, so K inherits the semiregular action of C. Let $x, y \in X$ lie in the same orbit of K and $g \in G$. Then $y = xk$ for some $k \in K$, so that $yg = (xk)g = (xg)k$, whence xg and yg lie in the same orbit of K. Thus the orbits of K form a block system for G. $\square$

Let G be a group and H a subgroup of $\mathrm{Sym}(n)$, where n is a positive integer. Recall that the *wreath product of G by H* is the group

$$G \wr H \;=\; \{(x_1, \ldots, x_n, h) \mid x_1, \ldots, x_n \in G, h \in H\}$$

with multiplication

$$(x_1,\ldots,x_n,h_1)(y_1,\ldots,y_n,h_2) \;=\; (x_1y_{1h_1},\ldots,x_ny_{nh_1},h_1h_2)\,.$$

Then

$$G\wr H \;\cong\; G^n \rtimes_\phi H$$

with respect to the homomorphism $\phi: H\to \mathrm{Aut}(G^n)$ defined by, for $h\in H$,

$$h\phi : (x_1,\ldots,x_n)\mapsto (x_{1h},\ldots,x_{nh})\,.$$

Consider the case where G is a subgroup of $\mathrm{Sym}(m)$, where m is a positive integer. Then $G\wr H$ may be regarded as a subgroup of $\mathrm{Sym}(X)\cong \mathrm{Sym}(mn)$ where $X=\{(i,j)\mid 1\le i\le m\,,\ 1\le j\le n\}$ with permutation action

$$(x_1,\ldots,x_n,h):(i,j)\mapsto (ix_j,jh)\,.$$

Then G and H may be identified with the subgroups

$$\{(g,1_G\ldots,1_G,1_H)\mid g\in G\}\quad\text{and}\quad \{(1_G,\ldots,1_G,h)\mid h\in H\}$$

respectively. Each permutation $g\in G$ (on m letters) may then be identified with the permutation (on mn letters) that maps $(i,1)$ to $(ig,1)$ for $1\le i\le m$, and fixes (i,j) if $j>1$, so that we may also identify G in a natural way with a subgroup of $\mathrm{Sym}(\{(1,1),\ldots,(m,1)\})$. Each permutation $h\in H$ (on n letters) then becomes identified with the permutation (on mn letters) that maps each (i,j) to (i,jh). If, further, G and H are transitive and $m\ge 2$, then this action is also transitive with blocks of imprimitivity $X_j=\{(i,j)\mid 1\le i\le m\}$ for $1\le j\le n$, and H may also be identified with a subgroup of $\mathrm{Sym}(\{X_1,\ldots,X_n\})$.

Lemma 2.4. *Let G be a transitive subgroup of $\mathrm{Sym}(X)$ such that $X=X_1\sqcup\ldots\sqcup X_n$, where $X_1,\ldots,X_n$ are blocks of imprimitivity. Put $\widetilde{G}=\{g_{|X_1}\mid g\in G \text{ and } X_1g=X_1\}$, a subgroup of $\mathrm{Sym}(X_1)$, and let $\overline{G}$ be the subgroup of $\mathrm{Sym}\{X_1,\ldots,X_n\}$ induced by G, regarding blocks as points. Then G embeds in the wreath product $\widetilde{G}\wr\overline{G}$. If $\mu(G)=|X|$ then $\mu(\widetilde{G})=|X_1|$.*

Proof. If $n=1$ then $G=\widetilde{G}$ and $X=X_1$, and the statements are trivially true. We may therefore suppose throughout that $n>1$.

Put $W=\widetilde{G}\wr\overline{G}$, which may be regarded as a subgroup of $\mathrm{Sym}(X)$. There is a natural mapping $^-: g\mapsto \overline{g}$ from G onto $\overline{G}$ with kernel

$$K \;=\; \{g\in G\mid X_ig=X_i \ \text{ for }\ 1\le i\le n\}\,.$$

Put $S=\{g\in G\mid X_1g=X_1\}$. Then $K\le S$, $|G:S|=|\overline{G}:\overline{S}|=n$ and

$$\mathrm{core}_{\overline{G}}(\overline{S}) \;=\; \overline{K} \;=\; \{\overline{1}\}\,.$$

For each $i\in\{1,\ldots,n\}$, there exists $x_i\in X_i$ and $h_i\in G$ such that $h_i: x_1\mapsto x_i$ (and we may take $h_1=1$, the identity permutation). For each i, put

$$G_i \;=\; \{g_{|X_i}\mid g\in G \ \text{ and }\ X_ig=X_i\} \;\cong\; \widetilde{G}\,,$$

and, for $g\in G$, put

$$g_i \;=\; (h_igh_{i\overline{g}}^{-1})|_{X_1} \;\in\; \mathrm{Sym}(X_1)\,.$$

Note that $G_1=\widetilde{G}$. It is well-known that the mapping

$$g\mapsto \big((g_1,\ldots,g_n),\overline{g}\big)\,,$$

for $g \in G$, is an embedding from G into W. We may regard

$$W \;=\; G_1 G_2 \ldots G_n H$$

as an internal semidirect product of an internal direct product $G_1G_2\ldots G_n$ of n copies of $\widetilde{G}$ by a group H, where $H \cong \overline{G}$ is the copy of $\overline{G}$ regarded as a group of permutations of X, in the manner described in the preamble to the lemma when introducing the wreath product. If $h \in H$ corresponds to $\overline{g} \in \overline{G}$, for $g \in G$, then, under these identifications, $G_i^h = G_{i\overline{g}}$, for each i. Let T be the subgroup of H corresponding to $\overline{S}$, so $|H : T| = n$ and T is core-free in W. Put

$$B \;=\; G_2 \ldots G_n T,$$

which is a subgroup of W, since S fixes X_1. By transitivity, it follows that B is also core-free in W.

