BLOCKS OF THE TRUNCATED q-SCHUR ALGEBRAS OF TYPE A

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ABSTRACT. This paper classifies the blocks of the truncated q-Schur algebras of type A which have as weight poset an arbitrary cosaturated set of partitions.

1. INTRODUCTION

In this paper we classify the blocks of the truncated q-Schur algebras of type A. The truncated Schur algebras are a natural family of quasi-hereditary algebras obtained from the q-Schur algebras by applying "Schur functors". These algebras include, as special cases, the usual Schur algebras $\mathcal{S}_{k,q}(n,r)$. Our main result gives a relatively quick and easy classification of the blocks of all of these algebras.

We think it quite remarkable that there is a uniform and relatively simple classification of the blocks of all of the truncated q-Schur algebras. As with the original classification of the blocks of the q-Schur algebra $\mathcal{S}_{k,q}(n,n)$ [**8**], the main tool that we use is the Jantzen sum formula, however, the new theme which permeates this paper is that the *Jantzen coefficients*, the integers which appear in the Jantzen sum formula, determine much of the representation theory. For example, the blocks correspond to the combinatorial "linkage classes" of partitions determined by the non-zero Jantzen coefficients.

The proof of the classification of blocks of the Schur algebras that we give is new, both for $\mathcal{S}_{k,q}(n,n)$ and more generally for the algebras $\mathcal{S}_{k,q}(n,r)$ considered in [**1**, **3**]. Throughout our focus is on the combinatorics of the Jantzen coefficients which has not been considered before. As with the arguments in [**1**,**3**,**8**], the strategy is to reduce to blocks which contain a unique maximal partition. Following Donkin, the arguments of [**1**, **3**] use translation functors to make these reductions whereas we achieve them, slightly more quickly and in greater generality, using just the combinatorics of the Jantzen coefficients. The key point from our perspective is to understand the partitions which contain only horizontal hooks (Definition 3.12). It turns out that these partitions classify the projective indecomposable Weyl modules for the truncated Schur algebras in characteristic zero.

To state our main result recall that a **partition** μ of r is a non-increasing sequence of non-negative integers such that $|\mu| = \mu_1 + \mu_2 + \cdots = r$. If μ is a partition then the **length** of μ is the smallest integer $\ell(\mu)$ such that $\mu_i = 0$ for $i > \ell(\mu)$. If $\ell = \ell(\mu)$ then we omit trailing zeros and write $\mu = (\mu_1, \dots, \mu_\ell)$.

If λ and μ are two partitions of r then λ **dominates** μ, and we write $\lambda \trianglerighteq \mu$, if

$$\sum_{i=1}^{s} \lambda_i \geq \sum_{i=1}^{s} \mu_i,$$

for all $s \geq 0$. Let Λ_r be the set of all partitions of r. A subset Λ of Λ_r is **cosaturated** if whenever $\lambda \in \Lambda$ and $\mu \trianglerighteq \lambda$ then $\mu \in \Lambda$.

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Let k be a field of characteristic $p \geq 0$ and suppose that $0 \neq q \in k$ and that Λ is a cosaturated set of partitions of r. For each partition μ of r there is a *permutation module* $M(\mu)$ for the Iwahori-Hecke algebra $\mathcal{H}_{k,q}(r)$ of the symmetric group $\mathfrak{S}_r$. (For more details see, for example, [**11**, Chapter 3].) The **truncated q-Schur algebra** with parameter $q \in k$ and weight poset Λ is the endomorphism algebra

$$\mathcal{S}_{k,q}(\Lambda) = \operatorname{End}_{\mathcal{H}_{k,q}(r)}\big(\bigoplus_{\mu\in\Lambda} M(\mu)\big).$$

The algebra $\mathcal{S}_{k,1}(\Lambda_r)$ is Morita equivalent to the "classical" Schur algebra [**6**].

The Schur algebras $\mathcal{S}_{k,q}(\Lambda)$ are cellular and quasi-hereditary by [**11**, Exercise 4.13], with weight poset Λ ordered by dominance. Thus, for each partition $\mu \in \Lambda$ there is a Weyl module, or standard module, $\Delta_k^\mu = \Delta_k^\mu(\Lambda)$. When $\mathcal{S}_{k,q}(\Lambda)$ is semisimple the Weyl modules $\{\, \Delta_k^\mu \mid \mu \in \Lambda \,\}$ are a complete set of pairwise non-isomorphic irreducible $\mathcal{S}_{k,q}(\Lambda)$-modules.

To describe the blocks of $\mathcal{S}_{k,q}(\Lambda)$ let $e \in \{0\} \cup \{2, 3, \dots\}$ be minimal such that $1 + q + \dots + q^{e-1} = 0$ and set $e = 0$ if no such integer exists. Set $\mathcal{P}_{e,p} = \{1, e, ep, ep^2, \dots\}$. If $\mu \in \Lambda$ let $\kappa = \operatorname{core}_e(\mu)$ be its e-core (see Section 3.2). Define a **length function** $\ell_\Lambda : \Lambda \longrightarrow \mathbb{N}$ by setting

$$\ell_\Lambda(\mu) = \min\{\, i \geq 0 \mid \lambda_j = \kappa_j \text{ whenever } j > i \text{ and } \lambda \in \Lambda \text{ has } e\text{-core } \kappa \,\},$$

where $\kappa = (\kappa_1, \kappa_2, \dots)$. If $\ell_\Lambda(\mu) \leq 1$ set $s_\Lambda(\mu) = 1$ and otherwise define

$$s_\Lambda(\mu) = \max\{\, s \in \mathcal{P}_{e,p} \mid \mu_i - \mu_{i+1} \equiv -1 \pmod{s} \text{ for } 1 \leq i < \ell_\Lambda(\mu) \,\}.$$

Finally, let $s = s_\Lambda(\mu)$ and define

$$\chi_\Lambda(\mu) = \big((\mu_1 - \kappa_1)/s, (\mu_2 - \kappa_2)/s, \dots\big).$$

It follows from Lemma 3.15 below that $\chi_\Lambda(\mu)$ is a partition.

Our main result is the following.

Main Theorem. *Suppose that Λ is a cosaturated set of partitions of r and that $\lambda, \mu \in \Lambda$. Then Δ_k^λ and Δ_k^μ are in the same block as $\mathcal{S}_{k,q}(\Lambda)$-modules if and only if the following three conditions are satisfied:*

a) *λ and μ have the same e-core;*
b) *$s_\Lambda(\lambda) = s_\Lambda(\mu)$; and,*
c) *if $s_\Lambda(\mu) > 1$ then $\chi_\Lambda(\lambda)$ and $\chi_\Lambda(\mu)$ have the same p-core.*

Note that the 0-core of the partition χ is χ.

Let $\Lambda_{n,r}$ be the set of partitions of r of length at most n, so that $\Lambda_r = \Lambda_{r,r}$. Set $\mathcal{S}_{k,q}(n,r) = \mathcal{S}_{k,q}(\Lambda_{n,r})$. Then Mod-$\mathcal{S}_{k,1}(n,r)$ is the category of homogeneous polynomial representations of the general linear group $\mathrm{GL}_n(k)$ of homogeneous degree r. The blocks of $\mathcal{S}_{k,q}(\Lambda_r)$ were classified by James and Mathas [**8**, Theorem 4.24]. The blocks of $S_{k,q}(n,r)$ were classified by Donkin [**3**, §4] (for $q = 1$), and Cox [**1**, Theorem 5.3] (for $q \neq 1$). We recover all of these results as special cases of our Main Theorem.

Finally, we remark that this paper grew out of our attempts to understand the blocks of the *baby Hecke algebras* $\mathcal{H}_\mu = \operatorname{End}_{\mathcal{H}_{k,q}(r)}(M(\mu))$, for μ a partition of r. Let Λ_μ be the set of partitions of r which dominate μ. Then Λ_μ is cosaturated and $\mathcal{H}_\mu \cong \varphi_\mu \mathcal{S}_{k,q}(\Lambda_\mu)\varphi_\mu$, where φ_μ is the identity map on $M(\mu)$. Hence, there is a natural *Schur functor*

$$\mathsf{F}_\mu : \text{Mod-}\mathcal{S}_{k,q}(\Lambda_\mu) \longrightarrow \text{Mod-}\mathcal{H}_\Lambda;\, X \mapsto X\varphi_\mu.$$

Our Main Theorem classifies the blocks of $\mathcal{S}_{k,q}(\Lambda_\mu)$, so this gives a necessary condition for two $\mathcal{H}_\mu$-modules to belong to the same block. Unfortunately, $\operatorname{End}_{\mathcal{S}_{k,q}(\Lambda_\mu)}(M(\mu))$ can be larger than $\mathcal{H}_\mu$, so the image of a block of $\mathcal{S}_{k,q}(\Lambda_\mu)$ under F_μ need not be indecomposable. Consequently, we are not able to describe the blocks of the algebras $\mathcal{H}_\mu$ completely.

2. Jantzen equivalence and blocks

This section develops a general theory for classifying blocks of quasi-hereditary (cellular) algebras using Jantzen filtrations. This theory is new in the sense that it does not appear in the literature, although everything that we do is implicit in [**9**] which develops these results in the special case of the cyclotomic Schur algebras.

We remark that it has long been known to people working in algebraic groups that Jantzen filtrations could be used to determine the blocks, however, the fact that the non-zero coefficients in the Jantzen sum formula actually classify the blocks (Proposition 2.9) surprised even experts in this field. For these reasons we think it is worthwhile to give a self contained treatment of this theory of quasi-hereditary (cellular) algebras.

2.1. **Cellular algebras.** We start by recalling Graham and Lehrer's definition of a cellular algebra [**5**]. Fix an integral domain $\mathcal{O}$.

