A new fusion procedure for the Brauer algebra and evaluation homomorphisms

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Abstract

We give a new fusion procedure for the Brauer algebra by showing that all primitive idempotents can be found by evaluating a rational function in several variables which has the form of a product of R-matrix type factors. In particular, this provides a new fusion procedure for the symmetric group involving an arbitrary parameter. The R-matrices are solutions of the Yang–Baxter equation associated with the classical Lie algebras $\mathfrak{g}_N$ of types B, C and D. Moreover, we construct an evaluation homomorphism from a reflection equation algebra $\mathrm{B}(\mathfrak{g}_N)$ to $\mathrm{U}(\mathfrak{g}_N)$ and show that the fusion procedure provides an equivalence between natural tensor representations of $\mathrm{B}(\mathfrak{g}_N)$ with the corresponding evaluation modules.

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1 Introduction

By an original observation of Jucys [14], all primitive idempotents of the symmetric group $\mathfrak{S}_n$ can be obtained by taking certain limit values of the rational function

$$\Phi(u_1,\dots,u_n)=\prod_{1\leqslant i<j\leqslant n}\Big(1-\frac{s_{ij}}{u_i-u_j}\Big), \tag{1.1}$$

where $s_{ij}\in\mathfrak{S}_n$ is the transposition of i and j, $u_1,\dots,u_n$ are complex variables and the product is calculated in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ in the lexicographic order on the pairs (i,j). This construction, which is commonly known as the *fusion procedure*, was also developed by Cherednik [5], while detailed proofs were given by Nazarov [22]. In the context of the quantum inverse scattering method developed by Faddeev's Leningrad school, the fusion procedure has been regarded as a way to construct new solutions of the Yang–Baxter equation out of old ones; see, e.g., [16]. The version of the fusion procedure found in [18] establishes its equivalence to the construction of the idempotents provided by Jucys [15] and Murphy [21] in terms of some special elements of the group algebra of $\mathfrak{S}_n$. It was shown in [11] that the fusion procedure for the Hecke algebra admits a similar interpretation; cf. [6], [25].

A fusion procedure for the Brauer algebra $\mathcal{B}_n(\omega)$ over $\mathbb{C}(\omega)$ was recently given by two of us in [9]. For this algebra (1.1) was replaced by the rational function

$$\Psi(u_1,\dots,u_n)=\prod_{1\leqslant i<j\leqslant n}\Big(1-\frac{\epsilon_{ij}}{u_i+u_j}\Big)\prod_{1\leqslant i<j\leqslant n}\Big(1-\frac{s_{ij}}{u_i-u_j}\Big) \tag{1.2}$$

previously considered in [24], with the ordered products as in (1.1); the elements ϵ_{ij} and s_{ij} of $\mathcal{B}_n(\omega)$ are defined in Sec. 2 below.

Recall that the irreducible representations of $\mathcal{B}_n(\omega)$ are indexed by all partitions of the nonnegative integers $n,n-2,n-4,\dots$. If λ is such a partition, then the *updown λ-tableaux* T parameterize basis vectors of the corresponding representation; see Sec. 2 for the definitions. This leads to an explicit isomorphism between $\mathcal{B}_n(\omega)$ and the direct sum of matrix algebras. The *primitive idempotents* E^λ_T are the elements of $\mathcal{B}_n(\omega)$ corresponding to the diagonal matrix units under this isomorphism; see [4], [23], [30].

By the main result of [9], given an updown λ-tableau T, the consecutive evaluations

$$(u_1-c_1)^{p_1}\dots(u_n-c_n)^{p_n}\,\Psi(u_1,\dots,u_n)\big|_{u_1=c_1}\big|_{u_2=c_2}\dots\big|_{u_n=c_n} \tag{1.3}$$

are well-defined and this value yields the corresponding primitive idempotent E^λ_T multiplied by a nonzero constant $f(T)$ which is calculated in an explicit form. Here the c_i are the contents of T and $p_1,\dots,p_n$ are certain integers depending on T, called its exponents.

The first main result of this paper is a new construction of all primitive idempotents of the Brauer algebra $\mathcal{B}_n(\omega)$; we use a different rational function in place of (1.2). Namely, set

$$\Omega(u_1,\dots,u_n)=\prod_{1\leqslant i<j\leqslant n}\Big(1+\frac{s_{ij}}{u_i+u_j-\omega/2+1}-\frac{\epsilon_{ij}}{u_i+u_j}\Big)$$
$$\times\prod_{1\leqslant i<j\leqslant n}\Big(1-\frac{s_{ij}}{u_i-u_j}+\frac{\epsilon_{ij}}{u_i-u_j-\omega/2+1}\Big) \qquad (1.4)$$

with both products taken in the lexicographic order on the pairs (i,j). Thus,

$$\Omega(u_1,\dots,u_n)=\prod_{1\leqslant i<j\leqslant n}\rho_{ij}(-u_i-u_j+\varkappa)\prod_{1\leqslant i<j\leqslant n}\rho_{ij}(u_i-u_j), \qquad (1.5)$$

where we use the notation

$$\rho_{ij}(u)=1-\frac{s_{ij}}{u}+\frac{\epsilon_{ij}}{u-\varkappa}, \qquad \varkappa=\frac{\omega}{2}-1. \qquad (1.6)$$

The rational functions $\rho_{ij}(u)$ satisfy the Yang–Baxter equation

$$\rho_{ij}(u)\,\rho_{ik}(u+v)\,\rho_{jk}(v)=\rho_{jk}(v)\,\rho_{ik}(u+v)\,\rho_{ij}(u), \qquad (1.7)$$

with any distinct indices i,j,k, where we set $\rho_{ji}(u)=\rho_{ij}(u)$ for $i<j$; see [32]. Also, for $i\neq j$ we have

$$\rho_{ij}(u)\,\rho_{ij}(-u)=\frac{u^2-1}{u^2}. \qquad (1.8)$$

The new version of the fusion procedure for $\mathcal{B}_n(\omega)$ takes the following form: the consecutive evaluations

$$(u_1-c_1)^{p_1}\dots(u_n-c_n)^{p_n}\,\Omega(u_1,\dots,u_n)\big|_{u_1=c_1}\big|_{u_2=c_2}\dots\big|_{u_n=c_n} \qquad (1.9)$$

are well-defined and this value yields the product $h(T)\,E_T^{\lambda}$ as with (1.3), where $h(T)$ is a constant calculated in a way similar to $f(T)$. To give an equivalent formulation, for any updown tableau T introduce a rational function $\Omega_T(u_1,\dots,u_n)$ which is obtained from $\Omega(u_1,\dots,u_n)$ by multiplying by a numerical rational function in $u_1,\dots,u_n$ depending on the contents of T; see (2.6). Then the fusion procedure can be reformulated as the relation

$$E_T=\Omega_T(u_1,\dots,u_n)\big|_{u_1=c_1}\big|_{u_2=c_2}\dots\big|_{u_n=c_n}. \qquad (1.10)$$

The existence of the two forms (1.3) and (1.9) (or, equivalently, (1.10)) of the fusion procedure can be explained by their connections with two different quantum algebras associated with a classical Lie algebra of type B, C or D. Consider the natural action of the Brauer algebra $\mathcal{B}_n(\omega)$ (with an appropriate specialization of the parameter ω) on the tensor product space

$$\mathbb{C}^N\otimes\mathbb{C}^N\otimes\dots\otimes\mathbb{C}^N, \qquad n\ \text{ factors}; \qquad (1.11)$$

see (4.3) and (4.4) below. Using the expression for the idempotent E_T^λ provided by (1.3) we find that the subspace $E_T^\lambda(\mathbb{C}^N)^{\otimes n}$ carries the structure of a representation of the *Olshanski twisted Yangian* $\mathrm{Y}'(\mathfrak{g}_N)$ associated with the orthogonal Lie algebra $\mathfrak{g}_N = \mathfrak{o}_N$ or symplectic Lie algebra $\mathfrak{g}_N = \mathfrak{sp}_N$, respectively, ($N$ is even for the latter) [24], [26]; see also [19, Ch. 2]. On the other hand, the representation of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$ on the space $E_T^\lambda(\mathbb{C}^N)^{\otimes n}$ arising from the Brauer–Schur–Weyl duality extends to the twisted Yangian $\mathrm{Y}'(\mathfrak{g}_N)$ via the evaluation homomorphism $\mathrm{Y}'(\mathfrak{g}_N) \to \mathrm{U}(\mathfrak{g}_N)$ [26]. Thus, the fusion procedure in the form associated with the evaluations (1.3) yields an equivalence of the two twisted Yangian actions on the space $E_T^\lambda(\mathbb{C}^N)^{\otimes n}$.

In the new version of the fusion procedure associated with the evaluations (1.9), the role of the twisted Yangian is now taken by the *reflection algebra* $\mathrm{B}(\mathfrak{g}_N)$. This algebra is defined by a reflection equation and it is closely related to the *Drinfeld Yangian* $\mathrm{Y}(\mathfrak{g}_N)$; see Definition 3.1 below. The Yangian $\mathrm{Y}(\mathfrak{g}_N)$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$ as a subalgebra; however, due to a result of Drinfeld [7], in contrast with the Yangian for $\mathfrak{gl}_N$, there is no homomorphism $\mathrm{Y}(\mathfrak{g}_N) \to \mathrm{U}(\mathfrak{g}_N)$, identical on the subalgebra $\mathrm{U}(\mathfrak{g}_N)$.

