# ON BRAUER GROUPS OF DOUBLE COVERS OF RULED SURFACES

*by*

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***Abstract.*** — Let $X$ be a smooth double cover of a geometrically ruled surface defined over a separably closed field of characteristic different from 2. The main result of this paper is a finite presentation of the 2-torsion in the Brauer group of $X$ with generators given by central simple algebras over the function field of $X$ and relations coming from the Néron-Severi group of $X$. The path to this result naturally involves a study of the 2-torsion Brauer classes of a smooth double cover of the projective line, yielding results of independent interest. Arithmetic applications are given for both curves and surfaces.

## Contents



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## Introduction

Let $X$ be a smooth, projective and geometrically integral variety over a field $k$. The Brauer group of $X$, denoted $\operatorname{Br} X$, is a generalization of the usual notion of the Brauer group of a field. Our results concern the 2-torsion in $\operatorname{Br} X$ for $X$ a desingularization of a double cover of a ruled surface. Up to birational equivalence, this class of varieties contains all surfaces with an elliptic or hyperelliptic fibration, all double covers of $\mathbb{P}^2$, and (at least over a separably closed field) all Enriques surfaces. Under fairly mild assumptions, we obtain a finite presentation of $\operatorname{Br} \overline{X}[2]$ in terms of generators given by unramified central simple algebras over $\mathbf{k}(\overline{X})$ and relations coming from the Néron-Severi group of $\overline{X}$, where $\overline{X}$ denotes the base change of $X$ to a separably closed field.

This result enables a study of the Galois action on $\operatorname{Br} \overline{X}[2]$, and, in some cases, computation of $\operatorname{Br} X[2]$; as a consequence, we expect it to have important arithmetic implications. More precisely, if $k$ is a global field, then, as Manin [Man71] observed, elements of the Brauer group can obstruct the existence of $k$-points, even when there is no local obstruction. Computation of such an obstruction requires explicit representations of the elements of $\operatorname{Br} X$; knowledge of the group structure alone does not suffice.

The key feature enabling our results is the fibration induced by the ruling on $S$, a fibration whose generic fiber is a double cover $C \to \mathbb{P}^1$. To understand the relevance of this, recall that the Brauer group admits a filtration, $\operatorname{Br}_0 X \subseteq \operatorname{Br}_1 X \subseteq \operatorname{Br} X$, where $\operatorname{Br}_0 X := \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X)$ is the subgroup of constant Brauer classes and $\operatorname{Br}_1 X := \ker\left(\operatorname{Br} X \to \operatorname{Br} \overline{X}\right)$ is the subgroup of algebraic Brauer classes. Using the Hochschild-Serre spectral sequence, the algebraic classes can be understood in terms of the Galois action on the Picard group of $\overline{X}$. In contrast, computation of transcendental Brauer classes, i.e. those surviving in the quotient $\operatorname{Br} X / \operatorname{Br}_1 X$, is usually much more difficult, with only a handful of articles addressing the problem [Har96, Wit04, SSD05, HS05, Ier10, KT11, HVAV11, SZ12, HVA13, Pre13].

In the presence of a fibration as above, one has $\operatorname{Br} X \subseteq \operatorname{Br} C$. Moreover, $\operatorname{Br} C$ is algebraic (over the function field of the base curve) by Tsen's theorem, and may thus be studied in terms of the Galois action on the geometric Picard group of $C$. We carry out such a study for an arbitrary double cover of $\mathbb{P}^1$ over a field $K$, obtaining an explicit presentation of $\operatorname{Br} C[2]$ when $K$ is a $C_1$ field (see Theorem I). In general we obtain a presentation of a subgroup of $(\operatorname{Br} C / \operatorname{Br}_0 C)[2]$ which we show to be large enough for interesting arithmetic applications.

That a fibration can be used in this way to compute Brauer classes on a surface is not new, but there are few classes of surfaces for which the method has been carried out in practice. Our work builds on that of Wittenberg [Wit04] and Ieronymou [Ier10] who each give an example of a nontrivial transcendental 2-torsion Brauer class on specific elliptic K3 surface. The surfaces they consider admit a genus one fibration such that the Jacobian fibration has full rational 2-torsion and such that the generic fiber is a double cover of $\mathbb{P}^1$. We formalize and generalize the technique to deal with fibrations of curves of arbitrary genus that are double covers of $\mathbb{P}^1$ and remove all assumptions on the Jacobian fibration. As an application, we give an explicit presentation of the Brauer group of any Enriques surface, and demonstrate the utility of this presentation by giving an example of an Enriques surface with a transcendental obstruction to weak approximation.

**Structure of the paper.** — The paper is divided into two parts in which we consider, respectively, double covers of $\mathbb{P}^1$ and double covers of ruled surfaces.

**Part I: Double covers of the projective line.** — We consider a smooth, irreducible double cover $\pi : C \to \mathbb{P}^1$ defined over a field $K$ of characteristic not equal to 2. Choosing a generator $x$ for $\mathbf{k}(\mathbb{P}^1)$ such that $\pi$ does not ramifiy above the pole of $x$, we obtain a model for $C$ of the form $y^2 = cf(x)$ with $c \in K^\times$ and $f(x) \in K[x]$ monic of even degree. Let $L = K[\theta]/f(\theta)$ and let $x - \alpha$ denote the image of $x - \theta$ in $\mathbf{k}(C_L) := L \otimes_K \mathbf{k}(C)$. The theory behind explicit descents developed in [PS97] shows that $x - \alpha$ induces a homorphism $\operatorname{Pic} C/2\operatorname{Pic} C \to L^\times/K^\times L^{\times 2}$. The main result of Part I is an explicit description of the 2-torsion in the Brauer group of $C$ in terms of the cokernel of this map.

***Theorem I.*** — *There is a complex*

$$\frac{\operatorname{Pic} C}{2\operatorname{Pic} C} \xrightarrow{x-\alpha} \mathfrak{L}_c \xrightarrow{\gamma} \left(\frac{\operatorname{Br} C}{\operatorname{Br}_0 C}\right)[2] \longrightarrow 0\,,$$

*where $\mathfrak{L}_c \subseteq L^\times/K^\times L^\times$ is the subset of classes that are represented by some $\ell \in L^\times$ such that $\operatorname{Norm}_{L/K}(\ell) \in (K^{\times 2} \cup cK^{\times 2})$, and $\gamma$ is induced by the map sending $\ell \in L^\times$ to the central simple algebra $\operatorname{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}((\ell, x-\alpha)_2)$ over $\mathbf{k}(C)$. If $\operatorname{Br} K[2] = 0$ or if $\pi$ has a $K$-rational branch point, then this complex is an exact sequence.*

Although $\gamma$ is not surjective in general (see Remark 5.4), we show that its image is large enough for interesting arithmetic applications (see Proposition 6.4, Remark 6.5 and Theorem 6.7). As an alternative to imposing conditions on $K$, assumptions on the Galois structure of the ramification locus can also be used to guarantee exactness (see Theorems 1.2 and 1.3). The reader will also note that Proposition 2.5 shows how $\operatorname{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}((\ell, x-\alpha)_2)$ may be written as a sum of quaternion algebras over K(x).

**Part II: Double covers of ruled surfaces.** — We consider a desingularization $X$ of a double cover $\pi : X^0 \to S$ of a ruled surface defined over a field $k$ of characteristic not 2. Let $\overline{k}$ be a separable closure of $k$. The ruling on $S$ is given by a map to a smooth curve $W$ such that the generic fiber is isomorphic to $\mathbb{P}^1_{\mathbf{k}(W)}$. If *every* fiber of $S/W$ is isomorphic to $\mathbb{P}^1$, we that $S$ is geometrically ruled. For the purposes of this introduction we will assume that the branch locus of $X/S$ is a reduced curve $B$ flat over $W$; see §7 for details on the general case.

The generic fiber of $X \to S \to W$ is a double cover $C \to \mathbb{P}^1_{\mathbf{k}(W)}$, and Theorem I gives a presentation of $\operatorname{Br} C_K[2]$ as the image of the surjective map, $\gamma : \mathfrak{L}_c \to \operatorname{Br} C_K[2]$, where $K = \mathbf{k}(\overline{W})$. The algebra $L$ may be identified with $\mathbf{k}(\overline{B})$, and by the purity theorem $\operatorname{Br} \overline{X} \subseteq \operatorname{Br} C_K$ is the subgroup unramified at all vertical divisors. In §9 we specify a finite set of functions $\ell \in \mathbf{k}(\overline{B})$ such that $\operatorname{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C_K)}((\ell, x-\alpha)_2)$ is unramified at all vertical divisors. With these functions we define a finite subgroup $\mathfrak{L}_{c,\mathcal{E}} \subseteq \mathfrak{L}_c$ and prove the following (see Corollary 9.5).

***Theorem II.*** — *Let $X$ be as above. Assume that $S$ is geometrically ruled and that $B$ has at worst simple singularities. Then there is an exact sequence of $\operatorname{Gal}(\overline{k}/k)$-modules*

$$\frac{\operatorname{Pic} C_K}{2\operatorname{Pic} C_K} \xrightarrow{x-\alpha} \mathfrak{L}_{c,\mathcal{E}} \xrightarrow{\gamma} \operatorname{Br} \overline{X}[2] \longrightarrow 0.$$

In §10 we show that this presentation can be used to determine the size of $\operatorname{Br} \overline{X}[2]$ without using the exponential sequence or knowledge of the Betti numbers. For example, a double cover of a quadric surface branched along a $(4,4)$ curve is a K3 surface for which we recover the well known fact that $\operatorname{Br} \overline{X}[2]$ has $\mathbb{F}_2$-dimension $22 - \operatorname{rank} \operatorname{NS} \overline{X}$. When the branch locus of the double cover has worse than simple singularities, we obtain a presentation for $\operatorname{Br}\overline{U}[2]$ where $\overline{U} \subset \overline{X}$ is a specified open subvariety (see §9 for more details).

If $S$ fails to be geometrically ruled, then these methods can still be used to obtain a presentation for $\operatorname{Br}\overline{X}[2]$ or $\operatorname{Br}\overline{U}[2]$, provided one has sufficient information about the singular fibers of $S \to W$. We demonstrate this in §11 by applying these methods to compute the nontrivial element in the Brauer group of an arbitrary Enriques surface, for which the ruled surface has 2 singular fibers. We then use this in §12 to give an explicit example of an Enriques surfaces with a transcendental Brauer-Manin obstruction to weak approximation.

**Notation.** — Let $K$ be a field, choose a separable closure $\overline{K}$ and let $G_K := \operatorname{Gal}(\overline{K}/K)$ be the absolute Galois group. If $M$ is a $G_K$-module (with the discrete topology) and $i \geq 0$, then $\mathrm{H}^i(K, M) := \mathrm{H}^i(G_K, M)$ denotes the $i$th Galois cohomology group. Similarly $C^i(K, M)$ and $Z^i(K, M)$ are, respectively, the groups of continuous $i$-cochains and $i$-cocycles. More generally, if $A$ is an étale $K$-algebra, then $\mathrm{H}^i(A, M)$ denotes the étale cohomology group $\mathrm{H}^i_{\text{ét}}(\operatorname{Spec} A, M)$. If $A \simeq \prod K_j$ for field extensions $K_j/K$, Shapiro's lemma shows that $\mathrm{H}^i(A, M) \simeq \prod \mathrm{H}^i(K_j, M)$. If $\mathcal{G}$ is an algebraic group defined over $K$ we define $\mathrm{H}^i(K, \mathcal{G}) := \mathrm{H}^i(K, \mathcal{G}(\overline{K}))$, and analogously for the other groups defined above.

The Brauer group of $X$ is the étale cohomology group $\operatorname{Br} X := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. Given invertible elements $a, b$ in an étale $K$-algebra $A$, we define the quaternion algebra $(a, b)_2 := A[i, j]/\langle i^2 = a,\ j^2 = b,\ ij = -ji\rangle$, which we often conflate with its class in $\operatorname{Br} A$. If $X$ and $S$ are $K$-schemes, we set $X_S := X \times_{\operatorname{Spec} K} S$. We also define $\overline{X} := X_{\overline{K}}$ and $X_A := X_{\operatorname{Spec} A}$, for a $K$-algebra $A$ of finite type. If $X$ is an integral $K$-scheme, $\mathbf{k}(X)$ denotes its function field. More generally, if $X$ is a finite union of integral $K$-schemes $X_i$, then $\mathbf{k}(X) := \prod \mathbf{k}(X_i)$ is the ring of global sections of the sheaf of total quotient rings. In particular, if $A \simeq \prod K_j$ is an étale $K$-algebra, then $X_A$ is a union of integral $K$-schemes and $\mathbf{k}(X_A) \simeq \prod \mathbf{k}(X_{K_j})$. For $r \geq 0$ we use $X^{(r)}$ to denote the set of codimension $r$ points on $X$.

Now suppose that $X$ is a smooth, projective and geometrically integral variety over $K$. Let $\operatorname{Pic} X$ be its Picard group and let $\operatorname{Pic}_X$ be its Picard scheme. Then $\operatorname{Pic} X = \operatorname{Div} X/\operatorname{Princ} X$, where $\operatorname{Div} X$ (resp. $\operatorname{Princ} X$) is the group of divisors (resp. principal divisors) of $X$ defined over $K$. If $D \in \operatorname{Div} X$, then $[D]$ denotes its class in $\operatorname{Pic} X$. There is a bijective map $(\operatorname{Pic}\overline{X})^{G_K} \to \operatorname{Pic}_X(K)$, but in general the map $\operatorname{Pic} X \to \operatorname{Pic}_X(K)$ is not surjective. Let $\operatorname{Pic}^0_X \subseteq \operatorname{Pic}_X$ denote the connected component of the identity, and use $\operatorname{Pic}^0 X$ to denote the subgroup of $\operatorname{Pic} X$ mapping into $\operatorname{Pic}^0_X(K)$. Then $\operatorname{NS} X := \operatorname{Pic} X/\operatorname{Pic}^0 X$ is the Néron-Severi group of $X$. If $\lambda \in (\operatorname{NS}\overline{X})^{G_K}$, let $\operatorname{Pic}^\lambda_X$ denote the corresponding component of the Picard scheme and use $\operatorname{Pic}^\lambda X$ and $\operatorname{Div}^\lambda X$ to denote the subsets of $\operatorname{Pic} X$ and $\operatorname{Div} X$ mapping into $\operatorname{Pic}^\lambda_X(K)$. We write $\operatorname{Alb}_X$ for the Albanese scheme of $X$ and, for $i \in \mathbb{Z}$, write $\operatorname{Alb}^i_X$ for the degree $i$ component of $\operatorname{Alb}_X$. Then $\operatorname{Alb}^i_X$ is a $K$-torsor under the abelian variety $\operatorname{Alb}^0_X$. When $X$ is a curve, $\operatorname{NS}\overline{X} = \mathbb{Z}$, $\operatorname{Pic}^i_X = \operatorname{Alb}^i_X$ for all $i \in \mathbb{Z}$ and $\operatorname{Jac}(X) := \operatorname{Pic}^0_X = \operatorname{Alb}^0_X$ is called the Jacobian of $X$.

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# PART I
# DOUBLE COVERS OF THE PROJECTIVE LINE

## 1. Introduction

Throughout Part I, $K$ is a field of characteristic different from 2 and $\pi : C \to \mathbb{P}^1$ is an irreducible double cover of the projective line defined over $K$ with Jacobian $J := \mathrm{Jac}(C)$. We say that $\pi$ is odd if $\pi$ is ramified above $\infty \in \mathbb{P}^1(K)$. Otherwise we say that $\pi$ is even. Provided $K$ has sufficiently many elements (e.g. if $K$ is infinite) a change of coordinates on $\mathbb{P}^1$ allows us to obtain an isomorphic double cover which is even. On the other hand, $\pi$ is isomorphic to an odd double cover if and only there is a $K$-rational ramification point. While there is thus no loss of generality in considering only even double covers, it is possible to obtain results that are sharper in the case of odd double covers (cf. Theorem 1.3 and Remark 2.8). We have chosen the notation below to allow the two cases to be treated in parallel.

By Kummer theory, $C$ has a model of the form $y^2 = cf(x)$ with $c \in K^\times$ and $f(x)$ a square free monic polynomial with coefficients in $K$. The degree of $f(x)$ is either $2g(C) + 2$ or $2g(C) + 1$, correspondingly as $C$ is even or odd. Moreover, when $C$ is odd, we can perform a change of coordinates to arrange that $c = 1$. Let $\Omega \subseteq C$ be the set of ramification points of $\pi$, and let $L = \mathrm{Map}_K(\Omega, \overline{K})$ denote the étale $K$-algebra corresponding to $\Omega$. When $C$ is even we may identify $K[\theta]/f(\theta)$ with $L$. When $C$ is odd, $K[\theta]/f(\theta)$ can be identified with the subalgebra $L_\circ \subseteq L$ consisting of elements $\ell \in L = \mathrm{Map}_K(\Omega, \overline{K})$ that take the value 1 at the ramification point above $\infty \in \mathbb{P}^1(K)$. In the odd case this gives a canonical isomorphism $L \cong L_\circ \times K$.

Set

$$\mathfrak{L} = \frac{L^\times}{K^\times L^{\times 2}}\,.$$

For $a \in K^\times$ and $\ell \in L^\times$, we use $\overline{a}$ and $\overline{\ell}$ to denote the corresponding classes in $K^\times/K^{\times 2}$ and $\mathfrak{L}$, and set

$$\mathfrak{L}_a = \left\{\overline{\ell} \in \mathfrak{L} \,:\, \mathrm{Norm}_{L/K}\left(\overline{\ell}\right) \in \langle \overline{a} \rangle\right\}\,,$$

where $\mathrm{Norm}_{L/K}$ denotes the map $\mathfrak{L} \to K^\times/K^{\times 2}$ induced by the norm on $L$. Note that when $C$ is odd we have a canonical isomorphism $\mathfrak{L} \cong L_\circ^\times/L_\circ^{\times 2}$ under which $\mathrm{Norm}_{L/K}$ coincides with the map induced by the norm on $L_\circ$.

Let $x - \alpha$ denote the image of $x - \theta$ in $\mathbf{k}(C_L) := L \otimes_K \mathbf{k}(C)$; in the odd case this means $x - \alpha$ is the image of $(x - \theta, 1)$ in $\mathbf{k}(C_{L_\circ}) \times \mathbf{k}(C)$. It is well known from the theory of explicit descents (see [Sch95, PS97]) that $x - \alpha$ induces a homorphism $\mathrm{Pic}\, C/2\,\mathrm{Pic}\, C \to \mathfrak{L}$. For a closed point $P \in C \setminus (\Omega \cup \pi^{-1}(\infty))$ one defines $x(P) - \alpha = \prod_{i=1}^{d}(x_i - \alpha) \in L^\times$, where $P(\overline{K}) = \{(x_1, y_1), \dots, (x_d, y_d)\}$. Every divisor class $[D] \in \mathrm{Pic}\, C$ can be represented by a sum $\sum_P n_P P$ of such closed points, and $(x - \alpha)([D])$ is defined to be the class of $\prod_P (x(P) - \alpha)^{n_P}$ in $\mathfrak{L}$.

Let $\gamma'$ be the map,

$$\gamma' \colon L^\times \to \mathrm{Br}\,\mathbf{k}(C), \quad \ell \mapsto \mathrm{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}((\ell, x - \alpha)_2)\,.$$

Proposition 2.5 shows how to write $\gamma'(\ell)$ as a sum of quaternion algebras over $\mathbf{k}(C)$.

***Theorem 1.1.*** — *The map $\gamma'$ induces a complex*

$$\frac{\operatorname{Pic} C}{2\operatorname{Pic} C} \xrightarrow{x-\alpha} \mathfrak{L}_c \xrightarrow{\gamma} \left(\frac{\operatorname{Br} C}{\operatorname{Br}_0 C}\right)[2]\,.$$

*If $\operatorname{Pic}^1 C = \operatorname{Pic}^1_C(K)$, then the sequence is exact. If $\operatorname{Br} K[2] = 0$, then the sequence is exact and $\gamma$ is surjective.*

The kernel of $x-\alpha$ is determined in Proposition 4.7, and although $\gamma$ is not surjective in general (see Remark 5.4), we show that its image is large enough for arithmetic applications. Namely, in §6 we recall the well known relationship between Brauer-Manin obstructions and the Cassels-Tate pairing, and show how the aforementioned results allow us to compute the pairing for certain elements of $Ш(J)[2]$. In a numerical example we carry out these computations for *all* quadratic twists of a specific curve, giving an infinite family of abelian surfaces over $\mathbb{Q}$ with nontrivial Shafarevich-Tate group.

As an alternative to imposing conditions on $K$, assumptions on the Galois structure of $\Omega$ can also be used to obtain a presentation of $(\operatorname{Br} C/\operatorname{Br}_0 C)[2]$.

***Theorem 1.2.*** — *If $\Omega$ admits a $G_K$-stable partition into two sets of odd cardinality, then $\gamma'$ induces an exact sequence*

$$0 \to J(K)/2J(K) \xrightarrow{x-\alpha} \mathfrak{L}_1 \xrightarrow{\gamma} \left(\frac{\operatorname{Br} C}{\operatorname{Br}_0 C}\right)[2] \to 0\,.$$

***Theorem 1.3.*** — *If $C$ is odd, then $\gamma'$ induces an exact sequence*

$$0 \to J(K)/2J(K) \xrightarrow{x-\alpha} \mathfrak{L}_1 \xrightarrow{\gamma} (\operatorname{Br}^0 C)[2] \to 0\,,$$

*where $\operatorname{Br}^0 C$ denotes the subgroup of $\operatorname{Br} C$ consisting of Brauer classes that evaluate to $0$ at the $K$-rational ramificiation point of $C$ lying above $\infty \in \mathbb{P}^1(K)$.*

The proofs of these theorems are inspired by the classical problem of 2-descents on Jacobians of hyperelliptic curves. Here one attempts to compute $J(K)/2J(K)$ by describing its image under the connecting homomorphism $\delta$ in the Kummer sequence,

$$0 \to J(K)/2J(K) \xrightarrow{\delta} \mathrm{H}^1(K, J[2]) \to \mathrm{H}^1(K, J)[2] \to 0\,. \tag{1.1}$$

To make use of this in practice, one requires concrete descriptions of $\mathrm{H}^1(K, J[2])$ and the map $\delta$. When $C$ is odd this is achieved in [Sch95] by giving an explicit isomorphism $\mathrm{H}^1(K, J[2]) \simeq \mathfrak{L}_1$ whose composition with $\delta$ is equal to the $x-\alpha$ map. Moreover, the existence of a rational point implies an isomorphism $\mathrm{H}^1(K, J)[2] \simeq \operatorname{Br}^0(C)[2]$. Together with (1.1) these isomorphisms imply the existence of an exact sequence as stated in Theorem 1.3, and the task is to verify that the description of $\gamma$ given is correct. This will ultimately be achieved by a cocycle computation. In the case that $C = J$ is an elliptic curve with rational 2-torsion this has been carried out in [Wit04, Prop. 2.2] (see also [Sko01, p.91]).

When $C$ is even, there are complications due to the fact that, in general, neither of the aforementioned isomorphisms exist. In the first instance we are forced to replace the isomorphism of [Sch95] with the *fake descent* setup of [PS97]. This implies the existence of an exact sequence,

$$\frac{\operatorname{Pic}^0 C}{2\operatorname{Pic}^0 C} \xrightarrow{x-\alpha} \mathfrak{L}_1 \xrightarrow{d} \frac{\mathrm{H}^1(K, J)[2]}{\langle \operatorname{Pic}^1_C \rangle}\,, \tag{1.2}$$

where under suitable hypothesis (e.g. if $\operatorname{Br} K[2] = 0$) the final map is surjective. When $C$ has no $K$-rational divisors of degree 1 the second isomorphism above must be replaced by an exact sequence,

$$0 \to \left(\frac{\operatorname{Br} C}{\operatorname{Br}_0 C}\right)[2] \xrightarrow{h_0} \left(\frac{\mathrm{H}^1(K,J)}{\langle \mathrm{Pic}^1_C\rangle}\right)[2] \to \mathrm{H}^3(K,\overline{K}^{\times})\,. \tag{1.3}$$

This means that, even under the assumption that $K$ is $C_1$, the image of the map $d$ in (1.2) may only correspond to an index 2 subgroup of $(\operatorname{Br} C/\operatorname{Br}_0 C)[2]$. Our solution to this problem is inspired by [Cre] where it is shown how the elements of $\mathfrak{L}_c \setminus \mathfrak{L}_1$ correspond to certain $\mathrm{Pic}^1_C$-torsors under $J[2]$. The natural images of these torsors in $\mathrm{H}^1(K,J)$ lie in the fiber above $\mathrm{Pic}^1_C$ under multiplication by 2. This allows one to deduce the existence of an exact sequence

$$\frac{\operatorname{Pic} C}{2\operatorname{Pic} C} \xrightarrow{x-\alpha} \mathfrak{L}_c \xrightarrow{d} \left(\frac{\mathrm{H}^1(K,J)}{\langle \mathrm{Pic}^1_C\rangle}\right)[2]\,, \tag{1.4}$$

which is compatible with (1.2), and again where the final map is surjective when $\operatorname{Br} K[2] = 0$. To prove Theorem 1.1 we must then show that these maps are compatible in the sense that $h_0 \circ \gamma = d$.