Suppose that $\mu(G) = |X|$ and let $\mathscr{C} = \{C_1, \ldots, C_k\}$ be a collection of subgroups of $\widetilde{G}$ that affords a minimal faithful representation of $\widetilde{G}$. Then $\mathscr{C}$ may be regarded as a collection of subgroups of G_1 in the identification of W with $G_1G_2\ldots G_nH$. Note that all elements of G_1 commute with all elements of $G_2\ldots G_nT$. Hence $C_1B, \ldots, C_kB$ are all subgroups of W. Put

$$\mathscr{D} \;=\; \{C_1B, \ldots, C_kB\}\ ,$$

and observe that, since the representation afforded by $\mathscr{C}$ is faithful,

$$\begin{aligned}
\bigcap_{j=1}^{k} \mathrm{core}_W(C_jB) \;&=\; \mathrm{core}_W\left(\bigcap_{j=1}^{k} \mathrm{core}_W(C_jB)\right) \;=\; \mathrm{core}_W\left(\bigcap_{j=1}^{k}\bigcap_{w\in W}(C_jB)^w\right) \\
&\subseteq\; \mathrm{core}_W\left(\bigcap_{j=1}^{k}\bigcap_{g\in G_1}(C_jB)^g\right) \;=\; \mathrm{core}_W\left(\bigcap_{j=1}^{k}\bigcap_{g\in G_1} C_j^g B\right) \\
&=\; \mathrm{core}_W\left(\left(\bigcap_{j=1}^{k}\mathrm{core}_{G_1}(C_i)\right)B\right) \;=\; \mathrm{core}_W(B) \;=\; \{1\}\ .
\end{aligned}$$

Hence the representation of W afforded by $\mathscr{D}$ is faithful. Further its degree is

$$\sum_{i=1}^{k}|W : C_iB| \;=\; \sum_{i=1}^{k}|W : G_1B||G_1B : C_iB| \;=\; |H : T|\sum_{i=1}^{k}|G_1 : C_i| \;=\; n\mu(\widetilde{G})\ .$$

But G embeds in W so that $\mu(G) \leq \mu(W) \leq n\mu(\widetilde{G})$, whence

$$\mu(\widetilde{G}) \;\geq\; \frac{\mu(G)}{n} \;=\; \frac{|X|}{n} \;=\; |X_1|\ .$$

But $\widetilde{G}$ is a subgroup of $\mathrm{Sym}(X_1)$, so $\mu(\widetilde{G}) \leq |X_1|$, and the lemma follows immediately. □

Lemma 2.5. *Let K and q be integers such that either $K \geq 2$ and $q \geq 5$, or $K \geq 5$ and $q \geq 3$. Then $q < K^{q-2}$.*

Proof. This follows by a simple induction on q. □

Lemma 2.6. *Let p and q be distinct primes and n any positive integer. Let s be the multiplicative order of p modulo q and suppose that $s \geq 2$. Then $p^nq < 2p^{ns}$.*

Proof. By [1, Lemma 4.3], $pq < 2p^s$. The result follows by a simple induction on n. □

3. Examples

The examples in this section are used to prove the backward direction of Theorem 4.1 below. Example 3.1 describes absorption of the smallest possible nonabelian group, namely Q_8. This is a special case of absorption of arbitrary powers of generalised quaternion 2-groups, described in Example 3.3. However, Example 3.1 is included, not just because it is interesting in its own right, but also to ease the transition for the reader in negotiating the delicate detail required to verify the claims made for the general class. If we take $p = 2$, $q = 5$ and $m = n = 1$ in Example 3.2 then we recover the smallest example of absorption described in [6], and it occurs within Sym(10). That no example of absorption can occur within Sym(9) follows by results in [8] and the main theorem of [3]. If we take $p = 5$, $q = 3$ and $m = n = 1$ in Example 3.2 then we recover the seminal example that appears in [10], the first published example demonstrating that equality can fail in (1).

Example 3.1. Put $X = \{x_{i,j} \mid 1 \leq i \leq 3\,,\ 1 \leq j \leq 8\}$, regarded as a set of 24 distinct letters. Consider the following permutations of X, for $1 \leq i \leq 3$, which together generate a copy of Q_8, the group of quaternions:

$$\alpha_i \;=\; (x_{i,1}\, x_{i,2}\, x_{i,3}\, x_{i,4})(x_{i,5}\, x_{i,6}\, x_{i,7}\, x_{i,8})\,,\quad \beta_i \;=\; (x_{i,1}\, x_{i,5}\, x_{i,3}\, x_{i,7})(x_{i,2}\, x_{i,8}\, x_{i,4}\, x_{i,6})\,.$$