The algebras $\mathrm{B}(\mathfrak{o}_N)$ and $\mathrm{B}(\mathfrak{sp}_{2n})$ were formally defined in [1] together with their super-version $\mathrm{B}(\mathfrak{osp}_{m|2n})$; however, they do not seem to have received much attention in the literature. Our second main result (Theorem 3.3) is a construction of an evaluation homomorphism $\mathrm{B}(\mathfrak{g}_N) \to \mathrm{U}(\mathfrak{g}_N)$. Furthermore, we show that the algebra $\mathrm{B}(\mathfrak{g}_N)$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$ as a subalgebra, and the evaluation homomorphism is identical on this subalgebra. These results also apply to the case of the Lie superalgebra $\mathfrak{osp}_{m|2n}$, and we indicate necessary changes in the notation in Remark 3.9.

The expression for the idempotent E_T^λ provided by (1.9) indicates that the subspace $E_T^\lambda(\mathbb{C}^N)^{\otimes n}$ carries the structure of a representation of the corresponding reflection algebra $\mathrm{B}(\mathfrak{g}_N)$. Moreover, we show that, as with the twisted Yangian, this representation factors through the evaluation homomorphism $\mathrm{B}(\mathfrak{g}_N) \to \mathrm{U}(\mathfrak{g}_N)$.

Taking the quotient of the Brauer algebra $\mathcal{B}_n(\omega)$ by the relations $\epsilon_{ij} = 0$ we come to a new version of the fusion procedure for the symmetric group $\mathfrak{S}_n$ involving an arbitrary parameter ω (Corollary 2.7). The standard version of the procedure associated with the rational function (1.1) is recovered in the limit $\omega \to \infty$. We show that the new version is related to the well-known evaluation homomorphism $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$ and to its restriction to the subalgebra of $\mathrm{Y}(\mathfrak{gl}_N)$ isomorphic to a reflection algebra.

Both the fusion procedure and the evaluation homomorphism described in Theorems 2.2 and 3.3 admit their natural quantum analogues. Namely, all primitive idempotents of the Birman–Murakami–Wenzl algebra can be obtained by evaluating a universal rational function. Moreover, it is possible to introduce a q-analogue $\mathrm{B}_q(\mathfrak{g}_N)$ of the reflection algebra and to construct a homomorphism from $\mathrm{B}_q(\mathfrak{g}_N)$ to the Drinfeld–Jimbo quantum group $\mathrm{U}_q(\mathfrak{g}_N)$ with the properties similar to (3.26). The details will appear in our forthcoming publication [10].

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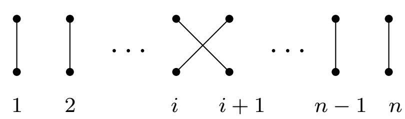
2 Fusion procedure

Let n be a positive integer and ω an indeterminate. An n-diagram d is a collection of $2n$ dots arranged into two rows with n dots in each row connected by n edges such that any dot belongs to only one edge. The product of two diagrams d_1 and d_2 is determined by placing d_1 above d_2 and identifying the vertices of the bottom row of d_1 with the corresponding vertices in the top row of d_2. Let s be the number of closed loops obtained in this placement. The product $d_1 d_2$ is given by ω^s times the resulting diagram without loops. The *Brauer algebra* $\mathcal{B}_n(\omega)$ is defined as the $\mathbb{C}(\omega)$-linear span of the n-diagrams with the multiplication defined above. The dimension of the algebra is $1 \cdot 3 \cdots (2n-1)$. The following presentation of $\mathcal{B}_n(\omega)$ is well-known; see, e.g., [3].

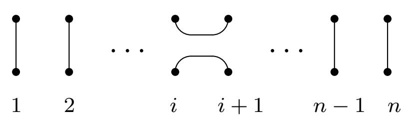
Proposition 2.1. *The Brauer algebra $\mathcal{B}_n(\omega)$ is isomorphic to the algebra with $2n-2$ generators $s_1, \dots, s_{n-1}, \epsilon_1, \dots, \epsilon_{n-1}$ and the defining relations*

$$
\begin{aligned}
&s_i^2 = 1, \qquad \epsilon_i^2 = \omega\, \epsilon_i, \qquad s_i \epsilon_i = \epsilon_i s_i = \epsilon_i, \qquad i = 1, \dots, n-1, \\
&s_i s_j = s_j s_i, \qquad \epsilon_i \epsilon_j = \epsilon_j \epsilon_i, \qquad s_i \epsilon_j = \epsilon_j s_i, \qquad |i-j| > 1, \\
&s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad \epsilon_i \epsilon_{i+1} \epsilon_i = \epsilon_i, \qquad \epsilon_{i+1} \epsilon_i \epsilon_{i+1} = \epsilon_{i+1}, \\
&s_i \epsilon_{i+1} \epsilon_i = s_{i+1} \epsilon_i, \qquad \epsilon_{i+1} \epsilon_i s_{i+1} = \epsilon_{i+1} s_i, \qquad i = 1, \dots, n-2.
\end{aligned}
$$

The generators s_i and ϵ_i correspond to the following diagrams respectively:



and



The subalgebra of $\mathcal{B}_n(\omega)$ generated over $\mathbb{C}$ by $s_1, \dots, s_{n-1}$ is isomorphic to the group algebra $\mathbb{C}[\mathfrak{S}_n]$ so that s_i can be identified with the transposition $(i, i+1)$. Then for any $1 \leqslant i < j \leqslant n$ the transposition $s_{ij} = (i, j)$ can be regarded as an element of $\mathcal{B}_n(\omega)$. Moreover, ϵ_{ij} will denote the element of $\mathcal{B}_n(\omega)$ represented by the diagram in which the i-th and j-th dots in the top row, as well as the i-th and j-th dots in the bottom row are connected by an edge, while the remaining edges connect the k-th dot in the top row with the k-th dot in the bottom row for each $k \neq i, j$. Equivalently, in terms of the presentation of $\mathcal{B}_n(\omega)$ provided by Proposition 2.1,

$$s_{ij} = s_i\, s_{i+1} \dots s_{j-2}\, s_{j-1}\, s_{j-2} \dots s_{i+1}\, s_i \qquad \text{and} \qquad \epsilon_{ij} = s_{i,j-1}\, \epsilon_{j-1}\, s_{i,j-1}.$$

We also set $\epsilon_{ji} = \epsilon_{ij}$ and $s_{ji} = s_{ij}$ for $i < j$. The Brauer algebra $\mathcal{B}_{n-1}(\omega)$ can be regarded as the subalgebra of $\mathcal{B}_n(\omega)$ spanned by all diagrams in which the n-th dots in the top and bottom rows are connected by an edge.

The *Jucys–Murphy elements* $x_1, \dots, x_n$ for the Brauer algebra $\mathcal{B}_n(\omega)$ are given by the formulas

$$x_r = \frac{\omega - 1}{2} + \sum_{i=1}^{r-1} (s_{ir} - \epsilon_{ir}), \qquad r = 1, \dots, n; \tag{2.1}$$

see [17] and [23], where, in particular, the eigenvalues of the x_r in irreducible representations were calculated. The element x_n commutes with the subalgebra $\mathcal{B}_{n-1}(\omega)$. This implies that the elements $x_1, \dots, x_n$ of $\mathcal{B}_n(\omega)$ pairwise commute. They can be used to construct a complete set of pairwise orthogonal primitive idempotents for the Brauer algebra following the approach of Jucys [15] and Murphy [21]. Namely, let λ be a partition of $n - 2f$ for some $f \in \{0, 1, \dots, \lfloor n/2 \rfloor\}$. We will identify partitions with their diagrams so that if the parts of λ are $\lambda_1, \lambda_2, \dots$ then the corresponding diagram is a left-justified array of rows of unit boxes containing λ_1 boxes in the top row, λ_2 boxes in the second row, etc. The box in row i and column j of a diagram will be denoted as the pair (i, j). An *updown λ-tableau* is a sequence $T = (\Lambda_1, \dots, \Lambda_n)$ of diagrams such that for each $r = 1, \dots, n$ the diagram Λ_r is obtained from Λ_{r-1} by adding or removing one box, where we set $\Lambda_0 = \varnothing$, the empty diagram, and $\Lambda_n = \lambda$. To each updown tableau T we attach the corresponding sequence of *contents* $(c_1, \dots, c_n)$, $c_r = c_r(T)$, where

$$c_r = \frac{\omega - 1}{2} + j - i \qquad \text{or} \qquad c_r = -\Big(\frac{\omega - 1}{2} + j - i\Big),$$

if Λ_r is obtained by adding the box (i, j) to Λ_{r-1} or by removing this box from Λ_{r-1}, respectively. The primitive idempotents $E_T = E_T^{\lambda}$ can now be defined by the following recurrence formula (we omit the superscripts indicating the diagrams since they are determined by the updown tableaux). Set $\mu = \Lambda_{n-1}$ and consider the updown μ-tableau $U = (\Lambda_1, \dots, \Lambda_{n-1})$. Let α be the box which is added to or removed from μ to get λ. Then

$$E_T = E_U\, \frac{(x_n - a_1) \dots (x_n - a_k)}{(c_n - a_1) \dots (c_n - a_k)}, \tag{2.2}$$

where $a_1, \dots, a_k$ are the contents of all boxes excluding α, which can be removed from or added to μ to get a diagram. When λ runs over all partitions of $n, n-2, \dots,$ and T runs over all updown λ-tableaux, the elements $\{E_T\}$ yield a complete set of pairwise orthogonal primitive idempotents for $\mathcal{B}_n(\omega)$. They have the properties

$$x_r\, E_T = E_T\, x_r = c_r(T)\, E_T, \qquad r = 1, \dots, n. \tag{2.3}$$

Moreover, given an updown tableau $U = (\Lambda_1, \dots, \Lambda_{n-1})$, we have the relation

$$E_U = \sum_T E_T, \tag{2.4}$$

the sum is over all updown tableaux of the form $T = (\Lambda_1, \dots, \Lambda_{n-1}, \Lambda_n)$; see e.g. [17] and [23]. Relation (2.2) implies

$$E_T = E_U \, \frac{(u - c_n)(u + x_n - \varkappa)}{(u - x_n)(u + c_n - \varkappa)} \Big|_{u=c_n}, \tag{2.5}$$

where u is a complex variable and we use notation (1.6). This relation follows by application of (2.4) and then (2.3).