Definition 2.1 (Graham and Lehrer [**5**])**.** *A cell datum for an associative $\mathcal{O}$-algebra $\mathcal{S}$ is a triple (Λ, T, C) where $\Lambda = (\Lambda, >)$ is a finite poset, $T(\lambda)$ is a finite set for $\lambda \in \Lambda$, and*

$$C\colon \coprod_{\lambda\in\Lambda} T(\lambda) \times T(\lambda) \longrightarrow \mathcal{S};\, (s,t) \mapsto C^\lambda_{st}$$

is an injective map (of sets) such that:

a) *$\{\, C^\lambda_{st} \mid \lambda \in \Lambda, s,t \in T(\lambda)\,\}$ is an $\mathcal{O}$ basis of $\mathcal{S}$;*
b) *For any $x \in \mathcal{S}$ and $t \in T(\lambda)$ there exist scalars $r_{tv}(x) \in \mathcal{O}$ such that, for any $s \in T(\lambda)$,*
$$C^\lambda_{st}x \equiv \sum_{v\in T(\lambda)} r_{tv}(x) C^\lambda_{sv} \pmod{\mathcal{S}^\lambda},$$
where $\mathcal{S}^\lambda$ is the $\mathcal{O}$–submodule of $\mathcal{S}$ with basis $\{\, C^\mu_{yz} \mid \mu > \lambda \text{ and } y,z \in T(\mu)\,\}$.
c) *The $\mathcal{O}$–linear map determined by $*\colon \mathcal{S} \longrightarrow \mathcal{S}; C^\lambda_{st} = C^\lambda_{ts}$, for all $\lambda \in \Lambda$ and $s,t \in T(\lambda)$, is an anti–isomorphism of $\mathcal{S}$.*

*Then $\mathcal{S}$ is a **cellular algebra** with **cellular basis** $\{\, C^\lambda_{st} \mid \lambda \in \Lambda \text{ and } s,t \in T(\lambda)\,\}$.*

Suppose that (Λ, T, C) is a cell datum for an $\mathcal{O}$-algebra $\mathcal{S}$. Following Graham and Lehrer [**5**, §2], for each $\lambda \in \Lambda$ define the cell module, or **standard module**, $\Delta^\lambda_{\mathcal{O}}$ to be the free $\mathcal{O}$–module with basis $\{\, C^\lambda_t \mid t \in T(\lambda)\,\}$ and with $\mathcal{S}$–action given by

$$C^\lambda_t x = \sum_{v\in T(\lambda)} r_{tv}(x) C^\lambda_v,$$

where $r_{tv}(x)$ is the scalar from Definition 2.1(b). As $r_{tv}(x)$ is independent of s this gives a well–defined $\mathcal{S}$–module structure on $\Delta^\lambda_{\mathcal{O}}$. The map $\langle\ ,\ \rangle_\lambda \colon \Delta^\lambda_{\mathcal{O}} \times \Delta^\lambda_{\mathcal{O}} \longrightarrow \mathcal{O}$ which is determined by

$$\langle C^\lambda_t, C^\lambda_u\rangle_\lambda C^\lambda_{sv} \equiv C^\lambda_{st} C^\lambda_{uv} \pmod{\mathcal{S}^\lambda}, \tag{2.2}$$

for $s,t,u,v \in T(\lambda)$, defines a symmetric bilinear form on $\Delta^\lambda_{\mathcal{O}}$. This form is associative in the sense that $\langle ax, b\rangle_\lambda = \langle a, bx^*\rangle_\lambda$, for all $a,b \in \Delta^\lambda_{\mathcal{O}}$ and all $x \in \mathcal{S}$.

It follows easily from the definitions that the framework above is compatible with base change. That is, if A is a commutative $\mathcal{O}$-algebra then $\{\, C^\lambda_{st} \otimes 1_A \mid \lambda \in \Lambda, s,t \in T(\lambda)\,\}$ is a cellular basis of the A-algebra $\mathcal{S}_A = \mathcal{S} \otimes_{\mathcal{O}} A$. Moreover, $\Delta^\lambda_A \cong \Delta^\lambda_{\mathcal{O}} \otimes_{\mathcal{O}} A$ for all $\lambda \in \Lambda$.

2.2. **Jantzen filtrations of cell modules.** In order to define the Jantzen filtrations of the standard modules we now assume that $\mathcal{O}$ is a discrete valuation ring with maximal ideal $\mathfrak{p}$ and we let K be the field of fractions of $\mathcal{O}$ and $k = \mathcal{O}/\mathfrak{p}$ be the residue field of $\mathcal{O}$. As remarked in the last paragraph, $\mathcal{S}_K = \mathcal{S} \otimes_{\mathcal{O}} K$ and $\mathcal{S}_k = \mathcal{S} \otimes_{\mathcal{O}} k$ are cellular algebras with, in essence, the same cell datum. In particular, $\Delta_K^\lambda \cong \Delta_{\mathcal{O}}^\lambda \otimes_{\mathcal{O}} K$ and $\Delta_k^\lambda \cong \Delta_{\mathcal{O}}^\lambda \otimes_{\mathcal{O}} k$, for $\lambda \in \Lambda$.

Henceforth, we assume that $\mathcal{S}_K$ is a semisimple algebra. Equivalently, by [5, Theorem 3.8], we assume that the bilinear form $\langle\ ,\ \rangle_\lambda$ for Δ_K^λ is non-degenerate, for all $\lambda \in \Lambda$. Thus, $\mathcal{S}_K$ is semisimple if and only if Δ_K^λ is irreducible for all $\lambda \in \Lambda$. Hence, $(K, \mathcal{O}, k)$ is a modular system for $\mathcal{S}_k$.

For $\lambda \in \Lambda$ and $i \geq 0$ define

$$J_i(\Delta_{\mathcal{O}}^\lambda) = \{\, x \in \Delta_{\mathcal{O}}^\lambda \mid \langle x, y\rangle_\lambda \in \mathfrak{p}^i \text{ for all } y \in \Delta_{\mathcal{O}}^\lambda \,\}.$$

Then, as the form $\langle\ ,\ \rangle_\lambda$ is associative, $\Delta_{\mathcal{O}}^\lambda = J_0(\Delta_{\mathcal{O}}^\lambda) \supseteq J_1(\Delta_{\mathcal{O}}^\lambda) \supseteq \dots$ is an $\mathcal{S}$-module filtration of $\Delta_{\mathcal{O}}^\lambda$.

Definition 2.3. *Suppose that $\lambda \in \Lambda$. The* ***Jantzen filtration*** *of Δ_k^λ is the filtration*

$$\Delta_k^\lambda = J_0(\Delta_k^\lambda) \supseteq J_1(\Delta_k^\lambda) \supseteq \dots,$$

where $J_i(\Delta_k^\lambda) = \big(J_i(\Delta_{\mathcal{O}}^\lambda) + \mathfrak{p}\Delta_{\mathcal{O}}^\lambda\big)/\mathfrak{p}\Delta_{\mathcal{O}}^\lambda$ for $i \geq 0$.

Notice that $J_i(\Delta_k^\lambda) = 0$ for $i \gg 0$ since Δ_k^λ is finite dimensional.

For each $\lambda \in \Lambda$ set $L_k^\lambda = \Delta_k^\lambda / J_1(\Delta_k^\lambda)$. By the general theory of cellular algebras [5, Theorem 3.4], L_k^λ is either zero or absolutely irreducible. Moreover, all of the irreducible $\mathcal{S}_k$-modules arise uniquely in this way. (Note that $L_K^\lambda = \Delta_K^\lambda$, for $\lambda \in \Lambda$, since $\mathcal{S}_K$ is semisimple.)

The definition of the Jantzen filtration makes sense for the standard modules of arbitrary cellular algebras, however, for the next Lemma we need to *assume that $\mathcal{S}_k$ is quasi-hereditary*. By Remark 3.10 of [5], $\mathcal{S}_k$ is quasi-hereditary if and only if $J_1(\Delta_{\mathcal{O}}^\lambda) \neq \Delta_{\mathcal{O}}^\lambda$ for all $\lambda \in \Lambda$. Equivalently, $\mathcal{S}_k$ is quasi-hereditary if and only if $L_k^\lambda \neq 0$, for all $\lambda \in \Lambda$.

A subset Γ of Λ is **cosaturated** if $\lambda \in \Gamma$ whenever $\lambda \in \Lambda$ and $\lambda > \gamma$ for some $\gamma \in \Gamma$. Let $\mathcal{S}^\Gamma$ be the subspace of $\mathcal{S}$ spanned by the elements $\{\, C_{st}^\lambda \mid \lambda \in \Gamma \text{ and } s, t \in T(\lambda) \,\}$. For future reference we note the following fact which follows easily from Definition 2.1 and the last paragraph.

Lemma 2.4. *Suppose that Γ is a cosaturated subset of Λ. Then:*

a) *The algebra $\mathcal{S}/\mathcal{S}^\Gamma$ is a cellular algebra with cellular basis*

$$\{\, C_{st}^\lambda + \mathcal{S}^\Gamma \mid \lambda \in \Lambda \backslash \Gamma \text{ and } s, t \in T(\lambda) \,\}.$$

b) *If $\mathcal{S}$ is a quasi-hereditary algebra then so is $\mathcal{S}/\mathcal{S}^\Gamma$.*

Let $K_0(\mathcal{S}_k)$ be the Grothendieck group of finite dimensional right $\mathcal{S}_k$-modules. If M is an $\mathcal{S}_k$-module let $[M]$ be its image in $K_0(\mathcal{S}_k)$.

If M is an $\mathcal{S}_k$-module and $\mu \in \Lambda$ let $[M : L_k^\mu]$ be the multiplicity of the simple module L_k^μ as a composition factor of M. In particular, if $\lambda, \mu \in \Lambda$ let $d_{\lambda\mu} = [\Delta_k^\lambda : L_k^\mu]$. Then, by [5, Proposition 3.6], $d_{\lambda\lambda} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \geq \mu$. Consequently, the decomposition matrix $\big(d_{\lambda\mu}\big)_{\lambda,\mu\in\Lambda}$ of $\mathcal{S}_k$ is a square unitriangular matrix, when its rows and columns are ordered in a way that is compatible with $>$. Therefore, the decomposition matrix of $\mathcal{S}_k$ is invertible over $\mathbb{Z}$ and as a consequence we obtain the following.

Lemma 2.5. *Suppose that $\mathcal{S}_k$ is a quasi-hereditary cellular algebra and $\lambda, \mu \in \Lambda$. Then*

a) $\{\,[\Delta_k^\lambda] \mid \lambda\in\Lambda\,\}$ *is a* $\mathbb{Z}$*-basis of* $K_0(\mathcal{S}_k)$.
b) *There exist integers* $J_{\lambda\mu}\in\mathbb{Z}$ *such that*

$$\sum_{i>0}[J_i(\Delta_k^\lambda)] = \sum_{\substack{\mu\in\Lambda\\ \lambda>\mu}} J_{\lambda\mu}[\Delta_k^\mu].$$

c) *If* $\mu\neq\lambda$ *then* $d_{\lambda\mu} = [J_1(\Delta_k^\lambda):L_k^\mu]$*. Consequently,* $d_{\lambda\mu}\neq 0$ *if and only if* $J'_{\lambda\mu}\neq 0$*, where*

$$J'_{\lambda\mu} = \sum_{i>0}[J_i(\Delta_k^\lambda):L_k^\mu] = \sum_{\lambda>\nu\geq\mu} J_{\lambda\nu}d_{\nu\mu}.$$

Moreover, if $\mu\neq\lambda$ *then* $d_{\lambda\mu}\leq J'_{\lambda\mu}$.