Now put

$$a_i \;=\; \alpha_i\alpha_{i+1}^{-1}\,,\quad b_i \;=\; \beta_i\beta_{i+1}^{-1}\,,$$

for $1 \leq i \leq 2$, which together again generate a copy of Q_8. For $1 \leq j \leq 8$, consider the following 3-cycle:

$$\gamma_j \;=\; (x_{1,j}\, x_{2,j}\, x_{3,j})\,,$$

and put $c = \gamma_1\gamma_2\ldots\gamma_8$. Put

$$A \;=\; \langle a_1\,,\, b_1\,,\, a_2\,,\, b_2\rangle \;\cong\; Q_8^2 \qquad \text{and} \qquad C \;=\; \langle c\rangle \;\cong\; C_3\,.$$

Consider the following subgroup G of $\mathrm{Sym}(X)$:

$$G \;=\; AC \;\cong\; Q_8^2 \rtimes C_3\,,$$

which may be regarded as an internal semidirect product of A by C, where the conjugation action of c is completely determined by

$$a_1 \;\mapsto\; a_2 \;\mapsto\; a_1^{-1}a_2^{-1} \;\mapsto\; a_1\,, \qquad b_1 \;\mapsto\; b_2 \;\mapsto\; b_1^{-1}b_2^{-1} \;\mapsto\; b_1\,.$$

We claim that $\mu(G) = 24$. Certainly, since $|X| = 24$, we have $\mu(G) \leq 24$. Observe first that

$$M \;=\; \langle a_1^2\,,\, a_2^2\rangle \;\cong\; C_2^2$$

is the unique minimal normal subgroup of G, on which c acts irreducibly by conjugation, and that $a_i^2 = b_i^2$ is contained in any nontrivial subgroup of $\langle a_i, b_i\rangle \cong Q_8$, for $1 \leq i \leq 2$. Any collection of subgroups of G affording a minimal faithful representation of G must contain a subgroup S that does not contain M. If S has order divisible by 3 then the order of S must be 3, for otherwise, by the irreducible conjugation action of c on M, S would contain M. But in this case, the index of S in G would be $64 > 24$, contradicting that $\mu(G) \leq 24$.

Hence S is a subgroup of A that intersects $\langle a_i, b_i\rangle$ trivially for some i. But then $|A : S| \geq 8$, so $|G : S| \geq 24$, which completes the proof that $\mu(G) = 24$. Put

$$\delta_i = (x_{i,1}\ x_{i,2}\ x_{i,3}\ x_{i,4})(x_{i,5}\ x_{i,6}\ x_{i,7}\ x_{i,8})^{-1}\ ,\ \ \varepsilon_i = (x_{i,1}\ x_{i,5}\ x_{i,3}\ x_{i,7})(x_{i,2}\ x_{i,8}\ x_{i,4}\ x_{i,6})^{-1}\ .$$

for $1 \leq i \leq 3$, which generate another copy of Q_8 and also commute with α_i and β_i. Finally, put

$$h_1 \ = \ \delta_1\delta_2\delta_3 \qquad \text{and} \qquad h_2 \ = \ \varepsilon_1\varepsilon_2\varepsilon_3\ ,$$

and

$$H \ = \ \langle h_1, h_2\rangle \ \cong \ Q_8\ .$$

Then elements of H commute with elements of G and, by a parity argument applied to elements of order 2 (since elements of M are even whilst the unique element of H of order two is an odd permutation), we have $H \cap G = \{1\}$. Thus $GH \cong G \times H \cong (Q_8^2 \rtimes C_3) \times Q_8$ is a subgroup of $\mathrm{Sym}(X)$ that is an internal direct product of G and H. Hence $\mu(GH) \leq |X| = 24 = \mu(G) \leq \mu(GH)$, whence

$$\mu(G \times H) \ = \ 24 \ = \ \mu(G)\ ,$$

exhibiting the property of absorption of H, a copy of Q_8. This, we believe, is the first published example of absorption of a nonabelian group. By Theorem 4.1 below, Q_8 is the smallest nonabelian group, up to isomorphism, that can be absorbed by another group. By Remark 4.3 below, this example is minimal also in the sense that it is impossible to find a subgroup of $\mathrm{Sym}(23)$ that absorbs a nonabelian group.

Example 3.2. Let p and q be distinct primes such that $q \geq 5$, or $q \geq 3$ and $p \geq 5$. Suppose that q does not divide $p - 1$. Let m and n be positive integers. Put

$$X \ = \ \{x_{i,j,k} \mid 1 \leq i \leq q\ ,\ 1 \leq j \leq p^n\ ,\ 1 \leq k \leq m\}\ ,$$

which we may regard as a set of mp^nq distinct letters. For $i = 1$ to q and $k = 1$ to m, consider the p^n-cycle

$$\sigma_{i,k} \ = \ (x_{i,1,k}\ x_{i,2,k}\ \ldots\ x_{i,p^n,k})\ .$$

For $j = 1$ to p^n and $k = 1$ to m, consider the q-cycle

$$\tau_{j,k} \ = \ (x_{1,j,k}\ x_{2,j,k}\ \ldots\ x_{q,j,k})\ .$$

For $k = 1$ to m, put

$$a_{1,k} = \sigma_{1,k}\sigma_{2,k}^{-1}\ ,\quad a_{2,k} = \sigma_{2,k}\sigma_{3k}^{-1}\ ,\quad \ldots\ ,\quad a_{q-1,k} = \sigma_{q-1,k}\sigma_{q,k}^{-1}\ ,$$

and

$$b = \tau_{1,1}\ldots\tau_{p^n,1}\ \tau_{1,2}\ldots\tau_{p^n,2}\ \ldots\ \tau_{1,m}\ldots\tau_{p^n,m}\ .$$