Given an updown tableau $T = (\Lambda_1, \dots, \Lambda_n)$ with the respective contents $c_1, \dots, c_n$, introduce a rational function in $u_1, \dots, u_n$ with values in the Brauer algebra $\mathcal{B}_n(\omega)$ by

$$\Omega_T(u_1, \dots, u_n) = \prod_{r=2}^{n} \frac{(u_r - c_r)(u_r + c_1 - \varkappa)}{(u_r - c_1)(u_r + c_r - \varkappa)} \prod_{i=1}^{r-1} \frac{(u_r - u_i)^2}{(u_r - u_i)^2 - 1} \, \Omega(u_1, \dots, u_n), \tag{2.6}$$

where $\Omega(u_1, \dots, u_n)$ is defined in (1.4) and (1.5). Note that if the indices i, j, k, l are distinct then the elements $\rho_{ij}(u)$ and $\rho_{kl}(v)$ commute. Together with (1.7) this implies the relation

$$\begin{aligned} &\rho_{1,n}(-u_1 - u_n + \varkappa) \dots \rho_{n-1,n}(-u_{n-1} - u_n + \varkappa) \prod_{1 \leqslant i < j \leqslant n-1} \rho_{ij}(u_i - u_j) \\ &\qquad = \prod_{1 \leqslant i < j \leqslant n-1} \rho_{ij}(u_i - u_j) \, \rho_{n-1,n}(-u_{n-1} - u_n + \varkappa) \dots \rho_{1,n}(-u_1 - u_n + \varkappa). \end{aligned}$$

An easy induction on n leads to the following equivalent expression for the rational function $\Omega(u_1, \dots, u_n)$:

$$\begin{aligned} \Omega(u_1, \dots, u_n) = \prod_{r=2}^{n} \, &\rho_{r-1,r}(-u_{r-1} - u_r + \varkappa) \dots \rho_{1,r}(-u_1 - u_r + \varkappa) \\ &\times \rho_{1,r}(u_1 - u_r) \dots \rho_{r-1,r}(u_{r-1} - u_r), \end{aligned} \tag{2.7}$$

where the factors are ordered in accordance with the increasing values of r.

Theorem 2.2. *The idempotent E_T is found by the consecutive evaluations*

$$E_T = \Omega_T(u_1, \dots, u_n)\big|_{u_1=c_1}\big|_{u_2=c_2} \dots \big|_{u_n=c_n}.$$

Proof. We use the induction on n. By the induction hypothesis, setting $u = u_n$ and using (2.7) we get

$$\begin{aligned} &\Omega_T(u_1, \dots, u_n)\big|_{u_1=c_1}\big|_{u_2=c_2} \dots \big|_{u_{n-1}=c_{n-1}} = \frac{(u - c_n)(u + c_1 - \varkappa)}{(u - c_1)(u + c_n - \varkappa)} \prod_{i=1}^{n-1} \frac{(u - c_i)^2}{(u - c_i)^2 - 1} \\ &\times E_U \, \rho_{n-1,n}(-c_{n-1} - u + \varkappa) \dots \rho_{1,n}(-c_1 - u + \varkappa) \, \rho_{1,n}(c_1 - u) \dots \rho_{n-1,n}(c_{n-1} - u), \end{aligned} \tag{2.8}$$

where U is the updown tableau $(\Lambda_1, \dots, \Lambda_{n-1})$.

Lemma 2.3. *We have the identity*

$$E_U\,\rho_{n-1,n}(-c_{n-1}-u+\varkappa)\dots\rho_{1,n}(-c_1-u+\varkappa)\,\rho_{1,n}(c_1-u)\dots\rho_{n-1,n}(c_{n-1}-u)$$
$$=\frac{u-c_1}{u+c_1-\varkappa}\prod_{i=1}^{n-1}\frac{(u-c_i)^2-1}{(u-c_i)^2}\,E_U\,\frac{u+x_n-\varkappa}{u-x_n}. \tag{2.9}$$

Proof. Embed the Brauer algebra $\mathcal{B}_n(\omega)$ into $\mathcal{B}_m(\omega)$ for some $m \geqslant n$ and prove a more general identity

$$E_U\,\rho_{n-1,m}(-c_{n-1}-u+\varkappa)\dots\rho_{1,m}(-c_1-u+\varkappa)\,\rho_{1,m}(c_1-u)\dots\rho_{n-1,m}(c_{n-1}-u)$$
$$=\frac{u-c_1}{u+c_1-\varkappa}\prod_{i=1}^{n-1}\frac{(u-c_i)^2-1}{(u-c_i)^2}\,E_U\,\frac{u+x_m^{(n-1)}-\varkappa}{u-x_m^{(n-1)}}, \tag{2.10}$$

where

$$x_m^{(k)}=\frac{\omega-1}{2}+\sum_{i=1}^{k}(s_{im}-\epsilon_{im}); \tag{2.11}$$

in particular, $x_n^{(n-1)}=x_n$; see (2.1). We use induction on n while keeping m fixed. In the case $n=1$ the identity is certainly true. Suppose that $n\geqslant 2$. By (2.4) we have $E_U=E_U\,E_W$, where W is the updown tableau $(\Lambda_1,\dots,\Lambda_{n-2})$ (in the case $n=2$ we set $E_W=1$). Hence, using the induction hypothesis we can write the left hand side of (2.10) as

$$E_U\,\rho_{n-1,m}(-c_{n-1}-u+\varkappa)$$
$$\times\frac{u-c_1}{u+c_1-\varkappa}\prod_{i=1}^{n-2}\frac{(u-c_i)^2-1}{(u-c_i)^2}\,E_W\,\frac{u+x_m^{(n-2)}-\varkappa}{u-x_m^{(n-2)}}\,\rho_{n-1,m}(c_{n-1}-u)$$

which equals

$$\frac{u-c_1}{u+c_1-\varkappa}\prod_{i=1}^{n-2}\frac{(u-c_i)^2-1}{(u-c_i)^2}\,E_U\,\rho_{n-1,m}(-c_{n-1}-u+\varkappa)\,\frac{u+x_m^{(n-2)}-\varkappa}{u-x_m^{(n-2)}}\,\rho_{n-1,m}(c_{n-1}-u). \tag{2.12}$$

Applying (1.8) we find that

$$\rho_{n-1,m}(c_{n-1}-u)=\frac{(u-c_{n-1})^2-1}{(u-c_{n-1})^2}\,\rho_{n-1,m}(u-c_{n-1})^{-1}.$$

Hence, comparing (2.12) with the right hand side of (2.10), we conclude that the lemma will be implied by the identity

$$E_U\,\rho_{n-1,m}(-c_{n-1}-u+\varkappa)\,\frac{u+x_m^{(n-2)}-\varkappa}{u-x_m^{(n-2)}}=E_U\,\frac{u+x_m^{(n-1)}-\varkappa}{u-x_m^{(n-1)}}\,\rho_{n-1,m}(u-c_{n-1}).$$

Since $x_m^{(n-1)}$ commutes with E_U, we can write the identity in the equivalent form

$$E_U\,(u - x_m^{(n-1)})\Big(1 + \frac{s_{n-1,m}}{c_{n-1}+u-\varkappa} - \frac{\epsilon_{n-1,m}}{c_{n-1}+u}\Big)\,(u + x_m^{(n-2)} - \varkappa)$$
$$- E_U\,(u + x_m^{(n-1)} - \varkappa)\Big(1 + \frac{s_{n-1,m}}{c_{n-1}-u} - \frac{\epsilon_{n-1,m}}{c_{n-1}-u+\varkappa}\Big)\,(u - x_m^{(n-2)}) = 0.$$

To verify the latter, note that by (2.3) and the relations in the Brauer algebra,

$$E_U\,x_m^{(n-1)}\,s_{n-1,m}\,x_m^{(n-2)} = E_U\,x_m^{(n-1)}\,x_{n-1}\,s_{n-1,m}$$
$$= E_U\,x_{n-1}\,x_m^{(n-1)}\,s_{n-1,m} = c_{n-1}\,E_U\,x_m^{(n-1)}\,s_{n-1,m},$$

while

$$E_U\,x_m^{(n-1)}\,\epsilon_{n-1,m}\,x_m^{(n-2)} = -E_U\,x_{n-1}\,\epsilon_{n-1,m}\,x_m^{(n-2)} = -c_{n-1}\,E_U\,\epsilon_{n-1,m}\,x_m^{(n-2)}.$$

Together with the relation $x_m^{(n-1)} = x_m^{(n-2)} + s_{n-1,m} - \epsilon_{n-1,m}$ this yields the identity in question and completes the proof of the lemma. □

Return to the proof of the theorem; we are left to verify that the rational function

$$E_U\,\frac{(u-c_n)(u+x_n-\varkappa)}{(u-x_n)(u+c_n-\varkappa)}$$

is well-defined at $u = c_n$ and the value coincides with E_T. But this holds due to (2.5). □

Example 2.4. (i) In the case of the Brauer algebra $\mathcal{B}_2(\omega)$ we have three updown tableaux

$$U_1 = (\square,\ \ \square\square), \qquad U_2 = \Big(\square,\ \ \begin{smallmatrix}\square\\ \square\end{smallmatrix}\Big), \qquad U_3 = (\square,\ \ \varnothing).$$

The respective contents (c_1, c_2) are given by

$$\Big(\frac{\omega-1}{2}, \frac{\omega+1}{2}\Big), \qquad \Big(\frac{\omega-1}{2}, \frac{\omega-3}{2}\Big), \qquad \Big(\frac{\omega-1}{2}, -\frac{\omega-1}{2}\Big).$$