**Outline of Part I.** — In §2 we compute the residues of an algebra of the form $\gamma'(\ell)$ and use this to show that $\gamma'$ induces a complex as stated in the theorems above. Then in section §3 we define the map $h_0$ and compute $h_0 \circ \gamma'$ in terms of cocycles. This is then related to the cohomological setup for 2-descents in §4. The results of the preceding sections are then utilized in §5 to prove the theorems above. §6 gives an arithmetic application of these results which is independent from the results in Part II.

## 2. Corestriction and residues

Recall that $\gamma'$ sends $\ell \in L^{\times}$ to $\mathrm{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}\left((\ell, x-\alpha)_2\right) \in \operatorname{Br}\mathbf{k}(C)$. We begin by computing the residues of such an algebra, and use the purity theorem to determine when it lies in the unramified subgroup $\operatorname{Br} C \subseteq \operatorname{Br}\mathbf{k}(C)$.

***Proposition 2.1***. — *Let $\ell \in L^{\times}$. If $C$ is odd, then $\gamma'(\ell) \in \operatorname{Br} C$. If $C$ is even, then $\gamma'(\ell) \in \operatorname{Br} C$ if and only if $\overline{\ell} \in \mathfrak{L}_c$.*

*Proof.* — Consider the following diagram:

$$\begin{tikzcd}
K_2^M(\mathbf{k}(C_L)) \arrow[rr, "\oplus\partial_w^M"] \arrow[dr, "h^2_{\mathbf{k}(C_L),2}"] \arrow[dd, "\mathrm{N}_{\mathbf{k}(C_L)/\mathbf{k}(C)}"'] & & \bigoplus_{w|v} K_1^M(\kappa(w)) \arrow[dr, "\oplus h^1_{\kappa(w),2}"] \arrow[dd, "\sum_{w|v}\mathrm{N}_{\kappa(w)/\kappa(v)}" near end] & \\
& \mathrm{H}^2(\mathbf{k}(C_L),\mu_2^{\otimes 2}) \arrow[rr, "\oplus\partial_w^2" near start] \arrow[dd, "\mathrm{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}" near start] & & \bigoplus_{w|v} \mathrm{H}^1(\kappa(w),\mu_2) \arrow[dd, "\sum_{w|v}\mathrm{Cor}_{\kappa(w)/\kappa(v)}"] \\
K_2^M(\mathbf{k}(C)) \arrow[rr, "\partial_v^M" near end] \arrow[dr, "h^2_{k(X),2}"] & & K_1^M(\kappa(v)) \arrow[dr, "h^1_{\kappa(v),2}"] & \\
& \mathrm{H}^2(\mathbf{k}(C),\mu_2^{\otimes 2}) \arrow[rr, "\partial_v^2"] & & \mathrm{H}^1(\kappa(v),\mu_2)
\end{tikzcd} \tag{2.1}$$

The back, side, top, and bottom squares are all commutative [GS06, Cor. 7.4.3 and Prop. 7.5.1 & 7.5.5]. Therefore, all ways of traversing from $K_2^M(\mathbf{k}(C_L))$ to $\mathrm{H}^1(\kappa(v), \mu_2)$ are equivalent. By the Merkurjev-Suslin theorem [GS06, Thm. 4.6.6], $h^2_{\mathbf{k}(C_L),2}$ is surjective so the front square is commutative. Using the purity theorem [Fuj02] and the commutativity of the front and right square of (2.1), we obtain the following commutative diagram where the bottom row is exact.

$$\begin{tikzcd} & & \operatorname{Br} \mathbf{k}(C_L)[2] \arrow[r, "\oplus_w \partial_w^2"'] \arrow[d, "\mathrm{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}"] & \bigoplus_v \left( \bigoplus_{w|v} \kappa(w)^\times / \kappa(w)^{\times 2} \right) \arrow[d, "\prod_{w|v} \mathrm{N}_{\kappa(w)/\kappa(v)}"] \\ 0 \arrow[r] & \operatorname{Br} C[2] \arrow[r] & \operatorname{Br} \mathbf{k}(C)[2] \arrow[r, "\oplus_v \partial_v^2"'] & \bigoplus_v \kappa(v)^\times / \kappa(v)^{\times 2} \end{tikzcd} \tag{2.2}$$

(Here, the direct sum $\oplus_v$ ranges over all valuations corresponding to prime divisors on $C$, and $\oplus_{w|v}$ ranges over all valuations corresponding to prime divisors in $C_L$ lying over $v$.)

Thus, $\mathrm{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}\left((\ell, x-\alpha)_2\right)$ is in $\operatorname{Br} C$ if and only if

$$\prod_{w|v} \mathrm{Cor}_{\kappa(w)/\kappa(v)}\left((-1)^{w(\ell)w(x-\alpha)} \ell^{w(x-\alpha)} (x-\alpha)^{w(\ell)}\right) \tag{2.3}$$

is a square in $\kappa(v)^\times$, for all valuations $v$. Since $\ell$ is a constant in $\mathbf{k}(C_L)$, $w(\ell) = 0$ for all valuations $w$. When $C$ is odd, $w(x-\alpha) \equiv 0 \bmod 2$, for all valuations $w$, so (2.3) is clearly a square. Hence we may assume that $C$ is even. Furthermore we can restrict our attention to valuations $v$ such that there exists a $w \mid v$ with $w(x-\alpha) \neq 0$.

For all valuations $w$ such that $w(x-\alpha)$ is positive, we have that $w(x-\alpha) \equiv 0 \bmod 2$ so (2.3) is clearly a square. Thus we may consider the valuations $v$ for which there exists a $w \mid v$ with $w(x-\alpha) < 0$. Such valuations $v$ correspond to the points at infinity on $C$, and, for every $w \mid v$, we have that $w(x-\alpha) = -1$. In this case (2.3) can be simplified to

$$\prod_{w|v} \mathrm{Cor}_{\kappa(w)/\kappa(v)}\left((-1)^{w(\ell)w(x-\alpha)} \ell^{w(x-\alpha)} (x-\alpha)^{w(\ell)}\right) = \prod_{w|v} \mathrm{Cor}_{\kappa(w)/\kappa(v)}\left(\ell^{-1}\right) = \mathrm{Norm}_{L/K}(\ell^{-1}).$$

This shows that $\gamma'(\ell) \in \operatorname{Br} C$ if and only if $\mathrm{Norm}_{L/K}(\ell) \in \kappa(v)^{\times 2}$. But $\kappa(v) = K(\sqrt{c})$, so this is equivalent to requiring that $\overline{\ell} \in \mathfrak{L}_c$. This completes the proof. □

***Lemma 2.2***. — *$\gamma'$ induces a homomorphism $\gamma \colon \mathfrak{L} \to \operatorname{Br} \mathbf{k}(C)/\operatorname{Br}_0 C$ such that the natural square commutes.*

*Proof.* — The map $\gamma'$ is clearly a homomorphism. It remains to show that $\gamma'$ sends $K^\times L^{\times 2}$ into $\operatorname{Br}_0 C$. Let $a \in K^\times$, $\ell \in L^\times$; we may expand $\gamma'(a\ell^2)$ as follows

$$\begin{aligned} \gamma'(a\ell^2) &= \mathrm{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}\left((a, x-\alpha)_2\right) + \mathrm{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}\left((\ell^2, x-\alpha)_2\right) \\ &= (a, \mathrm{Norm}_{\mathbf{k}(C_L)/\mathbf{k}(C)}(x-\alpha))_2 = (a, f(x))_2 = (a, y^2/c)_2 = (a, c)_2\,. \end{aligned}$$

This completes the proof since $(a, c)_2 \in \operatorname{Br}_0 C$. □

***Proposition 2.3***. — *If $P \in C \setminus (\Omega \cup \pi^{-1}(\infty))$ is a closed point, then $\gamma'((x(P)-\alpha)) \in \operatorname{Br}_0 C$.*

***Corollary 2.4.*** — *The sequence*

$$\operatorname{Pic} C \xrightarrow{x-\alpha} \mathfrak{L}_c \xrightarrow{\gamma} \operatorname{Br} C/\operatorname{Br}_0 C$$

*is a complex.*

*Proof.* — $[D] \in \operatorname{Pic} C$ is represented by a linear combination $\sum_P n_P P$ of points $P$ as in the proposition (see [PS97, §5]), and, by definition, $(x-\alpha)([D])$ is the class of $\prod(x(P)-\alpha)^{n_P}$ in $\mathfrak{L}$. □

*Proof of Proposition 2.3.* — If $P$ is the pullback of a point from $\mathbb{P}^1$, then $(x(P)-\alpha) \in L^{\times 2}$ and so $\gamma'(x(P)-\alpha) = 0$. Assume otherwise, and let $p(x) \in K[x]$ be the minimal polynomial of the $x$-coordinate of $P$. Then $\gamma'(x(P)-\alpha) = \operatorname{Cor}_{L(x)/K(x)}\left(((-1)^{\deg(P)}p(\alpha), x-\alpha)_2\right)$; we note this element is in $\operatorname{Br} K(x) = \operatorname{Br} \mathbf{k}(\mathbb{P}^1_K)$. We will show that the algebras $\gamma'(x(P)-\alpha)$ and $\mathcal{A} := (cf(x), (-1)^{\deg(P)}p(x))_2$ have the same residue at all points of $\mathbb{P}^1_K$. Since $\operatorname{Br} \mathbb{P}^1_K = \operatorname{Br} K$, this shows that $\gamma'(x(P)-\alpha)$ and $\mathcal{A}$ differ by a constant algebra. To complete the proof, we note that $\mathcal{A} \in \ker(\operatorname{Br} \mathbf{k}(\mathbb{P}^1_K) \to \operatorname{Br} \mathbf{k}(C))$.

Considered as an element of $\operatorname{Br} \mathbf{k}(\mathbb{P}^1)$, the algebra $\gamma'(x(P)-\alpha)$ has trivial residue away from the $\infty$ and the roots of $f(x)$. The residue at $\infty$ is $\operatorname{Norm}_{L/K}((-1)^{\deg(P)}p(\alpha))$ which is equal to $c^{\deg(P)}$ in $K^\times/K^{\times 2}$.

Now we compute the residue at the roots of $f(x)$. Let $f_Q$ be an irreducible factor of $f$ corresponding to a root $Q$ of $f(x)$, and let $\beta$ be the image of $\theta$ in $\kappa(Q) = K[\theta]/f_Q(\theta)$. There is a unique valuation on $\kappa(Q)(x) \subseteq L(x)$ lying above $Q$ such that $((-1)^{\deg(P)}p(\alpha), x-\alpha)_2$ has nontrivial residue, namely the valuation corresponding to the point $Q' = (\alpha : 1)$. Furthermore, the norm map $\kappa(Q') \to \kappa(Q)$ is an isomorphism which sends $\alpha$ to $\beta$. Therefore, using an analogue of (2.2), we see that the residue at $Q$ is $(-1)^{\deg(P)}p(\beta)$.

Now we consider the algebra $\mathcal{A}$; it has trivial residue away from $P$, $\infty$ and the zeros of $f(x)$. The residue at $P$ is equal to $cf(x(P))$, which is a square, the residue at $\infty$ is $(cf(\infty))^{\deg(P)}((-1)^{\deg(P)}p(\infty))^{-2g(C)-2} = c^{\deg(P)}$, and the residue at a zero $Q$ of $f(x)$ is $(-1)^{\deg(P)}p(\beta)$. Therefore, the residues of $\mathcal{A}$ and $\gamma'(x(P)-\alpha)$ are equal. □

**2.1. Corestriction as a sum of quaternion algebras.** — Using Rosset-Tate reciprocity, one can write the correstriction of a quaternion algebra over an extension as a sum of quaternion algebras over the base field. This is described in [GS06, Corollary 7.4.10 and Remark 7.4.12]. In our situation this allows us to write $\gamma'(\ell)$ as a sum of quaternion algebras over $K(x)$. We caution the reader that the $f$ and $g$ appearing in the proposition below are not to be confused with the $f$ and $g$ of [GS06, Corollary 7.4.10].

***Proposition 2.5.*** — *Suppose $\ell \in L^\times \setminus K^\times$ and let $g(x) \in K[x]$ be the minimal degree polynomial such that $g(\alpha) = \ell$. Set $r_0 = f(x)$, $r_1 = g(x)$, and for $i \geq 0$ define $r_{i+2}$ to be the unique polynomial of degree less than $\deg(r_{i+1})$ such that $r_{i+2} \equiv r_i \bmod r_{i+1}$. Then*

$$\operatorname{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}((\ell, x-\alpha)_2) = \left(\sum_{i=0}^{n}(r_{i+1}, r_i)_2\right) + \left(\sum_{i=0}^{n}(a_{i+1}, a_i)_2\right),$$

*where $a_i$ is the leading coefficient of $r_i$ and $n$ is the first integer such that $r_{n+2} = 0$.*

***Corollary 2.6.*** — *Modulo constant algebras, $\gamma'(\ell)$ may be written as a sum of $g(C)+1$ quaternion algebras over $K(x)$.*

*Proof.* — The proposition shows that, modulo constant algebras,

$$\gamma'(\ell)= \underbrace{(r_1,r_0)_2+(r_2,r_1)_2}_{=(r_1,r_0r_2)_2}+\cdots+\underbrace{(r_n,r_{n-1})_2+(r_{n+1},r_n)_2}_{=(r_n,r_{n-1}r_{n+1})_2}$$

is a sum of $\lceil n/2 \rceil$ quaternion algebras over $K(x)$. On the other hand, the $r_i$ are the remainders obtained by applying the Euclidean algorithm to $f(x)$ and $g(x)$, so $n \le \deg(f(x)) \le 2(g(C)+1)$. □

*Proof of Proposition 2.5.* — For $i \ge 0$, let $R_i(y) = r_i(x+y)$, considered as an element in the Euclidean ring $K(x)[y]$. Then, for all $i \ge 0$, the leading coefficient of $R_i(y)$ is $a_i$, and

$$R_{i+2}(y) \equiv R_i(y) \bmod R_{i+1}(y)\,.$$

Moreover, $R_0(-x+\alpha) = f(\alpha) = 0$ and $R_1(-x+\alpha) = g(\alpha) = \ell$, and $R_i$ and $R_j$ are relatively prime for all $i,j$ since $\ell \in L^\times$. In particular, $r_{n+1}$ and $R_{n+1}$ are nonzero constants. So by [GS06, Lemma 7.4.6 and Proposition 7.5.5],

$$\mathrm{Cor}_{L(x)/K(x)}\left((\ell, x-\alpha)_2\right) = (R_1(y)|R_0(y))_{\mathrm{RT}}\,,$$

where $(\cdot|\cdot)_{\mathrm{RT}}$ denotes the Rosset-Tate symbol. For any $i \ge 0$, the Rosset-Tate reciprocity law [GS06, Theorem 7.4.9] and the Merkurjev-Suslin theorem [GS06, Theorem 4.6.6] give

$$\begin{aligned}(R_{i+1}(y)|R_i(y))_{\mathrm{RT}} &= (R_{i+2}(y)|R_{i+1}(y))_{\mathrm{RT}} + (R_{i+1}(0), R_i(0))_2 + (a_{i+1}, a_i)_2\\ &= (R_{i+2}(y)|R_{i+1}(y))_{\mathrm{RT}} + (r_{i+1}, r_i)_2 + (a_{i+1}, a_i)_2\,.\end{aligned}$$

From this the result easily follows by induction. □

### 2.2. When $C$ is odd. —

***Lemma 2.7.*** — *Suppose that $C$ is odd. Then, for every $\ell \in L^\times$, $\gamma'(\ell)$ evaluates to $0$ at the point $\infty_C \in C(K)$ above $\infty \in \mathbb{P}^1(K)$.*

*Proof.* — Since $\deg f(x)$ is odd, the functions $(x-\alpha)$ and $\frac{(x-\alpha)^{\deg f(x)}}{y^2}$ represent the same class in $\mathbf{k}(C_L)^\times/\mathbf{k}(C_L)^{\times 2}$. The latter evaluates to 1 at $\infty_C$, from which it follows that $\gamma'(\ell)$ is trivial at $\infty_C$. □

***Remark 2.8.*** — For this lemma it is not enough to assume the existence of a rational ramification point; one must in fact have an odd double cover. For example, suppose $C$ is defined by $y^2 = x(x-a_1)(x-a_2)(x-a_3)$ with $a_i \in K^\times$. Then $\mathfrak{L}_1 \simeq (K^\times/K^{\times 2})\times(K^\times/K^{\times 2})$ and $\gamma'$ sends $(k_1,k_2) \in K^\times \times K^\times$ to $(k_1,(x-a_1)(x-a_3))_2+(k_2,(x-a_2)(x-a_3))_2$. Evaluating at the ramification point $\omega = (0,0)$ we have the algebra $(k_1, a_1a_3)_2 + (k_2, a_2a_3)_2$. The only conditions these must satisfy are $k_i, a_i \in K^\times$ and that the $a_i$ are distinct. Over say, $K = \mathbb{Q}$, one can easily find $k_i, a_i$ for which this algebra is nontrivial.

## 3. Computation of cocycles

Consider the following diagram:

$$\begin{tikzcd}
\operatorname{Br} K \arrow[r] \arrow[d, equal] & \operatorname{Br} C \arrow[r, "h"] \arrow[d, hook, "\phi"] & \mathrm{H}^1(K, \operatorname{Pic}_C) \arrow[r] \arrow[d, hook, "\rho"] & \mathrm{H}^3(K, \mathbb{G}_m) \arrow[d, equal] \\
\operatorname{Br} K \arrow[r] & \mathrm{H}^2(K, \mathbf{k}(\overline{C})^\times) \arrow[r, "j_*"] \arrow[d, "\operatorname{div}_*"] & \mathrm{H}^2(K, \mathbf{k}(\overline{C})^\times/\overline{K}^\times) \arrow[r] \arrow[d, "\operatorname{div}_*"] & \mathrm{H}^3(K, \mathbb{G}_m) \\
& \mathrm{H}^2(K, \operatorname{Div} \overline{C}) \arrow[r, equal] & \mathrm{H}^2(K, \operatorname{Div} \overline{C}) &
\end{tikzcd} \tag{3.1}$$

We claim that this diagram is commutative and that all rows and columns are exact. The existence and exactness of the morphisms in the top row can be deduced from exactness in the rest of the diagram. The second row and column come, respectively, from the Galois cohomology of the exact sequences,

$$1 \to \overline{K} \to \mathbf{k}(\overline{C})^\times \xrightarrow{j} \mathbf{k}(\overline{C})^\times/\overline{K}^\times \to 1\,,$$

and

$$1 \to \mathbf{k}(\overline{C})^\times/\overline{K}^\times \xrightarrow{\operatorname{div}} \operatorname{Div} \overline{C} \to \operatorname{Pic} \overline{C} \to 0\,.$$

The connecting homomorphism $\rho$ is injective since $\operatorname{Div} \overline{C}$ is a permutation module, which by Shapiro's lemma implies that $\mathrm{H}^1(K, \operatorname{Div} \overline{C}) = 0$. By Tsen's theorem the inflation map

$$\inf : \mathrm{H}^2(K, \mathbf{k}(\overline{C})^\times) \to \mathrm{H}^2(\mathbf{k}(C), \overline{\mathbf{k}(C)}^\times) = \operatorname{Br} \mathbf{k}(C)$$

is an isomorphism. The map $\phi$ is the composition of the inverse of this inflation map with the inclusion $\operatorname{Br} C \subset \operatorname{Br} \mathbf{k}(C)$. Exactness of the first column is proven in [CTS77, Lemme 14]. Commutativity of the bottom square is obvious. The other squares commute by definition, so the diagram is exact and commutative as claimed.

***Remark 3.1***. — The existence of an exact sequence as in the top row of (3.1) also follows from the spectral sequence $\mathrm{H}^p_{\text{ét}}(K, \mathrm{H}^q_{\text{ét}}(\overline{C}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^n_{\text{ét}}(C, \mathbb{G}_m)$. One can check that these coincide, at least up to sign. See [CTS77, Annexe].

It follows from the definition of $\phi$ that the map $\operatorname{Br} K \to \operatorname{Br} C$ in the top row of (3.1) is the natural map induced by the structure morphism of $C$. Hence, the map $h$ in the top row induces an injective homomorphism $h_0 : \operatorname{Br} C/\operatorname{Br}_0 C \to \mathrm{H}^1(K, \operatorname{Pic}_C)$. The goal of this section is to compute the composition

$$\mathfrak{L}_c \xrightarrow{\gamma} \frac{\operatorname{Br} C}{\operatorname{Br}_0 C} \xrightarrow{h_0} \mathrm{H}^1(K, \operatorname{Pic}_C) \tag{3.2}$$

explicitly. This is accomplished in Proposition 3.2 below, but first we need to fix some notation.

Given $\ell \in L^\times$, let $\chi_\ell \in Z^1(K, \mu_2(\overline{L}))$ be the corresponding quadratic character, i.e., fix a square root $m \in \overline{L}^\times$ of $\ell$, and define $\chi_\ell(\sigma) = \sigma(m)/m$. Composing $\chi_\ell$ with the bijection $\mu_2 \to \{0, 1\} \subseteq \mathbb{Z}$ sending $-1$ to 1, we obtain a map $\tilde{\chi}_\ell \in C^1(K, \mathbb{Z}^\Omega)$. For any $\tau \in G_K$, we may consider $\tilde{\chi}_\ell(\tau)$ as a map $\Omega \to \{0, 1\} \subseteq \mathbb{Z}$ whose value at $\omega \in \Omega$ will be denoted $\tilde{\chi}_\ell(\tau)_\omega$. Note that the action of an element $\sigma \in G_K$ on the map $\tilde{\chi}_\ell(\tau)$ is then given by $\sigma(\tilde{\chi}_\ell(\tau))_\omega = \tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega}$. The norm of $\chi_\ell$ is the quadratic character $\chi_a \in Z^1(K, \mu_2(\overline{K}))$

associated to $a = \mathrm{Norm}_{L/K}(\ell) \in K^\times$. We let $\tilde{\chi}_a \in C^1(K,\mathbb{Z})$ denote the corresponding map to $\{0,1\}$. We can then define a 1-cochain $g_\ell \in C^1(K,\mathbb{Z})$ by requiring that

$$\sum_{\omega\in\Omega} \tilde{\chi}_\ell(\sigma)_\omega = 2g_\ell(\sigma) + \tilde{\chi}_a(\sigma)\,, \text{ for all } \sigma \in G_K\,. \tag{3.3}$$

When $C$ is even we use $\infty^+$ and $\infty^-$ to denote the points on $C$ lying above $\infty \in \mathbb{P}^1$. When $C$ is odd we use both $\infty^+$ and $\infty^-$ to denote the unique point $\infty_C \in C(K)$ lying above $\infty \in \mathbb{P}^1(K)$. In both cases we set $\mathfrak{m} = (\infty^+ + \infty^-) \in \mathrm{Div}\, C$.

***Proposition 3.2***. — *Let $\xi_\ell \in C^1(K, \mathrm{Pic}_C)$ be the 1-cochain defined by*

$$\xi_\ell(\sigma) = \left(\sum_{\omega\in\Omega} \tilde{\chi}_\ell(\sigma)_\omega[\omega]\right) - g_\ell(\sigma)[\mathfrak{m}] - \tilde{\chi}_a(\sigma)[\infty^+]\,. \tag{3.4}$$

*If $\ell$ represents a class in $\mathfrak{L}_c$, then*

1. *$\xi_\ell$ is a cocycle, and*
2. *the image of $\xi_\ell$ in $\mathrm{H}^1(K,\mathrm{Pic}_C)$ is equal to $(h \circ \gamma')(\ell)$.*

To prove Proposition 3.2(2) we will explicitly compute the images of $\xi_\ell$ and $\gamma'(\ell)$ under the maps $\phi$ and $\rho$ of diagram (3.1). This will involve a rather technical computation with cocycles carried out in the lemmas below. Having accomplished this, the proposition will follow from a simple diagram chase.