The integers $J_{\lambda\mu}$ are the **Jantzen coefficients** of $\mathcal{S}_k$. By definition,

$$J'_{\lambda\mu} = [\bigoplus_{i>0} J_i(\Delta_k^\lambda):L_k^\mu] \geq [\operatorname{rad}\Delta_k^\lambda : L_k^\mu], \tag{2.6}$$

where $\operatorname{rad}\Delta_k^\lambda = J_1(\Delta_k^\lambda)$ is the radical of Δ_k^λ. We show in the next section that the Jantzen coefficients determine the blocks of $\mathcal{S}_k$. They also determine the irreducible standard modules.

Corollary 2.7. *Suppose that* $\lambda\in\Lambda$*. Then the following are equivalent:*
a) $\Delta_k^\lambda = L_k^\lambda$ *is an irreducible* $\mathcal{S}_k$*-module.*
b) $d_{\lambda\mu}=\delta_{\lambda\mu}$*, for all* $\mu\in\Lambda$ *(Kronecker delta).*
c) $J_{\lambda\mu}=0$*, for all* $\mu\in\Lambda$.

Proof. Parts (a) and (b) are equivalent essentially by definition and (b) and (c) are equivalent by Lemma 2.5(c). □

2.3. **Jantzen coefficients and the blocks of $\mathcal{S}_k$.** The algebra $\mathcal{S}_k$ decomposes in a unique way as a direct sum of indecomposable two-sided ideals $\mathcal{S}_k = B_1\oplus\cdots\oplus B_d$. If M is an $\mathcal{S}_k$-module then MB_i is a B_i-module. We say that M **belongs** to the block B_i if $MB_i = M$. Using an idempotent argument (cf. [2, Theorem 56.12]) it is easy to show that two indecomposable $\mathcal{S}_k$-modules P and Q belong to the same block if and only if they are in the same *linkage class*. That is, there exist indecomposable modules $P_1=P,\dots,P_l=Q$ such that P_i and P_{i+1} have a common irreducible composition factor, for $i=1,\dots,l-1$.

Definition 2.8. *Suppose that* $\lambda,\mu\in\Lambda$*. Then* λ *and* μ *are* ***Jantzen equivalent****, and we write* $\lambda\sim_J\mu$*, if there exist* $\lambda_1=\lambda,\lambda_2,\dots,\lambda_l=\mu\in\Lambda$ *such that either*

$$J_{\lambda_i\lambda_{i+1}}\neq 0 \quad\textit{or}\quad J_{\lambda_{i+1}\lambda_i}\neq 0,$$

for $1\leq i<l$.

The next result shows that the Jantzen equivalence classes and the blocks of $\mathcal{S}_k$ coincide. This is the main result of this section.

Proposition 2.9. *Suppose that* $\lambda,\mu\in\Lambda$*. Then* Δ_k^λ *and* Δ_k^μ *belong to the same block as* $\mathcal{S}_k$*–modules if and only if* $\lambda\sim_J\mu$.

Proof. We essentially repeat the argument of [9, Proposition 2.9]. Before we begin observe that if $\nu\in\Lambda$ then Δ_k^ν is indecomposable because L_k^ν is the simple head of Δ_k^ν. Consequently, all of the composition factors of Δ_k^ν belong to the same block.

Suppose, first, that $\lambda\sim_J\mu$. By definition $J_i(\Delta_k^\lambda)$ is a submodule of Δ_k^λ for all i, so all of the composition factors of $\bigoplus_{i>0}J_i(\Delta_k^\lambda)$ belong to the same block as Δ_k^λ by the last paragraph. Define Λ' to be the subset of Λ such that $\nu \in \Lambda'$ whenever Δ_k^ν and Δ_k^λ are in different blocks. Then $\sum_{\nu\in\Lambda'} J_{\lambda\nu}[\Delta_k^\nu] = 0$ by (2.6). Hence, $J_{\lambda\nu} = 0$ whenever $\nu \in \Lambda'$ by Lemma 2.5(a). It follows that Δ_k^λ and Δ_k^μ belong to the same block whenever $J_{\lambda\mu} \neq 0$.

To prove the converse it is sufficient to show that $\lambda \sim_J \mu$ whenever $d_{\lambda\mu} \neq 0$. By Lemma 2.5(c), if $d_{\lambda\mu} \neq 0$ then $J'_{\lambda\mu} \neq 0$. Therefore, there exists a partition $\nu_1 \in \Lambda$ such that $\lambda > \nu_1 \geq \mu$, $J_{\lambda\nu_1} \neq 0$ and $d_{\nu_1\mu} \neq 0$. If $\nu_1 = \mu$ then $\lambda \sim_J \mu$ and we are done. If $\nu_1 \neq \mu$ then $d_{\nu_1\mu} \neq 0$, so $J'_{\nu_1\mu} \neq 0$ and we may repeat this argument to find $\nu_2 \in \Lambda$ with $\nu_1 > \nu_2 \geq \mu$, $J_{\nu_1\nu_2} \neq 0$ and $d_{\nu_2\mu} \neq 0$. Continuing in this way we can find elements $\nu_0 = \lambda, \nu_1, \dots, \nu_l = \mu$ in Λ such that $J_{\nu_{i-1}\nu_i} \neq 0$, $d_{\nu_i\mu} \neq 0$, for $0 < i < l$, and $\lambda > \nu_1 > \cdots > \nu_l = \mu$. Note that we must have $\nu_l = \mu$ for some l since Λ is finite. Therefore, $\lambda \sim_J \nu_1 \sim_J \cdots \sim_J \nu_l = \mu$ as required. □

We have chosen to prove Proposition 2.9 using the formalism of cellular algebras, however, it can be proved entirely within the framework of quasi-hereditary algebras. Suppose that $\mathcal{O}$ is a (complete) discrete valuation ring with residue field k. Let A be a quasi-hereditary algebra which is free and of finite rank as an $\mathcal{O}$-module and set $A_k = A \otimes_{\mathcal{O}} k$. Following, for example, McNinch [**12**, §4.1] we can define Jantzen filtrations of the standard modules of the quasi-hereditary algebra A_k. The standard modules of a quasi-hereditary algebra are always indecomposable and they always give a basis for the Grothendieck group of A_k. Moreover, the decomposition matrix of A_k is unitriangular. Using these general facts, Proposition 2.9 (and Lemma 2.5 and Corollary 2.7), can be proved for A_k following the arguments above.

3. Combinatorics and Jantzen equivalence for Schur algebras

We are now ready to start proving our Main Theorem. We begin by recalling the combinatorics we need to describe the Jantzen coefficients for the algebras $\mathcal{S}_{k,q}(\Lambda)$ from the introduction. Recall from the introduction that $\Lambda_r = \Lambda_{r,r}$ is the set of all partitions of r.

As in the introduction, let Λ be a cosaturated set of partitions of r and fix a field k of characteristic $p \geq 0$ and a non-zero element $q \in k^\times$. Let $\mathcal{S}_{k,q}(\Lambda)$ be the q-Schur algebra over k with parameter q and weight poset Λ.

3.1. **Schur functors.** Recall that Λ_r is the set of all partitions of r and that $\mathcal{S}_{k,q}(\Lambda_r)$ is the q-Schur algebra with weight poset Λ_r. There is a natural embedding $\mathcal{S}_{k,q}(\Lambda) \hookrightarrow \mathcal{S}_{k,q}(\Lambda_r)$. Moreover, it is easy to see that if e_Λ is the identity element of $\mathcal{S}_{k,q}(\Lambda)$ then $\mathcal{S}_{k,q}(\Lambda) = e_\Lambda \mathcal{S}_{k,q}(\Lambda_r) e_\Lambda$.

For the next result, write $\Delta_k^\lambda(\Lambda_r)$ for the standard modules of $\mathcal{S}_{k,q}(\Lambda_r)$ and $\Delta_k^\lambda(\Lambda)$ for the standard modules of $\mathcal{S}_{k,q}(\Lambda)$. Then, by standard arguments (see for example, [**6**, §6] or [**4**, Proposition A3.11]), we obtain the following.

Lemma 3.1 (Schur functor)**.** *Right multiplication by e_Λ induces an exact functor F_Λ from the category of right $\mathcal{S}_{k,q}(\Lambda_r)$–modules to the category of right $\mathcal{S}_{k,q}(\Lambda)$-modules such that*

$$F_\Lambda(\Delta_k^\lambda(\Lambda_r)) \cong \begin{cases} \Delta_k^\lambda(\Lambda), & \text{if } \lambda \in \Lambda, \\ 0, & \text{if } \lambda \notin \Lambda. \end{cases}$$

Moreover, if $\lambda \in \Lambda$ then $\dim \Delta_k^\lambda(\Lambda) = \dim \Delta_k^\lambda(\Lambda_r)$.

The standard modules $\Delta_k^\lambda(\Lambda)$ are often called the **Weyl modules** of $\mathcal{S}_{k,q}(\Lambda)$. In view of Lemma 3.1 we now write $\Delta_k^\lambda = \Delta_k^\lambda(\Lambda)$.

Define the **quantum characteristic** of (k, q) to be

$$e = \min\{\, c \geq 1 \mid 1 + q + \cdots + q^{c-1} = 0 \,\}$$

and set $e = 0$ if no such integer exists. That is, $e = p$ if $q = 1$, $e = 0$ if q is not a root of unity and otherwise e is the multiplicative order of q.

By [**11**, Exercise 4.14] the algebra $\mathcal{S}_{k,q}(\Lambda_r)$ is semisimple if and only if $e = 0$ or $e > r$. Hence, applying the Schur functor of Lemma 3.1, $\mathcal{S}_{k,q}(\Lambda)$ is semisimple if $e = 0$ or $e > r$. Consequently, we assume henceforth that $0 < e \le r$.

3.2. **Jantzen coefficients for** $\mathcal{S}_{k,q}(\Lambda)$**.** To define the Jantzen filtrations of $\mathcal{S}_{k,q}(\Lambda)$ fix a modular system *with parameters* $(K, \mathcal{O}, k)_{t,q}$ such that

- $\mathcal{O}$ is a discrete valuation ring with maximal ideal $\mathfrak{p}$ and t is an invertible element of $\mathcal{O}$;
- K is the field of fractions of $\mathcal{O}$ and $\mathcal{S}_{K,t}(\Lambda) \cong S_{\mathcal{O},t}(\Lambda) \otimes_{\mathcal{O}} K$ is semisimple; and,
- $k \cong \mathcal{O}/\mathfrak{p}$, $q = t + \mathfrak{p}\mathcal{O}$ and $\mathcal{S}_{k,q}(\Lambda) \cong \mathcal{S}_{\mathcal{O},t}(\Lambda) \otimes_{\mathcal{O}} k$.