Now put

$$G \ = \ \langle a_{i,k}, b \mid 1 \leq i \leq q-1\ ,\ 1 \leq k \leq m\rangle\ ,$$

which is a subgroup of $\mathrm{Sym}(X)$. Here all of the $a_{i,k}$ have order p^n and pairwise commute, for $1 \leq i \leq q$ and $1 \leq k \leq m$. Further, b is an element of order q, acting by conjugation on the abelian subgroup

$$A = \langle a_{i,k} \mid 1 \leq i \leq q-1\ ,\ 1 \leq k \leq m\rangle \ \cong \ C_{p^n}^{(q-1)m}$$

in a manner completely determined by

$$a_{1,k} \mapsto a_{2,k} \mapsto \ldots \mapsto a_{q-1,k} \mapsto a_{1,k}^{-1}a_{2,k}^{-1}\ldots a_{q-1,k}^{-1} \mapsto a_{1,k} \ ,$$

for $1 \leq k \leq m$. Put $B = \langle b \rangle \cong C_q$. Thus $G = AB$ is a subgroup of $\mathrm{Sym}(X)$ that is an internal semidirect product of A by B, and G is isomorphic to a semidirect product $C_{p^n}^{(q-1)m} \rtimes C_q$. We claim that

$$\mu(G) \ = \ mp^n q \ .$$

By construction, $\mu(G) \leq mp^n q$, so, to verify the claim, it suffices to show $\mu(G) \geq mp^n q$. Suppose that $\mathscr{C}$ is a collection of subgroups of G affording a minimal faithful representation. In particular, $\mathscr{C}$ has a trivial core intersection. By [5, Lemma 11], we may suppose all elements of $\mathscr{C}$ are meet irreducible. Note that, if S is a subgroup of G with order divisible by q, then $S = TB^g$ is an internal semidirect product of a subgroup T, which is normal in G, by B^g for some $g \in G$, in which case TB is also a subgroup of G such that $\mathrm{core}(S) = \mathrm{core}(TB) = T$ and $|G : S| = |G : TB|$. Thus, without loss of generality, we may assume

$$\mathscr{C} \ = \ \{H_1, \ldots, H_\ell, T_1B, \ldots, T_tB\}$$

for some $\ell \geq 0$ and $t \geq 0$, where $H_1, \ldots, H_\ell$ are subgroups of A, and $T_1, \ldots, T_t$ are subgroups of A that are normal in G. (In fact, we will show $\ell = m$ and $t = 0$, so that $\mathscr{C}$ only involves subgroups of A.) If E is a subgroup of A, put

$$\overline{E} \ = \ \{e \in E \mid \text{the order of } e \text{ is } p \text{ or } 1\} \ ,$$

which is an elementary abelian subgroup of E. Now put

$$\overline{G} \ = \ \overline{A}B \ \cong \ C_p^{(q-1)m} \rtimes C_q \quad \text{and} \quad \overline{\mathscr{C}} \ = \ \{\overline{H_1}, \ldots, \overline{H_\ell}, \overline{T_1}B, \ldots, \overline{T_t}B\} \ .$$

Then $\overline{\mathscr{C}}$ also has trivial core intersection, so affords a faithful permutation representation of $\overline{G}$. It follows that subgroups in $\overline{\mathscr{C}}$ are also meet irreducible. We may identify $\overline{G}$ with the vector space semidirect product

$$V \rtimes M$$

in the sense of [1, Section 3], where $V = \overline{A} \cong C_p^{(q-1)m}$ is regarded as a vector space of dimension $(q-1)m$ over the field with p elements, and M is the matrix of multiplicative order q formed by taking a direct sum of m companion matrices of the polynomial $\pi(x) = 1 + x + \ldots + x^{q-1}$. Then $\pi(x)$ is the minimal polynomial of M and a product of distinct irreducible polynomials of degree s where s is the multiplicative order of p modulo q. Note that $s \geq 2$, since q does not divide $p-1$. Hence, by meet irreducibility, $\overline{H_1}, \ldots, \overline{H_\ell}$ may be identified with codimension 1 subspaces of V, and $\overline{T_1}, \ldots, \overline{T_t}$ with subspaces of V invariant under the action of M and of codimension s. By a dimension argument, when we make these identifications, $\mathrm{core}(\overline{H_i})$ is a subspace of V of codimension at most $q-1$, the degree of $\pi(x)$, for $i = 1$ to ℓ. Put

$$W \ = \ \bigcap_{i=1}^{\ell} \mathrm{core}(\overline{H_i}) \qquad \text{and} \qquad Z \ = \ \bigcap_{j=1}^{t} \mathrm{core}(\overline{T_j}B) \ = \ \bigcap_{j=1}^{t} \overline{T_j} \ .$$