***Lemma 3.3***. — *For any $\sigma,\tau \in G_K$ and $\omega \in \Omega$ we have*

1. *$\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega} + \tilde{\chi}_\ell(\sigma)_\omega - \tilde{\chi}_\ell(\sigma\tau)_\omega = 2\tilde{\chi}_\ell(\sigma)_\omega\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega}$, and*
2. *$g_\ell(\tau) + g_\ell(\sigma) - g_\ell(\sigma\tau) + \tilde{\chi}_a(\sigma)\tilde{\chi}_a(\tau) = \#\{\omega \in \Omega \,:\, \tilde{\chi}_\ell(\sigma)_\omega\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega} = 1\}$.*

*If, moreover, $\ell$ represents a class in $\mathfrak{L}_c$, then*

3. *$\sigma(\tilde{\chi}_a(\tau)\infty^+) + \tilde{\chi}_a(\sigma)\infty^+ - \tilde{\chi}_a(\sigma\tau)\infty^+ = \tilde{\chi}_a(\sigma)\tilde{\chi}_a(\tau)\mathfrak{m}$.*

*Proof.* — Since $\chi_\ell$ is a 1-cocycle, we have $\chi_\ell(\sigma\tau) = \sigma(\chi_\ell(\tau))\chi_\ell(\sigma)$. Evaluating at $\omega$ and rearranging we get $\chi_\ell(\tau)_{\sigma^{-1}\omega} = \chi_\ell(\sigma\tau)_\omega/\chi_\ell(\sigma)_\omega$. From this it follows that

$$\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega} \equiv \tilde{\chi}_\ell(\sigma)_\omega - \tilde{\chi}_\ell(\sigma\tau)_\omega \bmod 2\,.$$

Since all of the terms are either 0 or 1 we see that

$$\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega} + \tilde{\chi}_\ell(\sigma)_\omega - \tilde{\chi}_\ell(\sigma\tau)_\omega = \begin{cases} 2 & \text{if } \tilde{\chi}_\ell(\sigma)_\omega = \tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega} = 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This proves (1). To prove (2) we sum both sides of (1) over all $\omega \in \Omega$ and apply (3.3). This gives

$$2g_\ell(\tau) + 2g_\ell(\sigma) - 2g_\ell(\sigma\tau) + \tilde{\chi}_a(\tau) + \tilde{\chi}_a(\sigma) - \tilde{\chi}_a(\sigma\tau) = 2\#\{\omega \in \Omega \,:\, \tilde{\chi}_\ell(\sigma)_\omega\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega} = 1\}\,.$$

Using that

$$\tilde{\chi}_a(\tau) + \tilde{\chi}_a(\sigma) - \tilde{\chi}_a(\sigma\tau) = 2\tilde{\chi}_a(\sigma)\tilde{\chi}_a(\tau)$$

(which is proved by the same argument as above), and then removing the common factor of 2 gives (2).

If $\ell$ represents a class in $\mathfrak{L}_c$, then $a \in c^rK^{\times 2}$ for some $r \in \{0,1\}$. If $a \in K^{\times 2}$ then both sides of (3) are trivial, so to prove (3) we may assume $a \in cK^{\times 2}$. Under this assumption, the action of $G_K$ on $\infty^+$ is determined by the character $\chi_a$, so all of the terms in (3) are determined by the values of $\tilde{\chi}_a(\sigma)$ and $\tilde{\chi}_a(\tau)$. In each of the four possibilities, one can check directly that (3) holds. This completes the proof. $\square$

***Lemma 3.4***. — *Assume that $\ell$ represents a class in $\mathfrak{L}_c$, let $\xi'_\ell \in C^1(K, \operatorname{Div}\overline{C})$ denote the* 1*-cochain defined by*

$$\xi'_\ell(\sigma) = \left(\sum_{\omega\in\Omega} \tilde{\chi}_\ell(\sigma)_\omega \omega\right) - g_\ell(\sigma)\mathfrak{m} - \tilde{\chi}_a(\sigma)\infty^+ ,$$

*and let $\partial : C^1(K, \operatorname{Div}\overline{C}) \to C^2(K, \operatorname{Div}\overline{C})$ denote the coboundary map on cochains. Then for $(\sigma,\tau) \in G_K \times G_K$, we have*

$$\partial\xi'_\ell(\sigma,\tau) = \operatorname{div}\left(\operatorname{Norm}_{\mathbf{k}(C_L)/\mathbf{k}(C)}\left((x-\alpha)^{\tilde{\chi}_\ell(\sigma)\cdot\sigma(\tilde{\chi}_\ell(\tau))}\right)\right) .$$

*In particular, $\xi_\ell$ is a cocycle and the image of its class under $\rho$ is represented by the* 2*-cocycle*

$$(\sigma,\tau) \mapsto \operatorname{Norm}_{\mathbf{k}(C_L)/\mathbf{k}(C)}((x-\alpha)^{\tilde{\chi}_\ell(\sigma)\cdot\sigma(\tilde{\chi}_\ell(\tau))}).$$

*Proof.* — The second statement follows easily from the first.

To prove the first statement we compute $\partial\xi'_\ell$ explicitly. For $(\sigma,\tau) \in G_K \times G_K$ we have

$$(\partial\xi'_\ell)(\sigma,\tau) = \sum_{\omega\in\Omega} \left(\sigma(\tilde{\chi}_\ell(\tau)_\omega\omega) + \tilde{\chi}_\ell(\sigma)_\omega\omega - \tilde{\chi}_\ell(\sigma\tau)_\omega\omega\right) \tag{3.5}$$

$$- \left(g_\ell(\tau) + g_\ell(\sigma) - g_\ell(\sigma\tau)\right)\mathfrak{m} \tag{3.6}$$

$$- \left(\sigma(\tilde{\chi}_a(\tau)\infty^+) + \tilde{\chi}_a(\sigma)\infty^+ - \tilde{\chi}_a(\sigma\tau)\infty^+\right) . \tag{3.7}$$

Noting that $\sum_{\omega\in\Omega}\tilde{\chi}_\ell(\tau)_\omega\sigma(\omega) = \sum_{\omega\in\Omega}\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega}\omega$ and applying Lemma 3.3(1), (3.5) can be reduced to $\sum_{\omega\in\Omega}\tilde{\chi}_\ell(\sigma)_\omega\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega}2\omega$. Lemma 3.3(3) states that (3.7) is equal to $-\tilde{\chi}_a(\sigma)\tilde{\chi}_a(\tau)\mathfrak{m}$. Using these facts and then applying Lemma 3.3(2) we obtain,

$$\begin{aligned}
(\partial\xi'_\ell)(\sigma,\tau) &= \left(\sum_{\omega\in\Omega}\tilde{\chi}_\ell(\sigma)_\omega\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega}2\omega\right) - \left(g_\ell(\tau) + g_\ell(\sigma) - g_\ell(\sigma\tau) + \tilde{\chi}_a(\sigma)\tilde{\chi}_a(\tau)\right)\mathfrak{m} \\
&= \left(\sum_{\omega\in\Omega}\tilde{\chi}_\ell(\sigma)_\omega\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega}2\omega\right) - \#\left\{\omega\in\Omega \,:\, \tilde{\chi}_\ell(\sigma)_\omega\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega} = 1\right\}\mathfrak{m} \\
&= \sum_{\omega\in\Omega}\tilde{\chi}_\ell(\sigma)_\omega\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega}\left(2\omega - \mathfrak{m}\right) \\
&= \sum_{\omega\in\Omega}\operatorname{div}\left((x - x(\omega))^{\tilde{\chi}_\ell(\sigma)_\omega\tilde{\chi}_\ell(\tau)_{\sigma^{-1}\omega}}\right) \\
&= \operatorname{div}\left(\operatorname{Norm}_{\mathbf{k}(C_L)/\mathbf{k}(C)}\left((x-\alpha)^{\tilde{\chi}_\ell(\sigma)\cdot\sigma(\tilde{\chi}_\ell(\tau))}\right)\right) .
\end{aligned}$$

This completes the proof. $\square$

***Lemma 3.5***. — *Let $\epsilon \in C^2(K, \mathbf{k}(\overline{C})^\times)$ be the* 2*-cochain defined by*

$$\epsilon(\sigma,\tau) = \operatorname{Norm}_{\mathbf{k}(C_L)/\mathbf{k}(C)}\left((x-\alpha)^{\tilde{\chi}_\ell(\sigma)\cdot\sigma(\tilde{\chi}_\ell(\tau))}\right) .$$

*Then $\epsilon$ is a* 2*-cocycle and the map $\phi$ in* (3.1) *sends $\gamma'(\ell)$ to the class of $\epsilon$ in $\mathrm{H}^2(K, \mathbf{k}(\overline{C})^\times)$.*

*Proof.* — The composition $\inf \circ \phi : \operatorname{Br} C \to \operatorname{Br} \mathbf{k}(C)$ is the natural inclusion. If $\bar{\epsilon}$ denotes the cohomology class of $\epsilon$, then $\inf(\bar{\epsilon})$ is represented by the cocycle $\epsilon_{\mathbf{k}(C)}$ defined by

$$\epsilon_{\mathbf{k}(C)}(\sigma,\tau) = \operatorname{Norm}_{\mathbf{k}(C_L)/\mathbf{k}(C)}\left((x-\alpha)^{\tilde{\psi}_\ell(\sigma)\cdot\sigma(\tilde{\psi}_\ell(\tau))}\right),$$

where $\tilde{\psi}_\ell \in C^1\left(\mathbf{k}(C), \{0,1\}^\Omega\right)$ and $\psi_\ell \in Z^1\left(\mathbf{k}(C), \mu_2(\overline{L})\right)$ denote the lifts of $\tilde{\chi}_\ell$ and $\chi_\ell$, obtained by considering $\ell$ as an element of $\mathbf{k}(C_L)$. We want to show that $\epsilon_{\mathbf{k}(C)}$ represents $\operatorname{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}\left((\ell, x-\alpha)_2\right)$. We will instead show that $\operatorname{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}\left((x-\alpha,\ell)_2\right)$ is represented by the inverse of $\epsilon_{\mathbf{k}(C)}$. The result then follows from standard properties of the cup product (or because all elements in question are 2-torsion).

Standard cohomological arguments combined with Shapiro's lemma give a sequence of isomorphisms

$$\operatorname{Br} \mathbf{k}(C_L)[2] \simeq \mathrm{H}^2\left(\mathbf{k}(C_L), \mu_2^{\otimes 2}\right) \simeq \mathrm{H}^2\left(\mathbf{k}(C), \mu_2(\overline{L})^{\otimes 2}\right) \simeq \mathrm{H}^2\left(\mathbf{k}(C), \mu_2(\overline{L})\right),$$

under which $(x-\alpha,\ell)_2$ is represented by the cup product, $(\psi_{x-\alpha} \cup \psi_\ell) \in Z^2(\mathbf{k}(C), \mu_2(\overline{L})^{\otimes 2})$. Here $\psi_{x-\alpha}$ denotes the quadratic character $\psi_{x-\alpha} \in Z^1\left(\mathbf{k}(C), \mu_2(\overline{L})\right)$ associated to $x-\alpha$, i.e., if $s \in \overline{\mathbf{k}(C)}_L^\times := (\overline{\mathbf{k}(C)} \otimes_K L)^\times$ is a square root of $x-\alpha$, then $\psi_{x-\alpha}(\sigma) = \sigma(s)/s$. The image in $\mathrm{H}^2(\mathbf{k}(C), \mu_2(\overline{L}))$ of the cup product above is represented by the 2-cochain,

$$\begin{aligned}(\psi_{x-\alpha} \cup \psi_\ell)(\sigma,\tau) &= \psi_{x-\alpha}(\sigma) \otimes \sigma(\psi_\ell(\tau)) \\ &= \left(\frac{\sigma(s)}{s}\right)^{\sigma(\tilde{\psi}_\ell(\tau))} = \frac{\sigma\left(s^{\tilde{\psi}_\ell(\tau)}\right)}{s^{\sigma(\tilde{\psi}_\ell(\tau))}} \\ &= \left(\frac{\sigma\left(s^{\tilde{\psi}_\ell(\tau)}\right)s^{\tilde{\psi}_\ell(\sigma)}}{s^{\tilde{\psi}_\ell(\sigma\tau)}}\right)\left(\frac{s^{\tilde{\psi}_\ell(\sigma\tau)}}{s^{\sigma(\tilde{\psi}_\ell(\tau))}s^{\tilde{\psi}_\ell(\sigma)}}\right).\end{aligned}$$

We now note that the first factor is the coboundary of the 1-cochain

$$\left(\sigma \mapsto s^{\tilde{\psi}(\sigma)}\right) \in C^1\left(\mathbf{k}(C), \overline{\mathbf{k}(C)}_L^\times\right),$$

while using the obvious analog of Lemma 3.3(1) we can rewrite the second factor as

$$(x-\alpha)^{-\tilde{\psi}_\ell(\sigma)\cdot\sigma(\tilde{\psi}_\ell(\tau))}.$$

The norm of this expression is the inverse of $\epsilon_{\mathbf{k}(C)}$. This proves that $\epsilon$ is a cocycle, and that $\epsilon_{\mathbf{k}(C)}$ represents $\operatorname{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}((x-\alpha,\ell)_2)$ as required. $\square$

*Proof of Proposition 3.2.* — The first statement was proven in Lemma 3.4. For the second statement, suppose $\ell$ represents a class in $\mathfrak{L}_c$ and let $\bar{\xi}_\ell$ denote the class of $\xi_\ell$ in $\mathrm{H}^1(K, \operatorname{Pic}_C)$. Lemmas 3.4 and 3.5 show that

$$(j_* \circ \phi \circ \gamma')(\ell) = \rho(\bar{\xi}_\ell).$$

Since $\rho \circ h = j_* \circ \phi$ by (3.1) and $\rho$ is injective, this completes the proof. $\square$

## 4. Cohomological setup for 2-descent

In the previous section we explicitly computed the map $h \circ \gamma : \mathfrak{L}_c \to \mathrm{H}^1(K, \operatorname{Pic}_C)$. In this section we relate this to the map $\mathfrak{L}_1 \to \mathrm{H}^1(K, J)/\langle \operatorname{Pic}_C^1 \rangle$ coming from the theory of explicit 2-descents described in [PS97].

The 2-torsion subgroup of $J(\overline{K})$ may be identified (as a Galois module) with the set of even cardinality subsets of $\Omega$, modulo complements. Under this identification addition in $J[2]$ is given by the symmetric difference (i.e., the union of the sets minus their intersection), and the Weil pairing, denoted $e_2$, of two subsets is given by the parity of their intersection. By convention, for any $\omega \in \Omega$, the notation $\{\omega, \omega\}$ will be understood to mean the identity element.

For any $\omega_0 \in \Omega$, we may define a map

$$e_{\omega_0} : J[2] \to \mu_2(\overline{L}) = \mathrm{Map}(\Omega, \mu_2(\overline{K})), \quad P \mapsto (\omega \mapsto e_2(P, \{\omega, \omega_0\})) \ . \tag{4.1}$$

If $\omega_1 \in \Omega$, then, for every $P \in J[2]$, $e_{\omega_0}(P)$ and $e_{\omega_1}(P)$ differ by an element of $\mu_2(\overline{K}) \subseteq \mu_2(\overline{L})$, namely the constant map $\omega \mapsto e_2(P, \{\omega_0, \omega_1\})$. Therefore we obtain a map $e\colon J[2] \to \mu_2(\overline{L})/\mu_2(\overline{K})$ that is independent of the choice of $\omega_0 \in \Omega$. Nondegeneracy and Galois equivariance of the Weil pairing show that $e$ is an injective morphism of $G_K$-modules. On the other hand, $\sum_{\omega\in\Omega}\{\omega, \omega_0\} = 0 \in J[2]$. So $e$ fits into a short exact sequence,

$$0 \to J[2] \xrightarrow{e} \mu_2(\overline{L})/\mu_2(\overline{K}) \xrightarrow{\mathrm{Norm}_{L/K}} \mu_2(\overline{K}) \to 1\,. \tag{4.2}$$

***Remark 4.1***. — When $C$ is odd we may take $\omega_0$ to be the ramification point $\infty_C \in C(K)$. The identification of $L_\circ \subseteq L$ as the subalgebra of elements taking the value 1 at $\infty_C$ then induces a canonical isomorphism of short exact sequences of $G_K$-modules:

$$\begin{tikzcd}
0 \arrow[r] & J[2] \arrow[r, hook, "e_{\infty_C}"] \arrow[d, equal] & \mu_2(\overline{L}_\circ) \arrow[r, "\mathrm{Norm}_{L_\circ/K}"] \arrow[d, "\cong"] & \mu_2(\overline{K}) \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & J[2] \arrow[r, hook, "e"] & \frac{\mu_2(\overline{L})}{\mu_2(\overline{K})} \arrow[r, "\mathrm{Norm}_{L/K}"] & \mu_2(\overline{K}) \arrow[r] & 1\,.
\end{tikzcd}$$

Applying Galois cohomology to (4.2) gives an exact sequence,

$$\mu_2(K) \longrightarrow \mathrm{H}^1(K, J[2]) \xrightarrow{e_*} \mathrm{H}^1(K, \mu_2(\overline{L})/\mu_2(\overline{K})) \xrightarrow{(\mathrm{Norm}_{L/K})_*} \mathrm{H}^1(K, \mu_2)\,. \tag{4.3}$$

If $D \in \mathrm{Div}^1(\overline{C})$ is any divisor of degree 1 on $C$, then the 1-cocycle sending $\sigma \in G_K$ to $[\sigma(D) - D] \in \mathrm{Pic}^0(\overline{C}) = J(\overline{K})$ represents the class in $\mathrm{H}^1(K, J)$ of the torsor $\mathrm{Pic}^1_C$ parameterizing divisor classes of degree 1. Choosing $D = \omega$ for some $\omega \in \Omega$ gives a cocycle taking values in $J[2]$, whose class in $\mathrm{H}^1(K, J[2])$ does not depend on the choice for $\omega$. We will abuse notation slightly by denoting this class in $\mathrm{H}^1(K, J[2])$ also by $\mathrm{Pic}^1_C$. One can then check that $-1$ maps to $\mathrm{Pic}^1_C$ under the map $\mu_2(K) \to \mathrm{H}^1(K, J[2])$ in (4.3) (cf. [PS97, Lemma 9.1]).

***Lemma 4.2***. — *The following are equivalent:*

1. *The class of* $\mathrm{Pic}^1_C$ *in* $\mathrm{H}^1(K, J[2])$ *is trivial.*
2. $\Omega$ *admits an unordered* $G_K$*-stable partition into two sets of odd cardinality.*
3. $[\mathfrak{m}] \in 2\,\mathrm{Pic}\,C$.

*Proof.* — See [PS97, Lemma 11.2] ☐

***Remark 4.3***. — Note that these equivalent conditions are trivially satisfied when $C$ is odd. When $C$ is even they occur if and only if $f(x)$ has a factor of odd degree or if the genus of $C$ is even and there exists a quadratic extension $F$ of $K$ such $f(x)$ is the norm of a polynomial in $F[x]$.

Combining (4.3) with the Galois cohomology of

$$1 \to \mu_2(\overline{K}) \to \mu_2(\overline{L}) \xrightarrow{q} \mu_2(\overline{L})/\mu_2(\overline{K}) \to 1\,.$$

we obtain a commutative diagram with exact rows and columns,

$$\begin{tikzcd}
 & \frac{L^\times}{K^\times L^{\times 2}} \arrow[r, "\mathrm{Norm}_{L/K}"] \arrow[d, hook, "q_*"] & \frac{K^\times}{K^{\times 2}} \arrow[d, equal] \\
\frac{\mathrm{H}^1(K,J[2])}{\langle \mathrm{Pic}^1_C \rangle} \arrow[r, hook, "e_*"] \arrow[dr, "\Upsilon"'] & \mathrm{H}^1\left(K, \frac{\mu_2(\overline{L})}{\mu_2(\overline{K})}\right) \arrow[r, "(\mathrm{Norm}_{L/K})_*"] \arrow[d] & \mathrm{H}^1(K,\mu_2) \\
 & \mathrm{Br}\,K[2] &
\end{tikzcd} \tag{4.4}$$

The map labelled $\Upsilon$ sends $\xi \in \mathrm{H}^1(K, J[2])$ to the image of $\xi \cup \mathrm{Pic}^1_C$ under the map

$$\mathrm{H}^1(K, J[2] \otimes J[2]) \to \mathrm{H}^2(K, \mu_2) = \mathrm{Br}\,K[2]$$

induced by the Weil pairing [PS97, Proposition 10.3]. Exactness at the central term of (4.4) implies the existence of an exact sequence

$$1 \to \mathfrak{L}_1 \xrightarrow{d} \frac{\mathrm{H}^1(K, J[2])}{\langle \mathrm{Pic}^1_C \rangle} \xrightarrow{\Upsilon} \mathrm{Br}\,K[2]\,. \tag{4.5}$$

The exact sequence of $K$-group schemes

$$0 \to \mathrm{Pic}^0_C \to \mathrm{Pic}_C \xrightarrow{\deg} \mathbb{Z} \to 0$$

induces an isomorphism $\mathrm{H}^1(K, J)/\langle \mathrm{Pic}^1_C \rangle \simeq \mathrm{H}^1(K, \mathrm{Pic}_C)$. So, composing with $d$, the inclusions $J[2] \subseteq J = \mathrm{Pic}^0_C \subseteq \mathrm{Pic}_C$ induce maps from $\mathfrak{L}_1$ to $\mathrm{H}^1(K, J)/\langle \mathrm{Pic}^1_C \rangle$ and to $\mathrm{H}^1(K, \mathrm{Pic}_C)$. By abuse of notation we will use $d$ to denote any of these three maps. The following proposition, due to Poonen and Schaefer, relates (4.5) to the Kummer sequence (1.1).

***Proposition 4.4***. — *The composition $d \circ (x - \alpha)$ and the connecting homomorphism $\delta$ in* (1.1) *define the same map* $\mathrm{Pic}^0 C \to \mathrm{H}^1(K, J[2])/\langle \mathrm{Pic}^1_C \rangle$.

***Corollary 4.5***. — *There is an exact sequence*

$$\mathrm{Pic}^0 C \xrightarrow{x-\alpha} \mathfrak{L}_1 \xrightarrow{d} \mathrm{H}^1(K, \mathrm{Pic}_C)\,.$$

*Proof of Proposition 4.4*. — See [PS97, Theorem 9.4] when $C$ is even and [Sch95, Theorem 1.1] when $C$ is odd (see Remark 4.1). □

The following lemma gives an explicit description of the map $d$.

***Lemma 4.6***. — *Suppose $\ell \in L^\times$ represents a class $\overline{\ell} \in \mathfrak{L}_1$ and let $\tilde{\chi}_\ell, g_\ell \in C^1(K, \mathbb{Z})$ be as in* (3.3). *Then $d(\overline{\ell})$ is represented by the 1-cocycle $\xi''_\ell \in Z^1(K, J[2])$ defined by*

$$\xi''_\ell(\sigma) = \left( \sum_{\omega \in \Omega} \tilde{\chi}_\ell(\sigma)_\omega [\omega] \right) - g_\ell(\sigma)[\mathfrak{m}].$$

*Proof.* — The map $q_*$ in diagram (4.4) sends the class of $\ell$ to the class represented by $\chi_\ell$, while $e_*$ is induced by the map in (4.2), itself induced by the map $e_{\omega_0}$ of (4.1). To prove the lemma it is enough to show that, for every $\sigma \in G_K$, $e_{\omega_0}(\xi''_\ell(\sigma))$ and $\chi_\ell(\sigma)$ define the same element of $\mu_2(\overline{L})/\mu_2(\overline{K})$.

For any $\sigma \in G_K$,

$$g_\ell(\sigma)[\mathfrak{m}] = g_\ell(\sigma)[2\omega_0] = 2g_\ell(\sigma)[\omega_0] = \sum_{\omega\in\Omega} \tilde{\chi}_\ell(\sigma)_\omega[\omega_0]\,.$$

Since $[\omega] - [\omega_0] = \{\omega, \omega_0\}$, we may thus rewrite $\xi''_\ell(\sigma)$ as

$$\xi''_\ell(\sigma) = \sum_{\omega\in\Omega} \tilde{\chi}_\ell(\sigma)_\omega\{\omega, \omega_0\}.$$

Now let $\upsilon \in \Omega$. We may express $e_{\omega_0}(\xi''_\ell(\sigma))(\upsilon) = e_2(\xi''_\ell(\sigma), \{\upsilon, \omega_0\})$ as follows

$$e_2(\xi''_\ell(\sigma), \{\upsilon, \omega_0\}) = \prod_{\omega\in\Omega} e_2(\{\omega, \omega_0\}, \{\upsilon, \omega_0\})^{\tilde{\chi}_\ell(\sigma)_\omega} = \prod_{\omega\neq\upsilon,\omega_0} e_2(\{\omega, \omega_0\}, \{\upsilon, \omega_0\})^{\tilde{\chi}_\ell(\sigma)_\omega}.$$

Observing that $e_2(\{\omega, \omega_0\}, \{\upsilon, \omega_0\}) = -1$ unless $\omega = \upsilon$, $\omega = \omega_0$ or $\upsilon = \omega_0$, it follows that

$$e_{\omega_0}(\xi''_\ell(\sigma))(\upsilon) = e_2(\xi''_\ell(\sigma), \{\upsilon, \omega_0\}) = \prod_{\omega\neq\upsilon,\omega_0} \chi_\ell(\sigma)_\omega, \qquad \text{for any } \upsilon \neq \omega_0$$

Finally, we note that $\prod_{\omega\in\Omega} \chi_\ell(\sigma)_\omega = 1$ as $\overline{\ell} \in \mathfrak{L}_1$ and obtain the desired conclusion, that $e_{\omega_0}(\xi''_\ell(\sigma))(\upsilon) = \chi_\ell(\sigma)_\upsilon \chi_\ell(\sigma)_{\omega_0}$. $\square$

**4.1. The kernel and image of $(x - \alpha)$.** — The kernel of $(x - \alpha)$ on $\operatorname{Pic}^0 C$ is given by [PS97, Theorem 11.3]. Using this we derive the following.