In general, the Jantzen filtrations of $\mathcal{S}_{k,q}(\Lambda)$-modules may depend upon this choice of modular system. In this paper, however, we only need to know whether or not the Jantzen coefficients are zero and this is independent of the choice of modular system by Proposition 3.4 below.

Least the reader be concerned that a modular system with these properties need not always exist we note that if x is an indeterminate over k then we could let $\mathcal{O} = k[x]_{(x)}$ be the localization of $k[x]$ at the prime ideal (x), so that $\mathfrak{p} = x\mathcal{O}$ is the unique maximal ideal of $\mathcal{O}$. Then $\mathcal{O}$ is a discrete valuation ring with field of fractions $K = k(x)$. Set $t = x + q$ which, by abuse of notation, we consider as an invertible element of $\mathcal{O}$. Then, using the remarks above, the reader can check that $(K, \mathcal{O}, k)_{t,q}$ is a modular system with parameters for $\mathcal{S}_{k,q}(\Lambda)$.

As in section 2, for each partition $\lambda \in \Lambda$ define the Jantzen filtration $\{J_i(\Delta_k^\lambda)\}$ of the standard module Δ_k^λ of $\mathcal{S}_{k,q}(\Lambda)$. Define the **Jantzen coefficients** of $\mathcal{S}_{k,q}(\Lambda)$ to be the integers $J_{\lambda\mu}^\Lambda$ determined by the following equations in $K_0(\mathcal{S}_{k,q}(\Lambda))$:

$$\sum_{i>0} [J_i(\Delta_k^\lambda)] = \sum_{\substack{\mu\in\Lambda\\ \mu>\lambda}} J_{\lambda\mu}^\Lambda [\Delta_k^\mu].$$

Recall that Λ_r is the set of partitions of r. For $\lambda, \mu \in \Lambda_r$, set $J_{\lambda\mu} = J_{\lambda\mu}^{\Lambda_r}$. Applying the Schur functor (Lemma 3.1), shows that the Jantzen coefficients depend only on Λ_r in the following sense.

Corollary 3.2. *Suppose that* $\lambda, \mu \in \Lambda$*. Then* $J_{\lambda\mu}^\Lambda = J_{\lambda\mu}$*.*

Henceforth, we write $J_{\lambda\mu}^\Lambda = J_{\lambda\mu}$ for the Jantzen coefficients of $\mathcal{S}_{k,q}(\Lambda)$.

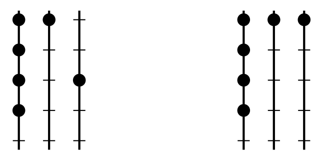
3.3. **Beta numbers, abaci and cores.** We now introduce the notation that we need to describe when the Jantzen coefficients of $\mathcal{S}_{k,q}(\Lambda)$ are non-zero. The bulk of the work has already been done in [**8**].

For any partition $\mu = (\mu_1, \mu_2, \dots)$ let $[\![\mu]\!] = \{\, (i,j) \mid 1 \le j \le \mu_i \,\}$ be the **diagram** of μ which we think of as a (left justified) collection of boxes in the plane. The e**-residue** of a node $(i,j) \in [\![\mu]\!]$ is the unique integer $\operatorname{res}_e(i,j)$ such that $0 \le \operatorname{res}_e(i,j) < e$ and $\operatorname{res}_e(i,j) \equiv j - i \pmod e$.

Fix any integer $l \ge \ell(\mu)$. For $1 \le i \le l$ set $\beta_i = \mu_i - i + l$. Then $\beta_1 > \beta_2 > \dots > \beta_l \ge 0$ are the l**-beta numbers** for μ. It is well-known and easy to prove that the beta numbers give a bijection between the set of partitions with at most l non-zero parts and the set of strictly increasing non-negative integer sequences of length l.

An e-**abacus** [7] is a Chinese abacus with e **runners** and with **bead positions** numbered $0, 1, 2, \dots$ from left to right and then top to bottom. (We will also need p-abaci and s-abaci.) Let $\beta_1 > \beta_2 > \cdots > \beta_l$ be the sequence of l-beta numbers for λ. The l-bead **abacus configuration** for λ is the abacus with beads at positions $\beta_1, \beta_2, \dots,$ and β_l. Any abacus configuration determines a set of beta numbers and hence corresponds to a unique partition. If $\beta \geq 0$ then bead position $\beta + 1$ is the position to the **right** of β and bead position $\beta - 1$ (if $\beta > 0$) is the position to the **left**. (In particular, the bead position to the left of a position on runner 0 is on runner $e - 1$ in the previous row.)

For example, taking $e = 3$ and $l = 6$ the abacus configurations for the partitions $\lambda = (4, 4, 3, 1)$ and $\kappa = (4, 2)$ are as follows:



The e-**core** of λ is the partition $\mathrm{core}_e(\lambda)$ whose abacus configuration is obtained from an abacus configuration of λ by pushing all beads as high as possible on their runner. If $e = 0$ then, by convention, $\mathrm{core}_0(\lambda) = \lambda$. If $e > 0$ then the e-**weight** of λ is the integer $(|\lambda| - |\mathrm{core}_e(\lambda)|)/e$ otherwise λ has e-weight zero. For example, if $\lambda = (4, 4, 3, 1)$, as in the example above, then $\mathrm{core}_e(\lambda) = (4, 2) = \kappa$ and λ has 3-weight 2.

Observe that, up to a constant shift, $\{\, \lambda_i - i \mid 1 \leq i \leq l \,\}$ is the set of beta numbers of λ. Therefore, two partitions λ and μ of r have the same e-core if and only if

$$\lambda_i - i \equiv \mu_{i^w} - i^w \pmod e,$$

for some $w \in \mathfrak{S}_r$.

Let $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ be the partition **conjugate** to λ, so that $\lambda'_j = \#\{\, i \geq 1 \mid \lambda_i \geq j \,\}$ for $j \geq 1$. If $(a, b) \in [\![\lambda]\!]$ then the (a, b)-**rim hook** of λ is the set of nodes

$$R^\lambda_{ab} = \{\, (i, j) \in [\![\lambda]\!] \mid a \leq i \leq \lambda'_b, b \leq j \leq \lambda_a \text{ such that } (i + 1, j + 1) \notin [\![\lambda]\!] \,\}.$$

The node (a, λ_a) is the **hand node** of R^λ_{ab}, $f^\lambda_{ab} = (\lambda'_b, b)$ is the **foot node** and $h^\lambda_{ab} = |R^\lambda_{ab}|$ is the **hook length** of R^λ_{ab}. The **foot residue** of R^λ_{ab} is $b - \lambda'_b \pmod e$, the residue of f^λ_{ab}. The hook R^λ_{ab} is an h-**hook** if $h = h^\lambda_{ab}$. Thus, R^λ_{ab} is the h-hook consisting of the set of nodes along the 'rim' of $[\![\lambda]\!]$ which connects the hand and foot nodes.

If μ is a partition and $[\![\mu]\!] = [\![\lambda]\!] \setminus R^\lambda_{ab}$, for some $(a, b) \in [\![\lambda]\!]$, then we say that μ is obtained from λ by **unwrapping** the rim hook R^λ_{ab} and that λ is obtained from μ by **wrapping on** this hook. A hook R^λ_{ab} is **removable** if $[\![\lambda]\!] \setminus R^\lambda_{ab}$ is the diagram of a partition.

Using the definitions, if $\{\beta_1, \dots, \beta_l\}$ is a set of beta numbers for λ and if μ is obtained from λ by unwrapping the rim hook R^λ_{ab} then $\{\beta_1, \dots, \beta_{a-1}, \beta_a - h^\lambda_{ab}, \beta_{a+1}, \dots, \beta_l\}$ is a set of beta numbers for μ. With a little extra care we obtain the following well-known fact; see, for example, [11, Lemma 5.26].

Lemma 3.3. *Suppose that λ is a partition. Then moving a bead h positions to the right in the abacus configuration of λ from runner f to runner f' corresponds to wrapping an h–rim hook with foot residue f onto λ. Equivalently, moving a bead h positions to the left, from runner f' to runner f corresponds to unwrapping an h–rim hook from λ with foot residue f.*

The following Lemma shows that the non-vanishing of the Jantzen coefficients is independent of the choice of modular system. First, however, we need a definition. Let

$\nu_{e,p} : \mathbb{N} \setminus \{0\} \longrightarrow \mathbb{N}$ be the map which sends $h > 0$ to

$$\nu_{e,p}(h) = \begin{cases} \nu_p(\frac{h}{e}) + 1, & \text{if } e \mid h, \\ 0, & \text{if } e \nmid h, \end{cases}$$

where ν_p is the usual p-adic valuation map on $\mathbb{N} \setminus \{0\}$ (and where we set $\nu_p(h) = 0$ for all $h \in \mathbb{N} \setminus \{0\}$ if $p = 0$). Note that $\nu_{p,p}(h) = \nu_p(h)$, for all $h \in \mathbb{N} \setminus \{0\}$.

For each integer $k \in \mathbb{Z}$, define the t-**quantum integer** $[k] = (t^k - 1)/(t-1) \in \mathbb{N}[t, t^{-1}]$.

The next result is a sharpening of results from [8]. The main point is to show that the Jantzen coefficients depend only on e and p.

Proposition 3.4. *Suppose that $\lambda, \mu \in \Lambda$. Then $J_{\lambda\mu} \neq 0$ if only if $\lambda \triangleright \mu$ and there exist nodes $(a, b), (a, c) \in [\![\lambda]\!]$ such that $b < c$ and $\nu_{e,p}(h^\lambda_{ab}) \neq \nu_{e,p}(h^\lambda_{ac})$ and μ is obtained from λ by unwrapping the rim hook R^λ_{ac} and then wrapping it back on with its hand node in column b.*

Proof. Let $\nu_\mathfrak{p}$ be the $\mathfrak{p}$-adic valuation map on $\mathcal{O}$. Then, by [8, Theorem 4.3],

$$J_{\lambda\mu} = \sum_{(a,b),(a,c) \in [\![\lambda]\!]} \pm\big(\nu_\mathfrak{p}([h^\lambda_{ab}]) - \nu_\mathfrak{p}([h^\lambda_{ac}])\big),$$

where the sum is over a collection of nodes $(a, b), (a, c) \in [\![\lambda]\!]$ which satisfy the assumptions of the Lemma. Using the abacus and Lemma 3.3, it is easy to see that there is at most one pair of nodes $(a, b), (a, c) \in [\![\lambda]\!]$ that allow us to obtain μ from λ by unwrapping and then wrapping on a hook, so the last equation becomes

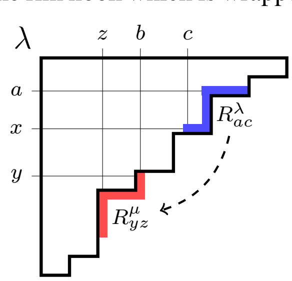
$$J_{\lambda\mu} = \pm\big(\nu_\mathfrak{p}([h^\lambda_{ab}]) - \nu_\mathfrak{p}([h^\lambda_{ac}])\big).$$

(The sign is determined by the parity of the sum of the leg lengths of the rim hooks involved.) By [8, Lemma 4.17], if a and b are any integers then $\nu_\mathfrak{p}([a]) = \nu_\mathfrak{p}([b])$ if and only if either (a) $e \nmid a$ and $e \nmid b$, or (b) $e \mid a$, $e \mid b$ and $\nu_p(a) = \nu_p(b)$. Putting these two statements together proves the lemma. □

Surprisingly, the next result appears to be new.