Then the codimensions of W and Z in V are at most $(q-1)\ell$ and $st \leq (q-1)t$ respectively. But $W \cap Z$ is trivial and the dimension of V is $(q-1)m$. Thus $(q-1)\ell + st \geq (q-1)m$, and so

$$t \;\geq\; \Big(\frac{q-1}{s}\Big)(m-\ell)\;. \tag{3}$$

Since $\overline{H_i}$ has codimension 1 in A, it follows that $|G : H_i| \geq p^n q$ for $i = 1$ to ℓ. Since $\overline{T_j}$ has codimension s in A, it follows that $|G : T_j B| \geq p^{ns}$ for $j = 1$ to t. Thus we have

$$mp^n q \;\geq\; \mu(G) \;=\; \sum_{S\in\mathscr{C}} |G : S| \;=\; \sum_{i=1}^{\ell} |G : H_i| + \sum_{j=1}^{t} |G : T_j B| \;\geq\; \ell p^n q + tp^{ns}\;,$$

and so

$$tp^{ns} \;\leq\; (m-\ell)p^n q\;. \tag{4}$$

Suppose $\ell < m$. From (3) and (4), we get

$$\Big(\frac{q-1}{s}\Big)p^{ns} \;\leq\; p^n q\;. \tag{5}$$

If $s = q - 1$ then (5) gives

$$p^{n(q-1)} \;\leq\; p^n q\;,$$

which contradicts Lemma 2.5. If $s \neq q - 1$, then s is a proper divisor of $q - 1$, so that (5) now gives

$$2p^{ns} \;\leq\; p^n q\;,$$

which contradicts Lemma 2.6. Hence $\ell \geq m$. If $\ell > m$ then the right-hand side of (4) is negative, which is impossible. Hence $\ell = m$, so that the right-hand side of (4) is zero, whence $t = 0$. Thus $\mathscr{C} = \{H_1, \ldots, H_\ell\}$ and

$$\mu(G) \;=\; \sum_{i=1}^{\ell} |G : H_i| \;\geq\; \ell p^n q \;=\; mp^n q\;,$$

whence $\mu(G) = mp^n q$, as claimed. Now, for each $k = 1$ to m, consider the permutation

$$h_k \;=\; \sigma_{1,k}\sigma_{2,k}\ldots\sigma_{q,k}\;,$$

which has order p^n, shifts letters only within the orbit

$$X_k \;=\; \{x_{i,j,k} \mid 1 \leq i \leq q\,,\; 1 \leq j \leq p^n\}$$

of G, and commutes with all elements of G. Put

$$H \;=\; \langle h_1, \ldots, h_m\rangle \;\cong\; C_{p^n}^m\;.$$

Note that all elements of A are products of elements of the form $\sigma_{1,k}^{\varepsilon_{1,k}} \ldots \sigma_{q,k}^{\varepsilon_{q,k}}$, which are permutations that shift only letters in X_k, where $\varepsilon_{1,k} + \ldots + \varepsilon_{q,k} = 0 \bmod p^n$, for $k = 1$ to m. Because $X = X_1 \cup \ldots \cup X_m$ is a disjoint union of orbits, it follows that $H \cap G = \{1\}$. Thus

$$GH \;\cong\; (C_{p^n}^{(q-1)m} \rtimes C_q) \times C_{p^n}^m$$

is a subgroup of $\mathrm{Sym}(X)$ that is an internal direct product of G and H. By construction $\mu(GH) \leq mp^n q = \mu(G) \leq \mu(GH)$, whence

$$\mu(G \times H) \;=\; mp^n q \;=\; \mu(G)\;,$$

exhibiting the property of absorption of H, a copy of $C_{p^n}^m$.

Example 3.3. Let $q \geq 5$ be a prime. Let m and n be positive integers, and put $M = 2^n$ and $N = 2M$. Put

$$X \;=\; \{x_{i,j,k} \mid 1 \leq i \leq q\,,\ 1 \leq j \leq N\,,\ 1 \leq k \leq m\}$$

and

$$Y \;=\; \{y_{i,j,k} \mid 1 \leq i \leq q\,,\ 1 \leq j \leq N\,,\ 1 \leq k \leq m\}\ ,$$

which we may regard as disjoint sets, each containing mqN distinct letters. For $i = 1$ to q and $k = 1$ to m, consider the following product of two disjoint N-cycles:

$$\alpha_{i,k} \;=\; (x_{i,1,k}\ x_{i,2,k}\ \ldots\ x_{i,N,k})(y_{i,1,k}\ y_{i,2,k}\ \ldots\ y_{i,N,k})\ ,$$

and the following product of M disjoint 4-cycles:

$$\begin{aligned}\beta_{i,k} \;=\; &(x_{i,1,k}\ y_{i,1,k}\ x_{i,M+1,k}\ y_{i,M+1,k})(x_{i,2,k}\ y_{i,N,k}\ x_{i,M+2,k}\ y_{i,M,k})\\ &\ldots(x_{i,M-1,k}\ y_{i,M+3,k}\ x_{i,N-1,k}\ y_{i,3,k})(x_{i,M,k}\ y_{i,M+2,k}\ x_{i,N,k}\ y_{i,2,k})\ ,\end{aligned}$$

which together generate a copy of the generalised quaternion 2-group Q_{4M} (which is easily checked, since $\alpha_{i,k}^M = \beta_{i,k}^2$ and $\alpha_{i,k}^{\beta_{i,k}} = \alpha_{i,k}^{-1}$). Now put

$$a_{i,k} \;=\; \alpha_{i,k}\alpha_{i+1,k}^{-1}\,,\quad b_{i,k} \;=\; \beta_{i,k}\beta_{i+1,k}^{-1}\ ,$$

for $1 \leq i \leq q-1$ and $1 \leq k \leq m$, which together again generate a copy of Q_{4M}. For $1 \leq j \leq N$ and $1 \leq k \leq m$, consider the following product of two disjoint q-cycles:

$$\gamma_{j,k} \;=\; (x_{1,j,k}\ x_{2,j,k}\ \ldots\ x_{q,j,k})(y_{1,j,k}\ y_{2,j,k}\ \ldots\ y_{q,j,k})\ .$$