***Proposition 4.7.*** — *Let $H$ be the kernel of the map $(x - \alpha) : \frac{\operatorname{Pic} C}{2\operatorname{Pic} C} \to \mathfrak{L}_c$.*

1. *If $\operatorname{Pic}^1_C(K) = \emptyset$, then $H = 0$.*
2. *If $\operatorname{Pic}^1_C(K) \neq \emptyset$ and $c \notin K^{\times 2}$, then $H$ is generated by $[\mathfrak{m}]$.*
3. *If $c \in K^{\times 2}$, then $H$ is generated by $[\mathfrak{m}]$ and $[\infty^+]$.*

*Furthermore, if $\operatorname{Pic} C = \operatorname{Pic}_C(K)$, then the $\mathbb{F}_2$-dimension of $(x - \alpha)(\operatorname{Pic} C)$ is equal to*

$$\operatorname{rank}(\operatorname{Pic} C) + (r - 2) + \dim_{\mathbb{F}_2}\left(\frac{K^\times \cap L^{\times 2}}{K^{\times 2}}\right) - \begin{cases} 1 & \text{if } c \in K^{\times 2}\,, \\ 0 & \text{if } c \notin K^{\times 2}\,, \end{cases}$$

*where $r$ denotes the number of $G_K$ orbits in $\Omega$.*

*Proof.* — Set

$$\Delta_1 = \begin{cases} 1 & \text{if } \operatorname{Pic}^1_C(K) \neq \emptyset, \\ 0 & \text{if } \operatorname{Pic}^1_C(K) = \emptyset\,, \end{cases}$$

$$\Delta_2 = \begin{cases} 1 & \text{if there is a } G_K\text{-stable partition of } \Omega \text{ into two sets of odd cardinality} \\ 0 & \text{otherwise}\,, \end{cases}$$

$$\Delta_c = \begin{cases} 1 & \text{if } c \in K^{\times 2} \\ 0 & \text{if } c \notin K^{\times 2}\,. \end{cases}$$

Let $\operatorname{Pic}^{(2)} C \subseteq \operatorname{Pic} C$ denote the subgroup of divisor classes of degree divisible by 2. It follows immediately from [PS97, Theorem 11.3] that the kernel of $(x - \alpha)$ on $\frac{\operatorname{Pic}^{(2)} C}{2\operatorname{Pic} C}$ has dimension $\Delta_1 - \Delta_2$ and contains $[\mathfrak{m}]$ if and only if $\Delta_1 \neq 0$. (Note that $\Delta_1 - \Delta_2$ is always non-negative by Lemma 4.2.) Clearly $(x-\alpha)$ maps divisors of degree $m$ to classes with norm in $c^m K^{\times 2}$. In particular, if $c \notin K^{\times 2}$, then the kernel of $(x-\alpha)$ does not contain any divisor classes of odd degree. On the other hand, if $c \in K^{\times 2}$, then $[\infty^+]$ is defined over $K$ and lies in $\ker(x-\alpha)$. It follows that $\dim_{\mathbb{F}_2} H = \Delta_1 - \Delta_2 + \Delta_c$, and the first three statements are clear.

It remains to compute the dimension of the image of $(x-\alpha)$. By assumption, $C$ has a $K$-rational divisor of odd degree if and only if $\Delta_1 = 1$, so

$$\dim_{\mathbb{F}_2} \frac{\operatorname{Pic} C}{2 \operatorname{Pic} C} = \operatorname{rank}(\operatorname{Pic} C) + (\Delta_1 - 1) + \dim_{\mathbb{F}_2} J(K)[2]\,.$$

Since $H$ has dimension $\Delta_1 - \Delta_2 + \Delta_c$, it suffices to show that

$$\dim_{\mathbb{F}_2} J(K)[2] = r - 1 + \dim_{\mathbb{F}_2}\left(\frac{K^\times \cap L^{\times 2}}{K^{\times 2}}\right) - \Delta_2\,. \tag{4.6}$$

As this does not depend on the model, we may assume $C$ is even. Recall that the elements of $J(K)[2]$ correspond to unordered $G_K$-stable partitions of $\Omega$ into two sets of even cardinality. These can arise in essentially two ways: from even degree factors of $f(x)$, and from quadratic extensions $F/K$ such that $f(x)$ is the norm of a polynomial over $F$. The partitions corresponding to even degree factors of $f(x)$ over $K$ generate a subgroup of $J(K)[2]$ of dimension equal to $r-2$ or $r-1$, correspondingly as $f(x)$ does or does not have any factors of odd degree. Partitions coming from a factorization over a quadratic extension only occur when the genus of $C$ is odd, and then only if $f(x)$ has no factor of odd degree, in which case they generate a subgroup of $J(K)[2]$ of dimension $\dim_{\mathbb{F}_2}(K^\times \cap L^{\times 2})/K^{\times 2}$. Thus, $\dim_{\mathbb{F}_2} J(K)[2]$ is equal to

$$\begin{cases} r-2 & \text{if } f(x) \text{ has a factor of odd degree}\,,\\ r-1 & \text{if } f(x) \text{ has no factor of odd degree and } g(C) \text{ is even}\,,\\ r-1+\dim_{\mathbb{F}_2}\left(\frac{K^\times \cap L^{\times 2}}{K^{\times 2}}\right) & \text{if } f(x) \text{ has no factor of odd degree and } g(C) \text{ is odd}\,. \end{cases}$$

When $f(x)$ has a factor of odd degree we clearly have $\dim_{\mathbb{F}_2}((K^\times \cap L^{\times 2})/K^{\times 2}) = 0$ and $\Delta_2 = 1$, so (4.6) holds. Now assume that $f(x)$ has no factors of odd degree. When the genus of $C$ is odd there cannot be a $G_K$-stable partition of $\Omega$ into two sets of odd cardinality because $\deg f(x) \equiv 0 \bmod 4$. When the genus of $C$ is even, $\deg f(x) \equiv 2 \bmod 4$, and so there can be at most one quadratic extension of $K$ contained in $L$. If such an extension exists, then it gives a $G_K$-stable partition of $\Omega$ into two sets of odd cardinality. Thus, when $f(x)$ has no odd degree factors,

$$\Delta_2 = \begin{cases} \dim_{\mathbb{F}_2}\left(\frac{K^\times \cap L^{\times 2}}{K^{\times 2}}\right) & \text{if } g(C) \text{ is even}\,,\\ 0 & \text{if } g(C) \text{ is odd}\,. \end{cases}$$

Combining this with the dimension of $J(K)[2]$ computed above gives (4.6) as desired. $\square$

## 5. Proofs of the main theorems

For $n \geq 2$ define

$$\operatorname{Br}_n C = \left\{ \mathcal{A} \in \operatorname{Br} C \,:\, h(\mathcal{A}) \in \operatorname{image}\left(\mathrm{H}^1(K, J[n]) \to \mathrm{H}^1(K, \operatorname{Pic}_C)\right) \right\}\,, \text{ and}$$
$$\operatorname{Br}_2^\Upsilon C = \left\{ \mathcal{A} \in \operatorname{Br} C \,:\, h(\mathcal{A}) \in \operatorname{image}\left(\ker(\Upsilon) \to \mathrm{H}^1(K, \operatorname{Pic}_C)\right) \right\}\,,$$

where $h : \operatorname{Br} C \to \mathrm{H}^1(K, \operatorname{Pic}_C)$ is as in (3.1), $\Upsilon : \mathrm{H}^1(K, J[2]) \to \operatorname{Br} K[2]$ is as in (4.5) and the map $\mathrm{H}^1(K, J[n]) \to \mathrm{H}^1(K, \operatorname{Pic}_C)$ is induced by the inclusion $J[n] \subseteq J = \operatorname{Pic}^0_C \subseteq \operatorname{Pic}_C$.

***Proposition 5.1***. — *There is an exact sequence*

$$\operatorname{Pic}^0 C \xrightarrow{x-\alpha} \mathfrak{L}_1 \xrightarrow{\gamma} \operatorname{Br}_2^{\Upsilon} C / \operatorname{Br}_0 C \to 0\,.$$

*Proof.* — From Proposition 3.2 and Lemma 4.6 it is clear that $h_0 \circ \gamma$ and $d$ give the same map $\mathfrak{L}_1 \to \mathrm{H}^1(K, \operatorname{Pic}_C)$. Since (4.5) is exact, $\operatorname{im}(d) = \ker(\Upsilon)$, so $\gamma(\mathfrak{L}_1) = \operatorname{Br}_2^{\Upsilon} C / \operatorname{Br}_0 C$. The exactness stated in the proposition now follows immediately from Corollary 4.5. □

***Lemma 5.2***. — $\operatorname{Br}_2 C / \operatorname{Br}_0 C = (\operatorname{Br} C / \operatorname{Br}_0 C)[2]$ *if and only if* $\operatorname{Pic}^1_C(K) \neq \emptyset$ *or* $\operatorname{Pic}^1_C \notin 2\,\mathrm{H}^1(K, J)$.

*Proof.* — Recall that $\mathrm{H}^1(K, \operatorname{Pic}_C) \simeq \mathrm{H}^1(K, J)/\langle \operatorname{Pic}^1_C \rangle$. Therefore

$$\frac{\operatorname{Br}_2 C}{\operatorname{Br}_0 C} \simeq \frac{\mathrm{H}^1(K, J)[2]}{\langle \operatorname{Pic}^1_C \rangle} \quad \text{and} \quad \left( \frac{\operatorname{Br} C}{\operatorname{Br}_0 C} \right)[2] \simeq \left( \frac{\mathrm{H}^1(K, J)}{\langle \operatorname{Pic}^1_C \rangle} \right)[2]\,.$$

The condition in the statement is that $\operatorname{Pic}^1_C$ is trivial or not divisible by 2 in $\mathrm{H}^1(K, J)$. These are precisely the situations in which

$$\frac{\mathrm{H}^1(K, J)[2]}{\langle \operatorname{Pic}^1_C \rangle} = \left( \frac{\mathrm{H}^1(K, J)}{\langle \operatorname{Pic}^1_C \rangle} \right)[2]\,.$$

□

***Lemma 5.3***. — *If* $\operatorname{Br} K[2] = 0$ *or* $\Omega$ *admits a* $G_K$*-stable unordered partition into two sets of odd cardinality, then* $\operatorname{Br}_2^{\Upsilon} C = \operatorname{Br}_2 C$.

*Proof.* — Either assumption implies that $\Upsilon = 0$. □

***Remark 5.4***. — In general one should not expect that $\operatorname{Br}_2^{\Upsilon} C = \operatorname{Br}_2 C$. For example, if $K$ is a $p$-adic field, $\operatorname{Pic}^1_C(K) \neq \emptyset$ and $\Omega$ does not admit a $G_K$-stable partition into two sets of odd cardinality, then $\operatorname{Br}_2^{\Upsilon} C \neq \operatorname{Br}_2 C$. To see this, recall that the the cup product on $\mathrm{H}^1(K, J[2])$ is nondegenerate (see [Tat63, §2]). The above assumptions therefore imply that there exists some $T \in \mathrm{H}^1(K, J[2])$ such that $\Upsilon(T) \neq 0$. Let $T' \in \mathrm{H}^1(K, \operatorname{Pic}_C)$ be the image of $T$. Then every lift of $T'$ to $\mathrm{H}^1(K, J[2])$ is of the form $\tilde{T} = T + \delta(P)$ for some $P \in J(K)$, and none of them lie in $\ker(\Upsilon)$ since $\Upsilon(\tilde{T}) = \Upsilon(T) + \Upsilon(\delta(P)) = \Upsilon(T) \neq 0$. Here $\Upsilon(\delta(P)) = 0$ since $\operatorname{Pic}^1_C$ lies in the image of $\delta$, which is self-orthogonal with respect to the pairing (ibid.). This shows that if $\mathcal{A} \in \operatorname{Br}_2 C$ is such that $h(\mathcal{A}) = T'$, then $\mathcal{A} \notin \operatorname{Br}_2^{\Upsilon} C$. Moreover, such an $\mathcal{A}$ exists as $\mathrm{H}^3(K, \mathbb{G}_m) = 0$.

**5.1. Proof of Theorems 1.2 and 1.3.** — In the odd case we have already seen that $\gamma$ maps $\mathfrak{L}_1$ to $\operatorname{Br}^0 C$ (Lemma 2.7). Using Lemmas 5.2 and 5.3 we see that the hypotheses imply that $\operatorname{Br}_2^{\Upsilon} C / \operatorname{Br}_0 C = (\operatorname{Br} C / \operatorname{Br}_0 C)[2]$, so the theorems follow from Proposition 5.1. □

### 5.2. Proof of Theorem 1.1. —

***Lemma 5.5.*** — *We have that* $\gamma(\mathfrak{L}_c \setminus \mathfrak{L}_1) \not\subseteq \mathrm{Br}_2 C/\mathrm{Br}_0 C$ *if and only if* $\mathrm{Pic}^1_C(K) = \emptyset$ *and* $\mathfrak{L}_c \neq \mathfrak{L}_1$*.*

*Proof.* — The statement is trivially true when $\mathfrak{L}_c = \mathfrak{L}_1$. So suppose $\ell \in L^\times$ is a representative for a class $\overline{\ell} \in \mathfrak{L}_c \setminus \mathfrak{L}_1$. Then $h_0 \circ \gamma(\overline{\ell})$ is represented by the cocycle $\xi_\ell \in C^1(K, \mathrm{Pic}_C)$ of Proposition 3.2. Using that $2[\omega] = [\mathfrak{m}]$ in $\mathrm{Pic}\,\overline{C}$ we have

$$\begin{aligned}
2\xi_\ell(\sigma) &= \left(\sum_{\omega\in\Omega} \tilde{\chi}_\ell(\sigma)_\omega 2[\omega]\right) - 2g_\ell(\sigma)[\mathfrak{m}] - 2\tilde{\chi}_c(\sigma)[\infty^+] \\
&= \tilde{\chi}_c(\sigma)[\mathfrak{m}] - \tilde{\chi}_c(\sigma)2[\infty^+] \\
&= \tilde{\chi}_c(\sigma)([\infty]^- - [\infty^+])\,.
\end{aligned}$$

This shows that, when considered as a cocycle taking values in $\mathrm{Pic}^0\,\overline{C} = J(\overline{K})$, $2\xi_\ell$ represents the class of $\mathrm{Pic}^1_C$ in $\mathrm{H}^1(K, J)$. This class is trivial if and only if $\mathrm{Pic}^1_C(K) \neq \emptyset$. The lemma now follows easily from the definition of $\mathrm{Br}_2 C$. □

***Proposition 5.6.*** — *The complex*

$$\mathrm{Pic}\,C \xrightarrow{x-\alpha} \mathfrak{L}_c \xrightarrow{\gamma} \left(\frac{\mathrm{Br}\,C}{\mathrm{Br}_0\,C}\right)[2]$$

*is exact except possibly if* $\mathrm{Pic}^1 C = \emptyset \neq \mathrm{Pic}^1_C(K)$ *and* $\bar{c} \in \mathrm{Norm}_{L/K}(\mathfrak{L})$*, in which case the image of* $(x-\alpha)$ *is a subgroup of index at most* 2 *in* $\ker(\gamma)$*.*

*Proof.* — Consider the following commutative diagram.

$$\begin{array}{ccccc}
\frac{\mathrm{Pic}^0 C}{2\,\mathrm{Pic}^0 C} & \xrightarrow{(x-\alpha)} & \mathfrak{L}_1 & \xrightarrow{\gamma} & \frac{\mathrm{Br}_2 C}{\mathrm{Br}_0 C} \\
\downarrow & & \downarrow & & \downarrow \\
\frac{\mathrm{Pic}\,C}{2\,\mathrm{Pic}\,C} & \xrightarrow{(x-\alpha)} & \mathfrak{L}_c & \xrightarrow{\gamma} & \frac{\mathrm{Br}\,C}{\mathrm{Br}_0\,C}
\end{array} \tag{5.1}$$

The top row is exact by Proposition 5.1, and the bottom row is a complex by Corollary 2.4.

Let us first consider the case when $\mathfrak{L}_c = \mathfrak{L}_1$. This happens if and only if $c \in K^{\times 2}$ or $\bar{c} \notin \mathrm{Norm}_{L/K}(\mathfrak{L})$. When $c \in K^{\times 2}$ we have $[\infty^+] \in \ker(x-\alpha)$, and when $\bar{c} \notin \mathrm{Norm}_{L/K}(\mathfrak{L})$ there are no $K$-rational divisor classes of odd degree [Cre, Corollary 4.4]. Both possibilities imply that $(x-\alpha)(\mathrm{Pic}\,C) = (x-\alpha)(\mathrm{Pic}^0\,C)$, and so exactness of the bottom row follows from exactness of the top row of (5.1).

Now we consider the case $\mathfrak{L}_c \neq \mathfrak{L}_1$, which implies that $\bar{c} \in \mathrm{Norm}_{L/K}(\mathfrak{L})$. Then $(x-\alpha)$ sends $K$-rational divisor classes of odd degree to $\mathfrak{L}_c \setminus \mathfrak{L}_1$ [Cre, Lemma 4.3]. If $[\mathfrak{L}_c : \mathfrak{L}_1] = \left[\frac{\mathrm{Pic}\,C}{2\,\mathrm{Pic}\,C} : \frac{\mathrm{Pic}^0\,C}{2\,\mathrm{Pic}^0\,C}\right] = 2$, then exactness follows from the fact that the top row is exact and the bottom row is a complex. So we may assume there are no $K$-rational divisors of odd degree. Then $(x-\alpha)(\mathrm{Pic}\,C) = (x-\alpha)(\mathrm{Pic}^0\,C) \subseteq \mathfrak{L}_1$, and exactness follows from exactness of the top row of (5.1), except possibly if $\gamma(\mathfrak{L}_c) \cap \gamma(\mathfrak{L}_1) \neq \emptyset$. Lemma 5.5 shows that this can only happen when $\mathrm{Pic}^1_C(K) \neq \emptyset$. □

To complete the proof of Theorem 1.1 it only remains to show that $\gamma(\mathfrak{L}_c) = (\mathrm{Br}\,C/\mathrm{Br}_0\,C)[2]$ when $\mathrm{Br}\,K[2] = 0$. By Proposition 5.1 and Lemma 5.3 the assumption on $\mathrm{Br}\,K[2]$ implies that $\gamma(\mathfrak{L}_1) = \mathrm{Br}_2 C/\mathrm{Br}_0 C$. Lemma 5.2 allows us to further assume that $\mathrm{Pic}^1_C$ is nonzero and divisible by 2 in $\mathrm{H}^1(K,J)$. Then $\mathrm{Br}_2 C/\mathrm{Br}_0 C$ has index 2 in $(\mathrm{Br}\, C/\mathrm{Br}_0 C)[2]$, so using Lemma 5.5 it suffices to show that $\mathfrak{L}_c \setminus \mathfrak{L}_1 \neq \emptyset$. We know that $c \notin K^{\times 2}$, otherwise $\mathrm{Pic}^1_C$ would be trivial in $\mathrm{H}^1(K,J)$. So we are reduced to showing that there exists some $\ell \in L^\times$ such that $\mathrm{Norm}_{L/K}(\ell) \in cK^{\times 2}$.

For this we will make use of the theory of torsors under groups of multiplicative type as described in [Sko01, Part I]. For $X = C$ or $X = \mathrm{Pic}^1_C$, let $\lambda_n$ denote the canonical embedding $\lambda_n : J[n] \cong \mathrm{Pic}^0_X[n] \subseteq \mathrm{Pic}_X$. An $n$-covering of $X$ is an $X$-torsor under $J[n]$ of type $\lambda_n$. Since $\mathrm{Pic}^1_C \in 2\,\mathrm{H}^1(K,J)$, there exists a 2-covering $T \to \mathrm{Pic}^1_C$ (see [Sko01, Proposition 3.3.5]). Pulling this back along the canonical embedding $C \to \mathrm{Pic}^1_C$ gives a 2-covering $\psi : Y \to C$. For any $\omega \in \Omega$ the pull back $\psi^*[\omega]$ is a $K$-rational divisor class on $Y$.

If $\psi^*[\omega] \in \mathrm{Pic}\, Y$, then it induces a projective embedding of $Y$ in which the pull backs of the ramification points on $C$ are hyperplane sections. Up to composition with the hyperelliptic involution on $C$, the 2-coverings of $C$ with a model of this type are parameterized by the elements in the set $\left\{\overline{\ell} \in \mathfrak{L} : \mathrm{Norm}_{L/K}(\ell) \in cK^{\times 2}\right\}$ (see [BS09, §3] or [Cre, Proposition 5.4]). In particular, it will suffice to show that $\psi^*[\omega] \in \mathrm{Pic}\, Y$, for then there exists some $\ell \in L^\times$ with norm in $cK^{\times 2}$.

The obstruction to a rational divisor class being represented by a rational divisor is given by a well known exact sequence, $0 \to \mathrm{Pic}\, Y \to \mathrm{Pic}_Y(K) \xrightarrow{\theta} \mathrm{Br}\, K$. In our situation, $2\psi^*[\omega] = \psi^*[2\omega] = \psi^*[\mathfrak{m}] \in \mathrm{Pic}\, Y$. So $\theta(\psi^*[\omega]) \in \mathrm{Br}\, K[2]$, which is trivial by assumption. This completes the proof.

***Remark 5.7***. — If one is willing to assume that $K$ is $C_1$, then the final argument of the proof above can be simplified: the equation $\mathrm{Norm}_{L/K}(\ell) = ca^{[L:K]}$ with $\ell \in L$ and $a \in K$ gives a homogeneous equation of degree $[L:K]$ in $[L:K]+1$ variables. If $K$ is $C_1$, then it must have a solution.

## 6. Relation to the Cassels-Tate pairing

Throughout this section $K$ is a number field. Let $X$ be a smooth, projective, and geometrically integral variety $X$ over $K$. There is a well known pairing due to Manin,

$$\langle\cdot,\cdot\rangle_{\mathrm{Br}} : \mathrm{Br}(X) \times X(\mathbb{A}_K) \longrightarrow \mathbb{Q}/\mathbb{Z}\,, \quad \langle \mathcal{A}, (P_v)\rangle_{\mathrm{Br}} \mapsto \sum_v \mathrm{inv}_v\, \mathrm{eval}_{P_v}(\mathcal{A})\,,$$

where the sum runs over all places of $K$. By the Hasse reciprocity law, the left kernel contains $\mathrm{Br}_0(X)$ and the right kernel contains the diagonal image of $X(K)$ in $X(\mathbb{A}_K)$. For any subgroup $B \subseteq \mathrm{Br}(X)$, we denote by $X(\mathbb{A}_K)^B$ the subset of $X(\mathbb{A}_K)$ which is orthogonal to $B$ with respect to the pairing. Define

$$\mathrm{Br}_{Ш} X = \left\{\mathcal{A} \in \mathrm{Br}\, X : h(\mathcal{A}) \in \mathrm{im}\left(Ш(\mathrm{Pic}^0_X) \to \mathrm{H}^1(K, \mathrm{Pic}_X)\right)\right\}\,,$$

where for an abelian variety $A$ over $K$, $Ш(A)$ denotes its Tate-Shafarevich group.

The following is a slight generalization of [Sko01, Theorem 6.2.3], which is due to Manin. Similar methods have been used to give a conditional proof that the Brauer-Manin obstruction to 0-cycles of degree 1 on smooth projective curves is the only one (see [ES08, Theorem 1.1], [CT99, Proposition 3.7], [Sai89, Theorem 8.4]). As a corollary we observe that [Sto07, Corollary 7.7] holds for all curves, not just those possessing a $K$-rational divisor class of degree 1.

