MULTIDIMENSIONAL STOCHASTIC BURGERS EQUATION

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ABSTRACT. We consider multidimensional stochastic Burgers equation on the torus $\mathbb{T}^d$ and the whole space $\mathbb{R}^d$. In both cases we show that for positive viscosity $\nu > 0$ there exists a unique strong global solution in L^p for $p > d$. In the case of torus we also establish a uniform in ν a priori estimate and consider a limit $\nu \searrow 0$ for potential solutions. In the case of $\mathbb{R}^d$ uniform with respect to ν a priori estimate established if a Beale-Kato-Majda type condition is satisfied.

CONTENTS

1. INTRODUCTION

The aim of this paper is to study the existence and uniqueness of solutions to the multidimensional stochastic Burgers equation of the following form:

$$\begin{cases} \frac{\partial u}{\partial t} = \nu\Delta u + u \cdot \nabla u + f + \xi, & t > 0, \ x \in \mathcal{O}, \\ u(0,x) = u_0(x), & x \in \mathcal{O}, \end{cases} \tag{1.1}$$

where $\mathcal{O} \subset \mathbb{R}^d$ with $d \geq 2$. In this paper we consider three examples of the domain $\mathcal{O}$. Either $\mathcal{O}$ is the whole space $\mathbb{R}^d$ or the torus $\mathcal{O} = \mathbb{T}^d$ or a bounded domain with a smooth boundary in which case we will supplement equation (1.1) with the zero Dirichlet boundary conditions. In the equation above f is a deterministic force and ξ is a multidimensional noise, white in time and correlated in space. We do not assume that u_0, f and ξ are of gradient form. The parameter $\nu > 0$ is known as viscosity. In this paper we will also study the limit of solutions to (1.1) when $\nu \to 0$.

Equation (1.1) has been proposed by Burgers [11] as a toy model for turbulence, see also E [20]. Later, numerous applications were found in Astrophysics and Statistical Physics. For an interesting review of applications and problems related to equation (1.1), see [4] and references therein. The Burgers equation with data of non-potential type arises in many areas of Physics, including gas dynamics and the theory of inelastic granular media, see for example [5]. The theory of equation (1.1) in the non-potential case is largely a terra incognita, see the review [4], where a variety of open problems can be found. This paper and the preceding work [25] by the second and third named authors are the first steps towards answering some of these questions.

One dimensional stochastic Burgers equation has been fairly well studied. Da Prato, Debussche, Temam [16], see also Bertini, Cancrini and Jona-Lasinio [6], showed the existence of a unique

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global solution for one dimensional Burgers equation with additive noise. The existence and uniqueness results have been extended to the case of multiplicative noise by Da Prato, Gatarek [17] and Gyöngy, Nualart [27].

Multidimensional Burgers equation has been studied much less comprehensively. Kiselev, Ladyzhenskaya [33] studied the deterministic Burgers equation with the Dirichlet boundary conditions on a bounded domain $\mathcal{O}$ and for small initial conditions proved the existence and uniqueness of a global solution in the class of functions $L^\infty(0,T;L^\infty(\mathcal{O}))\cap L^2(0,T;H_0^{1,2}(\mathcal{O}))$. The main idea of their proof is to apply the maximum principle to deduce a priori estimates similar to the a priori estimates for the Navier-Stokes equation. Ton [40] established convergence of solutions on small time interval when we take the limit $\nu\to 0$ and when the initial condition is zero.

The assumption that the initial condition and force have gradient form considerably simplifies analysis of the Burgers equation. It is well known that in this case one can apply the Hopf-Cole transformation (([29], [13])) to reduce the multidimensional Burgers equation either to the heat equation or to the Hamilton-Jacobi equation, see for example [19]. The number of works on the Hopf-Cole transformation is huge and we will not try to list them all here. We only mention Dermoune [18], where the Hopf-Cole transformation is used to show the existence of solution to the stochastic multidimensional Burgers equation with additive noise. Khanin et al [26] proved the existence of the so called quasi stationary solution by the Hopf-Cole transformation and Stochastic Lax formula, thus partially extending to many dimensions an important paper [21] by Sinai. This approach however has certain intrinsic problems. In particular, it seems difficult to find an a priori estimate for the solution without additional assumptions on the initial condition as in Dermoune [18] p. 303, Theorem 4.2. Hence, it is difficult to characterize functional spaces in which solution lies or to characterize quasi stationary solution, see Definition 1 in [26].

In this paper we consider multidimensional Burgers equation (1.1) in $L^p(\mathcal{O},\mathbb{R}^d)$ with $p>d\geq 2$. We prove, in Theorems 4.1, 4.3 and 6.3 respectively, the existence and uniqueness of global solutions for every initial condition $u_0\in L^p(\mathcal{O},\mathbb{R}^d)$ and establish a priori estimates. In particular, Theorem 4.3 holds in the case $\mathcal{O}=\mathbb{R}^d$ and $\xi=0$ thus improving our previous results from [25]. In the case of $\mathcal{O}=\mathbb{R}^d$ however, the a priori estimates are nonuniform with respect to ν. Theorems 4.1 and 4.3 extend all aforementioned results on the existence and uniqueness of solutions to (1.1) to the stochastic case.

In Theorem 4.5 we provide a general sufficient condition under which uniform with respect to ν estimates can be derived on $\mathbb{R}^d$ as well. It is interesting to note that this condition can be viewed as a modification and an extension to the stochastic case of the famous Beale-Kato-Majda condition assuring the existence of global solutions to the deterministic Navier-Stokes equation. Let us note here that contrary to the case of the Navier-Stokes equation, we are able to prove the global existence of smooth solutions to the stochastic (or deterministic) Burgers equation in any dimension. The main difference between the two equations is the availability of the Maximum Principle for the Burgers equation.

Finally, we apply our results to the gradient case. It is easy to see that in the gradient case the Beale-Kato-Majda condition holds and therefore the existence and uniqueness of global solutions follows from our general results. Morevoer, we obtain the estimates uniform in ν on the torus and on the whole space and as a consequence we show that there exists a vanishing viscosity limit for equation (1.1) for every $u_0\in L^p\left(\mathcal{O},\mathbb{R}^d\right)$.

In our proofs we extend the approach of [25], where the deterministic case $\xi=0$ was studied. We start with a proof of the local existence and uniqueness of mild solutions in $L^p(\mathcal{O},\mathbb{R}^d)$, $p\geq d$, following the argument of Weissler [41]. Then we find a priori estimates using the Maximum Principle and then show that the local solution is in fact global. We note here that this method was applied earlier to the deterministic Burgers equation by Kiselev, Ladyzhenskaya [33].

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2. Formulation of the problem and some auxiliary facts

We will study the Burgers equation (1.1) in a domain

$$\mathcal{O}\subset\mathbb{R}^d,\quad d\geq 2.$$

Everywhere with Exception of Section 6 we will assume that

$$\mathcal{O}\quad\text{is either } \mathbb{T}^d \text{ or } \mathbb{R}^d.$$

In both cases we will use the same notation Δ for the generator of the heat semigroup (S_t) in $\mathbb{L}^p(\mathcal{O}):=L^p\left(\mathcal{O},\mathbb{R}^d\right)$ for $p\in(1,\infty)$. Let us recall that

$$\mathrm{dom}_{\mathbb{L}^p(\mathcal{O})}(\Delta)=H^{2,p}\left(\mathbb{R}^d,\mathbb{R}^d\right)\quad\text{if}\quad\mathcal{O}=\mathbb{R}^d$$

and

$$\mathrm{dom}_{\mathbb{L}^p(\mathcal{O})}(\Delta)=H^{2,p}_{per}\left(\mathbb{T}^d,\mathbb{R}^d\right)\quad\text{if}\quad\mathcal{O}=\mathbb{T}^d.$$

We will use the standard notation $\mathbb{H}^{n,p}(\mathcal{O})=H^{n,p}\left(\mathcal{O},\mathbb{R}^d\right)$ for the Sobolev spaces of $\mathbb{R}^d$ -valued functions with the norm

$$|f|_{n,p}=|(I-\triangle)^{\frac{n}{2}}f|_{\mathbb{L}^p(\mathcal{O})}.$$

The dual space space of $\mathbb{H}^{n,p}(\mathcal{O})$ will be denoted by $\mathbb{H}^{-n,q}(\mathcal{O})$ with $q=\frac{p}{p-1}$.

Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ be a probability space with the filtration satisfying the usual conditions. For a given separable Banach space X we will denote by $M^p([0,T],X)$ the space of X-valued progressively measurable processes u endowed with the norm

$$\|u\|_{T,p,X}=\left(\mathbb{E}\int\limits_0^T|u(s)|^p_X\,ds\right)^{\frac{1}{p}}.$$

If $X=\mathbb{H}^{n,p}(\mathcal{O})$ then we will write $\|\cdot\|_{T,n,p}$ instead of $\|\cdot\|_{T,p,X}$. Let $(W_t)_{t\geq 0}$ be a standard cylindrical Wiener process on separable Hilbert space H defined on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$. Let us recall that for $p\geq 2$ the space $\mathbb{L}^p(\mathcal{O})$ is an M-type 2 Banach space and therefore the stochastic integration theory developed in [7] can be applied in this space. In order to give a meaning to equation (1.1) we will consider first its linearized version

$$\left\{\begin{array}{ll}\frac{\partial z}{\partial t}=\nu\Delta z+f+\xi, & t>0,\ x\in\mathcal{O},\\ z(0,x)=0, & x\in\mathcal{O}.\end{array}\right.\tag{2.1}$$

that will be understood as a stochastic evolution equation in the space $\mathbb{L}^p(\mathcal{O})$:

$$dz=(\nu\triangle z+f)\,dt+g\,dW_t,\quad z(0)=0.\tag{2.2}$$

To define solution to equation (2.2), let us recall that for a Banach space X and separable Hilbert space H, we denote by $\gamma(H,X)$ the Banach space of γ–radonifying operators from H to X, see [7] (and also [36, Definition 3.7]). If $g\in M^p\left([0,T];\gamma\left(H,\mathbb{L}^p(\mathcal{O})\right)\right)$ and $f\in M^p\left([0,T];\mathbb{H}^{-1,p}(\mathcal{O})\right)$ then solution to (2.2) is given by the formula

$$z(t)=\int_0^t S^\nu_{t-s}f(s)ds+\int_0^t S^\nu_{t-s}g(s)dW(s),$$

where (S^ν_t) denotes an extension to $\mathbb{H}^{-1,p}(\mathcal{O})$ of the heat semigroup generated in $\mathbb{L}^p(\mathcal{O})$ by the operator $\nu\Delta$. The regularity properties of the Ornstein-Uhlenbeck process were studied in a vast number of articles, see for instance [7] and references therein. The following theorem has been proved by Brzezniak [7] (Corollary 3.5) and Krylov [34] (Theorem 4.10 (i) and Theorem 7.2(i), chapter 5) for the case of whole space. The case of torus can be proved similarly.