The definition (2.6) reads

$$\Omega_T(u_1,u_2) = \frac{(u_2-c_2)(u_2+c_1-\varkappa)}{(u_2-c_1)(u_2+c_2-\varkappa)}\,\frac{(u_1-u_2)^2}{(u_1-u_2)^2-1}$$
$$\times\Big(1 + \frac{s_1}{u_1+u_2-\varkappa} - \frac{\epsilon_1}{u_1+u_2}\Big)\Big(1 - \frac{s_1}{u_1-u_2} + \frac{\epsilon_1}{u_1-u_2-\varkappa}\Big)$$

so that applying Theorem 2.2, we find

$$E_{U_1} = \frac{1+s_1}{2} - \frac{\epsilon_1}{\omega}, \qquad E_{U_2} = \frac{1-s_1}{2}, \qquad E_{U_3} = \frac{\epsilon_1}{\omega}.$$

(ii) In the case of $\mathcal{B}_3(\omega)$ consider the following two updown tableaux associated with the diagram $\lambda = (1)$:

$$T_1 = \big(\square, \quad \begin{smallmatrix}\square\\ \square\end{smallmatrix}, \quad \square\big), \qquad T_2 = \big(\square, \quad \varnothing, \quad \square\big).$$

The respective contents (c_1, c_2, c_3) are given by

$$\Big(\frac{\omega-1}{2}, \frac{\omega-3}{2}, -\frac{\omega-3}{2}\Big), \qquad \Big(\frac{\omega-1}{2}, -\frac{\omega-1}{2}, \frac{\omega-1}{2}\Big).$$

Then using Theorem 2.2 and (2.8) we find that E_{T_1} and E_{T_2} can be calculated by evaluating the respective rational functions

$$\frac{(u-c_3)(u+c_1-\varkappa)}{(u-c_1)(u+c_3-\varkappa)} \, \frac{(u-c_1)^2}{(u-c_1)^2-1} \, \frac{(u-c_2)^2}{(u-c_2)^2-1} \\ \times E_U \, \rho_{23}(-c_2-u+\varkappa) \, \rho_{13}(-c_1-u+\varkappa) \, \rho_{13}(c_1-u) \, \rho_{23}(c_2-u)$$

at $u = c_3$ with $U = U_2$ and $U = U_3$, respectively; see Example (i). Performing the calculation we get

$$E_{T_1} = \frac{(1-s_1)\,\epsilon_2\,(1-s_1)}{2\,(\omega-1)}, \qquad E_{T_2} = \frac{\epsilon_1}{\omega}.$$

(iii) For $\mathcal{B}_n(\omega)$ consider the only updown tableau $T = \big((1), (2), \dots, (n)\big)$ associated with the diagram $\lambda = (n)$. The corresponding contents $(c_1, \dots, c_n)$ are found by the formula $c_i = (\omega-1)/2 + i - 1$. Hence, by Theorem 2.2 the idempotent $S_n = E_T$ (the *symmetrizer* in the Brauer algebra) is given by

$$S_n = \frac{1}{n!} \prod_{r=2}^{n} \frac{\omega+2r-2}{\omega+4r-4} \prod_{1\leqslant i<j\leqslant n} \rho_{ij}(2-i-j-\omega/2) \prod_{1\leqslant i<j\leqslant n} \rho_{ij}(i-j)$$

with the products over the pairs (i, j) taken in the lexicographic order. Given any index $k \in \{1, \dots, n-1\}$, we can use (1.7) to reorder the factors in the last product in such a way that it would begin with the factor $\rho_{k,k+1}(-1)$. Since

$$\epsilon_k \, \rho_{k,k+1}(-1) = 0 \qquad \text{and} \qquad s_k \, \rho_{k,k+1}(-1) = \rho_{k,k+1}(-1),$$

we find that for any $k < l$

$$\rho_{kl}(u) \prod_{1\leqslant i<j\leqslant n} \rho_{ij}(i-j) = \frac{u-1}{u} \prod_{1\leqslant i<j\leqslant n} \rho_{ij}(i-j).$$

Hence the expression for the symmetrizer simplifies to

$$S_n = \frac{1}{n!} \prod_{1\leqslant i<j\leqslant n} \rho_{ij}(i-j);$$

cf. [9, Remark 3.8].

(iv) In the case of $\mathcal{B}_n(\omega)$ consider the only updown tableau $T = \big((1), (1^2), \dots, (1^n)\big)$ associated with the diagram $\lambda = (1^n)$. The corresponding contents $(c_1, \dots, c_n)$ are given by $c_i = (\omega - 1)/2 - i + 1$. Hence, by Theorem 2.2 the idempotent $A_n = E_T$ (the *anti-symmetrizer* in the Brauer algebra) is given by

$$A_n = \frac{1}{n!} \prod_{r=2}^{n} \frac{\omega - 2r + 2}{\omega - 4r + 4} \prod_{1 \leqslant i < j \leqslant n} \rho_{ij}(i + j - 2 - \omega/2) \prod_{1 \leqslant i < j \leqslant n} \rho_{ij}(j - i) \tag{2.13}$$

with the products over the pairs (i, j) taken in the lexicographic order. A different expression for the anti-symmetrizer is provided by the alternative fusion procedure in [9, Remark 3.8]:

$$A_n = \frac{1}{n!} \prod_{1 \leqslant i < j \leqslant n} \Big(1 - \frac{s_{ij}}{j - i}\Big) \tag{2.14}$$

with the product taken in the lexicographic order on the pairs (i, j). A more direct way to see the coincidence of the elements given by (2.13) and (2.14) is to regard the group algebra $\mathbb{C}[\mathfrak{S}_n]$ for the symmetric group as a natural subalgebra of $\mathcal{B}_n(\omega)$ and observe that (2.14) is the well-known factorization of the anti-symmetrizer in $\mathbb{C}[\mathfrak{S}_n]$. Indeed, these anti-symmetrizers satisfy the well-known recurrence relation $n\, A_n = A_{n-1}\big(1 - s_{1n} - \dots - s_{n-1,n}\big)$. Therefore, employing the induction on n and using the recurrence relations (2.2) in $\mathcal{B}_n(\omega)$ we come to checking the identity

$$A_{n-1} \frac{(x_n - c_1 - 1)(x_n + c_1 - n + 2)}{2c_1 - 2n + 3} = A_{n-1}\big(-1 + s_{1n} + \dots + s_{n-1,n}\big),$$

where, as before, $c_1 = (\omega - 1)/2$. The calculation is straightforward and it relies on the following relations in $\mathcal{B}_n(\omega)$ which hold for any two distinct indices $i, j \in \{1, \dots, n-1\}$:

$$A_{n-1}\, s_{in}\, s_{jn} = -A_{n-1}\, s_{in}, \quad A_{n-1}\, s_{in}\, \epsilon_{jn} = 0, \quad A_{n-1}\, \epsilon_{in}\, \epsilon_{jn} = A_{n-1}\, \epsilon_{in}\, s_{ij} = -A_{n-1}\, \epsilon_{jn}$$

and

$$A_{n-1}(\epsilon_{in} s_{jn} + \epsilon_{jn} s_{in}) = A_{n-1}(\epsilon_{in} s_{jn} - s_{ij} \epsilon_{jn} s_{in}) = A_{n-1}(\epsilon_{in} s_{jn} - \epsilon_{in} s_{jn}) = 0. \qquad \square$$

To make a connection of the version of the fusion procedure provided by Theorem 2.2 with that of [9], recall some parameters associated with updown tableaux; see [9]. Given an updown μ-tableau $U = (\Lambda_1, \dots, \Lambda_{n-1})$ we define two infinite matrices $m(U)$ and $m'(U)$ whose rows and columns are labelled by positive integers and only a finite number of entries in each of the matrices are nonzero. The entry m_{ij} of the matrix $m(U)$ (resp., the entry m'_{ij} of the matrix $m'(U)$) equals the number of times the box (i, j) was added (resp., removed) in the sequence of diagrams $(\varnothing = \Lambda_0, \Lambda_1, \dots, \Lambda_{n-1})$. For each integer k define the nonnegative integers $d_k = d_k(U)$ and $d'_k = d'_k(U)$ as the respective sums of the entries of the matrices $m(U)$ and $m'(U)$ on the k-th diagonal,

$$d_k = \sum_{j-i=k} m_{ij}, \qquad d'_k = \sum_{j-i=k} m'_{ij},$$

and set

$$g_k(U) = \delta_{k0} + d_{k-1} + d_{k+1} - 2\,d_k, \qquad g'_k(U) = d'_{k-1} + d'_{k+1} - 2\,d'_k. \tag{2.15}$$

Now the *exponents* $p_1, \dots, p_n$ of an updown λ-tableau $T = (\Lambda_1, \dots, \Lambda_n)$ are defined inductively, so that p_r depends only on the first r diagrams $(\Lambda_1, \dots, \Lambda_r)$ of T. Hence, it is sufficient to define p_n. Taking $U = (\Lambda_1, \dots, \Lambda_{n-1})$ we set

$$p_n = 1 - g_{k_n}(U) \qquad \text{or} \qquad p_n = 1 - g'_{k_n}(U), \tag{2.16}$$

respectively, if Λ_n is obtained from Λ_{n-1} by adding a box on the diagonal k_n or by removing a box on the diagonal k_n.

Define the constants $h(T)$ inductively by the formula

$$h(T) = h(U)\,\psi(U,T), \tag{2.17}$$

where $U = (\Lambda_1, \dots, \Lambda_{n-1})$, $T = (\Lambda_1, \dots, \Lambda_n)$, and $h(U) = 1$ if $U = (\Lambda_1)$. Here

$$\psi(U,T) = \frac{4k_n + \omega}{2k_n + \omega} \prod_{k \neq k_n} (k_n - k)^{g_k} \prod_k (k_n + k + \omega - 1)^{g'_k}$$

or

$$\psi(U,T) = \frac{4k_n + 6\omega - 4}{2k_n + \omega - 2} \prod_{k \neq k_n} (-k_n + k)^{g'_k} \prod_k (-k_n - k - \omega + 1)^{g_k},$$

if Λ_n is obtained from Λ_{n-1} by adding or removing a box on the diagonal k_n, respectively, where the products are taken over all integers k, while $g_k = g_k(U)$ and $g'_k = g'_k(U)$.