Now put

$$c \;=\; \gamma_{1,1}\ldots\gamma_{1,k}\ \gamma_{2,1}\ldots\gamma_{2,k}\ \ldots\ \gamma_{N,1}\ldots\gamma_{N,k}\ .$$

Put

$$A \;=\; \langle a_{i,k}\,,\ b_{i,k} \mid 1 \leq i \leq q-1\,,\ 1 \leq k \leq m\rangle \;\cong\; Q_{4M}^{q-1} \qquad \text{and} \qquad C \;=\; \langle c\rangle \;\cong\; C_q\ .$$

Consider the following subgroup G of $\mathrm{Sym}(X \cup Y)$:

$$G \;=\; AC \;\cong\; Q_{4M}^{q-1} \rtimes C_q\ ,$$

which may be regarded as an internal semidirect product of A by C, where the conjugation action of c is completely determined by, for $1 \leq k \leq m$,

$$\begin{aligned}a_{1,k} \;&\mapsto\; a_{2,k} \;\mapsto\; \ldots \;\mapsto\; a_{q-1,k} \;\mapsto\; a_{1,k}^{-1}a_{2,k}^{-1}\ldots a_{q-1,k}^{-1} \;\mapsto\; a_{1,k}\ ,\\ b_{1,k} \;&\mapsto\; b_{2,k} \;\mapsto\; \ldots \;\mapsto\; b_{q-1,k} \;\mapsto\; b_{1,k}^{-1}b_{2,k}^{-1}\ldots b_{q-1,k}^{-1} \;\mapsto\; b_{1,k}\ .\end{aligned}$$

We claim that

$$\mu(G) \;=\; |X \cup Y| \;=\; 2mqN\ .$$

By construction, $\mu(G) \leq 2mqN$, so, to verify the claim, it suffices to show $\mu(G) \geq 2mqN$. This follows, however, by exactly the same technique used in Example 3.2, by assuming the inequality is false, and then relying on dimension arguments involving action of C_q on an elementary abelian subgroup regarded as a vector space, this time over the field with two elements (relying on the fact that a generalised quaternion 2-group has a unique element of order two), and finally reaching contradictions by Lemmas 2.5 and 2.6. For $i = 1$ to q and $k = 1$ to m, consider the following product of two disjoint N-cycles:

$$\delta_{i,k} \;=\; (x_{i,1,k}\ x_{i,2,k}\ \ldots\ x_{i,N,k})(y_{i,1,k}\ y_{i,2,k}\ \ldots\ y_{i,N,k})^{-1}\ ,$$

and the following product of M disjoint 4-cycles:

$$\begin{aligned}\varepsilon_{i,k} \;=\; &(x_{i,1,k}\;\, y_{i,M+2,k}\;\, x_{i,M+1,k}\;\, y_{i,2,k})(x_{i,2,k}\;\, y_{i,M+3,k}\;\, x_{i,M+2,k}\;\, y_{i,3,k})\\ &\qquad\qquad \dots (x_{i,M-1,k}\;\, y_{i,N,k}\;\, x_{i,N-1,k}\;\, y_{i,M,k})(x_{i,M,k}\;\, y_{i,1,k}\;\, x_{i,N,k}\;\, y_{i,M+1,k})\;,\end{aligned}$$

which together generate a copy of the generalised quaternion 2-group Q_{4M} (which is easily checked, since $\delta_{i,k}^M = \varepsilon_{i,k}^2$ and $\delta_{i,k}^{\varepsilon_{i,k}} = \delta_{i,k}^{-1}$), and commute with $\alpha_{i,k}$ and $\beta_{i,k}$. Finally, put

$$h_{1,k} \;=\; \delta_{1,k}\dots\delta_{q,k}\;, \qquad h_{2,k} \;=\; \varepsilon_{1,k}\dots\varepsilon_{q,k}\;,$$

which also together generate a copy of Q_{4M}, for $1 \le k \le m$, and

$$H \;=\; \langle h_{1,1}, h_{2,1}, \dots, h_{1,m}, h_{2,m}\rangle \;\cong\; Q_{4M}^m\;.$$

Then elements of H commute with elements of G and, by a parity argument applied to the subgroup of A generated by the central elements of order 2, and considering separate orbits, we have $H \cap G = \{1\}$. Thus

$$GH \;\cong\; G \times H \;\cong\; (Q_{4M}^{(q-1)m} \rtimes C_q) \times Q_{4M}^m$$

is a subgroup of $\mathrm{Sym}(X \cup Y)$ that is an internal direct product of G and H. By construction $\mu(GH) \le 2mqN = \mu(G) \le \mu(GH)$, whence

$$\mu(G \times H) \;=\; 2mqN \;=\; \mu(G)\;,$$

exhibiting the property of absorption of H, a copy of Q_{4M}^m.