Theorem 2.1. *Assume $n \in \mathbb{Z}$ and $f \in M^p([0,T], \mathbb{H}^{n-1,p}(\mathcal{O}))$, $g \in M^p([0,T], \gamma(H, \mathbb{H}^{n,p}(\mathcal{O})))$, $p > 2$, $\frac{1}{2} > \beta > \alpha > \frac{1}{p}$. Then equation (2.2) has a unique solution $z \in C^{\alpha-\frac{1}{p}}([0,T], \mathbb{H}^{n+1-2\beta,p}(\mathcal{O}))$ a.s.*

Using the Hölder inequality it is easy to show that for every $p, q \in (1, \infty)$ such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1$$

the following bilinear mapping

$$\mathbb{L}^p(\mathcal{O}) \times \mathbb{H}^{1,q}(\mathcal{O}) \ni (\phi, \psi) \longrightarrow F(\phi, \psi) = (\phi\nabla)\psi \in \mathbb{L}^r(\mathcal{O}), \tag{2.3}$$

is well defined and bounded. If $\phi = \psi$ then we write simply $F(\phi)$ instead of $F(\phi, \phi)$. Now we will define the (mild) solution to the stochastic Burgers equation (1.1).

Definition 2.2. *Assume that $u_0 \in \mathbb{L}^p(\mathcal{O})$, $f \in M^p([0,T], \mathbb{H}^{-1,p}(\mathcal{O}))$, $g \in M^p([0,T], \gamma(H, \mathbb{L}^p(\mathcal{O})))$. An $\mathbb{L}^p(\mathcal{O})$-valued adapted and continuous process $u = (u(t))$, $t \in [0,T]$ is said to be a mild solution to the stochastic Burgers equation (1.1) with the initial condition u_0 iff*

$$u(t) \in \mathbb{H}^{1,p}(\mathcal{O}), \quad \textit{for a.a. } t \in (0,T], \tag{2.4}$$

$$\int_0^T |F(u(t))|_{\mathbb{L}^p(\mathcal{O})}\, dt < \infty, \quad \mathbb{P} - as., \tag{2.5}$$

and $u = v + z$, where $z : \Omega \to L^\infty(0,T; \mathbb{H}^{1,p}(\mathcal{O}))$ is defined by equation (2.2) and v satisfies equation

$$v(t) = S_t^\nu u_0 + \int_0^t S_{t-s}^\nu F(v(s) + z(s))\, ds, \quad t \in [0,T]. \tag{2.6}$$

Remark 2.3. We believe it is possible to define a weak solution to Burgers equation as in definition 8.5, p. 184 of [10]. Then it should be possible to prove that sufficiently regular process u is a weak solution iff it is a mild solution, i.e. solves

$$u(t) = S_t^\nu u_0 + \int_0^t S_{t-s}^\nu (F(u(s)))\, ds + z(s),\ t \in [0,T], \tag{2.7}$$

where z satisfies equation (2.2). In our paper we prove the existence and uniqueness of the solution of (2.7). This is done via a substitution

$$u = v + z. \tag{2.8}$$

For a process of the form (2.8) we can prove that it is a mild solution iff it is a strong solution according to the following definition.

Definition 2.4. *Assume that u_0, f, g satisfy the same assumptions as in Definition 2.2. We call progressively measurable process $u : \Omega \to L^\infty(0,T; \mathbb{L}^p(\mathcal{O}))$ a strong solution of stochastic Burgers equation (1.1) with the initial condition u_0 iff (2.4) and (2.5) hold and $u = v + z$ where $z : \Omega \to L^\infty(0,T; \mathbb{H}^{1,p}(\mathcal{O}))$ satisfies equation (2.2) and $v \in C^1((0,T]; \mathbb{L}^p(\mathcal{O}))$ satisfies equality*

$$\frac{\partial v}{\partial t}(t) = \nu \triangle v(t) + F(v(t) + z(t)),\ t \in [0,T] \tag{2.9}$$

$$v(0) = u_0. \tag{2.10}$$

Remark 2.5. It is possible to define in a similar fashion strong and mild solution of stochastic Burgers equation without referring to the Ornstein-Uhlenbeck process z. However, the definition given above has certain merit since it allows to transfer all noise effects to the process z and consider PDE with random coefficients instead of SPDE.

3. THE EXISTENCE OF A LOCAL SOLUTION TO THE STOCHASTIC BURGERS EQUATION

Theorem 2.1 allows us to work pathwise, i.e. we assume that a version of z of specified regularity is fixed.

The local existence of solution of Burgers equation in $\mathbb{L}^p(\mathcal{O})$ can be shown in the same way as for Navier-Stokes equation (see [24], [32], [31], [41], [22] and others). Here we only state main points of the proof following the work of Weissler [41].

We will use the following version of the abstract theorem proved in [41], p. 222, Theorem 2, see also [32] and [31].

Theorem 3.1. *Let W, X, Y, Z be Banach spaces continuously embedded into a topological vector space $\mathcal{X}$ and assume that $W \cap X$ is dense in W. Let (R_t) be a C_0-semigroup on X, that satisfies the following additional conditions*

(a1) *For each $t > 0$, R_t extends to a bounded map $W \to X$. Moreover, there exists $a > 0$, such that for any $T > 0$ there exists $C > 0$ with the property*

$$|R_t h|_X \leq C t^{-a} |h|_W, \quad h \in W,\ t \in (0, T]. \tag{3.1}$$

(a2) *For each $t > 0$, the mapping $R_t : X \to Y$ is well defined and bounded. There exists $b > 0$ such that for any $T > 0$ there exists $C > 0$ such that*

$$|R_t h|_Y \leq C t^{-b} |h|_X, \quad h \in X,\ t \in (0, T]. \tag{3.2}$$

Furthermore, for every $h \in X$ and $T > 0$ the function

$$(0, T] \ni t \to R_t h \in Y,$$

is continuous and

$$\lim_{t \searrow 0} t^b |R_t h|_Y = 0, \quad \forall h \in X. \tag{3.3}$$

(a3) *For each $t > 0$, the mapping $R_t : X \to Z$ is well defined and bounded. There exists $c > 0$ such that for any $T > 0$ there exists $C > 0$ such that*

$$|R_t h|_Z \leq C t^{-c} |h|_X, \quad h \in X,\ t \in (0, T]. \tag{3.4}$$

Furthermore, for every $h \in X$ the function

$$(0, T] \ni t \to R_t h \in Z$$

is continuous and

$$\lim_{t \searrow 0} t^c |R_t h|_Z = 0, \quad \forall h \in X. \tag{3.5}$$

Assume also that $G : Y \times Z \to W$ is a bounded bilinear map, $L \in L^\infty(0, T; \mathcal{L}(Y \cap Z, W))$, and let $G(u) = G(u, u), u \in Y \cap Z$, $f \in L^\infty(0, T; W)$. Assume also that $a + b + c \leq 1$. Then for each $u_0 \in X$ there exists $T > 0$ and a unique function $u : [0, T] \to X$ such that:

(a) $u \in C([0, T], X)$, $u(0) = u_0$.
(b) $u \in C((0, T], Y)$, $\lim_{t \searrow 0} t^b |u(t)|_Y = 0$.
(c) $u \in C((0, T], Z)$, $\lim_{t \searrow 0} t^c |u(t)|_Z = 0$.

(d)

$$u(t) = R_t u_0 + \int_0^t R_{t-\tau}(G(u(\tau)) + L(u(\tau)) + f(\tau))d\tau, \quad t \in [0,T]$$

Remark 3.2. Weissler [41] considers only the case of $L = f = 0$. The general case follows similarly (see also [25]).

In the next proposition we summarize some well known properties of the heat semigroup on $\mathcal{O}$, see for example books by Lunardi [35] or by Quittner, Souplet [38].