Consider now the rational function $\Omega(u_1, \dots, u_n)$ with values in the Brauer algebra $\mathcal{B}_n(\omega)$ defined by (1.4).

Corollary 2.5. *For any updown tableau $T = (\Lambda_1, \dots, \Lambda_n)$ the consecutive evaluations*

$$(u_1 - c_1)^{p_1} \dots (u_n - c_n)^{p_n}\, \Omega(u_1, \dots, u_n)\big|_{u_1=c_1}\big|_{u_2=c_2} \dots \big|_{u_n=c_n}$$

are well-defined. The corresponding value coincides with $h(T)\,E_T$.

Proof. We use induction on n as in the proof of Theorem 2.2. By the induction hypothesis, setting $u = u_n$ we get

$$(u_1 - c_1)^{p_1} \dots (u_n - c_n)^{p_n}\, \Omega(u_1, \dots, u_n)\big|_{u_1=c_1}\big|_{u_2=c_2} \dots \big|_{u_{n-1}=c_{n-1}} = h(U)\,(u - c_n)^{p_n}$$
$$\times\, E_U\, \rho_{n-1,n}(-c_{n-1} - u + \varkappa) \dots \rho_{1,n}(-c_1 - u + \varkappa)\, \rho_{1,n}(c_1 - u) \dots \rho_{n-1,n}(c_{n-1} - u), \tag{2.18}$$

where U is the updown tableau $(\Lambda_1,\dots,\Lambda_{n-1})$. Applying Lemma 2.3, we come to showing that the rational function

$$h(U)\,\frac{(u-c_1)(u+c_n-\varkappa)}{u+c_1-\varkappa}\prod_{r=1}^{n-1}\Big(1-\frac{1}{(u-c_r)^2}\Big)\,(u-c_n)^{p_n-1}\cdot E_U\,\frac{u-c_n}{u-x_n}\,\frac{u+x_n-\varkappa}{u+c_n-\varkappa}$$

is regular at $u=c_n$ and its value equals $h(T)\,E_T$. Using the parameters (2.15), we can write this expression as

$$h(U)\,\frac{u+c_n-\varkappa}{u+c_1-\varkappa}\prod_k\Big(u-\frac{\omega-1}{2}-k\Big)^{g_k}\prod_k\Big(u+\frac{\omega-1}{2}+k\Big)^{g'_k}\,(u-c_n)^{p_n-1}$$
$$\times\,E_U\,\frac{u-c_n}{u-x_n}\,\frac{u+x_n-\varkappa}{u+c_n-\varkappa},$$

where k runs over the set of integers. If the diagram Λ_n is obtained from Λ_{n-1} by adding or removing a box on the diagonal k_n, then the value of the content c_n is given by the respective formulas

$$c_n=\frac{\omega-1}{2}+k_n\qquad\text{or}\qquad c_n=-\Big(\frac{\omega-1}{2}+k_n\Big).$$

The argument is completed by recalling the definition of the exponents (2.16), and the constants $h(T)$ in (2.17) together with (2.5). □

An important particular case of Theorem 2.2 and Corollary 2.5 is the case where λ is a partition of n. Here the updown tableaux T of shape λ can be regarded as the *standard tableaux* which parameterize basis vectors of the corresponding representation of $\mathcal{B}_n(\omega)$. It was pointed out in [9, Corollary 3.7] that in this situation all exponents p_i are equal to zero and the constant $f(T)$ arising in the evaluations (1.3) depends only on λ and does not depend on the standard λ-tableau T. The new version of the fusion procedure associated with the rational function (1.4) possesses the same property as the next corollary shows. To formulate the result introduce the *content polynomial*

$$C_\lambda(z)=\prod_{\alpha\in\lambda}\big(z+\sigma(\alpha)\big),$$

where $\sigma(\alpha)=j-i$ if the box α of the diagram λ is in row i and column j.

Corollary 2.6. *If $T=(\Lambda_1,\dots,\Lambda_n)$ is an updown λ-tableau and $\lambda=\Lambda_n$ is a partition of n, then the consecutive evaluations give*

$$\Omega(u_1,\dots,u_n)\big|_{u_1=c_1}\big|_{u_2=c_2}\dots\big|_{u_n=c_n}=\frac{2^n\,C_\lambda(\omega/4)\,H(\lambda)}{C_\lambda(\omega/2)}\,E_T,$$

where $H(\lambda)$ is the product of the hooks of λ.

Proof. Arguing as in the proof of Corollary 2.5 and using the respective calculation of [18] we observe that the recurrence relation for the constants $h(T)$ now takes the form

$$h(T) = h(U)\,\frac{H(\lambda)}{H(\mu)}\,\frac{\omega+4\sigma_n}{\omega+2\sigma_n},$$

where μ is the shape of the tableau U and $\sigma_n = \sigma(\alpha)$ for the box α occupied by n. An obvious induction completes the proof. $\square$

As the group algebra $\mathbb{C}[\mathfrak{S}_n]$ can be regarded as the quotient of the Brauer algebra $\mathcal{B}_n(\omega)$ by the ideal generated by the elements $\epsilon_1, \dots, \epsilon_{n-1}$, Corollary 2.6 yields a new version of the fusion procedure for the symmetric group. Moreover, this procedure involves an arbitrary parameter ω inherited from $\mathcal{B}_n(\omega)$. To formulate the corresponding version introduce the rational function in variables $v_1, \dots, v_n$ with values in $\mathbb{C}[\mathfrak{S}_n]$ by

$$\Omega_\omega(v_1,\dots,v_n) = \prod_{1\leqslant i<j\leqslant n}\Big(1+\frac{s_{ij}}{v_i+v_j+\omega/2}\Big)\prod_{1\leqslant i<j\leqslant n}\Big(1-\frac{s_{ij}}{v_i-v_j}\Big) \qquad (2.19)$$

with both products taken in the lexicographic order on the pairs (i,j).

Corollary 2.7. *Suppose that λ is a partition of n and T is a standard λ-tableau. Let $\sigma_k = j-i$ if k occupies the box (i,j) in T. Then the consecutive evaluations give*

$$\Omega_\omega(v_1,\dots,v_n)\big|_{v_1=\sigma_1}\big|_{v_2=\sigma_2}\dots\big|_{v_n=\sigma_n} = \frac{2^n\,C_\lambda(\omega/4)\,H(\lambda)}{C_\lambda(\omega/2)}\,E_T,$$

where $H(\lambda)$ is the product of the hooks of λ and E_T is the primitive idempotent in $\mathbb{C}[\mathfrak{S}_n]$ associated with T.

Proof. This is immediate from Corollary 2.6, where the relation between the variables is given by $u_i = v_i + c_1$ for all i and we used the relation $2c_1 - \varkappa = \omega/2$. $\square$

This version of the fusion procedure for the symmetric group appears to be new. We will discuss its meaning in the context of evaluation homomorphisms below in Sec. 4. By taking the limit $\omega \to \infty$ we recover the standard fusion procedure originated in Jucys [14] in the form found in [18]; cf. [5] and [22].

3 Evaluation homomorphisms

Let $G = [g_{ij}]$ be a nonsingular symmetric or skew-symmetric $N \times N$ matrix with entries in $\mathbb{C}$ so that $G^t = \pm G$, where t denotes the standard matrix transposition. The skew-symmetric case may occur only if N is even. Consider the canonical basis $e_1, \dots, e_N$ of the vector space $\mathbb{C}^N$ and equip it with the bilinear form $\langle\,,\rangle$ by

$$\langle e_i, e_j\rangle = g_{ij}. \qquad (3.1)$$

The classical orthogonal and symplectic groups O_N and Sp_N consist of the $N\times N$ matrices $\mathbf{h}$ preserving the symmetric or skew-symmetric form, respectively,

$$\langle \mathbf{h}v, \mathbf{h}w\rangle = \langle v, w\rangle \qquad \text{for all} \quad v, w \in \mathbb{C}^N. \tag{3.2}$$

We will be using the following convention. Whenever the double sign $\pm$ or $\mp$ occurs, the upper sign will correspond to the symmetric case and the lower sign to the skew-symmetric case. For any $N \times N$ matrix A set

$$A' = GA^t\, G^{-1}. \tag{3.3}$$

We denote by E_{ij}, $1 \leqslant i, j \leqslant N$, the standard basis vectors of the Lie algebra $\mathfrak{gl}_N$. They satisfy the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj}. \tag{3.4}$$

For the entries of the inverse matrix of G we will write $G^{-1} = [\overline{g}_{ij}]$. Introduce the elements F_{ij} of the Lie algebra $\mathfrak{gl}_N$ by the formulas

$$F_{ij} = E_{ij} - \sum_{k,l=1}^{N} g_{ik}\,\overline{g}_{lj}\,E_{lk}. \tag{3.5}$$

The Lie subalgebra of $\mathfrak{gl}_N$ spanned by the elements F_{ij} is isomorphic to the orthogonal Lie algebra $\mathfrak{o}_N$ associated with O_N in the symmetric case and to the symplectic Lie algebra $\mathfrak{sp}_N$ associated with Sp_N in the skew-symmetric case. This Lie algebra will be denoted by $\mathfrak{g}_N$. Common choices of the matrix G in the symmetric case include the identity matrix $G = 1$ and the antidiagonal matrix $G = [\delta_{i,N-j+1}]$. In the skew-symmetric case with $N = 2n$ the entries of G are often chosen in the form $g_{ij} = \delta_{i,2n-j+1}$ for $i = 1, \dots, n$ and $g_{ij} = -\delta_{i,2n-j+1}$ for $i = n+1, \dots, 2n$. In all these cases, the summation in the formula (3.5) reduces to one term.