Corollary 3.5. *Suppose that $\lambda, \mu \in \Lambda$. Then $J_{\lambda\mu} \neq 0$ if and only if $\lambda \triangleright \mu$ and there exist nodes $(x, z), (y, z) \in [\![\mu]\!]$ such that $x < y$ and $\nu_{e,p}(h^\mu_{xz}) \neq \nu_{e,p}(h^\mu_{yz})$ and λ is obtained from μ by unwrapping the rim hook R^μ_{yz} and then wrapping it back on with its foot node in row x.*

Proof. Fix nodes $(a, b), (a, c) \in [\![\lambda]\!]$ as in Proposition 3.4 such that $b < c$ and μ is obtained from λ by unwrapping R^λ_{ac} and wrapping it back on with its hand node in column b. Let $(y, z) \in [\![\mu]\!]$ be the unique node such that $[\![\mu]\!] \setminus R^\mu_{yz} = [\![\lambda]\!] \setminus R^\lambda_{ac}$, as in the diagram below. Thus, $y = \lambda'_b + 1$ and R^μ_{yz} is the rim hook which is wrapped back on to $[\![\lambda]\!] \setminus R^\lambda_{ac}$ to form μ.



Set $x = \lambda'_c$. Then $x < y$ and $R^\mu_{xz} \sqcup R^\lambda_{ac} = R^\lambda_{ab} \sqcup R^\mu_{yz}$ (disjoint unions), where these sets are disjoint because $b < c$. Therefore, $h^\mu_{xz} = h^\lambda_{ab}$, since $h^\mu_{yz} = h^\lambda_{ac}$, and λ is obtained by unwrapping R^μ_{yz} from μ and then wrapping it back on with its foot node in row x. The result now follows by Proposition 3.4. $\square$

Hence, the coefficients $J_{\lambda\mu}$ are almost symmetric in λ and μ.

Corollary 3.6. *Suppose that $\lambda, \lambda', \mu, \mu' \in \Lambda_r$. Then $J_{\lambda\mu} \neq 0$ if and only if $J_{\mu'\lambda'} \neq 0$.*

Proof. This is immediate because the conditions in Proposition 3.4 and Corollary 3.5 are interchanged by taking conjugates of partitions. $\square$

It is well-known from [8] that the blocks of $\mathcal{S}_{k,q}(n,n)$ are determined by the e-cores of the partitions. The next lemma establishes the easy half of this classification within our framework.

Lemma 3.7. *Suppose that $\lambda, \mu \in \Lambda$ and that $J_{\lambda\mu} \neq 0$. Then $\lambda \triangleright \mu$ and λ and μ have the same e-core.*

Proof. By Proposition 3.4 since $J_{\lambda\mu} \neq 0$ there exist nodes $(a,b), (a,c) \in [\![\lambda]\!]$ such that $b < c$, $\nu_{e,p}(h^\lambda_{ab}) \neq \nu_{e,p}(h^\lambda_{ac})$ and μ is obtained from λ by unwrapping the rim hook R^λ_{ac} and then wrapping it back on with its hand node in column b. As the rim hook R^λ_{ac} is wrapped back onto λ lower down it follows $\lambda \triangleright \mu$. It remains to show that λ and μ have the same e-core.

By Lemma 3.3, the abacus configuration for μ is obtained from the abacus configuration for λ by moving one bead h^λ_{ac} positions to the left and another bead h^λ_{ac} positions to the right.

If $e \mid h^\lambda_{ac}$ then, by Lemma 3.3, the e-abacus configuration for μ is obtained by moving two beads on the same runner in the e-abacus configuration for μ. Hence, λ and μ have the same e-core.

On the other hand, if $e \nmid h^\lambda_{ac}$ then $\nu_{e,p}(h^\lambda_{ac}) = 0$, so that e divides h^λ_{ab} by Proposition 3.4. Therefore, the foot residues of the hooks being unwrapped and then wrapped back into $\lambda \backslash R^\lambda_{ac}$ coincide (since, modulo e, these residues differ by h^λ_{ab}). Hence, applying Lemma 3.3, the abacus configuration for μ is obtained from the abacus for λ by moving a bead h^λ_{ab} positions to the left to runner f, say, and then moving another bead h^λ_{ac} positions to the right from runner f. Consequently, the number of beads of each runner is unchanged, so that λ and μ have the same e-core. $\square$

One consequence of Lemma 3.7 is that we can weaken the definition of a cosaturated set of partitions.

Definition 3.8. *Suppose that Λ is a set of partitions of r. Then Λ is e-**cosaturated** if $\mu \in \Lambda$ whenever there exists a partition $\lambda \in \Lambda$ such that $\mu \trianglerighteq \lambda$ and λ and μ have the same e-core.*

Suppose that Λ is a cosaturated set of partitions and that κ is an e-core. Let Λ_κ be the set of partitions in Λ which have e-core κ. Then Λ_κ is e-cosaturated. For each $\lambda \in \Lambda$ let φ_λ be the identity map on $M(\lambda)$ and set

$$e^{(r)}_\kappa = \sum_{\lambda \in \Lambda_\kappa} \varphi_\lambda.$$

Then $e^{(r)}_\kappa$ is an idempotent in $\mathcal{S}_{k,q}(\Lambda)$. Hence, Lemma 3.7 and standard Schur functor arguments, as in Lemma 3.1, imply that the algebra

$$\mathcal{S}_{k,q}(\Lambda_\kappa) := e^{(r)}_\kappa \mathcal{S}_{k,q}(\Lambda) e^{(r)}_\kappa, \tag{3.9}$$

is a quasi-hereditary algebra with weight poset Λ_κ. The algebra $\mathcal{S}_{k,q}(\Lambda_\kappa)$ is a direct sum of blocks of $\mathcal{S}_{k,q}(\Lambda)$. In general, however, $\mathcal{S}_{k,q}(\Lambda_\kappa)$ is not indecomposable. In what follows it will sometimes be convenient to assume that $\Lambda = \Lambda_\kappa$ is e-cosaturated.

3.4. **Projective $\mathcal{S}_{k,q}(\Lambda)$-modules.** Up until now we have been recalling and slightly improving on results from the literature, but we now leave this well-trodden path. The main result of this section is Proposition 3.14 which is a very subtle characterisation of the partitions which contain only *horizontal* e-hooks. This result is the key to the main results of this paper. In particular, it motivates the combinatorial definitions which underpin our Main Theorem.

Let P_k^μ be the projective cover of L_k^μ. In Corollary 2.7 we used the Jantzen coefficients to classify the simple standard modules of a quasi-hereditary algebra. The next result, which is routine but puts the results below into context, shows that the Jantzen coefficients also classify the projective standard modules of $\mathcal{S}_{k,q}(\Lambda)$.

Lemma 3.10. *Suppose that $\mu \in \Lambda$. Then the following are equivalent:*

a) $\Delta_k^\mu = P_k^\mu$ *is a projective $\mathcal{S}_{k,q}(\Lambda)$-module.*
b) Δ_k^μ *is a projective $\mathcal{S}_{k,q}(\Lambda_r)$-module.*
c) $d_{\lambda\mu} = \delta_{\lambda\mu}$*, for all $\lambda \in \Lambda$.*
d) $J_{\lambda\mu} = 0$*, for all $\lambda \in \Lambda$.*
e) $\nu_{e,p}(h_{ac}^\mu) = \nu_{e,p}(h_{bc}^\mu)$*, for all nodes $(a,c), (b,c) \in [\![\mu]\!]$.*
f) $\Delta_k^{\mu'} = L_k^{\mu'}$ *is an irreducible $\mathcal{S}_{k,q}(\Lambda_r)$-module.*

Proof. Parts (a), (c) and (d) are equivalent exactly as in Corollary 2.7. Let P_k^μ be the projective cover of L_k^μ.

For part (c), it follows from the general theory of cellular algebras [**5**, §3] that P_k^μ has a Δ-filtration in which Δ_k^λ appears with multiplicity $d_{\lambda\mu}$. (Note that if $d_{\lambda\mu} \neq 0$ then $\lambda \trianglerighteq \mu$ so that $\lambda \in \Lambda$ since Λ is cosaturated.) Consequently, P_k^μ is also the projective cover of Δ_k^μ, so that Δ_k^μ is projective if and only if $d_{\lambda\mu} = \delta_{\lambda\mu}$. Hence, parts (b) and (c) are also equivalent.

Finally, note that $J_{\lambda\mu} \neq 0$ if and only if $J_{\mu'\lambda'} \neq 0$ by Corollary 3.6, so that (d) and (f) are equivalent by Corollary 2.7 and (d) and (e) are equivalent by Proposition 3.4. This completes the proof. □

Parts (a)–(d) of Lemma 3.10 are equivalent for any quasi-hereditary cellular algebra.