Proposition 3.3. *Assume that either $\mathcal{O} = \mathbb{T}^d$ or $\mathcal{O} = \mathbb{R}^d$ and $\triangle$ is a corresponding periodic (respectively free) Laplacian with domain of definition $\mathrm{dom}_{\mathbb{L}^p(\mathcal{O})}(\Delta) = H^{2,p}(\mathcal{O})$. Assume that $p \in (1,\infty)$ and $q \in [p,\infty)$ and r satisfy*

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}.$$

Then for any $T > 0$ there exists $c > 0$ such that for any $t \in (0,T]$

$$|\nabla^m e^{t\triangle} h|_{\mathbb{L}^q(\mathcal{O})} \le c t^{-\frac{m}{2} - \frac{d}{2r}} |h|_{\mathbb{L}^p(\mathcal{O})}, \quad h \in \mathbb{L}^p(\mathcal{O}). \tag{3.6}$$

Furthermore,

$$\lim_{t \searrow 0} t^{\frac{m}{2} + \frac{d}{2r}} |\nabla^m e^{t\triangle} h|_{\mathbb{L}^q(\mathcal{O})} = 0, \quad h \in \mathbb{L}^p(\mathcal{O}). \tag{3.7}$$

In particular, for any $p \in (1,\infty)$ and $T > 0$ there exists C, such that for any $h \in \mathbb{L}^p(\mathcal{O})$

$$|e^{t\triangle} h|_{\mathbb{H}^{1,p}(\mathcal{O})} \le C t^{-\frac{1}{2}} |h|_{\mathbb{L}^p(\mathcal{O})}, \quad t \in (0,T], \tag{3.8}$$

$$\lim_{t \searrow 0} t^{\frac{1}{2}} |e^{t\triangle} h|_{\mathbb{H}^{1,p}(\mathcal{O})} = 0. \tag{3.9}$$

Now we can formulate the following results about the existence and uniqueness of a local mild solution to the stochastic Burgers equation equation (1.1).

Theorem 3.4. *Assume that $p \ge d$. Assume also that $u_0 \in \mathbb{L}^p(\mathcal{O})$ and $z \in L^\infty(0,T;\mathbb{L}^{2p}(\mathcal{O}) \cap \mathbb{H}^{1,p}(\mathcal{O}))$. There there exists $T_0 = T_0(\nu, |u_0|_{\mathbb{L}^p(\mathcal{O})}, |z|_{L^\infty(0,T;\mathbb{L}^{2p}(\mathcal{O}) \cap \mathbb{H}^{1,p}(\mathcal{O}))}) > 0$ such that there exists a unique mild solution $u \in C([0,T_0];\mathbb{L}^p(\mathcal{O}))$ to the stochastic Burgers equation (1.1). Furthermore*

(a) *$u : (0,T_0] \to \mathbb{L}^{2p}(\mathcal{O})$ is continuous and $\lim_{t\to 0} t^{\frac{d}{4p}} |u(t)|_{\mathbb{L}^{2p}(\mathcal{O})} = 0$.*

(b) *$u : (0,T_0] \to \mathbb{H}^{1,p}(\mathcal{O})$ is continuous and $\lim_{t\to 0} t^{\frac{1}{2}} |u(t)|_{\mathbb{H}^{1,p}(\mathcal{O})} = 0$.*

Proof. We apply Theorem 3.1 to equation (2.6) with $X = \mathbb{L}^p(\mathcal{O})$, $Y = \mathbb{L}^{2p}(\mathcal{O})$, $Z = \mathbb{H}^{1,p}(\mathcal{O})$ and $W = \mathbb{L}^{\frac{2p}{3}}(\mathcal{O})$. Let $R_t = e^{t\triangle}$. We identify G with the bilinear mapping F defined by (2.3) and put $G(u) = F(u,u)$,

$$L\phi = F(z,\phi) + F(\phi,z) \quad \text{and} \quad f = F(z,z).$$

Clearly, the mappings f, G and L satisfy the assumption of Theorem (3.1), and $f \in L^\infty(0,T;W)$. Condition (3.1) is satisfied with $a = \frac{d}{4p}$ by estimate (3.6). Conditions (3.2), (3.3) are satisfied with $b = \frac{d}{4p}$ by (3.6) and (3.7). Conditions (3.4), (3.5) are satisfied with $c = \frac{1}{2}$ by (3.8) and (3.9). $\square$

Now we formulate the result about the existence of the local maximal solution to the auxiliary problem, see (3.10) below, related to the stochastic Burgers equation.

Theorem 3.5. *Let $p \geq d$, $\theta \in (0,1)$, $u_0 \in \mathbb{L}^p(\mathcal{O})$. Moreover, we assume that*

$$z \in \mathcal{Y} = L^\infty(0,T;\mathbb{H}^{1,2p}(\mathcal{O}) \cap \mathbb{H}^{1,p}(\mathcal{O})) \cap C^\theta((0,T],\mathbb{H}^{1,2p}(\mathcal{O})).$$

Then there exists $T_0 =$ depending only on ν, $|u_0|_{\mathbb{L}^p(\mathcal{O})}$ and $|z|_{\mathcal{Y}}$ and a unique function v such that

$$\begin{aligned} v \quad \in \quad & C^1\left((0,T_0];\mathbb{L}^p(\mathcal{O})\right) \cap C\left((0,T_0];\mathbb{H}^{2,p}(\mathcal{O})\right) \\ & \cap C^\theta_{loc}\left((0,T_0],\mathbb{H}^{2,p}(\mathcal{O})\right) \cap C^{1+\theta}_{loc}\left((0,T_0],\mathbb{L}^p(\mathcal{O})\right)) \end{aligned}$$

which is a strong solution to the equation

$$v' = \nu\triangle v + F(v+z), \quad v(0) = u_0. \tag{3.10}$$

Proof. We show first that there exists $T_0 \leq T$ such that $v \in C((0,T_0],\mathbb{H}^{1,2p}(\mathcal{O}))$ and $\lim\limits_{t\to 0} t^{\frac{1}{2}}|u(t)|_{\mathbb{H}^{1,2p}(\mathcal{O})} = 0$. We apply Theorem 3.1 with $X = Y = \mathbb{L}^p(\mathcal{O})$, $Z = \mathbb{H}^{1,2p}(\mathcal{O})$, $W = \mathbb{L}^{\frac{2p}{3}}(\mathcal{O})$.

Therefore, we can use again Theorem 3.1 identifying F with G and defining L and f in the same way as in the proof of Theorem 3.4. Condition (3.1) is satisfied with $a = \frac{d}{4p}$ by estimate (3.6). Condition (3.2),(3.3) is satisfied with arbitrary $b > 0$ because the heat semigroup is analytic on $\mathbb{L}^p(\mathcal{O})$. Conditions (3.4) and (3.5) are satisfied with $c = \frac{1}{2}$ by (3.8) and (3.9). Therefore, part (c) of Theorem 3.1 yields the existence of $T_0 \leq T$ such that $v \in C((0,T_0],\mathbb{H}^{1,2p}(\mathcal{O}))$ and $\lim\limits_{t\to 0} t^{\frac{1}{2}}|v(t)|_{\mathbb{H}^{1,2p}(\mathcal{O})} = 0$.

Invoking (2.3), where p is replaced with $2p$ and $q = 2p$ we obtain

$$\begin{aligned} |F(v+z)|_{L^1(0,T_0;\mathbb{L}^p(\mathcal{O}))} &\leq \int_0^{T_0} |v(s)+z(s)|_{\mathbb{L}^{2p}(\mathcal{O})}|\nabla(v+z)|_{\mathbb{L}^{2p}(\mathcal{O})}\,ds \\ &\leq \int_0^{T_0} \frac{1}{s^{\frac{d}{4p}+\frac{1}{2}}} \sup_s(s^{\frac{d}{4p}}|v(s)+z(s)|_{\mathbb{L}^{2p}}) \sup_s(s^{\frac{1}{2}}|v(s)+z(s)|_{\mathbb{H}^{1,2p}})ds \\ &\leq \sup_s(s^{\frac{d}{4p}}|v(s)+z(s)|_{\mathbb{L}^{2p}}) \sup_s(s^{\frac{1}{2}}|v(s)+z(s)|_{\mathbb{H}^{1,2p}})T_0^{\frac{1}{2}-\frac{d}{4p}} < \infty. \end{aligned} \tag{3.11}$$

If we show that for any $\varepsilon > 0$ the function $F(v(\cdot)+z(\cdot)) : [\varepsilon,T_0] \to \mathbb{L}^p(\mathcal{O})$ is Hölder continuous of order $\left(\frac{1}{2} - \frac{d}{4p}\right)$ then the result will follow from Theorem 4.3.4, p. 137 in [35] and inequality (3.11). Since $F : \mathbb{H}^{1,2p}(\mathcal{O}) \to \mathbb{L}^p(\mathcal{O})$ is locally Lipschitz it is enough to prove that $v : [\varepsilon,T_0] \to \mathbb{H}^{1,2p}(\mathcal{O})$ is Hölder continuous for any $\varepsilon > 0$. Since

$$v(t) = S_{t-\varepsilon}v(\varepsilon) - \int_\varepsilon^t S_{t-s}(F(v(s)+z(s)))ds, \quad t \in [\varepsilon,T_0], \tag{3.12}$$

it is enough to show that each term of this equation is Hölder continuous. Using (2.3) and arguments as in (3.11) we have

$$\sup_{t\in[0,T_0]} t^{\frac{1}{2}+\frac{d}{4p}}|F(v(t)+z(t))|_{\mathbb{L}^p} \leq \sup_s s^{\frac{d}{4p}}|v(s)+z(s)|_{\mathbb{L}^{2p}} \sup_s s^{\frac{1}{2}}|v(s)+z(s)|_{\mathbb{H}^{1,2p}} < \infty, \tag{3.13}$$

and it follows by Proposition 4.2.3 part (i), p.130 of [35] that

$$\int_0^t S_{t-s}F(v(s)+z(s))\,ds \in C^{\frac{1}{2}-\frac{d}{4p}}(0,T_0;\mathbb{L}^p).$$

□

Corollary 3.6. *Suppose that assumptions of Theorem 3.5 are satisfied. Assume also that* $z \in C^\theta((0,T],\mathbb{H}^{k+1,2p}(\mathcal{O}))$*, for some* $k \in \mathbb{N}$*. Assume that* $T_0 \in (0,T]$ *and* v *is the unique strong solution of equation* (3.10) *defined on an interval* $(0,T_0]$*. Then* v *satisfies the following condition*

$$v \in C^\theta((0,T_0],\mathbb{H}^{k+2,p}(\mathcal{O})) \cap C^{1+\theta}((0,T_0],\mathbb{H}^{k,p}(\mathcal{O})).$$