Consider the endomorphism algebra $\mathrm{End}\,\mathbb{C}^N$ and let $e_{ij} \in \mathrm{End}\,\mathbb{C}^N$ be the standard matrix units. We denote by F the $N \times N$ matrix whose ij-th entry is F_{ij}. We shall also regard F as the element

$$F = \sum_{i,j=1}^{N} e_{ij} \otimes F_{ij} \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{U}(\mathfrak{g}_N).$$

The definition (3.5) can be written in the matrix form as

$$F = E - E' = E - G\,E^t\,G^{-1}, \tag{3.6}$$

where

$$E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij} \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{U}(\mathfrak{gl}_N).$$

Consider the permutation operator

$$P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} \in \operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N \tag{3.7}$$

and its partial transpose P^t with respect to the first (or, equivalently, the second) copy of $\operatorname{End} \mathbb{C}^N$:

$$P^t = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ij}. \tag{3.8}$$

Furthermore, set

$$Q = G_1 \, P^t \, G_1^{-1} = G_2 \, P^t \, G_2^{-1}, \tag{3.9}$$

where the second equality follows from the relations $G_1 \, P^t = G_2^t \, P^t$ and $P^t \, G_1 = P^t \, G_2^t$.

Note that the operators P and Q satisfy the relations

$$P^2 = 1, \qquad PQ = QP = \pm Q, \qquad Q^2 = N \, Q. \tag{3.10}$$

The defining relations of the algebra $\mathrm{U}(\mathfrak{gl}_N)$ can be written in a matrix form as

$$E_1 \, E_2 - E_2 \, E_1 = E_1 \, P - P \, E_1,$$

where both sides are regarded as elements of the algebra $\operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N \otimes \mathrm{U}(\mathfrak{gl}_N)$ and

$$E_1 = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes E_{ij}, \qquad E_2 = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes E_{ij}. \tag{3.11}$$

The defining relations of the algebra $\mathrm{U}(\mathfrak{g}_N)$ then take the form

$$F_1 \, F_2 - F_2 \, F_1 = F_1 \, (P - Q) - (P - Q) \, F_1 \tag{3.12}$$

together with the relation $F + F' = 0$. The latter implies

$$Q \, (F_1 + F_2) = (F_1 + F_2) \, Q = 0. \tag{3.13}$$

Set

$$\kappa = N/2 \mp 1. \tag{3.14}$$

The R-matrix $R(u)$ is a rational function in a complex parameter u with values in the tensor product algebra $\operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N$ defined by

$$R(u) = 1 - \frac{P}{u} + \frac{Q}{u - \kappa}. \tag{3.15}$$

It is well known by [32] that $R(u)$ satisfies the Yang–Baxter equation

$$R_{12}(u) \, R_{13}(u+v) \, R_{23}(v) = R_{23}(v) \, R_{13}(u+v) \, R_{12}(u). \tag{3.16}$$

Here both sides take values in $\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N$ and the subscripts indicate the copies of $\operatorname{End}\mathbb{C}^N$ so that $R_{12}(u) = R(u) \otimes 1$ etc. Clearly, this is a recast of the properties of the functions $\rho_{ij}(u)$ defined in (1.6); we will discuss this relationship in more detail in Sec. 4.

Following the general approach of [8] and [27], define the *extended Yangian* $\mathrm{X}(\mathfrak{g}_N)$ as an associative algebra with generators $t_{ij}^{(r)}$, where $1 \leqslant i, j \leqslant N$ and $r = 1, 2, \dots$, satisfying certain quadratic relations. In order to write them down, introduce the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)} u^{-r} \in \mathrm{X}(\mathfrak{g}_N)[[u^{-1}]] \tag{3.17}$$

and set

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{X}(\mathfrak{g}_N)[[u^{-1}]]. \tag{3.18}$$

Consider the algebra $\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N \otimes \mathrm{X}(\mathfrak{g}_N)[[u^{-1}]]$ and introduce its elements $T_1(u)$ and $T_2(u)$ by

$$T_1(u) = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes t_{ij}(u), \qquad T_2(u) = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes t_{ij}(u). \tag{3.19}$$

The defining relations for the algebra $\mathrm{X}(\mathfrak{g}_N)$ can then be written in the form

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v). \tag{3.20}$$

The *Yangian* $\mathrm{Y}(\mathfrak{g}_N)$ is then defined as the quotient of the extended Yangian $\mathrm{X}(\mathfrak{g}_N)$ by the relation

$$T'(u+\kappa)\, T(u) = 1. \tag{3.21}$$

Although the matrix G is implicit in the definitions of $\mathrm{X}(\mathfrak{g}_N)$ and $\mathrm{Y}(\mathfrak{g}_N)$, the respective algebras associated with two different nonsingular symmetric (or skew-symmetric) $N \times N$ matrices G and $\widetilde{G}$ are isomorphic to each other so that each of them depends only on the Lie algebra $\mathfrak{g}_N$. Drinfeld defined the Yangians associated with simple Lie algebras by using different presentations; see [7, 8]. Explicit form of the defining relations for some standard choices of antidiagonal symmetric and skew-symmetric matrices G in terms of the series $t_{ij}(u)$ can be found in [1], where the equivalence of the R-matrix presentation and the Drinfeld presentation is explained; see also [2]. The Yangian $\mathrm{Y}(\mathfrak{g}_N)$ carries a Hopf algebra structure defined in a standard way; see e.g. *loc. cit.* for explicit formulas.

Following [29], use a reflection type equation to define algebras associated to $\mathfrak{g}_N$; see also [1], [20].

Definition 3.1. The *reflection algebra* $\mathrm{B}(\mathfrak{g}_N)$ corresponding to the Lie algebra $\mathfrak{g}_N$ is defined as the associative algebra generated by elements $s_{ij}^{(r)}$ with $1 \leqslant i,j \leqslant N$ and $r \geqslant 1$ subject to defining relations written in terms of the generating series

$$s_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} s_{ij}^{(r)}\, u^{-r}$$

as follows. Introduce the matrix

$$S(u) = \sum_{i,j=1}^{N} e_{ij} \otimes s_{ij}(u) \in \operatorname{End} \mathbb{C}^N \otimes \mathrm{B}(\mathfrak{g}_N)[[u^{-1}]]. \tag{3.22}$$

Then the defining relations have the form of the *reflection equation*

$$R(u-v)\, S_1(u)\, R(u+v)\, S_2(v) = S_2(v)\, R(u+v)\, S_1(u)\, R(u-v) \tag{3.23}$$

and the *unitary condition*

$$S(u)\, S(-u) = 1, \tag{3.24}$$

where we use the matrix notation as in (3.20). $\square$

Proposition 3.2. *The mapping*

$$S(u) \mapsto T(-u)^{-1}\, T(u) \tag{3.25}$$

defines an algebra homomorphism $\mathrm{B}(\mathfrak{g}_N) \to \mathrm{X}(\mathfrak{g}_N)$.

Proof. It is obvious that (3.24) holds when $S(u)$ is replaced by the matrix $T(-u)^{-1}\, T(u)$. Checking that the image satisfies (3.23) is standard and relies on (3.20) and the relation

$$T_1(u)\, R(u+v)\, T_2(-v)^{-1} = T_2(-v)^{-1}\, R(u+v)\, T_1(u)$$

implied by (3.20). $\square$

The Yangian $\mathrm{Y}(\mathfrak{g}_N)$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$ as a subalgebra. However, as shown by Drinfeld [7], in contrast with the Yangian for the Lie algebra $\mathfrak{gl}_N$, there is no homomorphism $\mathrm{Y}(\mathfrak{g}_N) \to \mathrm{U}(\mathfrak{g}_N)$, identical on the subalgebra $\mathrm{U}(\mathfrak{g}_N)$. Nevertheless, as the following Theorem 3.3 and Corollary 3.7 show, such a homomorphism from the reflection algebra does exist. This result is suggested by the solutions of the reflection equation associated with the affine Birman–Murakami–Wenzl algebras found in [12].

Theorem 3.3. *The mapping*

$$S(u) \mapsto \frac{u + F - N/4}{u - F + N/4} = 1 + 2 \sum_{r=1}^{\infty} \Big(F - \frac{N}{4}\Big)^{r} u^{-r} \tag{3.26}$$

defines an algebra homomorphism $\mathrm{B}(\mathfrak{g}_N) \to \mathrm{U}(\mathfrak{g}_N)$.

Proof. Set $\mathcal{F}(u) = u + F$. To make our formulas more readable, we will be using the notation $a = N/4$ throughout the proof. We will demonstrate that both relations (3.23) and (3.24) will hold when the matrix $S(u)$ is replaced by the product $\mathcal{F}(u-a)\mathcal{F}(-u-a)^{-1}$. This is obvious for (3.24), so we come to proving the identity

$$\begin{aligned} R(u-v)\, \mathcal{F}_1(u-a)\, \mathcal{F}_1(-u-a)^{-1}\, R(u+v)\, \mathcal{F}_2(v-a)\, \mathcal{F}_2(-v-a)^{-1} \\ = \mathcal{F}_2(v-a)\, \mathcal{F}_2(-v-a)^{-1}\, R(u+v)\, \mathcal{F}_1(u-a)\, \mathcal{F}_1(-u-a)^{-1}\, R(u-v). \end{aligned} \tag{3.27}$$

Lemma 3.4. *We have the identity*

$$R(u-v)\, \mathcal{F}_1(u)\, \mathcal{F}_2(v) - \mathcal{F}_2(v)\, \mathcal{F}_1(u)\, R(u-v) = \frac{1}{u-v-\kappa}\, U, \tag{3.28}$$

where

$$U = Q\,(F_1+\kappa)\, F_2 - F_2\,(F_1+\kappa)\, Q.$$

Proof. The left hand side of (3.28) simplifies to

$$F_1\, F_2 - F_2\, F_1 + P\, F_1 - F_1\, P + \frac{1}{u-v-\kappa}\Big(Q\, F_1\, F_2 - F_2\, F_1\, Q + (u-v)\big(Q\, F_2 - F_2\, Q\big)\Big),$$

where we used the relations $P\, F_1 = F_2\, P$ and (3.13). Now use (3.12) to write

$$F_1\, F_2 - F_2\, F_1 + P\, F_1 - F_1\, P = Q\, F_1 - F_1\, Q = -Q\, F_2 + F_2\, Q$$

which gives the required relation. □

We will now establish some simple properties of the element U to be used below.