3.11. **Example** Suppose that $(e,p) = (3,0)$ and let Λ be the set of partitions of 39 which dominate $(29,6,4)$ and which do not have empty 3-core. Then Λ is 3-cosaturated and contains the 10 partitions listed below. The reader may check that each of these partitions satisfies the equivalent conditions of Lemma 3.10. Consequently, the decomposition matrix of $\mathcal{S}_{\mathbb{C},\omega}(\Lambda)$, where $\omega = \exp(2\pi i/3)$, is the identity matrix and $\mathcal{S}_{\mathbb{C},\omega}(\Lambda)$ is semisimple. In contrast, if $(e,p) = (3,2)$ then, using [**10**] together with the Steinberg tensor product theorem via [**1**, Proposition 5.4] (see section 3.5), the decomposition matrix of $\mathcal{S}_{k,q}(\Lambda)$ is the following.

$$
\begin{array}{l|cccccccccc}
(34,5) & 1 \\
(31,5,3) & . & 1 \\
(31,8) & . & 1 & 1 \\
(37,2) & . & 1 & . & 1 \\
(30,7,2) & . & . & . & . & 1 \\
(33,4,2) & . & . & . & . & . & 1 \\
(29,6,4) & . & . & . & . & . & . & 1 \\
(29,9,1) & . & . & . & . & . & . & 1 & 1 \\
(32,6,1) & . & . & . & . & . & . & 1 & 1 & 1 \\
(35,3,1) & . & . & . & . & . & . & . & . & 1 & 1
\end{array}
$$

Therefore, when $p = 2$ the partitions in Λ which satisfy the equivalent conditions of Lemma 3.10 are $(34,5)$, $(31,8)$, $(37,2)$, $(30,7,2)$, $(33,4,2)$ and $(35,3,1)$. $\diamond$

The hook R^λ_{ab} is **horizontal** if it is entirely contained in row a of λ. Thus, R^λ_{ab} is horizontal if and only if $\lambda'_b = a$.

Definition 3.12. *Suppose that $\lambda \in \Lambda$. Then λ* ***only contains horizontal hooks*** *if λ is either an e-core or every removable e-hook of λ is horizontal and unwrapping any of these hooks gives a partition which only contains horizontal e-hooks.*

3.13. **Example** Suppose that $e = 3$ and $\mu = (7,4)$. Then μ does *not* contain only horizontal e-hooks because, even though all of the removable 3-hooks of μ are horizontal, $R^{(7,4)}_{15}$, $R^{(4^2)}_{13}$, $R^{(3,2)}_{12}$ is a sequence of e-hooks leading to its 3-core (1^2), however, only the first of these hooks is horizontal. $\diamond$

We now come to the main result of this section, which is both tricky to prove and pivotal for our Main Theorem. In particular, Proposition 3.14 shows that all of the partitions in Example 3.11 contain only horizontal 3-hooks.

Proposition 3.14. *Suppose that $\mu \in \Lambda$ and let κ be the e-core of μ. Then the following are equivalent:*

a) *If $(a,c),(b,c) \in [\![\mu]\!]$ then e divides h^μ_{ac} if and only if e divides h^μ_{bc}.*
b) *The partition μ only contains horizontal e-hooks.*
c) *$\mu_i - \mu_{i+1} \equiv -1 \pmod e$, whenever $i \geq 1$ and $\mu_{i+1} > \kappa_{i+1}$.*

Moreover, these three combinatorial conditions are all equivalent to $\Delta^\mu_k = P^\mu_k$ being projective as an $\mathcal{S}_{k,q}(\Lambda)$-module when $p = 0$.

Proof. By Lemma 3.10, part (a) is equivalent to Δ^μ_k being projective when $p = 0$, so it is enough to show that (a)–(c) are equivalent. Before we start, recall that a partition ν is an e-core if and only if e does not divide h^ν_{ab}, for all $(a,b) \in [\![\nu]\!]$. This is easily proved using Lemma 3.3.

Let w be the e-weight of μ. If $w = 0$ then μ is an e-core and (a)–(c) are equivalent because all three statements are vacuous by the remarks in the first paragraph. We now assume that $w > 0$ and argue by induction on w. Suppose that (a) holds and fix $(a,c) \in [\![\mu]\!]$ where a is maximal such that e divides h^μ_{ac}. By part (a), and the maximality of a, the node (a,c) must be at the bottom of its column. Hence, R^μ_{ac} is a horizontal hook and, by changing c if necessary, we may assume that $h^\mu_{ac} = e$. Let ν be the partition obtained from μ by unwrapping R^μ_{ac}. Since R^μ_{ac} is horizontal, and $h^\mu_{ac}=e$,

$$h^\nu_{xy}=\begin{cases} h^\mu_{xy}-1, & \text{if } c\le y<c+e \text{ and } x<a,\\ h^\mu_{xy}-e, & \text{if } y<c \text{ and } x\le a,\\ h^\mu_{xy}, & \text{otherwise.}\end{cases}$$

Therefore, ν satisfies the condition in part (a) of the Proposition. Hence, by induction on w, ν also satisfies condition (b) so that every removable e-hook contained in ν is horizontal. Now suppose that R^μ_{xy} is any removable fe-hook in μ, for $f\ge 1$. If $R^\mu_{xy}\cap R^\mu_{ac}=\emptyset$ then $R^\mu_{xy}\subseteq[\nu]$ so that it is a union of horizontal e-hooks by induction. If $R^\mu_{xy}\cap R^\mu_{ac}$ is non-empty then, $y\notin(c,c+e)$ by (a) since $h^\mu_{xy}=fe$. Therefore, $R^\mu_{ac}\subseteq R^\mu_{xy}$, so that $R^\mu_{xy}\backslash R^\mu_{ac}$ is a union of horizontal e-hooks in ν. Continuing in this way shows that every e-hook contained in μ is either equal to R^μ_{ac} or it is an e-hook contained in ν. Therefore, by induction, all of the e-hooks contained in μ are horizontal so that (b) holds.

Now suppose that (b) holds. By way of contradiction, suppose that $\mu_i-\mu_{i+1}\not\equiv -1 \pmod e$, for some i with $\mu_{i+1}>\kappa_{i+1}$. Without loss of generality, we may assume that i is maximal with this property. Let c be the unique integer such that $0<c<e$ and $e-c-1\equiv\mu_i-\mu_{i+1}$. Since i was chosen to be maximal, $(i+1,c')$ is at the bottom of its column, where $c'=\mu_{i+1}-c+1$, so that $R^\mu_{ic'}$ is a removable fe-hook which is not horizontal, for some $f\ge 1$. It follows that μ contains a non-horizontal e-hook, which is a contradiction. Hence, $\mu_i-\mu_{i+1}\equiv -1 \pmod e$ whenever $\mu_{i+1}>\kappa_{i+1}$ and (c) holds.

Finally, suppose that (c) holds. Suppose that i is maximal such that $\mu_{i+1}>\kappa_{i+1}$. Now because $\mu_i-\mu_{i+1}\equiv -1 \pmod e$, row $i+1$ of μ contains $(\mu_{i+1}-\kappa_{i+1})/e$ horizontal e-hooks and, moreover, $\mu_i>\kappa_i$. Hence, removing these horizontal e-hooks and arguing by induction it follows that $\mu_j-\mu_{j+1}\equiv -1 \pmod e$, for $1\le j\le i$.

Now let (a,c) and (b,c) be two nodes in $[\![\mu]\!]$ with $a<b$. If $(b,c)\in[\kappa]$ then $(a,c)\in[\kappa]$ so that e does not divide h^μ_{ac} and e does not divide h^μ_{bc}. If $(b,c)\notin[\kappa]$ then $\mu_b>\kappa_b$ and, by the last paragraph, $\mu_a>\kappa_a$ since $a<b$. Let $\mu'=(\mu'_1,\mu'_2,\dots)$ be the partition which is conjugate to μ. Then,

$$\begin{aligned} h^\mu_{ac}-h^\mu_{bc} &= (\mu_a-a+\mu'_c-c+1)-(\mu_b-b+\mu'_c-c+1)\\ &= \mu_a-\mu_b+b-a\equiv 0 \pmod e,\end{aligned}$$

where the last congruence follows because $\mu_j-\mu_{j+1}\equiv -1 \pmod e$, for $1\le j\le i$. Hence, (a) holds and the proof is complete. □

3.5. **The combinatorics of our Main Theorem.** Using Proposition 3.14 we can now properly define the combinatorics underpinning our Main Theorem. The main result of this section is Proposition 3.21 which gives a combinatorial reduction of the calculation of the Jantzen coefficients to the case when $s_\Lambda(\mu)=1$.

Recall from the introduction that $\mathscr{P}_{e,p}=\{1,e,ep,ep^2,\dots\}$. Suppose that $\mu\in\Lambda$ has e-core $\kappa=(\kappa_1,\kappa_2,\dots)$. Let Λ_κ be the set of partitions in Λ which have e-core κ and define the **length function** $\ell_\Lambda:\Lambda\longrightarrow\mathbb{N}$ by

$$\ell_\Lambda(\mu)=\min\{\, i\ge 0 \mid \lambda_j=\kappa_j \text{ whenever } j>i \text{ and } \lambda\in\Lambda_\kappa\,\}.$$

(By definition a partition is an infinite non-increasing sequence $\mu=(\mu_1,\mu_2,\dots)$ so this makes sense.) Observe that if $\Lambda\subseteq\Lambda_r$ then $\ell_\Lambda(\mu)<r$, for all $\mu\in\Lambda$. Moreover, $\ell_\Lambda(\mu)=0$ if and only if $\mu=\kappa$ is an e-core and $\ell_\Lambda(\mu)=1$ only if $\Lambda_\kappa=\{\mu\}$.

The reason why the length function ℓ_Λ is important is that if κ is an e-core and if μ is any partition in Λ_κ then $\mu_i=\kappa_i$, whenever $i>\ell_\Lambda(\mu)$. In particular, when applying the sum formula we can never move e-hooks below row $\ell_\Lambda(\mu)$.

Following the introduction, if $\mu \in \Lambda$ and $\ell_\Lambda(\mu) \le 1$ set $s_\Lambda(\mu) = 1$ and otherwise define

$$s_\Lambda(\mu) = \max\{\, s \in \mathcal{P}_{e,p} \mid \mu_i - \mu_{i+1} \equiv -1 \pmod{s}, 1 \le i < \ell_\Lambda(\mu)\,\}.$$

This definition is stronger than it appears.

Lemma 3.15. *Suppose that $\mu \in \Lambda$ and that $s' \in \mathcal{P}_{e,p}$ with $0 < s' \le s_\Lambda(\mu)$. Then*

$$\mu_i - \mu_{i+1} \equiv -1 \pmod{s'},$$

for $1 \le i < \ell_\Lambda(\mu)$. Moreover, every removable s'-hook contained in μ is horizontal and if $(a, c), (b, c) \in [\![\mu]\!]$ then s' divides h^μ_{ac} if and only if s' divides h^μ_{bc}.