Proof. We will show the result for $k=1$. General case follows similarly. We fix $\varepsilon>0$. Then part (a) of Theorem 3.4 yields $v(\varepsilon) \in \mathbb{L}^{2p}(\mathcal{O})$. As a result, by means of Theorem 3.5 we infer that

$$v \in C^\theta([\varepsilon,T_0],\mathbb{H}^{2,2p}(\mathcal{O})) \cap C^{1+\theta}([\varepsilon,T_0],\mathbb{L}^{2p}(\mathcal{O})). \tag{3.14}$$

Hence,

$$v+z \in C^\theta([\varepsilon,T_0],\mathbb{H}^{2,2p}(\mathcal{O})), \tag{3.15}$$

and therefore, we have following estimates for the nonlinearity F:

$$\begin{aligned} |F(v+z)|_{C^\theta([\varepsilon,T_0],\mathbb{L}^p(\mathcal{O}))} &\le |v+z|_{L^\infty(\varepsilon,T_0;\mathbb{L}^{2p}(\mathcal{O}))}|\nabla(v+z)|_{C^\theta([\varepsilon,T_0],\mathbb{L}^{2p}(\mathcal{O}))} \\ &\quad + |\nabla(v+z)|_{L^\infty(\varepsilon,T;\mathbb{L}^{2p}(\mathcal{O}))}|v+z|_{C^\theta([\varepsilon,T_0],\mathbb{L}^{2p}(\mathcal{O}))} < \infty, \end{aligned} \tag{3.16}$$

where we have used (3.15). Furthermore,

$$\begin{aligned} |\nabla F(v+z)|_{C^\theta([\varepsilon,T_0],\mathbb{L}^p(\mathcal{O}))} &\le C|\nabla(v+z)|_{L^\infty(\varepsilon,T_0;\mathbb{L}^{2p}(\mathcal{O}))}|\nabla(v+z)|_{C^\theta([\varepsilon,T_0],\mathbb{L}^{2p}(\mathcal{O}))} \\ &\quad + |v+z|_{L^\infty(\varepsilon,T_0;\mathbb{L}^{2p}(\mathcal{O}))}|\triangle(v+z)|_{C^\theta([\varepsilon,T_0],\mathbb{L}^{2p}(\mathcal{O}))} \\ &\quad + |\triangle(v+z)|_{L^\infty(\varepsilon,T_0;\mathbb{L}^{2p}(\mathcal{O}))}|v+z|_{C^\theta([\varepsilon,T_0],\mathbb{L}^{2p}(\mathcal{O}))} < \infty, \end{aligned} \tag{3.17}$$

where we have used (3.15). Thus, combining inequalities (3.16) and (3.17) we get

$$F(v+z) \in C^\theta([\varepsilon,T_0],\mathbb{H}^{1,p}(\mathcal{O})), \quad \forall \varepsilon>0.$$

Therefore by a maximal regularity result, see Theorem 4.3.1, p. 134 of [35], it follows that $v \in C^\theta([\varepsilon,T_0],\mathbb{H}^{3,p}(\mathcal{O})) \cap C^{1+\theta}([\varepsilon,T_0],\mathbb{H}^{1,p}(\mathcal{O}))$. □

In the next result we will show that the local solution defined in previous theorems is either global or it blows up. Let us denote by T_{max} the maximal existence time of solution in $\mathbb{L}^p(\mathcal{O})$.

Proposition 3.7. *Assume that* $u_0 \in \mathbb{L}^p(\mathcal{O})$*,* $z \in L^\infty([0,T],\mathbb{H}^{1,2p}(\mathcal{O}) \cap \mathbb{H}^{1,p}(\mathcal{O}))$*,* $z \in C^\theta(0,T;\mathbb{L}^p(\mathcal{O}))$*. Let* $p>d$*,* $\theta \in (0,1)$ *and* $T_{max} < T$*. Let* $u \in C([0,T_{max});\mathbb{L}^p(\mathcal{O}))$ *be a maximal local mild solution to the Burgers equation* (2.6)*. Then*

$$\limsup_{t\nearrow T_{max}} |u(t)|^2_{\mathbb{L}^p(\mathcal{O})} = \infty., \quad \mathbb{P}-a.s. \tag{3.18}$$

Question 3.8. Can Proposition 3.7 be strengthened to show $\lim$ instead of $\limsup$ in the equality (3.18)?

Question 3.9. It would be interesting to extend Proposition 3.7 to the case when $p=d$.

Proof. We will argue by contradiction. Assume that there exits $R>0$ such that

$$|u(t)|^2_{\mathbb{L}^p(\mathcal{O})} \le R, \ \ t \in [0,T_{max}]. \tag{3.19}$$

Let us denote

$$K_1 = \sup_{t\in[0,T_{max})} |u(t)|_{\mathbb{L}^p(\mathcal{O})} < \infty. \tag{3.20}$$

Since $z \in L^\infty([0,T],\mathbb{H}^{1,p}(\mathcal{O}))$ we have the similar bound for $v=u-z$:

$$\sup_{t\in[0,T_{max})} |v(t)|_{\mathbb{L}^p(\mathcal{O})} < K_1' = K_1 + |z|_{L^\infty([0,T_{max}],\mathbb{H}^{1,p}(\mathcal{O}))}. \tag{3.21}$$

Let us fix $\varepsilon \in (0,T_{max})$. We will show that there exist $C,\alpha>0$ such that

$$|u(t)-u(\tau)|_{\mathbb{L}^p(\mathcal{O})} \le C|t-\tau|^\alpha, \ \ t,\tau \in [\varepsilon,T_{max}). \tag{3.22}$$

Then it follows from inequalities (3.19) and (3.22) that there exist $y \in \mathbb{L}^p(\mathcal{O})$ such that

$$\lim_{t \nearrow T_{max}} |u(t) - y|_{\mathbb{L}^p(\mathcal{O})} = 0, \tag{3.23}$$

and we have a contradiction with the definition of T_{max}. Thus, we need to prove inequality (3.22). We will first show that

$$K_2 = \sup_{t \in [\varepsilon, T_{max})} |u(t)|_{\mathbb{H}^{1,p}(\mathcal{O})} < \infty. \tag{3.24}$$

It is enough to show

$$\sup_{t \in [\varepsilon, T_{max})} |\nabla v(t)|_{\mathbb{L}^p(\mathcal{O})} < \infty. \tag{3.25}$$

Indeed, the inequality (3.24) immediately follows from inequalities (3.21), (3.25) and the regularity of z. We have

$$\nabla v(t) = \nabla S_{t-\varepsilon} v(\varepsilon) - \int_\varepsilon^t \nabla S_{t-s}(F(v(s) + z(s)))\, ds. \tag{3.26}$$

Hence, for $t \in (\varepsilon, T_{max})$ we have

$$\begin{aligned}
|\nabla v(t)|_{\mathbb{L}^p(\mathcal{O})} &\le |\nabla S_{t-\varepsilon} v(\varepsilon)|_{\mathbb{L}^p(\mathcal{O})} + \int_\varepsilon^t |\nabla S_{t-s} F(v(s) + z(s))|_{\mathbb{L}^p(\mathcal{O})}\, ds \\
&\le \frac{C|v(\varepsilon)|_{\mathbb{L}^p(\mathcal{O})}}{(t-\varepsilon)^{1/2}} + C \int_\varepsilon^t \frac{|S_{(t-s)/2} F(v(s) + z(s))|_{\mathbb{L}^p(\mathcal{O})}}{|t-s|^{1/2}}\, ds \\
&\le \frac{C|u_0|_{\mathbb{L}^p(\mathcal{O})}}{t^{1/2}} + C \int_\varepsilon^t \frac{|F(v(s) + z(s))|_{\mathbb{L}^{p/2}(\mathcal{O})}}{|t-s|^{1/2+d/(2p)}}\, ds \\
&\le \frac{C|u_0|_{\mathbb{L}^p(\mathcal{O})}}{t^{1/2}} + C \int_\varepsilon^t \frac{|v(s) + z(s)|_{\mathbb{L}^p(\mathcal{O})}}{|t-s|^{1/2+d/(2p)}} |\nabla v(s) + \nabla z(s)|_{\mathbb{L}^p(\mathcal{O})}\, ds \\
&\le \frac{C|u_0|_{\mathbb{L}^p(\mathcal{O})}}{t^{1/2}} + C(K_1')^2 T + CK_1' \int_\varepsilon^t \frac{|\nabla v(s)|_{\mathbb{L}^p(\mathcal{O})}}{|t-s|^{1/2+d/(2p)}}\, ds < \infty,
\end{aligned} \tag{3.27}$$

where the second and third inequalities follow from the property (3.6) of the heat semigroup, fourth inequality follows from the Hölder inequality, assumption (3.20) is used in the fifth one and the last inequality follows from (c), Theorem 3.4. Now if $\frac{1}{2} + \frac{d}{2p} < 1$ (i.e. if $p > d$) we can use a version of the Gronwall inequality ([28], Lemma 7.1.1, p. 188) to conclude that estimate (3.25) holds. Thus we get an estimate (3.24).