***Theorem 6.1.*** — *Assume that $X(\mathbb{A}_K) \neq \emptyset$. Let $A = \mathrm{Alb}^0_X$, $V = \mathrm{Alb}^1_X$ and suppose $\mathcal{A} \in \mathrm{Br}_{Ш} X$ is such that $h(\mathcal{A})$ is the image of $W \in Ш(\mathrm{Pic}^0_X) = Ш(A^\vee)$. Then, for any adelic point $(P_v) \in X(\mathbb{A}_K)$,*

$$\langle \mathcal{A}, (P_v)\rangle_{\mathrm{Br}} = -\langle V, W\rangle_{\mathrm{CT}}\,,$$

*where $\langle\cdot,\cdot\rangle_{\mathrm{CT}}$ denotes the Cassels-Tate pairing on $Ш(A)\times Ш(A^\vee)$. In particular, $X(\mathbb{A}_K)^{\mathcal{A}}$ is either empty or equal to $X(\mathbb{A}_K)$, and $X(\mathbb{A}_K)^{\mathrm{Br}_{Ш} X} = \emptyset$ if and only if $\mathrm{Alb}^1_X$ is not divisible in $Ш(A)$.*

***Corollary 6.2.*** — *If $X$ is a smooth, projective, and geometrically integral curve, then for any $n$,*

$$X(\mathbb{A}_K)^{n\text{-ab}} = X(\mathbb{A}_K)^{\mathrm{Br}\, X[n]}\,,$$

*i.e., the adelic information coming from an $n$-descent is precisely the information coming from the $n$-torsion in the Brauer group.*

***Remark 6.3.*** — The set $X(\mathbb{A}_K)^{n\text{-ab}}$ is defined in [Sto07]; [Sko01, Theorem 6.1.2] shows that $X(\mathbb{A}_K)^{n\text{-ab}} = X(\mathbb{A}_K)^{\mathrm{Br}_n X}$, where $\mathrm{Br}_n X \subseteq \mathrm{Br}\, X$ is as defined at the beginning of §5. Thus the corollary can also be interpreted as saying that the elements of $\mathrm{Br}\, X[n] \setminus \mathrm{Br}_n X$ provide no additional information regarding the adelic points of $X$. In fact, the proof below shows that the elements of $\mathrm{Br}\, X[n] \setminus \mathrm{Br}_n X$ provide no information whatsoever.

*Proof of Corollary 6.2.* — As remarked above, $X(\mathbb{A}_K)^{\mathrm{Br}\, X[n]} \subseteq X(\mathbb{A}_K)^{\mathrm{Br}_n X} = X(\mathbb{A}_K)^{n\text{-ab}}$. So if $X$ has no locally solvable $n$-coverings, then both sets in question are empty. We may thus assume that $X$ has an everywhere locally solvable $n$-covering. This implies that $\mathrm{Pic}^1_X = nW$ for some $W \in Ш(J)$ and that $\langle\cdot,\cdot\rangle_{\mathrm{CT}}$ is alternating [PS99]. Now suppose $w \in \mathrm{Br}_{Ш} X$ has the same image in $\mathrm{H}^1(K, \mathrm{Pic}_X)$ as $W$. For any adelic point $(P_v) \in X(\mathbb{A}_K)$, applying the theorem gives:

$$\langle w, (P_v)\rangle_{\mathrm{Br}} = \langle \mathrm{Pic}^1_X, W\rangle_{\mathrm{CT}} = \langle nW, W\rangle_{\mathrm{CT}} = n\langle W, W\rangle_{\mathrm{CT}}\,,$$

which is trivial since the pairing is alternating. Hence, $X(\mathbb{A}_K)^w = X(\mathbb{A}_K)$.

In the exact sequence,

$$\mathbb{Z} \to \mathrm{H}^1(K, \mathrm{Pic}^0_X) \to \mathrm{H}^1(K, \mathrm{Pic}_X) \to 0\,,$$

$1 \in \mathbb{Z}$ maps to the class of $\mathrm{Pic}^1_X$. It follows that the quotient of $(\mathrm{Br}\, X/\mathrm{Br}_0 X)[n]$ by $\mathrm{Br}_n X/\mathrm{Br}_0 X$ is cyclic and generated by the image of $w$. The result follows since we have shown that $w$ does not obstruct any adelic points. □

*Proof of Theorem 6.1.* — For the case that $X$ is a torsor under an abelian variety (e.g., $X = V$) see [Man71, 6. Théorème] or [Sko01, Theorem 6.2.3]. To derive the general result from this, note that the canonical morphism $\phi : X \to V$ induces an isomorphism $\mathrm{Pic}^0_X \cong \mathrm{Pic}^0_V$, and consequently a commutative diagram,

$$\begin{tikzcd}
Ш(\mathrm{Pic}^0_V) \arrow[r, hook] \arrow[d, equal, "\phi^*"] & \mathrm{H}^1(K, \mathrm{Pic}^0_V) \arrow[r] \arrow[d, equal, "\phi^*"] & \mathrm{H}^1(K, \mathrm{Pic}_V) \arrow[r, equal] \arrow[d, "\phi^*"] & \mathrm{Br}_1 V/\mathrm{Br}_0 V \arrow[d, "\phi^*"] \\
Ш(\mathrm{Pic}^0_X) \arrow[r, hook] & \mathrm{H}^1(K, \mathrm{Pic}^0_X) \arrow[r] & \mathrm{H}^1(K, \mathrm{Pic}_X) \arrow[r, equal] & \mathrm{Br}_1 X/\mathrm{Br}_0 X\,.
\end{tikzcd}$$

Suppose $W \in Ш(\mathrm{Pic}^0_X)$ and $\mathcal{A} \in \mathrm{Br}_{Ш} X$ are as in the statement. From the diagram above it is clear that there exists $\mathcal{A}' \in \mathrm{Br}_{Ш} V$ such that $\phi^*\mathcal{A}' \equiv \mathcal{A} \bmod \mathrm{Br}_0 X$. Then we have

$$\langle \mathcal{A}, (P_v) \rangle_{\mathrm{Br}} = \langle \phi^* \mathcal{A}', (P_v) \rangle_{\mathrm{Br}} = \langle \mathcal{A}', \phi(P_v) \rangle_{\mathrm{Br}} = -\langle V, W \rangle_{\mathrm{CT}} \,,$$

since the theorem holds for $V$.

The final statement follows from the fact that the left and right kernels of the Cassels-Tate pairing are the maximal divisible subgroups [Tat63]. $\square$

**6.1. Computing Brauer-Manin Obstructions.** — The following proposition shows that, even though $\gamma$ may not be surjective, its image contains an arithmetically interesting subgroup of $(\operatorname{Br} C/\operatorname{Br}_0 C)[2]$.

***Proposition 6.4***. — *Let $C$ be a double cover of $\mathbb{P}^1_K$ with $C(\mathbb{A}_K) \neq \emptyset$. Then*

$$(\operatorname{Br}_{\text{Ш}} C/\operatorname{Br}_0 C)[2] \subseteq \gamma(\mathfrak{L}_c)\,.$$

*Proof.* — Set $\operatorname{Br}_{\text{Ш},2} C = (\operatorname{Br}_{\text{Ш}} C) \cap (\operatorname{Br}_2 C)$. By [PS97, Theorem 13.3], the subgroup of $\mathrm{H}^1(K, J[2])/\langle \operatorname{Pic}^1_C \rangle$ mapping into $\text{Ш}(J)[2]/\langle \operatorname{Pic}^1_C \rangle$ is contained in the kernel of $\Upsilon$. It follows that $\operatorname{Br}_{\text{Ш},2} C \subseteq \operatorname{Br}_2^{\Upsilon} C$, and so $\operatorname{Br}_{\text{Ш},2} C/\operatorname{Br}_0 C \subseteq \gamma(\mathfrak{L}_1)$ by Proposition 5.1. If $\operatorname{Br}_{\text{Ш}} C[2] \subseteq \operatorname{Br}_{\text{Ш},2} C$, then there is nothing more to prove. Hence we may assume that there exists some $w \in \operatorname{Br}_{\text{Ш}} C[2] \setminus \operatorname{Br}_{\text{Ш},2} C$. Then, as in the proof of Corollary 6.2, the quotient of $(\operatorname{Br}_{\text{Ш}} C/\operatorname{Br}_0 C)[2]$ by $\operatorname{Br}_{\text{Ш},2} C/\operatorname{Br}_0 C$ is of order 2.

The existence of $w$ implies that there exists $W \in \text{Ш}(J)$ such that $2W = \operatorname{Pic}^1_C \neq 0$. By [Cre, Theorem 4.6] this implies that there exists some $\overline{\ell} \in \mathfrak{L}_c \setminus \mathfrak{L}_1$ such that $\operatorname{res}_v(\overline{\ell}) \in (x-\alpha)(\operatorname{Pic}^1 C_{K_v})$, for every completion $K_v$ of $K$. Since $\gamma \circ (x-\alpha) = 0$ by Corollary 2.4, we must have $\gamma(\overline{\ell}) \in \operatorname{Br}_{\text{Ш}} C/\operatorname{Br}_0 C$. On the other hand, $\gamma(\overline{\ell}) \notin \operatorname{Br}_2 C/\operatorname{Br}_0 C$ by Lemma 5.2. Thus $\gamma(\overline{\ell})$ must generate the quotient of $(\operatorname{Br}_{\text{Ш}} C/\operatorname{Br}_0 C)[2]$ by $\operatorname{Br}_{\text{Ш},2} C/\operatorname{Br}_0 C$. Therefore, $(\operatorname{Br}_{\text{Ш}} C/\operatorname{Br}_0 C)[2] \subseteq \gamma(\mathfrak{L}_c)$. $\square$

***Remark 6.5***. — Regardless of whether $C$ is locally solvable or not, the proof of Corollary 6.2 shows that $C(\mathbb{A}_K)^{(\operatorname{Br}_{\text{Ш}} C)[2]} = C(\mathbb{A}_K)^{\operatorname{Br}_{\text{Ш},2} C}$. When $C$ has a $K_v$-rational divisor of degree 1 for every completion $K_v$ of $K$, then $\operatorname{Br}_{\text{Ш},2} C/\operatorname{Br}_0 C \subseteq \gamma(\mathfrak{L}_1)$. In this case the subgroup of $\mathfrak{L}_1$ mapping into $\operatorname{Br}_{\text{Ш}} C/\operatorname{Br}_0 C$ is the fake 2-Selmer group of $J$, denoted $\operatorname{Sel}^2_{\text{fake}}(J)$. An algorithm for computing it is described in [PS97]. Together with the following proposition, this gives a practical algorithm for computing the induced map

$$\operatorname{Sel}^2_{\text{fake}}(J) \to \frac{\text{Ш}(J)[2]}{\langle \operatorname{Pic}^1_C \rangle} \xrightarrow{\langle \operatorname{Pic}^1_C, \cdot \rangle} \mathbb{Q}/\mathbb{Z}\,,$$

at least when $C(\mathbb{A}_K) \neq \emptyset$.

***Proposition 6.6***. — *Suppose $C : y^2 = cf(x)$ is an even double cover of $\mathbb{P}^1$ defined over $K$ with $C(\mathbb{A}_K) \neq \emptyset$ and that the coefficients of $cf(x)$ are integral. Let $\beta \in \text{Ш}(J)$, and suppose $\ell$ represents $\overline{\ell} \in \mathfrak{L}_c$ such that $d(\overline{\ell})$ and $\beta$ give the same class in $\text{Ш}(J)/\langle \operatorname{Pic}^1_C \rangle$. Then, for any $(P_v) \in C(\mathbb{A}_K)$.*

$$\langle \operatorname{Pic}^1_C, \beta \rangle_{\mathrm{CT}} = \sum_{v \in S} \operatorname{inv}_v \operatorname{eval}_{P_v} \operatorname{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}(\ell, (x-\alpha))_2\,,$$

*The sum here runs over the primes in the finite set $S$ consisting of all primes of $K$ appearing with multiplicity greater or equal to 2 in $4c^2 \cdot \operatorname{disc}(f)$ and all archimedean primes.*

*Proof.* — If $v$ does not lie in $S$, then both $(x-\alpha)([P_v])$ and $\ell$ have even valuation at all primes $w$ above $v$, by [BS09, Lemma 4.3] and [Sto01, Proposition 5.10]. For such $v$ the invariant $\operatorname{inv}_v \operatorname{eval}_{P_v} \operatorname{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}(\ell,(x-\alpha))_2 = 0$. □

### 6.2. An Example. —

***Theorem 6.7.*** — *Let $c$ be a square free integer, let $C$ be the locally solvable double cover of $\mathbb{P}^1_{\mathbb{Q}}$ given by*

$$C : y^2 = c(x^2+1)(x^2+17)(x^2-17)\,.$$

*Then $(-1,x^2-17)_2 \in \mathrm{Br}_{\text{Ш}}\, C$, and if $W \in \text{Ш}(J)$ denotes a corresponding torsor, then*

$$\langle \mathrm{Pic}^1_C, W\rangle_{\mathrm{CT}} = \frac{\#\left\{p \mid c : p \text{ is an odd prime, and } \left(\frac{17}{p}\right) = \left(\frac{-1}{p}\right) = -1\right\}}{2} + \frac{\operatorname{sign}(c)-1}{4}$$

*Furthermore, if $\langle \mathrm{Pic}^1_C, W\rangle_{\mathrm{CT}} = 1/2$, then $\dim_{\mathbb{F}_2} \text{Ш}(J)[2] \geq 2$ and neither $W$ nor $\mathrm{Pic}^1_C$ is divisible by 2 in $\mathrm{H}^1(\mathbb{Q},J)$.*

*Proof.* — We first note that $(-1,x^2-17)_2 = \gamma'(\ell)$, for the element

$$\ell = (1,1,-1) \in \mathbb{Q}(\sqrt{-1}) \times \mathbb{Q}(\sqrt{-17}) \times \mathbb{Q}(\sqrt{17}) \simeq L.$$

It is easy to see that $C$ is locally solvable. In fact, it has a $\mathbb{Q}_p$-rational ramification point for every prime $p$. One can also check that $\operatorname{res}_p(\overline{\ell}) \in (L\otimes\mathbb{Q}_p)^\times/\mathbb{Q}_p^\times(L\otimes\mathbb{Q}_p)^{\times 2}$ is trivial for every prime $p$ (this is weaker than requiring $\operatorname{res}_p(\ell) \in \mathbb{Q}_p^\times(L\otimes\mathbb{Q}_p)^{\times 2}$ everywhere locally). This imples that $\gamma'(\ell) \in \mathrm{Br}_{\text{Ш}}\, C$. Consequently there is a torsor $W \in \text{Ш}(J)$ whose class in $\text{Ш}(J)/\langle \mathrm{Pic}^1_C\rangle$ is represented by $d(\overline{\ell})$. By Theorem 6.1 and Proposition 6.6, for any $(P_p) \in C(\mathbb{A}_{\mathbb{Q}})$, we have

$$\langle \mathrm{Pic}^1_C, W\rangle_{\mathrm{CT}} = \langle \gamma'(\ell),(P_p)\rangle_{\mathrm{Br}} = \sum_p \operatorname{inv}_p\left(\operatorname{eval}_{P_p}(-1,x^2-17)_2\right)\,, \tag{6.1}$$

To ease notation, let us set $\varepsilon_p = \operatorname{inv}_p\left(\operatorname{eval}_{P_p}(-1,x^2-17)_2\right)$. Note that, by Theorem 6.1, $\varepsilon_p$ depends on $c$, but not on the subsequent choice for $P_p$.

***Lemma 6.8.*** — *Let $p$ be an odd prime. Then*

$$\varepsilon_p = \begin{cases} 1/2 & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{-17}{p}\right) = -1 \textit{ and } p \mid c\,,\\ 0 & \text{if else}\,. \end{cases}$$

$$\varepsilon_2 = \begin{cases} 0 & \text{if } c \equiv 1,2 \textit{ or } 5 \mod 8\,,\\ 1/2 & \text{if } c \equiv 3,6 \textit{ or } 7 \mod 8\,; \end{cases}$$

$$\varepsilon_\infty = \begin{cases} 0 & \text{if } c > 0\,,\\ 1/2 & \text{if } c < 0\,; \end{cases}$$

The lemma is proved below; using it gives:

$$\begin{aligned}
\varepsilon_2 &= \#\left\{p \mid c : \left(\frac{-1}{p}\right) = -1\right\}/2 \\
&= \#\left\{p \mid c : \left(\frac{-1}{p}\right) = \left(\frac{17}{p}\right) = -1\right\}/2 + \#\left\{p \mid c : \left(\frac{-1}{p}\right) = \left(\frac{-17}{p}\right) = -1\right\}/2 \\
&= \#\left\{p \mid c : \left(\frac{-1}{p}\right) = \left(\frac{17}{p}\right) = -1\right\}/2 + \sum_{p \mid c} \varepsilon_p \,,
\end{aligned}$$

from which the formula in the theorem follows easily.

Now let us prove the final statement of the theorem. Since $C(\mathbb{A}_K) \neq \emptyset$, the pairing $\langle \cdot, \cdot \rangle_{\mathrm{CT}}$ is alternating [PS99]. Tate's proof that the left and right kernels of the pairing are the maximal divisible subgroups [Tat63, Theorem 3.2] shows that $\langle \cdot, \cdot \rangle_{\mathrm{CT}}$ induces a nondegenerate alternating pairing on $Ш(J)[2]/2Ш(J)[4]$. As is well known, this implies that the order of this group is a square. If $\langle \mathrm{Pic}^1_C, W \rangle_{\mathrm{CT}} = 1/2$, then the group is nontrivial, and hence has dimension at least 2. To show that this also implies that $\mathrm{Pic}^1_C \notin 2\,\mathrm{H}^1(K, J)$, we use [Cre13, Theorem 3], which states that an element of $Ш(J)$ is divisible by $n$ in $\mathrm{H}^1(K, J)$ if and only if it pairs trivially with the image of $Ш^1(K, J[n])$ in $Ш(J)[n]$. In our situation we know that $W$ lies in this image of $Ш^1(K, J[2]) \to Ш(J)$, because $\overline{\ell}$ is locally trivial. □

*Proof of Lemma 6.8.* — Suppose $p$ is odd. If $\left(\frac{17}{p}\right) = -1$, then $\ell$ is trivial since $-1$ is a square in $\mathbb{Q}(\sqrt{17}) \otimes \mathbb{Q}_p$. So suppose $\left(\frac{17}{p}\right) = 1$, let $a \in \mathbb{Q}_p$ be a square root of 17 and set $P_p = (a, 0) \in C(\mathbb{Q}_p)$. Then

$$\begin{aligned}
\varepsilon_p &= \mathrm{inv}_p\, \mathrm{eval}_{(a,0)}(-1, x^2 - 17)_2 \\
&= \mathrm{inv}_p\, \mathrm{eval}_{(a,0)}\left(-1, c(x^2 + 17)(x^2 + 1)\right)_2 \\
&= \mathrm{inv}_p(-1, c \cdot 2^2 \cdot 3^2 \cdot 17)_2 \\
&= \mathrm{inv}_p(-1, c)_2 \,,
\end{aligned}$$

which is nontrivial if and only if $p \mid c$ and $\left(\frac{-1}{p}\right) = -1$. To arrive at the statement in the lemma, note that if $\left(\frac{-1}{p}\right) = -1$ and $\left(\frac{17}{p}\right) = 1$, then $\left(\frac{-17}{p}\right) = -1$.

Clearly $\varepsilon_2$ depends only on the class of $c$ in $\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2}$. The table below gives, for each square class, a value $x(P_2) \in \mathbb{Z}$ for which $f(x(P_2)) \equiv c \bmod \mathbb{Q}_2^{\times 2}$, i.e., $x(P_2)$ is the $x$-coordinate of a $\mathbb{Q}_2$-point on the curve $y^2 = cf(x)$. The corresponding invariant is then

$$\varepsilon_2 = \mathrm{inv}_2(-1, x(P_2)^2 - 17)_2 \,,$$

The claim above follows immediately from the table.

| $c \bmod \mathbb{Q}_2^{\times 2}$ | 1 | 2 | 3 | 5 | 6 | 7 | 10 | 14 |
|---|---|---|---|---|---|---|---|---|
| $x(P_2)$ | 9 | 5 | 2 | 15 | 13 | 0 | 11 | 3 |
| $\varepsilon_2$ | 0 | 0 | 1/2 | 0 | 1/2 | 1/2 | 0 | 1/2 |

For any real point $P_\infty \in C(\mathbb{R}) \setminus \Omega$, $\varepsilon_\infty = \mathrm{inv}_\infty(-1, x(P_\infty)^2 - 17)_2$, which can be nonzero if and only if there are real points with $|x(P_\infty)| < \sqrt{17}$, which occurs if and only if $c < 0$. □

# PART II
# DOUBLE COVERS OF RULED SURFACES

## 7. Introduction

Let $k$ be a separably closed field of characteristic not equal to 2, and let $\varpi : S \to W$ be a ruled surface over $k$. Let $\pi\colon X^0 \to S$ be a double cover, and let $X$ be the desingularization of $X^0$ that is obtained by a canonical resolution (see [BHPVdV04, §III.7] for the definition). In this, the second part of the paper, we aim to give an explicit finite presentation of $\operatorname{Br} X[2]$ in terms of generators given by central simple algebras over $\mathbf{k}(X)$. Our main results in this direction are Theorems 9.3, 10.1 and 11.2. Note that since $\mathbb{P}^2$ blown up at a point is a ruled surface and the Brauer group is a birational invariant, the results in this part apply to double covers of $\mathbb{P}^2$.

**7.1. Notation.** — Let $C$ be the generic fiber of $X \to S \to W$; it is a double cover of $\mathbb{P}^1$ defined over $K = \mathbf{k}(W)$. As in Part I we may fix an even model for $C$ of the form $y^2 = cf(x)$, where $c \in K^\times$ and $f$ is squarefree and monic of degree $2g(C) + 2$. This implies that $C \to \mathbb{P}^1_K$ is not ramified above the point $\infty \in \mathbb{P}^1_K$. We let $\mathfrak{S}$ denote the flat closure of $\infty$ in $S$. The purity theorem [Fuj02] identifies $\operatorname{Br} X$ as the subgroup of $\operatorname{Br} C$ unramified at all vertical divisors. Since $k$ is separably closed, $\operatorname{Br} K = 0$ by Tsen's theorem. Hence, Theorem 1.1 gives a presentation of $\operatorname{Br} C[2]$ as the image of the surjective map $\gamma : \mathfrak{L}_c \to \operatorname{Br} C[2]$.

Let $B^0$ denote the union of the connected components of the branch locus of $X/S$ that map dominantly to $W$. We will assume that $B^0$ is nonempty. This is equivalent to assuming that $X$ is geometrically irreducible. The restriction of the map $\varpi : S \to W$ to $B^0$ may not be flat as $B^0$ may have vertical components. Let $B^{0,\mathrm{fl}} \subseteq B^0$ be the maximal subvariety such that the map $B^{0,\mathrm{fl}} \to W$ is flat, i.e., $B^{0,\mathrm{fl}}$ is the union of all irreducible components of $B$ that map dominantly to $W$. We write $B$ for the normalization of $B^0$ in $X$, and write $B^{\mathrm{fl}}$ for the normalization of $B^{0,\mathrm{fl}}$ in $X$. Note that $\mathbf{k}(B^{\mathrm{fl}}) = L$ and $\mathbf{k}(B) = L \times k(x)^n$, where $n$ equals the number of vertical components of $B^0$. We denote the normalization map $B \to B^0$ by $\nu$ and sometimes conflate $\nu$ with $\nu|_{B^{\mathrm{fl}}}$.

Since $X$ was obtained by a canonical resolution, the curve $B$ is smooth [BHPVdV04, §III.7]. In particular, $B$ is a disjoint union of integral curves $B_i$. We define $\operatorname{Div}(B) := \prod \operatorname{Div}(B_i)$ and $\operatorname{Jac}(B) := \prod \operatorname{Jac}(B_i)$. By convention we set $g(B) = \sum g(B_i) + 1 - h^0(B)$, where $h^0(B)$ is the number of connected components of $B$. The same notation will be used with $B^{\mathrm{fl}}$ in place of $B$.

If $b \in B^{0,\mathrm{fl}}$ is a point lying over $w \in W$ and $\ell \in L^\times$, we define

$$v_b(\ell) := \sum_{b' \in B^{\mathrm{fl}}, b' \mapsto b} v_{b'}(\ell), \quad e(b/w) := \sum_{b' \in B^{\mathrm{fl}}, b' \mapsto b} e(b'/w),$$

where $e(b'/w)$ denotes the ramification index of the map $B^{\mathrm{fl}} \to W$ at $b'$ and $v_{b'}(\ell)$ is the valuation of $\ell$ at $b'$.

Since the branch locus of $\pi$ is generically smooth and $S$ is smooth, $X^0$ is regular in codimension 1. Let $\mathcal{E}$ be the set of curves on $X$ that are either contracted to a point in $X^0$, or lie over some $w \in W$ such that $S_w$ is singular. Since $X^0$ is regular in codimension 1, the morphism $X \to X^0$ is an isomorphism away from $\mathcal{E}$. We say that an irreducible curve $F$ on $X$ is exceptional if $F \in \mathcal{E}$ and non-exceptional otherwise. If $F$ is a curve on $X^0$, we will often abuse notation and let $\partial_F$ denote the residue map at the strict transform of $F$ on $X$.