4. Main Theorem

Theorem 4.1. *If H is a group then $\mu(G \times H) = \mu(G)$ for some group G if and only if H is a subgroup of $A \times Q^k$ for some abelian group A of odd order, some generalised quaternion 2-group Q and some nonnegative integer k.*

Proof. Suppose that G and H are groups such that $\mu(G \times H) = \mu(G)$. We may suppose throughout that H is nontrivial (and of course this entails that G is nontrivial). Let $\mathscr{C} = \{S_1, \dots, S_\ell\}$ be a collection of subgroups of $G \times H = GH$ that affords a minimal faithful representation Ψ of GH of degree n, where GH is regarded as a subgroup of $\mathrm{Sym}(n)$ that is an internal direct product of G and H. For each i, denote by Ψ_i the ith transitive component of Ψ corresponding to S_i, so that the kernel of Ψ_i is the core of S_i. Put

$$\mathscr{C}_G \;=\; \{G \cap S_1, \dots, G \cap S_\ell\}\;.$$

Observe first that, for each i, any normal subgroup of G contained in $G \cap S_i$ is also normal in GH, since elements of G commute with elements of H, so that

$$G \cap \mathrm{core}_{GH}(S_i) \;=\; \mathrm{core}_G(G \cap S_i)\;.$$

By faithfulness, $\bigcap_{i=1}^{\ell} \mathrm{core}_{GH}(S_i) = \{1\}$, and it follows that $\bigcap_{i=1}^{\ell} \mathrm{core}_G(G \cap S_i) = \{1\}$. Hence $\mathscr{C}_G$ affords a faithful representation of G. Observe also that, for each i,

$$\begin{aligned} |H||G : G \cap S_i| \ &= \ |GH : G||G : G \cap S_i| \ = \ |GH : G \cap S_i| \\ &= \ |GH : S_i||S_i : G \cap S_i| \ = \ |GH : S_i||GS_i : G| \\ &\leq \ |GH : S_i||GH : G| \ = \ |H||GH : S_i| \ , \end{aligned}$$

so that $|G : G \cap S_i| \leq |GH : S_i|$. If $|G : G \cap S_i| < |GH : S_i|$ for any i then the degree of the representation of G afforded by $\mathscr{C}_G$ will be less than the degree of the representation of GH afforded by $\mathscr{C}$, which contradicts that $\mu(G) = \mu(G \times H)$. Hence, for each i,

$$|G : G \cap S_i| \ = \ |GH : S_i| \ ,$$

so $\mathscr{C}_G$ affords a minimal faithful representation Φ of G, and we have a bijection

$$\theta : G/G \cap S_i \to GH/S_i \ , \ \ (G \cap S_i)g \mapsto S_i g \quad \text{for} \quad g \in G \ ,$$

with respect to which $\Psi_i|_G$ and Φ_i become equivalent, where Φ_i is the transitive component of Φ corresponding to $G \cap S_i$, whose kernel is $\mathrm{core}_G(G \cap S_i)$.

Fix $i \in \{1, \ldots, \ell\}$. Then $\Psi_i|_H$ is equivalent to a representation Ξ_i of H by $\mathrm{Sym}(G/G \cap S_i)$ and $H\Xi_i$ centralises $G\Phi_i$. Note that $\mu(G\Phi_i) = |G : G \cap S_i|$, by Lemma 2.1. By Lemma 2.3, $H\Xi_i$ has a regular action and its orbits form a block system for $G\Phi_i$. Let B be one of the orbits of $H\Xi_i$ and put

$$G_B \ = \ \{g \in G \mid B(g\Phi_i) \subseteq B\}\Phi_i \ .$$

By Lemma 2.4, $G\Phi_i$ embeds in the wreath product $G_B \wr \overline{G}$, where $\overline{G}$ denotes the permutation group induced by $G\Phi_i$ acting on the blocks regarded as points, and $\mu(G_B) = |B|$. Further, G_B centralises $H\Xi_i$. But the action of $H\Xi_i$ on B is regular, so the stabiliser K of a point in B is trivial, so that $N_{H\Xi_i}(K)/K \cong H\Xi_i$. By Lemma 2.3, G_B is isomorphic to a subgroup of $H\Xi_i$. In particular, $\mu(G_B) \leq \mu(H\Xi_i)$. Moreover $|H\Xi_i| = |B| = \mu(G_B)$. It follows that $\mu(H\Xi_i) = |H\Xi_i|$. By Johnson's Theorem [5], $H\Xi_i$ is a cyclic group of prime power order, a Klein four-group or a generalised quaternion 2-group.

But $H\Psi$ is a subdirect product of $H\Xi_1, \ldots, H\Xi_\ell$. Thus $H \cong H\Psi$ is a subdirect product of a finite number of cyclic groups of prime-power order, Klein four-groups and generalised quaternion 2-groups. Let A be the direct product of all of the cyclic groups of odd order that arise and Q a sufficiently large generalised quaternion 2-group that contains the largest cycle of order a power of 2 that arises (in any of the cyclic 2-groups, Klein four-groups or generalised quaternion 2-groups that arise). Then H embeds in $A \times Q^k$ for some k. This completes the proof of the forward direction of the theorem.