Proof. If $0 \ne s' \in \mathcal{P}_{e,p}$ and $s_\Lambda(\mu) \ge s'$ then $\mu_i - \mu_{i+1} \equiv -1 \pmod{s'}$ since s' divides s. Applying Proposition 3.14 with $e = s'$ shows that every removable s'-hook contained in μ is horizontal and that s' divides h^μ_{ac} if and only if s' divides h^μ_{bc}, for $(a, c), (b, c) \in [\![\mu]\!]$. □

Armed with Lemma 3.15 we can now give a more transparent definition of the partition $\chi_\Lambda(\mu) = (\chi_1, \chi_2, \dots)$ from the introduction. That is, let χ_i be the number of horizontal s-hooks in row i of μ, where $s = s_\Lambda(\mu)$ and $1 \le i \le \ell_\Lambda(\mu)$. Hence, if $\kappa^{(s)}$ is the s-core of μ then

$$\mu_i \equiv \kappa^{(s)}_i \pmod{s}, \quad \text{for } 1 \le i \le \ell_\Lambda(\mu).$$

Therefore, since μ is a partition which contains only horizontal s-hooks, the number of s-hooks in row i of μ is greater than that number of s-hooks in row $i+1$. Hence, $\chi_\Lambda(\mu)$ is a partition.

We illustrate all of the definitions in this section in Example 3.17 below.

By definition, if λ and μ are two partitions in Λ which have the same e-core then $\ell_\Lambda(\lambda) = \ell_\Lambda(\mu)$. This observation accounts for the dependence of the integer $s_\Lambda(\mu)$ and the partition $\chi_\Lambda(\mu)$ upon the poset Λ.

Definition 3.16. *Define $\sim_\Lambda$ to be the equivalence relation on Λ such that $\lambda \sim_\Lambda \mu$, for $\lambda, \mu \in \Lambda$, if*

a) *λ and μ have the same e-core;*
b) *$s_\Lambda(\lambda) = s_\Lambda(\mu)$; and,*
c) *if $s_\Lambda(\mu) > 1$ then $\chi_\Lambda(\lambda)$ and $\chi_\lambda(\mu)$ have the same p-core.*

For part (c), recall that if $p = 0$ then the 0-core of the partition ν is ν.

Thus, our Main Theorem says that if $\lambda, \mu \in \Lambda$ then Δ^λ_k and Δ^μ_k are in the same block if and only if $\lambda \sim_\Lambda \mu$. We prove this in the next section. First, however, we give an example and begin to investigate the combinatorics of the equivalence relation $\sim_\Lambda$.

3.17. **Example** Suppose that $(e, p) = (3, 2)$ and let Λ be the set of partitions of 39 which dominate $(29, 6, 4)$ and which do not have empty 3-core. Then Λ is 3-cosaturated and it contains the 10 partitions in the table below which describes the equivalence $\sim_\Lambda$.

μ	$\mathrm{core}_3(\mu)$	$\ell_\Lambda(\mu)$	$s_\Lambda(\mu)$	$\chi_\Lambda(\mu)$	$\mathrm{core}_2(\chi_\Lambda(\mu))$
$(35,3,1)$	$(5,3,1)$	3	3	(10)	(0)
$(32,6,1)$	$(5,3,1)$	3	3	$(9,1)$	(0)
$(29,9,1)$	$(5,3,1)$	3	3	$(8,2)$	(0)
$(29,6,4)$	$(5,3,1)$	3	3	$(8,1,1)$	(0)
$(33,4,2)$	$(6,4,2)$	2	6	(4)	(0)
$(30,7,2)$	$(6,4,2)$	2	24	(0)	(0)
$(37,2)$	$(4,2)$	3	3	(11)	(1)
$(31,8)$	$(4,2)$	3	3	$(9,2)$	(1)
$(31,5,3)$	$(4,2)$	3	3	$(9,1,1)$	(1)
$(34,5)$	$(4,2)$	3	6	(4)	(0)

The different regions in the table give the $\sim_\Lambda$ equivalence classes in Λ. By our Main Theorem these regions label the blocks of $\mathcal{S}_{k,q}(\Lambda)$. The reader can check that this agrees with Example 3.11 which gives the decomposition matrix for $\mathcal{S}_{k,q}(\Lambda)$. This example shows that ℓ_Λ need not be constant on Λ.

Continuing Example 3.11, if $(e,p)=(3,0)$ then $\ell_\Lambda(\mu)$ is as given above but $s_\Lambda(\mu)=3$ for all $\mu\in\Lambda$. Therefore, by our Main Theorem, all of these partitions are in different blocks because the partitions $\mathrm{core}_0\big(\chi_\Lambda(\mu)\big)=\chi_\Lambda(\mu)$ are distinct, for $\mu\in\Lambda$. Once again, this agrees with the block decomposition of $\mathcal{S}_{k,q}(\Lambda)$ given in Example 3.11 when $(e,p)=(3,0)$. $\diamond$

The following results establish properties of the equivalence relation $\sim_\Lambda$ that we need to prove our Main Theorem.

Lemma 3.18. *Suppose that $\mu\in\Lambda$ and $s\in\mathcal{P}_{e,p}$. Then $s_\Lambda(\mu)\ge s$ if and only if the last $\ell_\Lambda(\mu)$ beads on an s-abacus configuration for μ are all on the same runner.*

Proof. Observe that $\mu_i-\mu_{i+1}\equiv -1 \pmod s$, for $1\le i<\ell$, if and only if

$$\mu_{i+1}-(i+1)\equiv \mu_i - i \pmod s,$$

whenever $1\le i<\ell$. For any positive integer m the m-beta numbers for μ are $m+\mu_j-j$, for $1\le j\le m$. Hence, using Lemma 3.15, it follows that $s_\Lambda(\mu)\ge s$ if and only if the last ℓ beads on any abacus configuration for μ with s runners all lie on the same runner. $\square$

Lemma 3.19. *Suppose that $\mu\in\Lambda$ and that $(a,b)\in[\chi]$ is a node in $\chi=\chi_\Lambda(\mu)$ and let $s=s_\Lambda(\mu)$. Then $h^\chi_{ab}=\frac{1}{s}h^\mu_{aB}$, where column B of μ is the b^{th} column of μ with hook lengths divisible by s, reading from left to right.*

Proof. By Proposition 3.14 and the definition of $s=s_\Lambda(\mu)$, all of the removable s-hooks in μ are horizontal so the definition of B makes sense. The Lemma follows from the observation that the nodes in χ correspond to the removable s-hooks in μ and that the nodes in the rim rook R^χ_{ab} correspond to the removable s–hooks which make up the rim hook R^μ_{aB} in μ. $\square$

Lemma 3.20. *Suppose that λ and μ have the same e-core and that $s_\Lambda(\lambda)=s_\Lambda(\mu)$, for partitions $\lambda,\mu\in\Lambda$. Then $\lambda\trianglerighteq\mu$ if and only if $\chi_\Lambda(\lambda)\trianglerighteq\chi_\Lambda(\mu)$.*

Proof. By Proposition 3.14 the partitions λ and μ only contain horizontal s-hooks, where $s = s_\Lambda(\mu)$. The Lemma follows using this observation and the correspondence between the nodes in $\chi_\Lambda(\lambda)$ and $\chi_\Lambda(\mu)$ and the horizontal s-hooks in λ and μ, respectively. $\square$

The following result is a key reduction step for understanding the blocks of $\mathcal{S}_{k,q}(\Lambda)$. Proposition 3.21 can be interpreted as saying that the Steinberg tensor product theorem preserves Jantzen equivalence — note, however, that its proof requires no knowledge of Steinberg. This result will allow us to reduce Jantzen equivalence to the case where $s_\Lambda(\mu) = 1$. Recall that $\mathcal{S}_{k,1}(\Lambda)$ is the Schur algebra with parameter $q = 1$.

Proposition 3.21. *Suppose that $\lambda, \mu \in \Lambda_\kappa$, where κ is an e-core, are partitions with $s = s_\Lambda(\lambda) = s_\Lambda(\mu) > 1$. Let $\Gamma = \{\, \chi_\Lambda(\nu) \mid \nu \in \Lambda_\kappa \text{ and } s_\Lambda(\nu) = s \,\}$. Then Γ is an cosaturated set of partitions and*

$$J^\Lambda_{\lambda\mu} \neq 0 \quad \textit{if and only if} \quad J^\Gamma_{\chi_\Lambda(\lambda)\chi_\Lambda(\mu)} \neq 0,$$

where $J^\Gamma_{\chi_\Lambda(\lambda)\chi_\Lambda(\mu)}$ is a Jantzen coefficient for the algebra $\mathcal{S}_{k,1}(\Gamma)$. Moreover, if $p = 0$ then $J^\Lambda_{\lambda\mu} = 0$.

Proof. By Lemma 3.20, Γ is an e-cosaturated set of partitions. In order to compare the Jantzen coefficients of the algebras $\mathcal{S}_{k,q}(\Lambda)$ and $\mathcal{S}_{k,1}(\Gamma)$ write $s = ep^d$, for some $d \geq 0$ (with $d = 0$ if $p = 0$). By Lemma 3.15, if $(x, z) \in [\![\mu]\!]$ then $\nu_{e,p}(h^\mu_{xz}) \neq 0$ only if s divides h^μ_{xz}. Moreover, using the notation of Lemma 3.19, if s divides h^μ_{aB}, for $(a, B) \in [\![\mu]\!]$, then

$$\nu_{e,p}(h^\mu_{aB}) = \nu_{e,p}(sh^{\chi_\Lambda(\mu)}_{ab}) = \nu_{e,p}(ep^d h^{\chi_\Lambda(\mu)}_{ab}) = \begin{cases} d + \nu_{p,p}(h^{\chi_\Lambda(\mu)}_{ab}), & \text{if } p > 0, \\ 1 & \text{if } p = 0. \end{cases}$$

Hence, $J^\Lambda_{\lambda\mu} \neq 0$ if and only if $J^\Gamma_{\chi_\Lambda(\lambda)\chi_\Lambda(\mu)} \neq 0$ by Corollary 3.5. Finally, if $p = 0$ then $J^\Lambda_{\lambda\mu} = 0$ by Corollary 3.5 because, by what we have shown, if $s_\Lambda(\mu) > 1$ then $\nu_{e,0}(h^\mu_{xy})$ is constant on the columns of μ. $\square$

3.6. **The Main Theorem.** We are now ready to prove our main theorem. We start by settling the case when $s_\Lambda(\mu) = 1$.

Recall from after Definition 3.8 that if κ is an e-core then Λ_κ is the set of partitions in Λ with e-core κ. By (3.9) the algebra $\mathcal{S}_{k,q}(\Lambda_\kappa)$ is a direct summand of $\mathcal{S}_{k,q}(\Lambda)$.