***Remark 7.1.*** — There are some curves in $\mathcal{E}$ which are not "exceptional" in the usual sense, i.e., are not the exceptional divisor of some blow-up. Such curves are all components of $X_w^0$ for some $w \in W$ such that $S_w$ is not smooth. In particular, if $S$ is geometrically ruled, i.e., *every* fiber is isomophic to $\mathbb{P}^1$, then every curve in $\mathcal{E}$ is the exceptional divisor of some blow-up.

***Remark 7.2.*** — The surface $X$ considered is in fact birational to a double cover of a *geometrically* ruled surface of the form $\mathbb{P}^1 \times W$. However, allowing for a more general ruled surface enables us to choose a model where the branch locus has milder singularities. This will be used when we consider the Brauer group of an Enriques surface in §11.

***Remark 7.3.*** — Many of the arguments below are simpler and more intuitive when $B^0 = B^{0,\mathrm{fl}}$, and even more so under the additional assumption that $B^0$ is smooth and irreducible. As many of the results are of equal interest in these cases, the reader may wish to make these assumptions on a first reading.

**Outline of Part II.** — In §8, we determine necessary and sufficient conditions for a Brauer class $\gamma(\overline{\ell})$ to be unramified outside $\mathcal{E}$, and show that $\operatorname{Br} X[2] = \operatorname{Br}(X \setminus \mathcal{E})[2]$ when $S$ is geometrically ruled and $B^0$ has at worst simple singularities. In §9, we exhibit functions satisfying these conditions and prove that, when $S$ is geometrically ruled, every class $\operatorname{Br}(X \setminus \mathcal{E})[2]$ can be represented using a product thereof. In the same section, we also explain how the results can be used to give a presentation of $\operatorname{Br} X'[2]$ when $X'$ is a smooth double cover of $\mathbb{P}^2$. Next, in §10, we use our presentation to determine the size of $\operatorname{Br} X[2]$. In §11, we compute the non-trivial Brauer class for any Enriques surface, and in §12 use this class in an example to give a transcendental Brauer-Manin obstruction to weak approximation.

## 8. Residues at vertical divisors

### 8.1. The non-exceptional curves. —

***Proposition 8.1.*** — *Fix $w \in W$ such that $S_w$ is smooth and fix $\ell \in L^\times$ such that $\overline{\ell} \in \mathfrak{L}_c$.*

1. *If $X_w^0$ is reduced and irreducible, then $\partial_{X_w^0}(\gamma(\overline{\ell})) \in \kappa(X_w^0)^{\times 2}$ if and only if*
$$e(b'/w)v_b(\ell) \equiv e(b/w)v_{b'}(\ell) \bmod 2\,, \quad \text{for all } b, b' \in B_w^0 \setminus (B_w^0 \cap \mathfrak{S}). \tag{8.1}$$
2. *If $X_w^0$ is reduced and reducible and $S_w$ is smooth, then $\partial_F(\gamma(\overline{\ell})) \in \kappa(F)^{\times 2}$ for all irreducible components $F \subseteq X_w^0$ if and only if*
$$v_b(\ell) \equiv 0 \bmod 2\,, \quad \text{for all } b \in B_w^0 \setminus (B_w^0 \cap \mathfrak{S}). \tag{8.2}$$
3. *If $S_w \subseteq B^0$, then $\partial_{(X_w^0)_{\mathrm{red}}}(\gamma(\overline{\ell})) \in \kappa((X_w^0)_{\mathrm{red}})^{\times 2}$ for all $\ell \in L^\times$.*

***Corollary 8.2.*** — *Let $\overline{\ell} \in \mathfrak{L}_c$. Then $\gamma(\overline{\ell}) \in \operatorname{Br}(X \setminus \mathcal{E})$ if and only if some (equivalently every) representative of $\overline{\ell}$ satisfies* (8.1) *at every $w \in W$ such that $X_w^0$ is reduced and irreducible and $S_w$ is smooth and satisfies* (8.2) *at every $w \in W$ such that $X_w^0$ is reduced and reducible and $S_w$ is smooth.*

*Proof.* — Every non-exceptional vertical curve maps dominantly to a smooth and irreducible $S_w$ for some $w \in W$. If $S_w$ is smooth and irreducible, then $X^0_w$ is reduced if and only if $S_w \subseteq B^0$. Therefore, for every $F \in X^{(1)} \setminus \mathcal{E}$, Proposition 8.1 gives necessary and sufficient conditions for $\partial_F(\ell) \in \kappa(F)^{\times 2}$. This is exactly the content of the Corollary. $\square$

*Proof of Proposition 8.1.* — Fix $\ell \in L^\times$ such that $\overline{\ell} \in \mathfrak{L}_c$ and let $F \subseteq X^0_w$ be a reduced and irreducible curve. Using (2.2), we see that

$$\partial_F(\gamma(\overline{\ell})) = \prod_{\substack{F' \in (X'_{B^{\mathrm{fl}}})^{(1)} \\ F' \mapsto F \text{ dominantly}}} \mathrm{Norm}_{\kappa(F')/\kappa(F)}(\ell^{w'(x-\alpha)}(x-\alpha)^{w'(\ell)}). \tag{8.3}$$

Here $X'_{B^{\mathrm{fl}}}$ denotes the desingularization of $X_{B^{\mathrm{fl}}} := X \times_W B^{\mathrm{fl}}$, and $w'$ denotes the valuation associated to $F'$. The surface $X \times_W B^{\mathrm{fl}}$ is regular at all codimension 1 points lying over $w \in W$ such that $X^0_w$ is reduced.

Assume that $X^0_w$ is not reduced, or, equivalently, that $S_w \subset B^0$. Then the map on residues $\mathrm{H}^1(\kappa(S_w), \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(\kappa(X^0_w), \mathbb{Q}/\mathbb{Z})$ is identically zero on 2-torsion classes. Since $\gamma'(\ell) \in \mathrm{im}\,(\mathrm{res}\,\mathrm{Br}\,\mathbf{k}(S) \to \mathrm{Br}\,\mathbf{k}(X))$, the residue $\partial_{(X^0_w)_{\mathrm{red}}}(\ell) \in \kappa((X^0_w)_{\mathrm{red}})^{\times 2}$ for all $\ell \in L^\times$.

Henceforth, we assume that $X^0_w$ is reduced, or, equivalently, that $B^{0,\mathrm{fl}}_w = B^0_w$. Then, since $X \times_W B^{\mathrm{fl}}$ is regular at all codimension 1 points above $w$, the prime divisors of $X'_{B^{\mathrm{fl}}}$ that map dominantly to $F$ are exactly the prime divisors of $X_{B^{\mathrm{fl}}}$ that map dominantly to $F$.

To compute the residues at $X^0_w$, we will need to have a model of the fiber. For this, we will use the following lemma.

***Lemma 8.3.*** — *For every $w \in W$ such that $S_w$ is smooth, there exists an open set $U \subset W$ containing $w$ and constants $a, b \in K$ such that*

$$S_U \xrightarrow{\sim} \mathbb{P}^1_k \times U, s \mapsto (ax(s) + b, \varpi(s)).$$

*Proof.* — By the Noether-Enriques theorem [Bea96, Thm. III.4], there is an isomorphism $\varphi\colon S_U \to \mathbb{P}^1 \times U$ which commutes with the obvious morphisms to $U$. After possibly composing with an automorphism of $\mathbb{P}^1_k$, we may assume that $\mathfrak{S}$ maps to $\{\infty\} \times U$. To complete the proof we observe that $\varphi$ must induce an automorphism of $\mathbb{P}^1_K$ that preserves $\infty$. $\square$

Fix $U \subset W$, $a, b \in K$ as in the lemma. Note that, the algebra $\mathrm{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}\left((\ell, a)_2\right)$ is constant, and hence trivial. Therefore, $\mathrm{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}\left((\ell, ax+b-(a\alpha+b))_2\right) = \gamma'(\ell)$.

Hence, by replacing $x$ with $ax+b$, $\alpha$ with $a\alpha+b$ and $f$ with $f(x/a - b/a)$ if necessary, we may assume that $x$ is a horizontal function, i.e. that it has no zeros or poles along any fibers of $U$, and that it restricts to a non-constant function along any fiber of $U$. Then the function $x - \alpha$ has non-positive valuation along any fiber of $X_{B^{\mathrm{fl}}_U}$, and it has negative valuation on the fibers of $X_{B^{\mathrm{fl}}_U}$ where $\alpha$ has negative valuation.

Assume that $B_w \cap \mathfrak{S} = \emptyset$; then $w'(x-\alpha) = 0$ for all $F'$ lying over $F$. Moreover, the norm map, $\mathrm{Norm}_{\kappa(F')/\kappa(F)}$, is an isomorphism for all $F'$, so (8.3) gives

$$\partial_F(\gamma(\overline{\ell})) = \prod_{\substack{P \in B_w \\ v_P(\ell) \equiv 1 \bmod 2}} (x - \alpha(P)).$$

If $X^0_w$ is irreducible, then (8.1) fails if and only if there exist points $b, b' \in B^0_w$ with odd ramification index over $w$ and such that $v_b(\ell)$ and $v_{b'}(\ell)$ have different parity. Using this equivalence, the assumption that that $\overline{\ell} \in \mathfrak{L}_c$, as well as the defining equation for $X_w^0$, we see that $\partial_F(\gamma(\overline{\ell})) \notin \kappa(F)^{\times 2}$ if and only if (8.1) fails. If $X_w^0$ is reducible, then $F$ is rational and $\kappa(F) = k(x)$. One can check that (8.2) fails if and only if there is a $b \in B_w$ such that $v_b(\ell) \equiv 1 \bmod 2$ which in turn occurs if and only if $\partial_F(\gamma(\overline{\ell})) \notin \kappa(F)^{\times 2}$.

It remains to consider the case when $B_w^0 \cap \mathfrak{S} \neq \emptyset$. In this case, there is a unique valuation $w'_\infty$ such that $w'_\infty(x-\alpha) < 0$; $w'(x-\alpha) = 0$ for all other $F'$ lying over $F$. Indeed, this unique valuation $w'_\infty$ corresponds to the unique point of intersection $B_w^0 \cap \mathfrak{S}$. Since $w'_\infty(x) = 0$, the function $(x-\alpha)^{w'_\infty(\ell)}/\ell^{w'_\infty(x-\alpha)}$ reduces to a constant in $\kappa(F'_\infty)$. Therefore,

$$\partial_F(\gamma(\overline{\ell})) = \prod_{\substack{b \in B_w^0 \setminus (B_w^0 \cap \mathfrak{S}) \\ v_b(\ell) \equiv 1 \bmod 2}} (x - \alpha(P)).$$

Using the same reasoning as above, we see that $\partial_F(\gamma(\overline{\ell})) \notin \kappa(F)^{\times 2}$ if and only if (8.1) or (8.2) fails, depending whether $X_w^0$ is irreducible, respectively reducible. □

### 8.2. Exceptional curves lying over simple singularities. —

***Proposition 8.4.*** — *Let $F$ be a $(-2)$-curve lying over a simple singularity of $X^0$. If $A \in \operatorname{Br} \mathbf{k}(X)[2]$ is unramified at all curves $F' \subset X$ that intersect $F$ and that are not contracted in $X^0$, then $A$ is unramified at $F$. In particular, if $S$ is geometrically ruled and $B^0$ has at worst simple singularities, then* $\operatorname{Br} X[2] = \operatorname{Br}(X \setminus \mathcal{E})[2]$.

*Proof of Proposition 8.4.* — The canonical resolution of a simple singularity consists of a series of a blow-ups. Therefore, the preimage of a simple singularity $P \in X^0$ is a tree of $(-2)$-curves. For the sake of exposition, we will fix a curve $F_0$ as the root of the tree. Consider any $(-2)$-curve $F$ which is a leaf of the tree (i.e., has valence 1) and let $Q \in F$ be the (unique) point which intersects another curve in the tree.

As a special case of the Bloch-Ogus arithmetic complex [Kat86, §1, Prop. 1.7], we have the complex

$$\operatorname{Br} \mathbf{k}(X)[2] \xrightarrow{\oplus \partial_{F'}} \bigoplus_{F' \in X^{(1)}} \frac{\kappa(F')^{\times}}{\kappa(F')^{\times 2}} \longrightarrow \bigoplus_{P \in X^{(2)}} \mathbb{Z}/2\mathbb{Z}.$$

Therefore, for every codimension 2 point $P \in X$, we have

$$\sum_{\substack{F' \in X^{(1)} \\ \text{with } P \in \overline{F'}}} v_P(\partial_{F'}(A)) \equiv 0 \bmod 2.$$

By assumption $\partial_{F'}(A) \in \kappa(F')^{\times 2}$ for all $F' \in X^{(1)}$ whose closure intersects $F$ and is not contracted in $X^0$. Hence $v_P(\partial_F(A)) \equiv 0 \bmod 2$ for all $P \in X^{(2)}$ such that $P \in F$ and $P \neq Q$. Since $F$ is rational, this implies $\partial_F(A) \in \kappa(F)^{\times 2}$. Therefore $A$ is unramified at all $(-2)$-curves which are leaves of the tree.

The same proof then shows that $A$ is unramified at all $(-2)$-curves $F$ such that all the children of $F$ are leaves. Then we apply the same argument to all curves $F$ such that all of the grandchildren of $F$ are leaves, and so on, until we have shown that $A$ is unramified at all curves in the tree.

For the final claim, we note that if $S$ is geometrically ruled and $B^0$ has at worst simple singularities, then $\mathcal{E}$ consists of $(-2)$-curves lying over simple singularities of $X^0$. □

## 9. The presentation of $\operatorname{Br} X[2]$

Corollary 8.2 gives an explicit criterion for determining when a Brauer class of the form $\gamma(\overline{\ell})$ lies in $\operatorname{Br}(X \setminus \mathcal{E})$. In this section we identify the functions $\ell \in L^\times = \mathbf{k}(B^{\mathrm{fl}})^\times$ that satisfy this condition and use them to obtain a presentation for the 2-torsion in the Brauer group.

**9.1. Candidate functions.** — The condition of Corollary 8.2 depends only on the set $\{b \in B^{0,\mathrm{fl}} : v_b(\ell) \equiv 1 \bmod 2\}$. In particular, it is obviously satisfied if this set is empty. Functions for which this is the case arise in two ways: from the 2-torsion in $\operatorname{Jac}(B)$, and from cycles on the dual graph $\Gamma$ of $B^0$.

Since the Jacobians of the vertical components of $B$ are trivial, every class of $\operatorname{Jac}(B)[2]$ can be represented by a divisor $D \in \operatorname{Div}(B^{\mathrm{fl}})$ such that $2D$ is principal. Conversely, if $\ell \in L^\times$ is such that $\operatorname{div}(\ell) = 2D$, then $[D] \in \operatorname{Jac}(B)[2]$. Moreover, if $\operatorname{div}(\ell) = 2D$ and $\operatorname{div}(\ell') = 2D'$, then $[D] = [D']$ if and only if $\ell/\ell' \in L^{\times 2}$. This means that there is an injective homomorphism $\operatorname{Jac}(B)[2] \to L^\times/L^{\times 2}$ whose image consists precisely of those classes represented by functions whose divisors are doubles. For each $[D] \in \operatorname{Jac}(B)[2]$, let us fix a representative $D$ and a function $\ell_D \in L^\times$ such that $\operatorname{div}(\ell_D) = 2D$.

When $B^0$ is singular, it is possible to construct functions $\ell \in L^\times$ such that $v_b(\ell) \equiv 0 \bmod 2$, for every $b \in B^{0,\mathrm{fl}}$, but $\operatorname{div}(\ell) \in \operatorname{Div}(B)$ is not a double. The construction can be formalized by introducing the dual graph $\Gamma$ of $B^0$. For every singular point $b \in B^0$, fix an ordering of the preimages $b'_0, \ldots, b'_s \in B$ of $b$. We define the vertices of $\Gamma$ to be in one-to-one correspondence with the irreducible components of $B$, and define the edges of $\Gamma$ by the following rule: for every singular point $b \in B^0$ and every $1 \leq i \leq s$, there is an edge $e_{b,i}$ joining the vertices corresponding to the irreducible components containing $b'_{i-1}$ and $b'_i$.

***Remark 9.1.*** — Strictly speaking it is not correct to refer to *the* dual graph of $B$, since $\Gamma$ depends on the ordering chosen above. However, its fundamental group does not; as this is all we are really concerned with below, we will allow ourselves this abuse of language.

Now suppose $\mathcal{C}$ is a cycle on $\Gamma$ consisting of edges $e_{b_1,i_1}, \ldots, e_{b_n,i_n}$, and consider the corresponding divisor $D_{\mathcal{C}} = \sum_{j=1}^n (b_j)'_{i_n-1} + (b_j)'_{i_n}$. On every irreducible component of $B$, the degree of $D_{\mathcal{C}}$ is even (since a cycle has even degree at every vertex). Thus, since the Jacobian of $B$ is divisible, there exist functions $a_{\mathcal{C},i} \in k(x)^\times$ and a function $\ell_{\mathcal{C}} \in L^\times$ such that

$$\operatorname{div}((\ell_{\mathcal{C}}, a_{\mathcal{C},1}, \ldots, a_{\mathcal{C},n})) \equiv D_{\mathcal{C}} \bmod 2\operatorname{Div}(B)$$

By convention we consider the empty set to be a cycle and set $\ell_\emptyset = 1$. The fact that $\mathcal{C}$ is a cycle implies that $v_b(\ell) \equiv 0 \mod 2$, for every $b \in B^{0,\mathrm{fl}}$. Although the functions $\ell_{\mathcal{C}}$ are not uniquely defined, any two choices differ by a function whose divisor is a double. Furthermore, $\operatorname{div}(\ell_{\mathcal{C}})$ is a double if and only if the class of $\mathcal{C}$ in $\pi_1(\Gamma) \otimes \mathbb{Z}/2\mathbb{Z}$ is trivial. Moreover, if $\mathcal{C}$ and $\mathcal{C}'$ are two cycles with a vertex in common and $\ell_{\mathcal{C}\mathcal{C}'}$ is the function corresponding to the concatenation of the cycles, then the divisor of $\ell_{\mathcal{C}\mathcal{C}'}/\ell_{\mathcal{C}}\ell_{\mathcal{C}'}$ is a double. Therefore, the subspace of $L^\times/L^{\times 2}$ generated by the classes of functions in the set

$$\{\ell_{\mathcal{C}} : \mathcal{C} \subseteq \Gamma \text{ is a cycle}\} \cup \{\ell_D : [D] \in \operatorname{Jac}(B)\}$$

has $\mathbb{F}_2$-dimension $\dim_{\mathbb{F}_2} \operatorname{Jac}(B)[2] + b_1(\Gamma)$ and it does not depend on the choices for the $\ell_{\mathcal{C}}$ and $\ell_D$.

Now we consider functions satisfying Corollary 8.2 for which the set $\{b \in B^{0,\mathrm{fl}} : v_b(\ell) \equiv 1 \bmod 2\}$ is not necessarily empty. If $v_{b'}(\ell) = e(b'/w)$, for every $w \in W$ and $b' \in B^{\mathrm{fl}}_w$, then $v_b(\ell) = e(b/w)$, for every $w \in W$ and $b \in B^{0,\mathrm{fl}}_w$, in which case it is easy to check that the condition in Corollary 8.2 is satisfied. In particular, this holds if $\operatorname{div}(\ell) \equiv B^{\mathrm{fl}}_w \bmod 2\operatorname{Div}(B^{\mathrm{fl}})$ for some $w \in W$. For every $w \in W$, the degree of $B^{\mathrm{fl}}_w$ is even, so, since the Jacobian of $B$ is divisible, there exists a function $\ell_w \in L^\times$ such that

$$\operatorname{div}(\ell_w) \equiv B^{\mathrm{fl}}_w \bmod 2\operatorname{Div}(B^{\mathrm{fl}})\,.$$

Again, $\ell_w$ is not uniquely determined by this condition, but the ratio of any two choices is a function whose divisor is a double. Moreover, if $w, w' \in W$, then there exists some $a \in K^\times$ such that $a\cdot\ell_w/\ell_{w'}$ is a double. Let us fix a function $\ell_1 = \ell_{w_1} \in L^\times$ corresponding to some point $w_1 \in W$.

Finally, we note that Corollary 8.2 imposes no restriction on the valuations at points of $B^{0,\mathrm{fl}} \cap \mathfrak{S}$. In particular, if the valuation of $\ell$ is even outside $\nu^{-1}(B^{0,\mathrm{fl}} \cap \mathfrak{S})$, then it satisfies the condition of the Corollary. In light of Proposition 2.1 we are only interested in functions $\ell$ such that $\operatorname{div}(\operatorname{Norm}_{L/K}(\ell)) \equiv \operatorname{div}(c) \bmod 2\operatorname{Div}(W)$. Hence we should choose a function $\ell_c \in L^\times$ such that

$$\operatorname{div}(\ell_c) \equiv (B^{\mathrm{fl}} \cap \tilde{\mathfrak{S}}) \bmod 2\operatorname{Div}(B^{\mathrm{fl}})\,,$$

where $\tilde{\mathfrak{S}}$ denotes the strict transform of $\mathfrak{S}$. Such a function exists because $(B^{\mathrm{fl}} \cap \tilde{\mathfrak{S}})$ has even degree and $\operatorname{Jac}(B)$ is divisible; the choice is again unique up to functions whose divisors are doubles.

**9.2. The presentation.** — We claim that if $S$ is geometrically ruled, then every class in $\operatorname{Br}(X \setminus \mathcal{E})[2]$ can be represented as the image under $\gamma'$ of a product of the functions defined in §9.1. More precisely, let $\mathfrak{L}_{\mathcal{E}} \subseteq \mathfrak{L}$ be the subgroup generated by the classes of functions in the set

$$\{\ell_1, \ell_c\} \cup \{\ell_C : \mathcal{C} \subseteq \Gamma \text{ is a cycle}\} \cup \{\ell_D : [D] \in \operatorname{Jac}(B)[2]\}\,,$$

and define $\mathfrak{L}_{c,\mathcal{E}} = \mathfrak{L}_c \cap \mathfrak{L}_{\mathcal{E}}$.

***Remark 9.2.*** — For any function $\ell \in L^\times$ representing a class in $\mathfrak{L}_{\mathcal{E}}$, we have that $\operatorname{div}(\operatorname{Norm}_{L/K}(\ell)) \equiv n\operatorname{div}(c) \bmod 2\operatorname{Div}(W)$, for some $n \in \{0,1\}$. When $S$ is rational (i.e., $W = \mathbb{P}^1$), this implies that $\mathfrak{L}_{c,\mathcal{E}} = \mathfrak{L}_{\mathcal{E}}$.

***Theorem 9.3.*** — *If $S$ is geometrically ruled, then there is an exact sequence*

$$\frac{\operatorname{Pic} C}{2\operatorname{Pic} C} \xrightarrow{x-\alpha} \mathfrak{L}_{c,\mathcal{E}} \xrightarrow{\gamma} \operatorname{Br}(X \setminus \mathcal{E})[2] \longrightarrow 0\,.$$

***Remark 9.4.*** — Suppose $\pi\colon X \to S$ is defined over a subfield $k_0$ of $k$. Then all abelian groups in Theorem 9.3 have an action of $\operatorname{Gal}(k/k_0)$ and the maps in the exact sequence are morphisms of Galois modules. The same statement holds for the corollaries below.

***Corollary 9.5.*** — *If $S$ is geometrically ruled and $B^0$ has at worst simple singularities, then there is an exact sequence*

$$\frac{\operatorname{Pic} C}{2\operatorname{Pic} C} \xrightarrow{x-\alpha} \mathfrak{L}_{c,\mathcal{E}} \xrightarrow{\gamma} \operatorname{Br} X[2] \longrightarrow 0\,.$$

*Proof.* — Apply Proposition 8.4. □

***Corollary 9.6***. — *Let $\pi' \colon X' \to \mathbb{P}^2$ be a double cover branched over a smooth irreducible curve $B'$. Then there is a short exact sequence*

$$0 \to \frac{\operatorname{Pic} X'}{\langle [H]\rangle + 2\operatorname{Pic} X'} \to \left(\frac{\operatorname{Pic} B'}{K_{B'}}\right)[2] \to \operatorname{Br} X'[2] \to 0\,.$$

*where $[H] \in \operatorname{Pic} X'$ is the pullback of the hyperplane class on $\mathbb{P}^2$ and $K_{B'}$ is the canonical divisor on B'. In particular, we have $\dim_{\mathbb{F}_2} \operatorname{Br} X'[2] = 2 + 2(2d-1)(d-1) - \operatorname{rk}\operatorname{Pic} X'$, where $2d = \deg(B')$.*

***Remark 9.7***. — If $B'$ has at worst simple singularities, then Theorem 9.3 still applies to give a presentation of $\operatorname{Br} X'[2]$; however, the presentation cannot solely be given in terms of a quotient of $(\operatorname{Pic} B'/K_{B'})\,[2]$.