Now we can turn to the proof of uniform continuity condition (3.22). Since $u = v + z$ and $z \in C^\theta(0, T; \mathbb{L}^p(\mathcal{O}))$ it is enough to show (3.22) with v instead of u. We have

$$v(t) - v(\tau) = S_{t-\tau} v(\tau) - v(\tau) - \int_\tau^t S_{t-s}(F(v(s) + z(s)))\, ds. \tag{3.28}$$

Then

$$
\begin{aligned}
|v(t)-v(\tau)|_{\mathbb{L}^p(\mathcal{O})} &\le |S_{t-\tau}v(\tau)-v(\tau)|_{\mathbb{L}^p(\mathcal{O})} \\
&\quad + \Big|\int_\tau^t S_{t-s}F(v(s)+z(s))\,ds\Big|_{\mathbb{L}^p(\mathcal{O})} \\
&= (I)+(II).
\end{aligned} \tag{3.29}
$$

The first term can be estimated as follows, where the sup is taken over the set $\{\phi \in C_0^\infty(\mathcal{O}) : |\phi|_{\mathbb{L}^q(\mathcal{O})}=1\}$,

$$
\begin{aligned}
(I) &= \sup|\langle S_{t-\tau}v(\tau)-v(\tau),\phi\rangle| = \sup|\langle v(\tau), S_{t-\tau}\phi-\phi\rangle| \\
&= \sup|\langle v(\tau),\nu\int_\tau^t \triangle S_{s-\tau}\phi ds\rangle| = \nu\sup|\langle \nabla v(\tau),\int_\tau^t \nabla S_{s-\tau}\phi ds\rangle| \\
&\le \nu\sup|\nabla v(\tau)|_{\mathbb{L}^p(\mathcal{O})}\int_\tau^t |\nabla S_{s-\tau}\phi|_{\mathbb{L}^q(\mathcal{O})}\,ds \\
&\le \nu\sup|\nabla v(\tau)|_{\mathbb{L}^p(\mathcal{O})}C\int_\tau^t \frac{|\phi|_{\mathbb{L}^q(\mathcal{O})}}{|s-\tau|^{1/2}}\,ds \le \nu CK_2|t-\tau|^{1/2}.
\end{aligned}
$$

For the second term, by using the property (3.6) of heat semigroup and the Hölder inequality we have

$$
\begin{aligned}
(II) &\le \int_\tau^t \frac{|F(v(s)+z(s))|_{\mathbb{L}^{p/2}(\mathcal{O})}}{|t-s|^{\frac{d}{2p}}}\,ds \le \int_\tau^t \frac{|u(s)|_{\mathbb{L}^p(\mathcal{O})}|\nabla u(s)|_{\mathbb{L}^p(\mathcal{O})}}{|t-s|^{\frac{d}{2p}}}\,ds \\
&\le CK_2^2|t-\tau|^{1-\frac{d}{2p}}.
\end{aligned} \tag{3.30}
$$

Finally, the last inequality follows from estimate (3.24).

Combining inequalities (3.30) and (3.30) we get (3.22). ☐

4. The existence of a global solution to Stochastic Burgers equation

In this section we continue to work pathwise. We will state and proof the global existence results for both the cases of the torus and the full space. We begin with the former case. Let us note here that the proof of Theorem 4.4 can be adapted to the case of the torus as well. We prefer however to provide an independent and simpler proof below.

Theorem 4.1. *Assume that $p > d$. Assume also that $u_0 \in \mathbb{L}^p(\mathbb{T}^d)$ a.s., $f \in M^{2p}_{loc}([0,\infty),\mathbb{H}^{3,2p}(\mathbb{T}^d))$ and $g \in M^{2p}_{loc}([0,\infty),\gamma(H,\mathbb{H}^{4,2p}(\mathbb{T}^d)))$. Then the following holds.*
(1) There exists a unique strong global $\mathbb{L}^p(\mathbb{T}^d)$-valued solution u of the stochastic Burgers equation.
(2) Assume that $q \ge p$. Then for any $T>0$, almost surely,

$$
|u(t)|^q_{\mathbb{L}^p(\mathbb{T}^d)} \le C(|u_0|^q_{\mathbb{L}^p(\mathbb{T}^d)} + |z|^{2q/p}_{L^\infty(0,T;\mathbb{H}^{2,p}(\mathbb{T}^d))})e^{(q/p)|\nabla z|_{L^1(0,T;\mathbb{L}^\infty(\mathbb{T}^d))}}, \quad t\in[0,T]. \tag{4.1}
$$

(3) If f,g are deterministic, then for any $T>0$, $q\ge 1$ and u_0 such that $\mathbb{E}\,|u_0|^q_{\mathbb{L}^p(\mathbb{T}^d)}<\infty$,

$$
\mathbb{E}\sup_{t\in[0,T]}|u(t)|^q_{\mathbb{L}^p(\mathbb{T}^d)} < \infty. \tag{4.2}
$$

Proof. We begin with the proof of part (2) of the Theorem. To this end let us fix the initial data u_0 and $T > 0$. We can assume that $f \in M^{2p}(0, T, \mathbb{H}^{3,2p}(\mathbb{T}^d))$ and $g \in M^{2p}(0, T, \gamma(H, \mathbb{H}^{4,2p}(\mathbb{T}^d)))$.

Let $u = (u(t))$, $t \in [0, T_*)$, where $T_* = T_{\max} \in (0, T]$, be the unique maximal local solution of the stochastic Burgers equation (1.1) whose existence has been shown in Theorem 1.1. We need to prove that $T_* = T$. Suppose by contradiction that $T_* < T$. Let us put $v = u - z$. According to Proposition 3.7 it is enough to find an estimate for v in the $\mathbb{L}^p(\mathbb{T}^d)$ norm. Since on torus[1] $L^\infty(\mathbb{T}^d) \subset \mathbb{L}^p(\mathbb{T}^d)$, it is enough to find an estimate for v in the $\mathbb{L}^\infty(\mathbb{T}^d)$ norm. Therefore, it is enough to prove that for any fixed $0 < \delta < T_*$, we have

$$|v|_{L^\infty((0,T_*;\mathbb{L}^\infty(\mathbb{T}^d))} \le (|v(\delta)|_{\mathbb{H}^{1,p}(\mathbb{T}^d)} + |z|^2_{L^\infty(0,T_*;\mathbb{H}^{2,p}(\mathbb{T}^d))}) e^{T_* + |\nabla z|_{L^1(0,T_*;\mathbb{L}^\infty(\mathbb{T}^d))}}. \tag{4.3}$$

To prove (4.3) we note first that the local solution v satisfies the equation

$$v' = \nu \triangle v - (v+z)\nabla v - v\nabla z - (z\nabla)z.$$

Let us define a function ϕ by formula

$$\phi(t) = v(t) e^{-\int_0^t (1+|\nabla z|_{\mathbb{L}^\infty})\, ds} - |z|^2_{L^\infty(0,T_*;\mathbb{H}^{2,p}(\mathbb{T}^d))}, \quad t \in [0, T_*).$$

Then

$$\phi' = \nu\triangle\phi - (v+z)\nabla\phi + (\phi + |z|^2_{L^\infty(0,T_*;\mathbb{H}^{2,p}(\mathbb{T}^d))})(-\nabla z - |\nabla z|_{\mathbb{L}^\infty} - 1) - (z\nabla)z e^{-\int_0^\cdot (1+|\nabla z|_{\mathbb{L}^\infty})\, ds}$$

or, equivalently,

$$\begin{aligned} &\nu\triangle\phi - (v+z)\nabla\phi + \phi(-\nabla z - |\nabla z|_{\mathbb{L}^\infty} - 1) - \phi' \\ &\quad = |z|^2_{L^\infty(0,T_*;\mathbb{H}^{2,p}(\mathbb{T}^d))}(\nabla z + |\nabla z|_{\mathbb{L}^\infty} + 1) + (z\nabla)z e^{-\int_0^\cdot (1+|\nabla z|_{\mathbb{L}^\infty})\, ds} \ge 0. \end{aligned}$$

Now (4.3) follows from the maximum principle, see for instance [37, Theorem 7, p. 174]. Finally, (4.2) follows immediately from inequality (4.1) and the Fernique Theorem since the process z is Gaussian. ∎

Remark 4.2. We remark that in the a priori estimate above we can take the limit $\nu \to 0$ under appropriate assumptions for the noise. Hence, here as well as in the next theorem, we cannot take the limit $\nu \to 0$ in the a priori estimate above (and below).

In order to prove the global existence for the case $\mathcal{O} = \mathbb{R}^d$ we will need the following

Lemma 4.3. *For every $\varepsilon > 0$ and sufficiently regular function $v : \mathbb{R}^d \to \mathbb{R}^d$ one has*

$$\int_{\mathbb{R}^d} |v(x)|^p |\nabla v(x)|\, dx \le |v|_{L^\infty} \Big[\frac{1}{4\varepsilon} |v|^p_{L^p} + \varepsilon \int_{\mathbb{R}^d} |v(x)|^{p-2} |\nabla v(x)|^2\, dx \Big]. \tag{4.4}$$

Proof. Let us fix ε and v as in the statement. Then by the Cauchy-Schwartz and Hölder inequalities we get

$$\begin{aligned} \int_{\mathbb{R}^d} |v(x)|^p |\nabla v(x)|\, dx &\le |v|_{L^\infty} \int_{\mathbb{R}^d} |v(x)|^{p/2} |v(x)|^{p/2-1} |\nabla v(x)|\, dx \\ &\le |v|_{L^\infty} \Big(\frac{1}{4\varepsilon} \int_{\mathbb{R}^d} |v(x)|^p\, dx + \varepsilon \int_{\mathbb{R}^d} |v(x)|^{p-2} |\nabla v(x)|^2\, dx \Big). \end{aligned}$$

The proof is complete. ∎

In the next theorem we prove the global existence of solutions to the Burgers equation on $\mathbb{R}^d$. Let us note that in this case we do not have an embedding $\mathbb{L}^\infty \subset \mathbb{L}^p$, hence estimate in $\mathbb{L}^\infty$ does not imply estimate in $\mathbb{L}^p$.