Lemma 3.22. *Suppose that $s_\Lambda(\tau) = 1$, where $\tau \in \Lambda$ has e-core κ. Then*

$$\{\, \mu \in \Lambda \mid \mu \sim_\Lambda \tau \,\} = \Lambda_\kappa = \{\, \mu \in \Lambda \mid \mu \sim_J \tau \,\}.$$

In particular, $\mu \sim_\Lambda \tau$ if and only if $\mu \sim_J \tau$.

Proof. By definition, if $\mu \in \Lambda$ then $\mu \sim_\Lambda \tau$ if and only if $\mu \in \Lambda_\kappa$ and $s_\Lambda(\mu) = 1$. Suppose, by way of contradiction, that $s_\Lambda(\mu) > 1$ for some $\mu \in \Lambda_\kappa$. Taking $s' = e$ in Lemma 3.15, it follows that

$$-1 \equiv \mu_i - \mu_{i+1} \pmod e, \qquad \text{for } 1 \leq i < \ell_\Lambda(\mu).$$

Combining parts (b) and (c) of Proposition 3.14, this last equation is equivalent to

$$-1 \equiv \kappa_i - \kappa_{i+1} \pmod e, \qquad \text{for } 1 \leq i < \ell_\Lambda(\mu).$$

By the same argument, since $\ell_\Lambda(\tau) = \ell_\Lambda(\mu)$, this implies that

$$-1 \equiv \tau_i - \tau_{i+1} \pmod e, \qquad \text{for } 1 \leq i < \ell_\Lambda(\tau).$$

This implies that $s_\Lambda(\tau) \geq e$, a contradiction! Therefore, $s_\Lambda(\mu) = 1$, for all $\mu \in \Lambda_\kappa$. Hence, $\Lambda_\kappa = \{\, \mu \in \Lambda \mid \mu \sim_\Lambda \tau \,\}$, giving the left hand equality of the Lemma.

We now show that $\mu \sim_J \tau$ if and only if $\mu \in \Lambda_\kappa$. If $\mu \sim_J \tau$ then $\mu \in \Lambda_\kappa$ by Lemma 3.7. To prove the converse, let γ be the unique partition with e-core κ which has $(|\tau| - |\kappa|)/e$ horizontal e-hooks in its first row. Then $\gamma \trianglerighteq \mu$ for all $\mu \in \Lambda_\kappa$. To complete the proof it is enough to show that $\mu \sim_J \gamma$, whenever $\mu \in \Lambda_\kappa$. If $\mu = \gamma$ there is nothing to prove, so suppose that $\mu \neq \gamma$. If $J_{\lambda\mu} \neq 0$ for some $\lambda \in \Lambda$ then $\lambda \triangleright \mu$ by Lemma 3.7 so that $\mu \sim_J \lambda \sim_J \gamma$ by induction on dominance.

We have now reduced to the case when $J_{\lambda\mu} = 0$ for all $\lambda \in \Lambda$. Consequently, $\nu_{e,p}(h^\mu_{ac}) = \nu_{e,p}(h^\mu_{bc})$, for all $(a,c),(b,c) \in [\![\mu]\!]$ by Lemma 3.10(e). Hence, μ contains only horizontal e-hooks by Proposition 3.14. On the other hand, since $s_\Lambda(\mu) = 1$ by the last paragraph, there exists an integer i such that $\mu_i - \mu_{i+1} \not\equiv -1 \pmod e$ and $1 \leq i < \ell_\Lambda(\mu)$. Fix i which is minimal with this property and notice that we must have $\mu_{i+1} = \kappa_{i+1}$ by Proposition 3.14(c). Recalling that all of the e-hooks in μ are horizontal, let λ be the partition obtained by unwrapping the lowest removable e-hook from μ and then wrapping it back on with its foot node in row $i+1$. Then λ is a partition because i is minimal such that $\mu_i - \mu_{i+1} \not\equiv -1 \pmod e$. Moreover, since Λ is cosaturated, $\lambda \in \Lambda_\kappa$ because $\ell_\Lambda(\mu) > i$, $\mu_{i+1} = \kappa_{i+1}$ and all of the e-hooks in μ are horizontal. Next observe that $J_{\mu\lambda} \neq 0$ by Proposition 3.4 because the valuations of the corresponding hook lengths are different since all of the e-hooks in μ are horizontal. Now let σ be the partition obtained by unwrapping this same hook from λ and wrapping it back on as a horizontal hook in the first row. By construction all of the e-hooks in σ are horizontal so, as before, $J_{\sigma\lambda} \neq 0$ by Corollary 3.5 (note, however, that $J_{\sigma\mu} = 0$). Hence, $\mu \sim_J \lambda \sim_J \sigma \sim_J \gamma$, with the last equivalence following by induction since $\sigma \triangleright \mu$. This completes the proof. $\square$

Lemma 3.23. *Suppose that $\lambda \sim_J \mu$, for $\lambda, \mu \in \Lambda$. Then $s_\Lambda(\lambda) = s_\Lambda(\mu)$.*

Proof. Let $s = s_\Lambda(\mu)$ and let $\ell = \ell_\Lambda(\mu)$. By Lemma 3.22 we may assume that $s > 1$ and hence that $p > 0$ since $s_\Lambda(\mu) \in \{1, e\}$ if $p = 0$. It is enough to show that $s_\Lambda(\lambda) = s$ whenever $J_{\lambda\mu} \neq 0$. By Corollary 3.5, $J_{\lambda\mu} \neq 0$ if and only if there exist nodes $(x,z),(y,z) \in [\![\mu]\!]$ such that $x < y \leq \ell$, $\nu_{e,p}(h^\mu_{xz}) \neq \nu_{e,p}(h^\mu_{yz})$ and λ is obtained from μ by unwrapping R^μ_{yz} and wrapping it back on with its foot node in row x. Therefore, s divides both of h^μ_{xz} and h^μ_{yz} and λ is obtained from μ by moving a union of s-hooks.

By Lemma 3.18 the last ℓ beads are always on the same runner in any s-abacus. Therefore, by the last paragraph, an s-abacus for λ is obtained from the s-abacus configuration for μ by moving two beads on the same runner. That is, the abacus configuration for λ is obtained from an s-abacus for μ by moving one bead up fs positions and another bead down fs-positions, for some $f \geq 1$. Hence, $s_\Lambda(\lambda) \geq s_\Lambda(\mu) = s$ by Lemma 3.18 (since $\ell_\Lambda(\lambda) = \ell$ by Lemma 3.7).

By symmetry, using Proposition 3.4 instead of Corollary 3.5, $s_\Lambda(\mu) \geq s_\Lambda(\lambda)$. Hence, $s_\Lambda(\mu) = s_\Lambda(\lambda)$ as required. $\square$

We can now prove our Main Theorem.

Proof of the Main Theorem. By Proposition 2.9 we need to prove that $\lambda \sim_J \mu$ if and only if $\lambda \sim_\Lambda \mu$, for $\lambda, \mu \in \Lambda$. If $p = 0$ then the result follows from Proposition 3.21 and Lemma 3.22, so assume that $p > 0$.

First suppose that $\lambda \sim_\Lambda \mu$, for $\lambda, \mu \in \Lambda$. To show that $\lambda \sim_J \mu$ we argue by induction on $s = s_\Lambda(\mu)$. If $s = 1$ the result is just Lemma 3.22, so suppose that $s > 1$. As in Proposition 3.21 let $\Gamma = \{\, \chi_\Lambda(\nu) \mid \nu \in \Lambda_\kappa \text{ and } s_\Lambda(\nu) = s \,\}$, an e-cosaturated set of partitions. By definition, $s_\Gamma(\chi_\Lambda(\lambda)) = 1 = s_\Gamma(\chi_\Lambda(\mu))$ and, since $\lambda \sim_\Lambda \mu$, the partitions $\chi_\Lambda(\lambda)$ and $\chi_\Lambda(\mu)$ have the same p-core. Therefore, $\chi_\Lambda(\lambda) \sim_{J^\Gamma} \chi_\Lambda(\mu)$ by Lemma 3.22. Hence, by Proposition 3.21, $\lambda \sim_J \mu$ as required.

To prove the converse it is enough to show that $\lambda \sim_\Lambda \mu$ whenever $J_{\lambda\mu} \neq 0$. By Lemma 3.7, λ and μ have the same e-core. Moreover, $s_\Lambda(\lambda) = s_\Lambda(\mu)$, by Lemma 3.23. Finally, $\chi_\Lambda(\lambda) \sim_{J^\Gamma} \chi_\Lambda(\mu)$ are Jantzen equivalent for $\mathcal{S}_{k,1}(\Gamma)$ by Proposition 3.21 since $\lambda \sim_J \mu$. Consequently, $\chi_\Lambda(\lambda)$ and $\chi_\Lambda(\mu)$ have the same p-core by Lemma 3.7. Hence, $\lambda \sim_\Lambda \mu$ as we wanted to show. □

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References

[1] A. Cox, *The blocks of the q–Schur algebra*, J. Algebra, **207** (1998), 306–325. [Pages 1, 2, and 11]
[2] C. W. Curtis and I. Reiner, *Methods of Representation Theory*, Vols. I and II, John Wiley, New York, 1987. [Page 5]
[3] S. Donkin, *On Schur algebras and related algebras IV. The blocks of the Schur algebras*, J. Algebra, **168** (1994), 400–429. [Pages 1 and 2]
[4] ———, *The q–Schur algebra*, Lond. Math. Soc. Lecture Notes, **253**, CUP, Cambridge, 1999. [Page 6]
[5] J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math., **123** (1996), 1–34. [Pages 3, 4, and 11]
[6] J. A. Green, *Polynomial representations of GL_n*, SLN, **830**, Springer–Verlag, New York, 1980. [Pages 2 and 6]
[7] G. D. James, *Some combinatorial results involving Young diagrams*, Proc. Camb. Phil. Soc., **83** (1978), 1–10. [Page 8]
[8] G. D. James and A. Mathas, *A q–analogue of the Jantzen–Schaper Theorem*, Proc. Lond. Math. Soc. (3), **74** (1997), 241–274. [Pages 1, 2, 7, 9, and 10]
[9] S. Lyle and A. Mathas, *Blocks of cyclotomic Hecke algebras*, Adv. Math., **216** (2007), 854–878. [Pages 3 and 5]
[10] A. Mathas, *Decomposition matrices of Hecke algebras of type* **A**, in Gap: groups, algorithms and programming, 3.4.4, M. Schönert et al., eds., RWTH Aachen, 1997. [Page 11]
[11] ———, *Hecke algebras and Schur algebras of the symmetric group*, Univ. Lecture Notes, **15**, Amer. Math. Soc., 1999. [Pages 2, 7, and 8]
[12] G. J. McNinch, *Filtrations and positive characteristic Howe duality*, Math. Z., **235** (2000), 651–685. [Page 6]

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