*Proof of Corollary 9.6.* — The group $(\operatorname{Pic} B'/K_{B'})\,[2]$ consists of the 2-torsion classes in $\operatorname{Jac}(B')$ and the theta charactistics, and so it has $\mathbb{F}_2$-dimension $1 + 2g(B') = 1 + 2(2d-1)(d-1)$. Therefore, the second claim follows easily from the first.

Fix a point $P \in \mathbb{P}^2 \setminus B'$ such that no line through $P$ is tritangent to $B'$. Let $S := \operatorname{Bl}_P\mathbb{P}^2$, $X := \operatorname{Bl}_{\pi'^{-1}(P)}X'$, and let $B$ denote the strict transform of $B'$ in $S$, or equivalently, $X$. Projection away from $P$ gives a geometric ruling on $S$, thus we may apply the theorem. We may choose coordinates such that $\mathfrak{S}$ is the exceptional curve above $P$; then $c = 1$ and the two exceptional curves above $\pi'^{-1}(P)$ correspond to $\infty^+$ and $\infty^-$. Hence, by Proposition 4.7 and Theorem 9.3, we have a short exact sequence

$$0 \to \frac{\operatorname{Pic} C}{\langle [\infty^+], [\infty^-]\rangle + 2\operatorname{Pic} C} \xrightarrow{x-\alpha} \mathfrak{L}_{c,\mathcal{E}} \xrightarrow{\gamma} \operatorname{Br} X[2] \longrightarrow 0\,.$$

By adjunction, $K_{B'} = (2d-3)[l]|_{B'}$, and $[l]|_{B'} = [B_{w_1}]$ in $\operatorname{Pic}(B)$, where $[l] \in \operatorname{Pic}\mathbb{P}^2$ is the class of a line. (Note that $\pi'^*[l] = [H]$.) So the theta characteristics are in bijection with functions $\ell \in L^\times$, considered up to squares, such that $\operatorname{div}(\ell) = B_{w_1} + 2D$ for some $D \in \operatorname{Div}(B)$. This, together with our assumption on the point $P$, implies that $\left(\frac{\operatorname{Pic} B'}{K_{B'}}\right)[2]$ is isomorphic to $\mathfrak{L}_{c,\mathcal{E}}$. Furthermore, our assumptions on the point $P$ also implies that we have an isomorphism $\operatorname{Pic} X'/[H] \xrightarrow{\sim} \operatorname{Pic} C/\langle [\infty^+], [\infty^-]\rangle$ obtained by composing the pullback map with restriction to the generic fiber. This completes the proof. □

*Proof of Theorem 9.3.* — Let $\ell \in L^\times$ be such that $\gamma(\overline{\ell}) \in \operatorname{Br}(X \setminus \mathcal{E}) \subseteq \operatorname{Br} C$. Proposition 2.1 implies that $\overline{\ell} \in \mathfrak{L}_c$, and by Theorem 1.1 it suffices to show that $\overline{\ell} \in \mathfrak{L}_{\mathcal{E}}$. Since $\overline{\ell} \in \mathfrak{L}_c$, the divisor $\operatorname{div}(\operatorname{Norm}_{L/K}(\ell))$ is of the form $n\operatorname{div}(c) + 2D$ for some $n \in \{0,1\}$ and $D \in \operatorname{Div}(W)$. After possibly multiplying $\ell$ by some power of $\ell_c$, we may assume that $\operatorname{div}(\operatorname{Norm}_{L/K}(\ell)) \in 2\operatorname{Div}(W)$.

Consider the set of points $Z \subseteq B^{\mathrm{fl}}$ where $\ell$ has odd valuation. Using the criteria given in Proposition 8.1, we see that after possibly multiplying $\ell$ by an element of $K^\times$ and a power of $\ell_1$, we may assume that $Z \subseteq \nu^{-1}(\mathfrak{S} \cap B^{0,\mathrm{fl}}) \cup \nu^{-1}(B^{0,\mathrm{fl}}_{\mathrm{sing}})$ and that $\sum_{b'\mapsto b, b'\in B} v_{b'}(\ell)$ is even for all $b \in B^{0,\mathrm{fl}}$ away from $\mathfrak{S}$ and the vertical components of $B^0$. Since we have already assumed that $\operatorname{div}(\operatorname{Norm}_{L/K}(\ell)) \in 2\operatorname{Div}(W)$, we may further conclude that $Z \subseteq \nu^{-1}(B^0_{\mathrm{sing}})$.

Write $\{b_1, \dots, b_r\} = \nu(Z \cap \nu^{-1}(B^0_{\mathrm{sing}}))$. For each $b_i$ such that $\nu(b_i)$ is not on a vertical component, there must be an even number of points in $Z \cap \nu^{-1}(b_i)$. Moreover, since $\operatorname{div}(\operatorname{Norm}_{L/K}(\ell)) \in 2\operatorname{Div}(W)$, for each vertical component $F$ of $B$, there is an even number of indices $i$ such that $Z \cap \nu^{-1}(b_i)$ is odd and such that $\nu^{-1}(b_i) \cap F \neq \emptyset$. Therefore, since principal divisors have degree 0, there are an even number of points in $Z \cap \nu^{-1}(B^0_{\mathrm{sing}})$ that lie on a particular irreducible component of $B$. The combination of these facts implies that there is a cycle $\mathcal{C} \subseteq \Gamma$ such that $v_b(\ell) + v_b(\ell_{\mathcal{C}})$ is even for all $b \in B$. Thus, after replacing $\ell$ with $\ell\ell_{\mathcal{C}}$, we may assume that $\ell$ has even valuation at all points of $B$. Therefore $\ell = \ell_D$ for some divisor $D$ whose class in $\mathrm{Jac}(B)$ is 2-torsion, and so $\ell \in \mathfrak{L}_{\mathcal{E}}$. $\square$

***Remark 9.8***. — The proof of Theorem 9.3 also has consequences when $k$ is not necessarily separably closed: given an algebra $\gamma'(\ell) \in \operatorname{Br} \mathbf{k}(X)$, there exists a separable extension $k'/k$ such that $\gamma'(\ell)$ is unramified over $k'$ (i.e., represents a class in $\operatorname{Br} X_{k'}$) if and only if the class of $\ell$ in $\overline{L}^\times/\overline{K}^\times\overline{L}^{\times 2}$ lies in the subgroup $\mathfrak{L}_{c,\mathcal{E}} \subseteq \overline{L}^\times/\overline{K}^\times\overline{L}^{\times 2}$.

## 10. The dimension of $\operatorname{Br} X[2]$

***Theorem 10.1***. — *Assume that $S$ is geometrically ruled and that $B^0$ is connected with at worst simple singularities. Then*

$$\dim_{\mathbb{F}_2} \operatorname{Br} X[2] \leq 4 + 2g(B^0) - 2g(W) - \operatorname{rank} \operatorname{NS} X\,, \tag{10.1}$$

*with equality if and only if $\mathfrak{L}_{\mathcal{E}} = \mathfrak{L}_{c,\mathcal{E}}$.*

***Corollary 10.2***. — *Let $\varpi\colon S \to \mathbb{P}^1$ be a rational geometrically ruled surface with invariant $e \geq 0$, and let $Z$ be a section of $\varpi$ with self-intersection $-e$. Suppose that $B^0$ is a connected curve of type $(a,b) \in \operatorname{Pic}(S) \simeq \mathbb{Z}S_{w_1} \times \mathbb{Z}Z$ with at worst simple singularities. Then*

$$\dim_{\mathbb{F}_2} \operatorname{Br} X[2] = 2(a-1)(b-1) - a(a-1)e + 4 - \operatorname{rank}(\operatorname{NS} X)\,.$$

*Proof.* — If $S$ is rational then, as noted in Remark 9.2, $\mathfrak{L}_{\mathcal{E}} = \mathfrak{L}_{c,\mathcal{E}}$, so the upper bound in the theorem is sharp. The formula now follows from the adjunction formula, which gives $g(B^0) = (a-1)(b-1) - a(a-1)e/2$. $\square$

***Example 10.3***. — *If $e = 0$ and $(a,b) = (4,4)$, then $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $X$ is a K3 surface. We recover the well known fact that $\dim_{\mathbb{F}_2} \operatorname{Br} X[2] = 22 - \operatorname{rank}(\operatorname{NS} X)$. Note that this argument does not require knowing that $b_2(X) = 22$ or that $\mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}} = \{1\}$.*

*Proof of Theorem 10.1.* — Corollary 9.5 readily yields,

$$\dim_{\mathbb{F}_2} \operatorname{Br} X[2] = \dim_{\mathbb{F}_2} \mathfrak{L}_{c,\mathcal{E}} - \dim_{\mathbb{F}_2} \operatorname{im}(x-\alpha) \leq \dim_{\mathbb{F}_2} \mathfrak{L}_{\mathcal{E}} - \dim_{\mathbb{F}_2} \operatorname{im}(x-\alpha)\,.$$

We will show that $\dim_{\mathbb{F}_2} \mathfrak{L}_{\mathcal{E}} - \dim_{\mathbb{F}_2} \operatorname{im}(x-\alpha) = 4 + 2g(B^0) - 2g(W) - \operatorname{rank} \operatorname{NS} X$. As we will make a similar argument later under the weaker assumption that $S$ is ruled, but not necessarily geometrically ruled, we will take care to point out when the geometrically ruled hypothesis is used; it will not come in until the end of the proof.

Noting that $h^0(B^{\mathrm{fl}})$ is the number of $G_K$ orbits in $\Omega$, Proposition 4.7 states that

$$\dim_{\mathbb{F}_2} \operatorname{im}(x-\alpha) = \operatorname{rank}(\operatorname{Pic} C) + h^0(B^{\mathrm{fl}}) - 2 + \dim_{\mathbb{F}_2}\left(\frac{K^\times \cap L^{\times 2}}{K^{\times 2}}\right) - \begin{cases} 1 & \text{if } c \in K^{\times 2} \\ 0 & \text{if } c \notin K^{\times 2} \end{cases}.$$

It therefore suffices to show that

$$\begin{aligned}\dim_{\mathbb{F}_2} \mathfrak{L}_{\mathcal{E}} = {} & 2g(B^0) - 2g(W) + 2 - \operatorname{rank} \operatorname{NS} X \\ & + \operatorname{rank}(\operatorname{Pic} C) + h^0(B^{\mathrm{fl}}) + \dim_{\mathbb{F}_2}\left(\frac{K^\times \cap L^{\times 2}}{K^{\times 2}}\right) - \begin{cases} 1 & \text{if } c \in K^{\times 2} \\ 0 & \text{if } c \notin K^{\times 2} \end{cases}.\end{aligned}$$

Recall that $\mathfrak{L}_{\mathcal{E}}$ is generated by classes of functions in the set

$$\{\ell_1, \ell_c\} \cup \{\ell_C : \mathcal{C} \subseteq \Gamma \text{ is a cycle}\} \cup \{\ell_D : [D] \in \mathrm{Jac}(B)[2]\}.$$

Clearly these generate a subspace of $L^\times/L^{\times 2}$ of dimension at most $\dim_{\mathbb{F}_2} \mathrm{Jac}(B)[2] + b_1(\Gamma) + 2 = 2g(B) + 2h^0(B) + b_1(\Gamma)$. For this counting argument, we may assume the point $w_1 \in W$ used to construct $\ell_1$ is such that $e(b/w_1)$ is odd for some $b \in B^0_{w_1} \setminus B^0_{\mathrm{sing}}$ and that $S_{w_1}$ is smooth and not contained in $B^0$. Then, by considering the parity of $v_b(\ell)$ for all $b \in B$, it is clear that the functions $\ell_1$, $\ell_{\mathcal{C}}$ and $\ell_D$ generate a subspace, modulo squares, of $\mathbb{F}_2$-dimension *equal* to $-1 + 2g(B) + 2h^0(B) + b_1(\Gamma)$. Furthermore, also by considering the parity of $v_b(\ell)$ for all $b \in B$, we see that $\ell_c$ is in this subspace if and only if $c \in K^{\times 2}$. Therefore, the dimension of the subspace of $L^\times/L^{\times 2}$ of elements of this form is

$$2g(B) + 2h^0(B) + b_1(\Gamma) - \begin{cases} 1 & \text{if } c \in K^{\times 2} \\ 0 & \text{if } c \notin K^{\times 2} \end{cases}.$$

Now we must determine which functions of the form $\ell_1^{n_1} \ell_c^{n_c} \ell_{\mathcal{C}} \ell_D$ are in $K^\times L^{\times 2}$. Since $K^\times L^{\times 2} \subseteq \mathfrak{L}_1$, and $\mathrm{div}(\mathrm{Norm}_{L/K}(\ell_c)) \in 2\,\mathrm{Div}(W)$ if and only if $\overline{\ell}_c \in \mathrm{Span}(\overline{\ell}_1, \overline{\ell}_{\mathcal{C}}, \overline{\ell}_D)$, it suffices to consider when $\ell_1^{n_1} \ell_{\mathcal{C}} \ell_D$ is in $K^\times L^{\times 2}$. We claim that modulo $L^{\times 2}$ the subspace of $L^\times/L^{\times 2}$ generated by such functions has $\mathbb{F}_2$-dimension equal to:

$$2g(W) + \#\{w \in W : 2|e(b/w)\ \forall b \in B^{0,\mathrm{fl}}_w \text{ or } S_w \subseteq B^0\} - \dim_{\mathbb{F}_2}\left(\frac{K^\times \cap L^{\times 2}}{K^{\times 2}}\right). \tag{10.2}$$

First note that (10.2) is always non-negative. Indeed, if an element $a \in K^\times \setminus K^{\times 2}$ is equal to $\ell^2$ for some $\ell \in L$, then $\mathrm{div}_B(a) \in 2\,\mathrm{Div}(B)$ and so $e(b/w)$ is even for all $w$ in the support of $a$ and all $b \in B^{0,\mathrm{fl}}_w$.

Let $\ell := \ell_1^{n_1} \ell_{\mathcal{C}} \ell_D$ for some integer $n_1$, cycle $\mathcal{C} \subseteq \Gamma$, and divisor $D \in \mathrm{Div}(B)$ whose class is 2-torsion. Then, by construction, $v_b(\ell)$ is even for all points $b \in B^{0,\mathrm{fl}}$ away from $w_1$ and the vertical fibers. Moreover $v_b(\ell)$ is odd for some point $b \in B^0_{w_1}$ only if $n_1 = 1$. From this description, it is clear that if $\ell = a \in K^\times$, then for all $w \in W \setminus \{w_1\}$ such that $S_w \not\subseteq B^0$ either $v_w(a)$ is even or $e(b/w)$ is even for all $b \in B^0_w$. Therefore,

$$\mathrm{div}(a) = n_{w_1} w_1 + \sum_{\substack{w \in W \\ 2|e(b/w)\ \forall b \in B^0_w \\ \text{or } S_w \subseteq B^0}} n_w w + 2D,$$

for some $D \in \mathrm{Div}(W)$ and integers $n_w \in \{0, 1\}$. Furthermore, such functions $a$, modulo squares, are in one-to-one correspondence with elements of

$$\mathrm{Jac}(W)[2] \times \{w \in W : 2|e(b/w)\ \forall b \in B^0_w, \text{ or } S_w \subseteq B^0\},$$

and every such function is of the form $\ell_1^{n_1} \ell_{\mathcal{C}} \ell_D$, for some $n_1, \mathcal{C}, D$. This proves that the kernel has dimension at most $2g(W) + \#\{w \in W : 2|e(b/w)\ \forall b \in B^0_w, \text{ or } S_w \subseteq B^0\}$. To complete the proof of the claim, we note that such a function $a$ does not contribute to the kernel if and only if $a = \ell^2$.

Thus far we have shown that

$$\begin{aligned}\dim_{\mathbb{F}_2}(\mathfrak{L}_{\mathcal{E}}) =& 2g(B) + 2h^0(B) + b_1(\Gamma) - 2g(W) + \dim_{\mathbb{F}_2}((K^\times \cap L^{\times 2})/K^{\times 2}) \\ & - \#\{w \in W : 2|e(b/w)\ \forall b \in B^{0,\mathrm{fl}}_w \text{ or } S_w \subseteq B^0\} - \begin{cases} 1 & \text{if } c \in K^{\times 2} \\ 0 & \text{if } c \notin K^{\times 2} \end{cases}.\end{aligned}$$

It remains to show that this expression simplifies to the desired form.

Let $a_n, d_n, e_6, e_7, e_8 \in \mathbb{Z}$ denote the number of $A_n, D_n, E_6, E_7$, and $E_8$ singularities on $B^0$ respectively (for definitions see [BHPVdV04, §II.8]). Recall that the $\delta$-invariant of a singular point $P$ is the difference between the genus of the singular curve and the genus of the curve obtained by resolving the singularity at $P$. It can be computed using the Milnor number and the number of branches of the singularity [Mil68, Thm. 10.5]. Since

$$\delta(A_n) = \left\lfloor \frac{n+1}{2} \right\rfloor, \ \delta(D_n) = \left\lfloor \frac{n+2}{2} \right\rfloor, \ \delta(E_6) = 3, \ \delta(E_7) = \delta(E_8) = 4,$$

the genus of $B$ equals

$$g(B^0) - \sum_n \left( a_n \left\lfloor \frac{n+1}{2} \right\rfloor + d_n \left\lfloor \frac{n+2}{2} \right\rfloor \right) - 3e_6 - 4(e_7 + e_8).$$

Furthermore, singularities of type $A_{2k+1}, D_{2k+1}$ or $E_7$ each contribute exactly one edge to $\Gamma$, and singularities of type $D_{2k}$ each contribute two edges to $\Gamma$ [BHPVdV04, Table 1, p.109]. Moreover, $\Gamma$ has $h^0(B)$ vertices and, since $B^0$ is connected, $\Gamma$ has 1 connected component. Therefore, $b_1(\Gamma) = \sum_k (a_{2k+1} + d_{2k+1} + 2d_{2k}) + e_7 + 1 - h^0(B)$. Combining these facts, we have

$$2g(B) + h^0(B) + b_1(\Gamma) = 2g(B^0) - \sum_n n(a_n + d_n) - 6e_6 - 7e_7 - 8e_8 + 1.$$

Since $S$ is geometrically ruled, $\mathcal{E}$ consists *only* of exceptional curves obtained by blowing up singularities of $B^0$, and so $\#\mathcal{E} = \sum_n n(a_n + d_n) + 6e_6 + 7e_7 + 8e_8$. Therefore, we have

$$\begin{aligned} \dim_{\mathbb{F}_2}(\mathfrak{L}_{\mathcal{E}}) =& 2g(B^0) + h^0(B) + 1 - \#\mathcal{E} - 2g(W) + \dim_{\mathbb{F}_2}((K^\times \cap L^{\times 2})/K^{\times 2}) \\ & - \#\{w \in W : 2|e(b/w) \ \forall b \in B_w^{0,\mathrm{fl}} \text{ or } X_w^0 \text{ non-reduced}\} - \begin{cases} 1 & \text{if } c \in K^{\times 2} \\ 0 & \text{if } c \notin K^{\times 2} \end{cases}. \end{aligned}$$

Recall that there is a surjective homomorphism from $\operatorname{Pic} X \to \operatorname{Pic} C$. By the Lang-Néron theorem [LN59], $\operatorname{Pic} C$ is finitely generated, so $\operatorname{Pic}^0 X$ is contained in the kernel of the map $\operatorname{Pic} X \to \operatorname{Pic} C$. Therefore, we have a surjective homomorphism $\operatorname{NS} X \to \operatorname{Pic} C$. Since $S$ is geometrically ruled, we have

$$\operatorname{rank} \ker (\operatorname{NS} X \to \operatorname{Pic} C) = \#\mathcal{E} + 1 + \#\{w \in W : 2|e(b/w) \ \forall b \in B_w^{0,\mathrm{fl}}, \ S_w \not\subseteq B^0\}.$$

Finally, to complete the proof, we note that $h^0(B) = h^0(B^{\mathrm{fl}}) + \#\{w \in W : S_w \subseteq B^0\}$. □

## 11. The Brauer group of an Enriques surface

An Enriques surface is a smooth projective minimal surface $E$ with nontrivial 2-torsion canonical divisor and with irregularity $h^1(\mathcal{O}_E) = 0$. Equivalently, an Enriques surface is a quotient of a K3 surface by a fixed point free involution. The Brauer group of any Enriques surface (over a separably closed field) is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [HS05, p. 3223]. In this section we give a complete description of $\operatorname{Br} E$.

We shall see below that every Enriques surface is birational to a double cover of a ruled surface whose branch locus has at worst simple singularities. This implies that every Enriques surface is birational to a double cover of a *geometrically* ruled surface; however, the branch locus of this double cover may have worse singularities. We will find it more convenient to adapt the methods of the previous sections to ruled surfaces which fail to be geometrically ruled.

**11.1. Horikawa's representation of Enriques surfaces.** — Let $E$ be an Enriques surface and let $\tilde{E}$ be its K3 double cover. Horikawa's representation of Enriques surfaces [BHPVdV04, Chap VIII, Props. 18.1, 18.2] shows that $\tilde{E}$ is the minimal resolution of a double cover of a quadric surface $\tilde{S}^0 \subseteq \mathbb{P}^3$ branched over a reduced curve $\tilde{B}^0$, which has at worst simple singularities, and which is obtained by intersecting a quartic hypersurface with $\tilde{S}^0$. Furthermore, the covering involution $\sigma\colon \tilde{E} \to \tilde{E}$ for the quotient $\tilde{E} \to E$ descends to the involution

$$\tau\colon \mathbb{P}^3 \to \mathbb{P}^3\,, \quad (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : -z_1 : -z_2 : z_3)\,.$$

Therefore, $\tilde{B}^0$ is invariant and fixed point free under the action of $\tau$. Under the Horikawa representation, we may take $\tilde{S}^0$ to be the quadric cone $V(z_0z_3 - z_1^2)$ if $E$ is special and $\tilde{S}^0 = V(z_0z_3 - z_1z_2)$ otherwise. (An Enriques surface is special if it is endowed with the structure of an elliptic pencil together with a $(-2)$-curve which is a 2-section, and nonspecial otherwise.) If $\tilde{S}^0$ is non-singular, then the morphism

$$\tilde{S}^0 \to \mathbb{P}^1_{\tilde{t}}\,, \quad \vec{z} \mapsto (z_1 : z_0) = (z_3 : z_2)$$

shows that $\tilde{S} := \tilde{S}^0$ is a rational geometrically ruled surface. If $\tilde{S}^0$ is the quadric cone, then the rational map

$$\tilde{S}^0 \dashrightarrow \mathbb{P}^1_{\tilde{t}}\,, \quad \vec{z} \mapsto (z_1 : z_0) = (z_3 : z_1)\,.$$

shows that the blow up $\tilde{S} := \mathrm{Bl}_{(0:0:1:0)}(\tilde{S}^0)$ is a rational geometrically ruled surface. In either case, $\tilde{E}$ is birational to the double cover $\tilde{X}^0$ of $\tilde{S}$ branched over $\tilde{B}^0$ (where we abuse notation using $\tilde{B}^0$ to denote its strict transform in $\tilde{S}$).

We may embed $\tilde{S}/\tau$ in $\mathbb{P}^4$ as the vanishing of $V(w_0w_3 - w_4^2, w_1w_2 - w_4^2)$. Under this embedding, the morphism $\tilde{S} \to \tilde{S}/\tau$ is given by $(z_0 : z_1 : z_2 : z_3) \mapsto (z_0^2 : z_1^2 : z_2^2 : z_3^2 : z_0z_3)$. The ruling on $\tilde{S}$ induces a map $\varpi\colon \tilde{S}/\tau \to \mathbb{P}^1_t, \mathbf{w} \mapsto (w_1 : w_0) = (w_3 : w_2)$ giving $\tilde{S}/\tau$ the structure of a ruled surface. Note that the coordinates $\tilde{t}$ and $t$ on the two copies of $\mathbb{P}^1$ are related by $\tilde{t} = \sqrt{t}$.

Let $S := \mathrm{Bl}_{\mathrm{Sing}(\tilde{S}/\tau)}(\tilde{S}/\tau)$. Then $E$ is birational to the double cover $X^0$ of $S$ branched over the exceptional divisors on $S$ and the strict transform $B^0$ of $\tilde{B}^0/\tau$. We let $X$ and $\tilde{X}$ be the desingularizations of $X^0$ and $\tilde{X}^0$ obtained by canonical resolutions. Observe that, in agreement with the convention set in §7.1, $B^0$ contains all connected components of the branch locus that map dominantly to $\mathbb{P}^1$. We may thus avail ourselves of the notation and results established in the previous sections for both $X/S$ and $\tilde{X}/\tilde{S}$, using tildes to denote objects corresponding to $\tilde{X}$.

***Remark 11.1***. — As a caution, we note that $S$ is not geometrically ruled; the fibers above 0 and $\infty$ consist of a chain of three arithmetic genus 0 curves with the center curve appearing with multiplicity 2.