[1] This is the reason why the proof of Theorem 4.1 is not transferable to the whole space case of Theorem 4.4

Theorem 4.4. *Fix $p > d$. Assume also that $u_0 \in \mathbb{L}^p(\mathbb{R}^d)$ a.s., $f \in M^{2p}_{loc}([0,\infty), \mathbb{H}^{3,2p}(\mathbb{R}^d))$, $g \in M^{2p}_{loc}([0,\infty), \gamma(H, \mathbb{H}^{4,2p}(\mathbb{R}^d)))$. Then the following holds.*
(1) There exists a unique strong global $\mathbb{L}^p(\mathbb{R}^d)$-valued solution u of the stochastic Burgers equation.
(2) Assume that $q \geq p$. Then there exist constants C, C_1, C_p such that for any $T > 0$, almost surely

$$
\begin{aligned}
|u(t)|^p_{\mathbb{L}^p(\mathbb{R}^d)} \leq C_1(|u_0|^q_{\mathbb{L}^p(\mathbb{R}^d)} + |F \circ z|^{q/p}_{L^1(0,t;\mathbb{L}^p(\mathbb{R}^d))}) \exp\Big\{ C_p \Big(t|z|_{L^\infty(0,t;\mathbb{H}^{2,p}(\mathbb{R}^d))} + t \\
+ \frac{t}{\nu}(C|u_0|^{2p}_{\mathbb{L}^p(\mathbb{R}^d)} + |z|^4_{L^\infty(0,t;\mathbb{H}^{2,p}(\mathbb{R}^d))}) \Big) e^{2t|z|_{L^\infty(0,t;\mathbb{H}^{2,p}(\mathbb{R}^d))}} \Big\}, \quad t \geq 0.
\end{aligned}
\tag{4.5}
$$

(3) Moreover, if f, g are deterministic, for any $T > 0$ and $q \geq 1$ then

$$
\mathbb{E} \sup_{t\in[0,T]} |u(t)|^q_{\mathbb{L}^p(\mathbb{R}^d)} < \infty,
\tag{4.6}
$$

provided $\mathbb{E}\, |u_0|^q_{\mathbb{L}^p(\mathbb{R}^d)} < \infty$.

Proof. First of all we note that assertions (1) and (3) of the theorem are consequences of the 2nd part and the Fernique Theorem. Thus we only need to prove assertion (2).

We begin the proof of this assertion as we did in the proof of Theorem 4.4, i.e. we fix the initial data u_0 and $T > 0$. We can assume that $f \in M^{2p}(0, T, \mathbb{H}^{3,2p}(\mathbb{R}^d))$ and $g \in M^{2p}(0, T, \gamma(H, \mathbb{H}^{4,2p}(\mathbb{R}^d)))$.

Let $u = (u(t))$, $t \in [0, T_*)$, where $T_* = T_{\max} \in (0, T]$, be the unique maximal local solution of the stochastic Burgers equation (1.1) whose existence has been shown in Theorem 1.1. We need to prove that $T_* = T$. Suppose by contradiction that $T_* < T$. Let us put $v = u - z$. According to Proposition 3.7 it is enough to find an estimate for v in the $\mathbb{L}^p(\mathbb{R}^d)$ norm. Since v is a strong solution of equation (3.10). In particular, the L^p-norm of v is differentiable and for $t \in (0, T_*)$

$$
\begin{aligned}
\frac{1}{p}\frac{d}{dt}|v(t)|^p_{\mathbb{L}^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |v(t,x)|^{p-2} \sum_i \langle v^i(t,x), \dot{v}^i(t,x)\rangle\, dx \\
&= \int_{\mathbb{R}^d} |v(t,x)|^{p-2} \Big(\nu \sum_i v^i(t,x)\Delta v^i(t,x) + \langle v(t,x), F(v(t,x) + z(t,x))\rangle \Big)\, dx.
\end{aligned}
$$

Let us observe that for $v \in \mathbb{H}^{2,p}(\mathbb{R}^d)$

$$
\begin{aligned}
-\sum_i \int_{\mathbb{R}^d} |v|^{p-2} v^i \Delta v^i\, dx &= \sum_i \int_{\mathbb{R}^d} |v|^{p-2} |\nabla v^i|^2\, dx \\
&\quad + (p-2) \sum_i \int_{\mathbb{R}^d} |v|^{p-4} v^i \sum_j v^j \left\langle \nabla v^i, \nabla v^j \right\rangle\, dx \\
&= \int_{\mathbb{R}^d} \left[|v|^{p-2}|\nabla v|^2 + (p-2)|v|^{p-4}|(\nabla v)v|^2 \right] dx.
\end{aligned}
$$

Therefore, for any fixed $t_0 \in (0, T_*)$ and $t \in [t_0, T_*)$

$$
\begin{aligned}
\frac{1}{p}|v(t)|^p_{\mathbb{L}^p(\mathbb{R}^d)} + \nu \int_{t_0}^{t} \int_{\mathbb{R}^d} \left[|v(s,x)|^{p-2}|\nabla v(s,x)|^2 + (p-2)|v(s,x)|^{p-4}|(\nabla v(s,x))v(s,x)|^2 \right] dx ds \\
= \frac{1}{p}|v(t_0)|^p_{\mathbb{L}^p(\mathbb{R}^d)} + \int_{t_0}^t \int_{\mathbb{R}^d} |v(s,x)|^{p-2} \langle v(s,x), F(v(s,x) + z(s,x))\rangle\, dx\, ds.
\end{aligned}
\tag{4.7}
$$

Let us also observe that for $v, z \in \mathbb{H}^{2,p}(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d} |v|^{p-2}\langle v, F(v+z)\rangle\, dx &\le \int_{\mathbb{R}^d} |v|^{p-1}|F(v)|\, dx \\ &+ \int_{\mathbb{R}^d} |v|^{p-1}|F(z)|\, dx + \int_{\mathbb{R}^d} |v|^{p-1}|F(v,z)|\, dx \\ &+ \int_{\mathbb{R}^d} |v|^{p-1}|F(z,v)|\, dx\end{aligned}$$

Let us now fix $\varepsilon > 0$. Then we estimate the first term above by means of inequality (4.4) from Lemma 4.3, i.e.

$$\begin{aligned}\int_{\mathbb{R}^d} |v(x)|^{p-1}|F(v(x))|\, dx &\le \int_{\mathbb{R}^d} |v(x)|^{p}|\nabla v(x)|\, dx \\ &\le \frac{1}{4\varepsilon}|v|_{L^\infty}|v|_{L^p}^p + \varepsilon |v|_{L^\infty}\int_{\mathbb{R}^d} |v(x)|^{p-2}|\nabla v(x)|^2\, dx.\end{aligned} \tag{4.8}$$

By applying the Hölder and Young inequalities we get the following estimate for the second term

$$\int_{\mathbb{R}^d} |v(x)|^{p-1}|F(z(x))|\, dx \le \frac{p-1}{p}|v|_{L^p}^p + \frac{1}{p}|F\circ z|_{L^p}^p. \tag{4.9}$$

The third term we estimate by applying the Hölder and Young inequalities and get

$$\int_{\mathbb{R}^d} |v(x)|^{p-1}|F(v(x),z(x))|\, dx \le \int_{\mathbb{R}^d} |v(x)|^{p}|\nabla z(x)|\, dx \le |v|_{L^\infty}|v|_{L^p}^p + |v|_{L^\infty}|\nabla z|_{L^p}^p \tag{4.10}$$

The fourth term we estimate in a somehow similar way to the 1st one.

$$\begin{aligned}&\int_{\mathbb{R}^d} |v(x)|^{p-1}|F(z(x),v(x))|\, dx \le \int_{\mathbb{R}^d} |v(x)|^{p-1}|z(x)||\nabla v(x))|\, dx \\ &\le |v|_{L^\infty}\int_{\mathbb{R}^d} |v(x)|^{\frac{p}{2}-1}|z(x)||v(x)|^{\frac{p}{2}-1}|\nabla v(x))|\, dx \\ &\le |v|_{L^\infty}\Big[\frac{C}{\varepsilon}\big(|v|_{L^p}^p + |z|_{L^p}^p + \varepsilon|v|_{L^\infty}\int_{\mathbb{R}^d} |v(x)|^{p-2}|\nabla v(x)|^2\, dx\Big] \\ &= \frac{C}{\varepsilon}|v|_{L^\infty}\big(|v|_{L^p}^p + |z|_{L^p}^p\big) + \varepsilon|v|_{L^\infty}\int_{\mathbb{R}^d} |v(x)|^{p-2}|\nabla v(x)|^2\, dx\end{aligned} \tag{4.11}$$