Lemma 3.5. *We have the relations*

$$P\, U = U\, P = \pm U, \tag{3.29}$$

$$Q\, U + U\, Q = N\, U, \tag{3.30}$$

$$(F_1 + F_2)\, U = U\, Q. \tag{3.31}$$

Proof. We have

$$P\, U = P\, Q\,(F_1+\kappa)\, F_2 - P\, F_2\,(F_1+\kappa)\, Q = \pm Q\,(F_1+\kappa)\, F_2 \mp F_1\,(F_2+\kappa)\, Q.$$

Applying (3.10) and (3.12), we get

$$\begin{aligned} F_1\, F_2\, Q &= \big(F_2\, F_1 + F_1\,(P-Q) - (P-Q)\, F_1\big)\, Q \\ &= F_2\, F_1\, Q - (N \mp 1)\, F_1\, Q \mp F_2\, Q = F_2\, F_1\, Q - 2\,\kappa\, F_1\, Q, \end{aligned}$$

where we also used (3.13) and the easily verified relation $Q\, F_1\, Q = 0$. This implies $P\, U = \pm U$. The second relation in (3.29) is verified by essentially the same calculation.

Relation (3.30) is immediate from the identity $QF_1F_2Q = QF_2F_1Q$. To prove (3.31) use (3.13) to write

$$\begin{aligned}(F_1+F_2)\,U &= -(F_1+F_2)\,F_2\,(F_1+\kappa)\,Q = -[F_1+F_2, F_2\,(F_1+\kappa)]\,Q\\ &= -[F_1,F_2]\,(F_1+\kappa)\,Q + F_2\,[F_1,F_2]\,Q.\end{aligned}$$

Due to (3.12) we may replace $[F_1, F_2]$ by $[F_1, P-Q]$ and complete the calculation as in the proof of (3.29) to show that the expression coincides with $U\,Q$. □

Multiplying both sides of (3.28) by $\mathcal{F}_2(v)^{-1}$ from the left and the right, we come to the relation

$$\mathcal{F}_1(u)\,R(u-v)\,\mathcal{F}_2(v)^{-1} - \mathcal{F}_2(v)^{-1}\,R(u-v)\,\mathcal{F}_1(u) = -\frac{1}{u-v-\kappa}\,\mathcal{F}_2(v)^{-1}\,U\,\mathcal{F}_2(v)^{-1}. \quad (3.32)$$

By conjugating its both sides by P and using Lemma 3.5, we get

$$\mathcal{F}_1(v)^{-1}\,R(u-v)\,\mathcal{F}_2(u) - \mathcal{F}_2(u)\,R(u-v)\,\mathcal{F}_1(v)^{-1} = \frac{1}{u-v-\kappa}\,\mathcal{F}_1(v)^{-1}\,U\,\mathcal{F}_1(v)^{-1}. \quad (3.33)$$

We will need one more relation obtained from (3.33) by multiplying both sides by $\mathcal{F}_2(u)^{-1}$ from the left and the right:

$$\begin{aligned}&R(u-v)\,\mathcal{F}_1(v)^{-1}\,\mathcal{F}_2(u)^{-1} - \mathcal{F}_2(u)^{-1}\,\mathcal{F}_1(v)^{-1}\,R(u-v)\\ &\qquad = -\frac{1}{u-v-\kappa}\,\mathcal{F}_2(u)^{-1}\,\mathcal{F}_1(v)^{-1}\,U\,\mathcal{F}_1(v)^{-1}\,\mathcal{F}_2(u)^{-1}. \quad (3.34)\end{aligned}$$

Our strategy in proving (3.27) is to use Lemma 3.4 and relations (3.32)–(3.34) to transform the expression on the left hand side to that on the right hand side and then to show that the additional terms arising in the process will add up to zero. We start by applying (3.33) with the substitution $u \mapsto v-a$ and $v \mapsto -u-a$. This yields the expression

$$R(u-v)\,\mathcal{F}_1(u-a)\,\mathcal{F}_2(v-a)\,R(u+v)\,\mathcal{F}_1(-u-a)^{-1}\,\mathcal{F}_2(-v-a)^{-1}$$

together with the additional term

$$\frac{1}{u+v-\kappa}\,R(u-v)\,\mathcal{F}_1(u-a)\,\mathcal{F}_1(-u-a)^{-1}\,U\,\mathcal{F}_1(-u-a)^{-1}\,\mathcal{F}_2(-v-a)^{-1}. \quad (3.35)$$

Now use (3.28) with the substitution $u \mapsto u-a$ and $v \mapsto v-a$ to get

$$\mathcal{F}_2(v-a)\,\mathcal{F}_1(u-a)\,R(u-v)\,R(u+v)\,\mathcal{F}_1(-u-a)^{-1}\,\mathcal{F}_2(-v-a)^{-1}$$

with the additional term

$$\frac{1}{u-v-\kappa}\,U\,R(u+v)\,\mathcal{F}_1(-u-a)^{-1}\,\mathcal{F}_2(-v-a)^{-1}. \quad (3.36)$$

Furthermore, since $R(u-v)\,R(u+v) = R(u+v)\,R(u-v)$, applying (3.34) with the substitution $u \mapsto -v-a$ and $v \mapsto -u-a$ we get the expression

$$\mathcal{F}_2(v-a)\,\mathcal{F}_1(u-a)\,R(u+v)\,\mathcal{F}_2(-v-a)^{-1}\,\mathcal{F}_1(-u-a)^{-1}\,R(u-v)$$

together with the additional term

$$-\frac{1}{u-v-\kappa}\,\mathcal{F}_2(v-a)\,\mathcal{F}_1(u-a)\,R(u+v)\,\mathcal{F}_2(-v-a)^{-1} \\ \times\,\mathcal{F}_1(-u-a)^{-1}\,U\,\mathcal{F}_1(-u-a)^{-1}\,\mathcal{F}_2(-v-a)^{-1}. \quad (3.37)$$

Finally, the application of (3.32) with the substitution $u \mapsto u-a$ and $v \mapsto -v-a$ yields the right hand side of (3.27) with the additional term

$$-\frac{1}{u+v-\kappa}\,\mathcal{F}_2(v-a)\,\mathcal{F}_2(-v-a)^{-1}\,U\,\mathcal{F}_2(-v-a)^{-1}\,\mathcal{F}_1(-u-a)^{-1}\,R(u-v). \quad (3.38)$$

We need to show that the sum of the four additional terms (3.35)–(3.38) is zero. To do this, note that the sum of the terms (3.37) and (3.38) equals the sum of the terms

$$-\frac{1}{u-v-\kappa}\,\mathcal{F}_2(v-a)\,\mathcal{F}_2(-v-a)^{-1}\,R(u+v)\,\mathcal{F}_1(u-a) \\ \times\,\mathcal{F}_1(-u-a)^{-1}\,U\,\mathcal{F}_1(-u-a)^{-1}\,\mathcal{F}_2(-v-a)^{-1} \quad (3.39)$$

and

$$-\frac{1}{u+v-\kappa}\,\mathcal{F}_2(v-a)\,\mathcal{F}_2(-v-a)^{-1}\,U\,R(u-v)\,\mathcal{F}_1(-u-a)^{-1}\,\mathcal{F}_2(-v-a)^{-1}. \quad (3.40)$$

Indeed, this follows by applying (3.32) with the substitution $u \mapsto u-a$ and $v \mapsto -v-a$ to (3.37), and applying (3.34) with the substitution $u \mapsto -v-a$ and $v \mapsto -u-a$ to (3.38). Clearly, the new additional terms arising in these transformations cancel.