**11.2. The Brauer group.** —

***Theorem 11.2***. — *For every $[D] \in \mathrm{Jac}(B^{\mathrm{fl}})[2]$ and cycle $\mathcal{C}$ on the dual graph of $B^0$, the algebra $\gamma'(\ell_{\mathcal{C}}\ell_D)$ lies in* $\mathrm{Br}\, X$. *Moreover, these elements generate* $\mathrm{Br}\, X$.

*Proof.* — Fix a cycle $\mathcal{C}$ and a divisor $D$ on $B^{\mathrm{fl}}$ whose divisor class is 2-torsion. We claim that $\partial_F(\gamma'(\ell_{\mathcal{C}}\ell_D)) \in \kappa(F)^{\times 2}$ for all reduced and irreducible curves $F \subseteq X$, and thus that $\gamma(\ell_{\mathcal{C}}\ell_D) \in \mathrm{Br}\, X$. If $F$ is a horizontal curve then this follows from Proposition 2.1, since

$\overline{\ell_{\mathcal{C}}\ell_D} \in \mathfrak{L}_c$. Now assume that $F$ is a vertical curve. If $F$ does not map dominantly to the reduced part of a component of $S_0$ or $S_\infty$, then the preimage of $F$ in $\tilde{X}$ consists of exactly two curves $F_1, F_2$, each of which are isomorphic to $F$. We will show that $\partial_{F_i}(\mathrm{Res}(\gamma'(\ell_{\mathcal{C}}\ell_D))) \in \kappa(F_i)^{\times 2}$, and thus conclude that $\partial_F(\gamma'(\ell_{\mathcal{C}}\ell_D)) \in \kappa(F)^{\times 2}$. Since $\mathbf{k}(\tilde{B}) = \mathbf{k}(\tilde{B}/\tau) \otimes_{k(t)} k(\sqrt{t})$, we have

$$\mathrm{Res}(\gamma'(\ell_{\mathcal{C}}\ell_D)) = \mathrm{Cor}_{\mathbf{k}(\tilde{X}_{\tilde{B}})/\mathbf{k}(\tilde{X})}\left(\mathrm{Res}\left((x-\alpha, \ell_{\mathcal{C}}\ell_D)_2\right)\right).$$

We may choose our coordinates $x$ and $\alpha$ on the Enriques surface so that $\mathrm{Res}(x) = \tilde{x}/\tilde{t}$ and $\mathrm{Res}(\alpha) = \tilde{\alpha}/\tilde{t}$, where $\tilde{x}$ and $\tilde{\alpha}$ are the functions on $\tilde{E}$ and $\tilde{B}$. Therefore

$$\mathrm{Cor}_{\mathbf{k}(\tilde{E}_{\tilde{B}})/\mathbf{k}(\tilde{E})}\left(\mathrm{Res}\left((x-\alpha, \ell_{\mathcal{C}}\ell_D)_2\right)\right) = \tilde{\gamma}'(\mathrm{Res}(\ell_{\mathcal{C}}\ell_D)) + (\tilde{t}, \mathrm{Cor}(\ell_{\mathcal{C}}\ell_D))_2.$$

Since $\ell_{\mathcal{C}}\ell_D \in \mathfrak{L}_1$, the algebra $(\tilde{t}, \mathrm{Cor}(\ell_{\mathcal{C}}\ell_D))_2$ is trivial in the Brauer group. Furthermore, by construction $\mathrm{Res}(\ell_{\mathcal{C}}\ell_D) = \ell_{\tilde{\mathcal{C}}}\ell_{\tilde{D}}$ for some cycle $\tilde{\mathcal{C}}$ on the dual graph of $\tilde{B}$ and some two-torsion divisor $\tilde{D}$ on $\tilde{B}$. Hence, by applying Corollary 9.5 to $\tilde{X}/\tilde{S}$ it follows that $\partial_{F_i}(\gamma'(\ell_{\mathcal{C}}\ell_D)) \in \kappa(F_i)^{\times 2}$

Now assume that $F$ maps dominantly to the reduced part of a component of $S_0$. If $F$ maps dominantly to an exceptional divisor of $S$, then $x - \alpha$ has trivial valuation and reduces to a constant on all curves $F'$ that lie above $F$ in the desingularization of $X \times_{\mathbb{P}^1} B$. Therefore, $\partial_F(\gamma'(\ell_{\mathcal{C}}\ell_D)) \in \kappa(F)^{\times 2}$. Now consider the case when $F$ maps dominantly to the reduced part of $(\tilde{S}/\tau)_0$. If the singular locus of $B^0$ is supported away from the fibers of 0, then, after adjusting $D$ by a principal divisor, we may assume that $v_b(\ell_{\mathcal{C}}\ell_D) = 0$ for all $b \in B \cap (\varpi^{-1}(0))$. Then $\partial_F(\gamma'(\ell_{\mathcal{C}}\ell_D))$ is a constant and so it is clear that $\partial_F(\gamma'(\ell_{\mathcal{C}}\ell_D)) \in \kappa(F)^{\times 2}$. If there is a singularity of $B^0$ lying over 0, then $F$ must be rational. Since there are no nontrivial étale covers of a rational curve, the preimage of $F$ in $\tilde{X}$ consists of exactly two curves $F_1, F_2$, each of which are isomorphic to $F$, and we may apply the same argument used above. The case where $F$ maps dominantly to a component of $(S_\infty)_{\mathrm{red}}$ follows similarly.

We have shown that the subspace of $\mathfrak{L}_c$ generated by the $\ell_{\mathcal{C}}$ and the $\ell_D$ maps into $\mathrm{Br}\, X$; now we will use a cardinality argument to show that the image of this subspace is all of $\mathrm{Br}\, X$. Arguing as in the proof of Theorem 10.1 we see that the functions $\ell_{\mathcal{C}}$ and $\ell_D$ generate a subspace of $\mathfrak{L}' \subseteq \mathfrak{L}$ of $\mathbb{F}_2$-dimension

$$\begin{aligned}\dim_{\mathbb{F}_2} \mathfrak{L}' = {} & 2g(B) + 2h^0(B) + b_1(\Gamma) - 1 + \dim_{\mathbb{F}_2}\left(\frac{K^\times \cap L^{\times 2}}{K^{\times 2}}\right) \\ & - \#\{w \in \mathbb{P}^1 : 2|e(b/w)\ \forall b \in B_w^{0,\mathrm{fl}} \text{ or } S_w \subseteq B^0\}.\end{aligned}$$

Using that $h^0(B) = h^0(B^{\mathrm{fl}}) + \#\{w \in \mathbb{P}^1 : S_w \subseteq B^0\}$, we get

$$\begin{aligned}\dim_{\mathbb{F}_2} \mathfrak{L}' = {} & 2g(B) + h^0(B) + b_1(\Gamma) - 1 + h^0(B^{\mathrm{fl}}) + \dim_{\mathbb{F}_2}\left(\frac{K^\times \cap L^{\times 2}}{K^{\times 2}}\right) \\ & - \#\{w \in \mathbb{P}^1 : 2|e(b/w)\ \forall b \in B_w^{0,\mathrm{fl}} \text{ and } S_w \not\subseteq B^0\}.\end{aligned}$$

Since $\tilde{B}^0$ is connected and has at most simple singularities, the same is true for $B^0$. Therefore, the same argument as in the proof of Theorem 10.1 shows that $2g(B) + h^0(B) + b_1(\Gamma) - 1 = 2g(B^0) - \#\mathcal{E}'$, where $\mathcal{E}'$ is the set of exceptional curves obtained by blowing up the singularities of $B^0$. (We have $\#(\mathcal{E} \setminus \mathcal{E}') = 4$ corresponding to the extra irreducible components of $S_0$ and $S_w$ – see Remark 7.1.) Therefore

$$\begin{aligned}\dim_{\mathbb{F}_2} \mathfrak{L}' = 2g(B^0) + h^0(B^{\mathrm{fl}}) + \dim_{\mathbb{F}_2}\left(\frac{K^\times \cap L^{\times 2}}{K^{\times 2}}\right) \\ - \mathcal{E}' - \#\{w \in \mathbb{P}^1 : 2|e(b/w)\ \forall b \in B_w^{0,\mathrm{fl}} \text{ and } S_w \not\subseteq B^0\}\,.\end{aligned}$$

Now noting that the rank of $\mathrm{NS}\,X$ is

$$\#\mathcal{E}' + 4 + 1 + \#\{w \in \mathbb{P}^1 \setminus \{0,\infty\} : 2|e(b/w)\ \forall b \in B_w^{0,\mathrm{fl}} \text{ and } S_w \not\subseteq B^0\} + \operatorname{rank}(\operatorname{Pic} C)\,,$$

we conclude that

$$\begin{aligned}\dim \mathfrak{L}' = 2g(B^0) + h^0(B^{\mathrm{fl}}) + \dim_{\mathbb{F}_2}\left(\frac{K^\times \cap L^{\times 2}}{K^{\times 2}}\right) + 5 + \operatorname{rank}(\operatorname{Pic} C) \\ - \operatorname{rank} \mathrm{NS}\,X - \Delta_0 - \Delta_\infty,\end{aligned}$$

where $\Delta_w$ equals 1 if $e(b/w)$ is even for all $b \in B_w^{\mathrm{fl}}$ and $(S_w)_{\mathrm{red}} \not\subseteq B^0$ and 0 otherwise. Since $X_0$ and $X_\infty$ are not reduced and $\tilde{B}^0$ did not contain the fixed points of $\tau$, both $\Delta_0$ and $\Delta_\infty$ are 1. In addition, $\operatorname{rank}(\mathrm{NS}\,X) = \operatorname{rank}(\mathrm{NS}\,E) + 4 = 10 + 4$, and since $B^0$ is the quotient of a genus 9 curve by a fixed-point free involution, $g(B^0) = 5$. Hence, rearranging and using Proposition 4.7 we obtain,

$$\begin{aligned}\dim \mathfrak{L}' &= h^0(B^{\mathrm{fl}}) - 1 + \dim_{\mathbb{F}_2}\left(\frac{K^\times \cap L^{\times 2}}{K^{\times 2}}\right) + \operatorname{rank}(\operatorname{Pic} C) \\ &\geq \dim_{\mathbb{F}_2} \operatorname{im}(x - \alpha) + 1\,.\end{aligned}$$

Therefore, some element of $\mathfrak{L}'$ has nontrivial image in $\operatorname{Br} \mathbf{k}(X)$ and so must be equal to the unique nontrivial element of $\operatorname{Br} X$. $\square$

***Remark 11.3***. — If $B^0$ is smooth, it is possible to prove Theorem 11.2 without using that the Néron-Severi group of an Enriques surface has rank 10 and that the Brauer group of an Enriques surface is $\mathbb{Z}/2\mathbb{Z}$. One can instead prove that all other functions in $\mathfrak{L}_{c,\mathcal{E}}$ are ramified along some vertical divisor. However, this proof is more complicated as it requires a detailed study of the desingularization (or at least the normalization) of the fiber product $X \times_{\mathbb{P}^1} B^{\mathrm{fl}}$.

## 12. An Enriques surface failing weak approximation

We demonstrate the the previous results are amenable to explicit computation by exhibiting an Enriques surface with a transcendental Brauer-Manin obstruction to weak approximation. We note that it was already known that Enriques surfaces need not satisfy weak approximation due to work of Harari and Skorobogatov [HS05], who constructed an Enriques surface whose étale-Brauer set was strictly smaller than its Brauer set.

**12.1. Construction of the Enriques surface.** — Let $\tilde{S}$ denote the quadric surface $V(z_0z_3-z_1z_2) \subseteq \mathbb{P}^3$ and let $\tilde{B}^0 \subseteq \tilde{S}$ be the (reducible) quartic curve given by the vanishing of

$$F(\mathbf{z}) := \left(z_3^2 - 3z_2^2 - 3z_1^2 - 2z_0^2 + 3z_3z_0\right)^2 - \left(z_3z_1 - 2z_3z_2 + 4z_0z_2 + z_0z_1\right)^2.$$

We let $\tilde{E}$ denote the minimal resolution of the double cover

$$\tilde{E}^0 := V(y^2 - F(\mathbf{z}), z_0z_3 - z_1z_2) \subseteq \mathbb{P}(1,1,1,1,2);$$

note that $\tilde{E}$ is a K3 surface defined over $\mathbb{Q}$. There is a fixed point free involution $\sigma^0 \colon \tilde{E}^0 \to \tilde{E}^0$, $\quad (z_0 : z_1 : z_2 : z_3 : y) \mapsto (z_0 : -z_1 : -z_2 : z_3 : -y)$ that can be lifted to a fixed point free involution $\sigma$ on $\tilde{E}$.

***Proposition 12.1***. — *Let $E$ denote the Enriques surface $\tilde{E}/\sigma$. There exists a number field $k$ such that*

$$E(\mathbb{A}_k)^{\operatorname{Br} E_k} \subsetneq E(\mathbb{A}_k)^{\operatorname{Br}_1 E_k}.$$

***Remark 12.2***. — The surface $X$ from the previous section is a blow-up of the Enriques surface $E$. Since the Brauer group is a birational invariant of smooth projective surfaces, $\operatorname{Br} \overline{E} = \operatorname{Br} \overline{X}$. We will use this equality throughout this section.

**12.2. Coordinates on the generic fiber.** — The map $\mathbb{P}^4 \to \mathbb{P}^1, \mathbf{w} \mapsto (w_1 : w_0)$, induces a morphism $\varpi \colon \tilde{S}/\tau \to \mathbb{P}^1$ whose generic fiber is isomorphic to $\mathbb{P}^1_{\mathbb{Q}(t)}$. The generic fiber of $E \to \tilde{S}/\tau \to \mathbb{P}^1$ has a model of the form $v^2 = cf(x)$, where

$$f(x) = x^4 + \frac{10t-2}{t-10+9/t}x^3 + \frac{-7t^2+19t-4}{t-10+9/t}x^2 + \frac{-20t^2-20t}{t-10+9/t}x + \frac{9t^3+11t^2+4t}{t-10+9/t}$$

and $c = t - 10 + 9/t$. The isomorphism between this model and $E$ identifies $t$ with $w_3/w_2 = w_1/w_0 = w_4^2/(w_0w_2)$ and identifies $x$ with $w_4/w_0 = w_3/w_4 = tw_2/w_4$.

**12.3. Geometry of the branch curve.** — Let $E^0 := \tilde{E}^0/\sigma^0$; he morphism $E^0 \to \tilde{S}/\tau$ is branched over the four singular points of $\tilde{S}/\tau$ and the irreducible curve $B^0 := \tilde{B}^0/\tau$. The singular locus of $B^0$ is a degree 5, 0-dimensional reduced subscheme. It consists of one $\mathbb{Q}$-point, $P_0 = (1:1:1:1:1)$, which corresponds to the singular point of an irreducible component of $B^0$, and a degree 4 point which is irreducible over $\mathbb{Q}$. The curve $B^0$ is embedded in $\mathbb{P}^4$ as the complete intersection of 3 quadrics so it has arithmetic genus 5. One can check that each singularity is an ordinary double point, thus $B^0$ is geometrically rational. A naive point search quickly finds smooth $\mathbb{Q}$-points, so $B^0$ is birational to $\mathbb{P}^1_{\mathbb{Q}}$ and $L \cong \mathbb{Q}(s)$. We fix the following isomorphism between $L$ and $\mathbb{Q}(s)$ :

$$\frac{w_1}{w_0} = \left(\frac{2s^2-16s+41}{-s^2-s+29}\right)^2, \quad \frac{w_2}{w_0} = \left(\frac{2s^2-7s+32}{-s^2+8s+20}\right)^2,$$

$w_4/w_0 = \sqrt{w_1w_2/w_0^2}$, and $w_3/w_0 = w_1w_2/w_0^2$. Using the above expression for $f(x)$, we see that this isomorphism sends

$$\alpha \mapsto \frac{4s^4 - 46s^3 + 258s^2 - 799s + 1312}{s^4 - 7s^3 - 57s^2 + 212s + 580}.$$

**12.4. Representing a transcendental Brauer class.** — Let $k_1$ denote the residue field of the singular degree 4 $\mathbb{Q}$-subscheme, and let $P_1$ denote a $k_1$-point of the singular locus different from $P_0$. (In fact, $k_1$ is an $S_4$ extension, so there is a unique such $P_1$.) We write $x_1 := w_4/w_0(P_1)$ and $t_1 := w_1/w_0(P_1)$. Let $\ell$ be the monic quadratic separable $k_1$-polynomial in $s$ whose zeros lie above $P_1$. By Theorem 11.2, $\gamma'(\ell) \otimes_k \overline{k}$ is contained in $\operatorname{Br} \overline{E}$. We claim that it represents the nontrivial element in $\operatorname{Br} \overline{E}$.

Using linear algebra and elimination ideals, we find equations for curves on $\tilde{S}$ which pass through an even number of the $\overline{\mathbb{Q}}$ points that are Galois conjugate to $P_1$, and meet $B^0$ with even multiplicity at every point of intersection. There are finitely many such curves $Z$, and one checks that for every set of singular points of $B^0$ containing an even number of the $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates of $P_1$, there is such a reduced and irreducible curve $Z$ such that $\pi^{-1}(Z)$ is reducible. By computing intersection numbers, one sees that these curves, the rulings on $\tilde{S}$, and the exceptional curves generate $\operatorname{Pic} \overline{E}$. Therefore, the above curves and the horizontal ruling on $S$ generates $\operatorname{Pic} \overline{C}_{\overline{\mathbb{Q}(t)}}$. By the construction of these curves, any function in the image of $x - \alpha$ that has odd valuation at both points in $\nu^{-1}(P_1)$ will have odd valuation at both points in $\nu^{-1}(P_i)$ for an odd number of points $P_i$ that are Galois conjugate to $P_1$. Therefore $\overline{\ell} \notin \operatorname{im}(x - \alpha)$, and so, by Theorem 11.2, $\gamma'(\ell) \otimes_k \overline{k}$ is nontrivial in $\operatorname{Br} \overline{E}$.

Now we will compute a number field $k$ such that $\gamma'(\ell) \otimes_k \overline{k} \in \operatorname{im}(\operatorname{Br} E_k \to \operatorname{Br} \overline{E})$. Since $\ell$ is defined over $k_1$, we have $\gamma'(\ell) \in \operatorname{Br} \mathbf{k}(E_{k_1})$. However, a direct computation shows that $\gamma'(\ell)$ is ramified at all of the $(-2)$-curves. There are two linearly independent quadratic extensions of $k_1$ over which the residues at the exceptional curve above $P_0$ and $P_1$ respectively become trivial. Then there is a degree 4 extension over the composite of these quadratic extensions over which the residue at the exceptional curve above the degree three singular point becomes trivial.(1) We let $k$ denote this degree 16 extension over $k_1$.

We claim that $\gamma'(\ell) \otimes_k \overline{k} \in \operatorname{im}(\operatorname{Br} E_k \to \operatorname{Br} \overline{E})$, or, more precisely, that the algebra

$$A := \gamma'(\ell) + (t - 1, \ell(10))_2 + (t - 9, \ell(-2))_2$$

lies in $\operatorname{Br} E_k$. The algebras $(t - 1, \ell(10))_2$ and $(t - 9, \ell(-2))_2$ are algebraic, so it is clear that the second claim implies the first. To prove that $A \in \operatorname{Br} E_k$, we must show that $\partial_D(A) \in \kappa(D)/\kappa(D)^{\times 2}$ for all prime divisors $D$ on $X$. From the definition of $A$ and $k$ and the proofs of Propositions 2.1 and 8.1, this is certainly true except possibly for the fibers above $t = 0, \infty$ and for $\mathfrak{S}_\infty$. One can directly check that $\operatorname{Norm}_{k_1(s)/k_1(t)}(\ell) \in k_1(t)^{\times 2}$, so by Proposition 2.1, $\gamma'(\ell)$, and therefore $A$, is unramified at $\mathfrak{S}_\infty$. To compute the residue at the fibers above $t = 0, \infty$, we note that $(\tilde{S} \times_{\mathbb{P}^1_{\sqrt{t}}} \tilde{B})/\tau$ is a smooth birational model of $\tilde{S}/\tau \times_{\mathbb{P}^1_t} B$. Then a direct computation using (8.3) shows that $\tilde{\gamma}'(\ell)$ and $\gamma'(\ell)$ are unramified at the fibers above $t = 0, \infty$. Thus $A \in \operatorname{Br} E_k \setminus \operatorname{Br}_1 E_k$.

**12.5. Computing the obstruction.** — In this section we will construct a $k$-adelic point on $E$ that is not orthogonal to $A$. First note that $\tilde{E}$, and therefore $E$, has $\mathbb{Q}$-rational points. Indeed, $(1 : 0 : 0 : 0 : \pm 2) \in \tilde{E}^0(\mathbb{Q})$, and since $\tilde{E} \to \tilde{E}^0$ is an isomorphism when $y \neq 0$, $Q := \psi(1 : 0 : 0 : 0 : \pm 2) \in E(\mathbb{Q})$ (here $\psi \colon \tilde{E} \to E$ denotes the quotient map). Hence, if we find a place $v$ of $k$ and a point $Q_v \in E(k_v)$ such that $\operatorname{inv}_v A(Q_v) \neq \operatorname{inv}_v A(Q)$,

(1) A `Magma` [BCP97] script verifying these claims and all other computational claims in this section can be found with the arXiv distribution of this article.

then the adelic point that is equal to $Q$ for all places $w \neq v$ and equal to $Q_v$ at $v$ is orthogonal to $A$.

We will take $v$ to be a place lying over 2. We note that 2 splits completely in $k_1$, and of these four places lying over 2, there is a unique place and a unique extension $v$ of that place to $k$ such that $k_v = \mathbb{Q}_2(i)$. Since $A \in \operatorname{Br} \mathbf{k}(E_{k_1})$ and $A$ is unramified on an open set of $E_{k_1}$ containing $Q$, $A(Q) \in \operatorname{Br} k_1$ and, consquently, $A(Q) \otimes_k k_v \in \operatorname{im}(\operatorname{Br} \mathbb{Q}_2 \to \operatorname{Br} \mathbb{Q}_2(i))$. Thus $\operatorname{inv}_v A(Q) = 0$. Let

$$Q_v := \psi(1 : -6 : 1 + i : -6 - i : 2\sqrt{4255 - 4160i}).$$

The point $Q_v$ lies over $t = 36$ and $B_t$ consists of 2 $k_v$-points $R_1$ and $R_2$ and one quadratic point $R$; they have $s$ values

$$\frac{19}{4}, \quad \frac{-7}{2}, \quad \text{and} \quad \frac{-11 + 3\sqrt{109}}{4}.$$

Since these points are unramified in $B \to \mathbb{P}^1$ and are away from the support of $\ell$ and $\alpha$, the cocycle description of $\gamma'(\ell)$ in Lemma 3.5 shows that

$$\begin{aligned} A(Q_v) &= (\ell(-2), 25)_2 + (\ell(-10), 35)_2 + \gamma'(\ell)(Q_v) \\ &= (\ell(10), 35)_2 + \operatorname{Cor}_{\mathbb{Q}_2(i,\sqrt{109})/\mathbb{Q}_2(i)} ((x(Q_v) - \alpha(R), \ell(R))_2) \\ &\qquad + \sum_{j=1}^{2} (x(Q_v) - \alpha(R_j), \ell(R_j))_2. \end{aligned}$$

A computation shows that $(x(Q_v) - \alpha(R_j), \ell(R_j))_2$ is trivial in $\operatorname{Br} \mathbb{Q}_2(i)$ for all $i$, that $(\ell(10), 35)_2$ is trivial in $\operatorname{Br} \mathbb{Q}_2(i)$, and that $\operatorname{Cor}_{\mathbb{Q}_2(i,\sqrt{109})/\mathbb{Q}_2(i)} ((x(Q_v) - \alpha(R), \ell(R))_2)$ is nontrivial in $\operatorname{Br} \mathbb{Q}_2(i)$. Therefore $\operatorname{inv}_v A(Q_v) = 1/2$.

**12.6. Determining the algebraic Brauer classes.** — It remains to prove that this failure of weak approximation is not accounted for by algebraic Brauer classes. In 12.4, we outlined how to obtain generators for $\operatorname{Pic} \overline{E}$. Given these generators, a standard, although involved, computation shows that $\mathrm{H}^1(G_k, \operatorname{Num} \overline{E}) = 0$, and so $\mathrm{H}^1(G_k, \langle K_E \rangle) \twoheadrightarrow \mathrm{H}^1(G_k, \operatorname{Pic} \overline{E})$. Then, the isomorphism from the Hochschild-Serre spectral sequence and [Sko01, Thm. 6.2.1] shows that $\psi(\tilde{E}(\mathbb{A}_k)) \subseteq E(\mathbb{A}_k)^{\mathrm{Br}_1}$. Since the adelic point considered in the previous section is contained in $\psi(\tilde{E}(\mathbb{A}_k))$, this shows that the adelic point above lies in $E(\mathbb{A}_k)^{\mathrm{Br}_1}$, as desired.

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