Therefore, we infer that

(4.12)

$$\frac{1}{p}|v(t)|^p_{\mathbb{L}^p(\mathbb{R}^d)}+\nu\int_{t_0}^{t}\int_{\mathbb{R}^d}|v(s,x)|^{p-2}|\nabla v(s,x)|^2\,dx\,ds\le\frac{1}{p}|v(t_0)|^p_{\mathbb{L}^p(\mathbb{R}^d)}$$
$$+\int_{t_0}^t\Big[\frac{1}{4\varepsilon}|v(s)|_{L^\infty}|v(s)|^p_{L^p}+\varepsilon|v(s)|_{L^\infty}\int_{\mathbb{R}^d}|v(s,x)|^{p-2}|\nabla v(s,x)|^2\,dx+\frac{p-1}{p}|v(s)|^p_{L^p}$$
$$+\frac{1}{p}|F\circ z(s)|^p_{L^p}+|v(s)|_{L^\infty}|v(s)|^p_{L^p}+|v(s)|_{L^\infty}|\nabla z(s)|^p_{L^p}+\frac{C}{\varepsilon}|v(s)|_{L^\infty}\big(|v(s)|^p_{L^p}+|z(s)|^p_{L^p}\big)$$
$$+\varepsilon|v(s)|_{L^\infty}\int_{\mathbb{R}^d}|v(s,x)|^{p-2}|\nabla v(s,x)|^2\,dx\Big]\,ds\le\frac{1}{p}|v(t_0)|^p_{\mathbb{L}^p(\mathbb{R}^d)}$$
$$+2\varepsilon|v|_{L^\infty(t_0,t;L^\infty)}\int_{\mathbb{R}^d}|v(s,x)|^{p-2}|\nabla v(s,x)|^2\,dx$$
$$+|v|_{L^\infty(t_0,t;L^\infty)}\int_{t_0}^t\big(\frac{1}{4\varepsilon}+\frac{p-1}{p}+1+\frac{C}{\varepsilon}\big)|v(s)|^p_{L^p}\,ds$$
$$+|v|_{L^\infty(t_0,t;L^\infty)}\int_{t_0}^t\Big[\frac{1}{p}|F\circ z(s)|^p_{L^p}+|\nabla z(s)|^p_{L^p}+\frac{C}{\varepsilon}|z(s)|^p_{L^p}\Big]\,ds$$

We can estimate $\mathbb{L}^\infty$ norm of v using the maximum principle, see for instance [37, Theorem 7, p. 174] in the same way as in the torus case above i.e. we have

$$\begin{aligned}|v|_{L^\infty([\delta,t];\mathbb{L}^\infty(\mathbb{R}^d))}\le&(|v(\delta)|_{\mathbb{H}^{1,p}(\mathbb{R}^d)}+|z|^2_{L^\infty(0,t;\mathbb{H}^{2,p}(\mathbb{R}^d))})e^{|\nabla z|_{L^1(0,t;\mathbb{L}^\infty(\mathbb{R}^d))}}\\ =&:Q(\delta,t),\end{aligned}\tag{4.13}$$

for any fixed $0<\delta\le t<T_*$. Put $\varepsilon=\frac{\nu p}{4Q(t_0,T_*)}$. Then, for $t\in[t_0,T_*)$ we have

$$|v(t)|^p_{\mathbb{L}^p(\mathbb{R}^d)}+\frac{\nu p}{2}\int_{t_0}^{t}\int_{\mathbb{R}^d}|v|^{p-2}(s,x)|\nabla v(s,x)|^2\,dx\,ds$$
$$\le|v(t_0)|^p_{\mathbb{L}^p(\mathbb{R}^d)}+\big(\frac{\hat C}{\varepsilon}+2\big)Q(t_0,T_*)\int_{t_0}^{t}|v(s)|^p_{L^p}\,ds$$
$$+Q(t_0,T_*)\int_{t_0}^t\Big[\frac{1}{p}|F\circ z(s)|^p_{L^p}+|\nabla z(s)|^p_{L^p}+\frac{C}{\varepsilon}|z(s)|^p_{L^p}\Big]\,ds.\tag{4.14}$$

Now we apply the Gronwall Lemma to conclude that

$$|v(t)|^p_{\mathbb{L}^p(\mathbb{R}^d)}+\frac{\nu p}{2}\int_{t_0}^{t}\int_{\mathbb{R}^d}|v|^{p-2}(s,x)|\nabla v(s,x)|^2dxds$$
$$\le\Big(|v(t_0)|^p_{\mathbb{L}^p(\mathbb{R}^d)}++Q(t_0,T_*)\int_{t_0}^t\Big[\frac{1}{p}|F\circ z(s)|^p_{L^p}+|\nabla z(s)|^p_{L^p}+\frac{C}{\varepsilon}|z(s)|^p_{L^p}\Big]\,ds\Big)\tag{4.15}$$
$$\cdot\exp\Big[\big(\frac{\hat C}{\varepsilon}+2\big)Q(t_0,T_*)\Big].\tag{4.16}$$

Thus we get the desired a'priori estimate for $|v(t)|^p_{\mathbb{L}^p(\mathbb{R}^d)}$ for $t\in[t_0,T_*)$. The proof is complete. □

In the next Theorem we will show that if a Beale-Kato-Majda type condition is satisfied i.e. vorticity is bounded then a priori estimate holds uniformly in $\nu\ge 0$.

Theorem 4.5. *Fix $p > d$ and let $\theta \in (0,1)$. Assume that $u_0 \in \mathbb{L}^p(\mathbb{R}^d)$, $f \in M^{2p}([0,T], \mathbb{H}^{3,2p}(\mathbb{R}^d))$, $g \in M^{2p}([0,T], \gamma(H, \mathbb{H}^{4,2p}(\mathbb{R}^d)))$. Let $u \in L^\infty([0,T_{max}); \mathbb{L}^p(\mathbb{R}^d))$ be a strong maximal local solution of Burgers equation. Assume also that a.s.*

$$\operatorname{curl} u \in L^\infty(0, T_{max}; \mathbb{L}^\infty(\mathbb{R}^d)), \tag{4.17}$$

and there exists $t_0 \in (0,T)$ such that

$$\operatorname{div} u(t_0, \cdot) \in \mathbb{L}^\infty(\mathbb{R}^d). \tag{4.18}$$

Then $T_{max} = T$ and we have a.s.

$$\begin{aligned} |u(t)|^p_{\mathbb{L}^p(\mathbb{R}^d)} \leq\ & C(|u_0|^p_{\mathbb{L}^p(\mathbb{R}^d)} + |F(z)|_{L^1(0,t;\mathbb{L}^p(\mathbb{R}^d))}) \\ & \exp\Big\{C(t-t_0)(|\operatorname{div} u(t_0,\cdot)|_{\mathbb{L}^\infty(\mathbb{R}^d)} + |\operatorname{curl} u|_{\mathbb{L}^\infty((0,t]\times\mathbb{R}^d)} \\ & + |z|^2_{L^\infty(0,T;\mathbb{H}^{3,p}(\mathbb{R}^d))}(1 + |u_0|_{L^p} + |z|^2_{\mathbb{H}^{2,p}(\mathbb{R}^d)})e^{t|z|_{\mathbb{H}^{2,p}(\mathbb{R}^d)}})\Big\}. \end{aligned} \tag{4.19}$$

Proof. The proof follows the lines of Theorem 2.2 in [25] and is omitted. □

Remark 4.6. It is possible to construct a random dynamical system corresponding to the solution of the stochastic Burgers equation following the argument of the first named auhor and Yuhong Li [8].

5. Gradient case

In this section we will consider a particular case when the initial condition and force are potential.

Corollary 5.1. *Fix $p > d$. Assume that $\psi_0 \in H^{1,p}(\mathcal{O})$ a.s., $U \in M^{2p}([0,T], H^{4,2p}(\mathcal{O}))$, $V \in M^{2p}([0,T], \gamma(H, H^{5,2p}(\mathcal{O})))$. Then there exists unique global solution $u \in C(0,T; \mathbb{L}^p(\mathcal{O}))$ a.s. of equation*

$$\begin{cases} du + (u\nabla)u dt = (\nu \triangle u + \nabla U)\, dt + \nabla V\, dw_t \\ u(0) = \nabla \psi_0. \end{cases}$$

Furthermore, if ψ_0, U, V are non random then for $\mathcal{O} = \mathbb{T}^d$ we have

$$\mathbb{E} \sup_{s\in[0,t]} |u(s)|^p_{\mathbb{L}^p(\mathcal{O})} \leq C(|\psi_0|_{\mathbb{H}^{1,p}}, |U|_{L^1([0,t],\mathbb{H}^{2,p})}, |V|_{L^2([0,t],\gamma(H,\mathbb{H}^{3,p}))}),$$

and for $\mathcal{O} = \mathbb{R}^d$ we have

$$\mathbb{E} \log(1 + \sup_{s\in[0,t]} |u(s)|^p_{\mathbb{L}^p(\mathcal{O})}) \leq C(|\psi_0|_{\mathbb{H}^{1,p}}, |U|_{L^1([0,t],\mathbb{H}^{2,p})}, |V|_{L^2([0,t],\gamma(H,\mathbb{H}^{3,p}))}),$$

Proof. The assertion (1) follows immediately from Theorem 4.1 and Theorem 4.5. The assertion (3) follows from estimates (4.1), (4.19) and the Fernique Theorem. Indeed, if U, V are non random then the Ornstein-Uhlenbeck process z has Gaussian distribution in $L^2([0,T], \mathbb{H}^{1,p}(\mathcal{O})) \subset L^1([0,T], \mathbb{L}^\infty(\mathcal{O}))$. □

Consequently, since u is a gradient of a certain function provided the initial condition and the force are gradients we can deduce the following corollary.

Corollary 5.2. *Fix $p > d$ and $\nu > 0$. Assume that $\psi_0 \in H^{1,p}(\mathcal{O})$ a.s., $U \in M^{2p}([0,T], H^{4,2p}(\mathcal{O}))$, $V \in M^{2p}([0,T], \gamma(H, H^{5,2p}(\mathcal{O})))$. Then there exists a unique global solution $\psi^\nu \in C(0,T; H^{1,p}(\mathcal{O}))$ a.s. of the equation*

$$\begin{cases} d\psi^\nu + |\nabla \psi^\nu|^2 dt = (\nu \triangle \psi^\nu + U)\, dt + V dW_t \\ \psi^\nu(0) = \psi_0. \end{cases} \tag{5.1}$$

Furthermore, if ψ_0, U, V are non random then for $\mathcal{O} = \mathbb{T}^d$ we have

(5.2)
$$\mathbb{E}\sup_{s\in[0,t]}|\psi^\nu(s)|^p_{H^{1,p}(\mathcal{O})} \le C(|\psi_0|_{H^{1,p}(\mathcal{O})}, |U|_{L^1([0,t],H^{2,p}(\mathcal{O}))}, |V|_{L^2([0,t],\gamma(H,H^{2,p}(\mathcal{O})))}),\ t\ge 0.$$

and for $\mathcal{O} = \mathbb{R}^d$ we have

(5.3)
$$\mathbb{E}\log(1+\sup_{s\in[0,t]}|\psi^\nu(s)|^p_{H^{1,p}(\mathcal{O})}) \le C(|\psi_0|_{H^{1,p}(\mathcal{O})}, |U|_{L^1([0,t],H^{2,p}(\mathcal{O}))}, |V|_{L^2([0,t],\gamma(H,H^{2,p}(\mathcal{O})))}),\ t\ge 0.$$

We note that estimates (5.2), (5.3) are uniform w.r.t. ν. This leads us to the following Corollary.