Suppose, conversely, that H is a subgroup of $A \times Q^k$ for some abelian group A of odd order, generalised quaternion 2-group Q and positive integer k. We may assume A is nontrivial. Then there exist distinct odd primes $p_1, \ldots, p_\ell$ and positive integers $k_1, \ldots, k_\ell$ such that H is a subgroup of

$$K \ = \ C_{p_1}^{k_1} \times \ldots \times C_{p_\ell}^{k_\ell} \times Q^k \ .$$

Choose a prime q larger than $p_1, \ldots, p_\ell$. By Example 3.2, there exist a p_i-group A_i and semidirect product of A_i by C_q, with nontrivial action, such that

$$\mu(A_i \rtimes C_q) \times C_{p_i}^{k_i} \;=\; \mu(A_i \rtimes C_q)\;, \tag{6}$$

for $1 \le i \le \ell$. By Example 3.3, there exist a 2-group B and semidirect product of B by C_q, with nontrivial action, such that

$$\mu(B \rtimes C_q) \times Q^k \;=\; \mu(B \rtimes C_q)\;. \tag{7}$$

Put

$$G \;=\; (A_1 \times \ldots \times A_\ell \times B) \rtimes C_q$$

where the semidirect product action of C_q on the base group is induced from the actions on the base groups of $A_1 \rtimes C_q, \ldots, A_\ell \rtimes C_q, B \rtimes C_q$, in the sense described in the preamble to [1, Theorem 2.7]. Observe also that, for $1 \le i \le \ell$,

$$(A_i \rtimes C_q) \times C_{p_i}^{k_i} \;\cong\; (A_i \times C_{p_i}^{k_i}) \rtimes C_q\;, \tag{8}$$

and

$$(B \rtimes C_q) \times Q^k \;\cong\; (B \times Q^k) \rtimes C_q\;, \tag{9}$$

where, in each case, on the right-hand side, the action of C_q on the base group is nontrivial. By iterating the last case of the formula in [1, Theorem 2.7], and by (6),(7), (8) and (9), we have

$$\begin{aligned}
\mu(G \times K) \;&=\; \mu\Big((A_1 \times \ldots \times A_\ell \times B) \rtimes C_q) \times (C_{p_1}^{k_1} \times \ldots \times C_{p_\ell}^{k_\ell} \times Q^k\Big) \\
&=\; \mu\Big(\Big((A_1 \times C_{p_1}^{k_1}) \times \ldots \times (A_\ell \times C_{p_\ell}^{k_\ell}) \times (B \times Q^k)\Big) \rtimes C_q\Big) \\
&=\; \mu\Big((A_1 \times C_{p_1}^{k_1}) \rtimes C_q\Big) + \ldots + \mu\Big((A_\ell \times C_{p_\ell}^{k_\ell}) \rtimes C_q\Big) + \mu\Big((B \times Q^k) \rtimes C_q\Big) \\
&=\; \mu\Big((A_1 \rtimes C_q) \times C_{p_1}^{k_1}\Big) + \ldots + \mu\Big((A_\ell \rtimes C_q) \times C_{p_\ell}^{k_\ell}\Big) + \mu\Big((B \rtimes C_q) \times Q^k\Big) \\
&=\; \mu(A_1 \rtimes C_q) + \ldots + \mu(A_\ell \rtimes C_q) + \mu(B \rtimes C_q) \\
&=\; \mu\Big((A_1 \times \ldots \times A_\ell \times B) \rtimes C_q\Big) \;=\; \mu(G)\;.
\end{aligned}$$

But then

$$\mu(G) \;\le\; \mu(G \times H) \;\le\; \mu(G \times K) \;=\; \mu(G)\;,$$

whence $\mu(G \times H) = \mu(G)$, and absorption of H has been achieved, completing the proof of the backward direction of the theorem. □

The following consequence appears to be nontrivial and tells us that no direct power of any nontrivial group G can absorb another copy of G.

Corollary 4.2. *If G is a group and n a nonnegative integer then $\mu(G^{n+1}) = \mu(G^n)$ if and only if G is trivial.*

Proof. Suppose G is a group and n an integer such that $\mu(G^{n+1}) = \mu(G^n)$. By Theorem 4.1, G is a subgroup of $A \times Q^k$ for some abelian group A, generalised quaternion 2-group Q and integer k. In particular, G is nilpotent, so $n\mu(G) = \mu(G^n) = \mu(G^{n+1}) = (n+1)\mu(G)$ by the main result of [10], which is impossible unless $\mu(G) = 0$, that is, G is trivial. The reverse direction is obvious. □

Remark 4.3. In Example 3.1, we exhibited a subgroup G of $\mathrm{Sym}(24)$ such that $\mu(G\times Q_8) = \mu(G)$. In fact, 24 is minimal in the following sense: suppose that G is a subgroup of $\mathrm{Sym}(n)$ such that $\mu(G \times H) = \mu(G)$ for some nonabelian group H. We prove that $n \geq 24$. By Theorem 4.1, H must have a subgroup isomorphic to Q_8, so there is no loss in generality in assuming $H \cong Q_8$. Because Q_8 is subdirectly irreducible, we may suppose G is transitive (by Lemma 2.1). From the forward direction of the proof of Theorem 4.1, G embeds in a wreath product of Q_8 by some permutation group K acting on k letters, such that $n = 8k$. If $k = 1$ or 2 then G embeds in a 2-group, so that $\mu(G) = \mu(G\times H) = \mu(G)+\mu(H) = \mu(G)+8$, by Wright's theorem [10], since both G and H are nilpotent, which is impossible. Hence $k \geq 3$, so $n \geq 24$, and the proof is complete.

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