Corollary 5.3. *Fix $p > d$. Assume that $\psi_0 \in H^{1,p}(\mathcal{O})$ a.s., $U \in L^1([0,T], H^{2,p}(\mathcal{O}))$, $V \in L^p(0,T;\gamma(H,H^{2,p}(\mathcal{O})))$. Then there exists a unique global viscosity solution $\psi \in C(0,T;H^{1,p}(\mathcal{O}))$ of the equation*

$$\begin{cases} d\psi + |\nabla\psi|^2 dt = Udt + VdW_t \\ \psi(0) = \psi_0. \end{cases} \tag{5.4}$$

Furthermore, if $\mathcal{O} = \mathbb{T}^d$, respectively $\mathcal{O} = \mathbb{R}^d$, then we have, for all $t \ge 0$

$$\mathbb{E}\sup_{s\in[0,t]}|\psi(s)|^p_{H^{1,p}(\mathcal{O})} \le C(|\psi_0|_{H^{1,p}}, |U|_{L^1([0,t],H^{2,p})}, |V|_{L^2([0,t],\gamma(H,H^{2,p}))}), \tag{5.5}$$

$$\mathbb{E}\log(1+\sup_{s\in[0,t]}|\psi(s)|^p_{H^{1,p}(\mathcal{O})}) \le C(|\psi_0|_{H^{1,p}}, |U|_{L^1([0,t],H^{2,p})}, |V|_{L^2([0,t],\gamma(H,H^{2,p}))}). \tag{5.6}$$

Remark 5.4. Corollaries 5.2 and 5.3 differ from results of [19], where viscosity solutions are studied in the space of continuous functions while we consider solutions in $\mathbb{H}^{1,p}(\mathcal{O})$, $p > d$.

Proof. Let $\{\psi^\nu\}_{\nu>0} \in C(0,T;\mathbb{H}^{1,p}(\mathcal{O})) \cap C^{1,2}((0,T]\times\mathcal{O})$ be sequence of solutions of the equation (5.1). Since $\mathbb{H}^{1,p}(\mathcal{O}) \subset C(\mathcal{O},\mathbb{R}^d), p > d$ and estimate (4.1) (corr. estimate (4.19) if $\mathcal{O} = \mathbb{R}^d$) we have uniform w.r.t. ν estimate $\mathbb{P}$-a.s.

$$|\psi^\nu|^p_{C(0,T;C(\mathcal{O},\mathbb{R}^d))} \le K(T,\psi_0,h,d), T>0, p>d. \tag{5.7}$$

Then according to Theorem 1.1, p. 175 in [2] there exist uniformly bounded upper continuous subsolution $\psi^* = \overset{*}{\limsup_{\nu\to 0}}\psi^\nu$ $\mathbb{P}$-a.s. and uniformly bounded lower continuous supersolution $\psi_* = \overset{*}{\liminf_{\nu\to 0}}\psi^\nu$ $\mathbb{P}$-a.s. of equation (5.4). Therefore, by comparison principle for viscosity solutions of Hamilton-Jacobi equations (see Theorem 2, p. 585 and Remark 3, p. 593 of [15]), $\psi^* \le \psi_*$ and $\psi = \psi^* = \psi_*$. Thus, ψ^ν locally uniformly converges to unique viscosity solution ψ of equation (5.4) $\mathbb{P}$-a.s. Estimate (5.2) implies that ψ satisfies (5.5).

□

6. Case of Dirichlet boundary conditions

The results obtained for the case of $\mathcal{O} = \mathbb{T}^d$ can be obtained in a similar way when $\mathcal{O}$ is an open connected bounded domain with C^2 boundary and the Burgers equation is supplanted with the Dirichlet boundary conditions. We will use standard notation $\mathbb{H}^{n,p}_0(\mathcal{O}) = H^{n,p}_0(\mathcal{O},\mathbb{R}^d)$ for the closure in the norm of $\mathbb{H}^{n,p}(\mathcal{O})$ of the set of smooth functions with compact support in $\mathcal{O}$. Let Δ be the generator of the heat semigroup (S_t) in $\mathbb{L}^p(\mathcal{O}) := L^p(\mathcal{O},\mathbb{R}^d)$ for $p\in(1,\infty)$ with the domain

$$\mathrm{dom}_{\mathbb{L}^p(\mathcal{O})}(\Delta) = \mathbb{H}^{2,p}(\mathcal{O}) \cap \mathbb{H}^{1,p}_0(\mathcal{O}).$$

Let us recall, that the ultracontractive estimates from Lemma 3.3 are still satisfied and therefore the result of proposition 3.3 holds as well. Hence, the following theorem on the existence of local mild solution to equation (2.6).

Theorem 6.1. *Assume that $p > d$. Then for all $u_0 \in \mathbb{L}^p(\mathcal{O})$, $z \in L^\infty(0,T;\mathbb{L}^{2p}(\mathcal{O}) \cap \mathbb{H}_0^{1,p}(\mathcal{O}))$, there exists $T_0 = T_0(\nu, |u_0|_{\mathbb{L}^p(\mathcal{O})}, |z|_{L^\infty(0,T;\mathbb{L}^{2p}(\mathcal{O})\cap\mathbb{H}_0^{1,p}(\mathcal{O}))}) > 0$ such that there exists unique mild solution $u \in L^\infty(0,T_0;\mathbb{L}^p(\mathcal{O}))$ of equation (2.6). Furthermore*

(a) *$u : (0,T_0] \to \mathbb{L}^{2p}(\mathcal{O})$ is continuous and $\lim_{t\to 0} t^{\frac{d}{4p}} |u(t)|_{\mathbb{L}^{2p}(\mathcal{O})} = 0$.*

(b) *$u : (0,T_0] \to \mathbb{H}_0^{1,p}(\mathcal{O})$ is continuous and $\lim_{t\to 0} t^{\frac{1}{2}} |u(t)|_{\mathbb{H}_0^{1,p}(\mathcal{O})} = 0$.*

Similarly to the torus case we can show that the local solution is either global or blows up in finite time i.e. we have

Lemma 6.2. *Let $p > d$ and let $\theta \in (0,1)$. Assume that $u_0 \in \mathbb{L}^p(\mathcal{O})$,*

$$z \in L^\infty([0,T], \mathbb{H}_0^{1,2p}(\mathcal{O}) \cap \mathbb{H}^{1,p}(\mathcal{O})) \cap C^\theta(0,T;\mathbb{L}^p(\mathcal{O})),$$

and $T_{max} < T$. Let $u \in C([0,T_{max});\mathbb{L}^p(\mathcal{O}))$ be a maximal local mild solution to the Burgers equation (2.6). Then

$$\limsup_{t\nearrow T_{max}} |u(t)|^2_{\mathbb{L}^p(\mathcal{O})} = \infty. \tag{6.1}$$

Consequently, as in the case of torus, an a priori $\mathbb{L}^\infty$ estimate for the solution allows us to deduce the global existence.

Theorem 6.3. *Fix $p > d$. Assume that $u_0 \in \mathbb{L}^p(\mathcal{O})$ a.s., $f \in M^{2p}([0,T], \mathbb{H}_0^{3,2p}(\mathcal{O}))$, and $g \in M^{2p}([0,T], \gamma(H, \mathbb{H}_0^{4,2p}(\mathcal{O})))$. Then there exists a unique strong global $\mathbb{L}^p(\mathcal{O})$-valued solution u of the Burgers equation. Moreover,*

$$|u(t)|^p_{\mathbb{L}^p(\mathcal{O})} \le C(|u_0|^p_{\mathbb{L}^p(\mathcal{O})} + |z|^2_{L^\infty(0,T;\mathbb{H}_0^{2,p}(\mathcal{O}))}) e^{|\nabla z|_{L^1(0,T;\mathbb{L}^\infty(\mathcal{O}))}}, t \in [0,T]. \tag{6.2}$$

In the case of potential force and potential initial condition we obtain the following

Corollary 6.4. *Fix $p > d$. Assume that $\psi_0 \in H^{1,p}(\mathcal{O})$ a.s., $U \in M^{2p}([0,T], H_0^{4,2p}(\mathcal{O}))$, $V \in M^{2p}([0,T], \gamma(H, H_0^{5,2p}(\mathcal{O})))$. Then there exists unique global solution $u \in C(0,T;\mathbb{L}^p(\mathcal{O}))$ a.s. of equation*

$$\begin{cases} du + (u\nabla)udt = (\nu\triangle u + \nabla U)\,dt + \nabla V\,dw_t \\ u(0) = \nabla\psi_0. \end{cases}$$

Furthermore, if ψ_0, U, V are non random then we have

$$\mathbb{E} \sup_{s\in[0,t]} |u(s)|^p_{\mathbb{L}^p(\mathcal{O})} \le C(|\psi_0|_{H^{1,p}}, |U|_{L^1([0,t],H_0^{2,p})}, |V|_{L^2([0,t],\gamma(H,H_0^{3,p}))}),$$

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