Now multiply each of the expressions in (3.35), (3.36), (3.39) and (3.40) by the product $\mathcal{F}_2(-v-a)\,\mathcal{F}_1(-u-a)$ from the right and by $(u-v-\kappa)(u+v-\kappa)\,\mathcal{F}_2(-v-a)$ from the left. Adding up the resulting expressions we get

$$\begin{aligned} &(u-v-\kappa)\,\mathcal{F}_2(-v-a)\,R(u-v)\,\mathcal{F}_1(u-a)\,\mathcal{F}_1(-u-a)^{-1}\,U \\ &+(u+v-\kappa)\,\mathcal{F}_2(-v-a)\,U\,R(u+v) \\ &-(u+v-\kappa)\,\mathcal{F}_2(v-a)\,R(u+v)\,\mathcal{F}_1(u-a)\,\mathcal{F}_1(-u-a)^{-1}\,U \\ &-(u-v-\kappa)\,\mathcal{F}_2(v-a)\,U\,R(u-v). \end{aligned}$$

Using the definition (3.15) of $R(u)$ we bring this sum to the form

$$\begin{aligned} &2v\,\Big(P-Q-\mathcal{F}_2(u-\kappa-a)\Big)\mathcal{F}_1(u-a)\,\mathcal{F}_1(-u-a)^{-1}\,U \\ &+2v\Big(U(P-Q)+\mathcal{F}_2(-u+\kappa-a)\,U\Big) \\ &+\frac{2v\kappa}{u^2-v^2}\,\Big(\mathcal{F}_2(-u-a)\,P\,\mathcal{F}_1(u-a)\,\mathcal{F}_1(-u-a)^{-1}\,U-\mathcal{F}_2(u-a)\,UP\Big). \end{aligned}$$

By Lemma 3.9 we have

$$P\,\mathcal{F}_1(u-a)\,\mathcal{F}_1(-u-a)^{-1}\,U=\mathcal{F}_2(u-a)\,\mathcal{F}_2(-u-a)^{-1}\,U\,P$$

so that the last summand vanishes. Thus, it remains to verify that

$$\Big(P-Q-\mathcal{F}_2(u-\kappa-a)\Big)\mathcal{F}_1(u-a)\,\mathcal{F}_1(-u-a)^{-1}\,U \\ +U(P-Q)+\mathcal{F}_2(-u+\kappa-a)\,U=0. \quad (3.41)$$

However, by (3.12), the term $P-Q-\mathcal{F}_2(u-\kappa-a)$ commutes with F_1. Therefore, when multiplied from the left by $\mathcal{F}_1(-u-a)$, the left hand side of (3.41) becomes

$$\mathcal{F}_1(u-a)\,\Big(P-Q-\mathcal{F}_2(u-\kappa-a)\Big)\,U+\mathcal{F}_1(-u-a)\,\Big(U(P-Q)+\mathcal{F}_2(-u+\kappa-a)\,U\Big),$$

which simplifies to

$$u\Big(N\,U+(P-Q)\,U-U\,(P-Q)-2(F_1+F_2)\,U\Big)+(F_1-a)\Big((P-Q)\,U+U\,(P-Q)-2\kappa\,U\Big).$$

Lemma 3.5 implies that this expression is zero thus proving (3.41) and the theorem. □

Remark 3.6. The evaluation homomorphism of Theorem 3.3 can be "lifted" to the *affine Brauer algebra* defined in [23] (under the name *degenerate affine Wenzl algebra*). The precise statement will be given in [10] in the context of the affine Birman–Murakami–Wenzl algebras in the spirit of the approach to the representation theory of these algebras developed in [13]. □

In the following we denote by $s'_{ij}(u)$ the entries of the matrix $S'(u)=G\,S^t(u)\,G^{-1}$, and write

$$s'_{ij}(u)=\delta_{ij}+\sum_{r=1}^{\infty}s'^{(r)}_{ij}\,u^{-r}.$$

Corollary 3.7. *The assignment*

$$F_{ij}\mapsto\frac{1}{4}\left(s^{(1)}_{ij}-s'^{(1)}_{ij}\right) \quad (3.42)$$

defines an embedding $\mathrm{U}(\mathfrak{g}_N)\hookrightarrow\mathrm{B}(\mathfrak{g}_N)$. *Moreover, the homomorphism* (3.26) *is identical on* $\mathrm{U}(\mathfrak{g}_N)$.

Proof. Write (3.42) in a matrix form

$$F\mapsto\frac{1}{4}\,(S-S'), \quad (3.43)$$

with $S=[s^{(1)}_{ij}]$ and $S'=[s'^{(1)}_{ij}]$, and verify that this assignment defines a homomorphism. Obviously, the relation $F+F'=0$ is preserved by the assignment so we are left to show that (3.12) is preserved as well. Expand the rational functions in u and v involved in (3.23) into series in v^{-1} by

$$\frac{1}{v-a} = v^{-1} + a v^{-2} + \dots$$

and compare the coefficients of $u^{-1}v^{-1}$ on both sides. This yields

$$S_1\, S_2 - S_2\, S_1 = 2\, S_1\, (P-Q) - 2\, (P-Q)\, S_1.$$

Now apply partial transpositions with respect to the first and the second copies of $\mathrm{End}\,\mathbb{C}^N$ to get

$$\begin{aligned} S_1'\, S_2 - S_2\, S_1' &= 2\, S_1'\, (P-Q) - 2\, (P-Q)\, S_1', \\ S_1\, S_2' - S_2'\, S_1 &= 2\, (P-Q)\, S_1 - 2\, S_1\, (P-Q), \\ S_1'\, S_2' - S_2'\, S_1' &= 2\, (P-Q)\, S_1' - 2\, S_1'\, (P-Q). \end{aligned}$$

This implies that (3.12) holds under the assignment (3.43) and thus the latter defines a homomorphism.

Furthermore, under the homomorphism (3.26) we have $S \mapsto 2\,F - N/2$ and so its composition with (3.43) (applying (3.43) first) is the identity map on $\mathrm{U}(\mathfrak{g}_N)$. Hence the kernel of the homomorphism (3.43) is zero. □

Remark 3.8. Due to relation (3.21), the homomorphism (3.25) provides a homomorphism $\mathrm{B}(\mathfrak{g}_N) \to \mathrm{Y}(\mathfrak{g}_N)$ which can be written as

$$S(u) \to T'(-u+\kappa)\, T(u). \tag{3.44}$$

As pointed out in [1], this brings up a connection of the reflection algebra $\mathrm{B}(\mathfrak{g}_N)$ with the Olshanski *twisted Yangians* [26], see also [19, Ch. 2]. The homomorphism (3.44) is not injective and its kernel can be described by a symmetry-type relations; cf. *loc. cit.* For instance, in the case $G = 1$ the elements $s_{ii}^{(1)}$ belong to the kernel. However, $s_{ii}^{(1)} \neq 0$ in $\mathrm{B}(\mathfrak{g}_N)$, as follows from Theorem 3.3. In other words, if we define the twisted Yangian $\mathrm{Y}^{\mathrm{tw}}(\mathfrak{g}_N)$ as the quotient of $\mathrm{B}(\mathfrak{g}_N)$ by the kernel of the homomorphism (3.44), then the evaluation homomorphism of Theorem 3.3 will not factor through the natural epimorphism $\mathrm{B}(\mathfrak{g}_N) \to \mathrm{Y}^{\mathrm{tw}}(\mathfrak{g}_N)$. □

Remark 3.9. Most of the results of this paper admit natural super-analogues where the Lie algebra $\mathfrak{g}_N$ is replaced by the orthosymplectic Lie superalgebra $\mathfrak{osp}_{m|2n}$. Indeed, our calculations are performed in a matrix language so that to apply the same approach in the super case we only need to set up appropriate matrix notation taking care of the sign rules. Here we will indicate the necessary changes to be made in the notation and formulas. The $\mathbb{Z}_2$-degree (or parity) of the basis elements E_{ij} of the Lie superalgebra $\mathfrak{gl}_{m|2n}$ is given by $\deg(E_{ij}) = \bar{\imath} + \bar{\jmath}$, where $\bar{\imath} = 0$ for $1 \leqslant i \leqslant m$ and $\bar{\imath} = 1$ for $m+1 \leqslant i \leqslant m+2n$. The commutation relations in $\mathfrak{gl}_{m|2n}$ have the form

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj} (-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})},$$

where the square brackets denote the super-commutator. We will work with square matrices $A=[A_{ij}]$ of size $m+2n$. Any such matrix with entries in a superalgebra $\mathcal{A}$ will be identified with the element

$$\sum_{i,j=1}^{m+2n} e_{ij}\otimes A_{ij}(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}}\in \operatorname{End}\mathbb{C}^{m|2n}\otimes\mathcal{A}.$$

The signs here are necessary to preserve matrix multiplication rules as we work with graded tensor products of superalgebras. Fix a block-diagonal matrix G whose upper-left $m\times m$ block is a nonsingular symmetric matrix while the lower-right $2n\times 2n$ block is nonsingular skew-symmetric matrix. Given a matrix A we define its transpose associated with G by the formula (3.3), where t now denotes a standard matrix super-transposition, $(A^t)_{ij} = A_{ji}(-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}}$. Note that in contrast with the standard super-transposition, the transposition defined by (3.3) is involutive. We will also regard t as a linear map

$$t:\operatorname{End}\mathbb{C}^{m|2n}\to\operatorname{End}\mathbb{C}^{m|2n},\qquad e_{ij}\mapsto e_{ji}(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}}. \tag{3.45}$$

In the case of multiple tensor products of the superalgebras $\operatorname{End}\mathbb{C}^{m|2n}$ we will indicate by t_a the map (3.45) acting on the a-th copy. Introduce the square matrix $E=[E_{ij}(-1)^{\bar{\jmath}}]$ of size $m+2n$ and define the elements F_{ij} of $\mathfrak{gl}_{m|2n}$ by the formula (3.6), where F is the matrix $F=[F_{ij}(-1)^{\bar{\jmath}}]$. The Lie superalgebra spanned by the elements F_{ij} is isomorphic to the orthosymplectic Lie superalgebra $\mathfrak{osp}_{m|2n}$. The matrix form of the commutation relations in $\mathfrak{osp}_{m|2n}$ coincides with (3.12) together with $F+F'=0$; the latter implies (3.13). This time both sides of (3.12) are elements of the superalgebra

$$\operatorname{End}\mathbb{C}^{m|2n}\otimes\operatorname{End}\mathbb{C}^{m|2n}\otimes\mathrm{U}(\mathfrak{osp}_{m|2n}),$$

while the definitions of the operators P and Q are modified by

$$P=\sum_{i,j=1}^{m+2n} e_{ij}\otimes e_{ji}(-1)^{\bar{\jmath}}\in\operatorname{End}\mathbb{C}^{m|2n}\otimes\operatorname{End}\mathbb{C}^{m|2n}$$

and

$$Q=G_1\,P^{t_1}\,G_1^{-1}=G_2\,P^{t_2}\,G_2^{-1}.$$

Instead of (3.10) we have

$$P^2=1,\qquad PQ=QP=Q,\qquad Q^2=(m-2n)\,Q.$$

The R-matrix $R(u)$ has the form (3.15) with $\kappa=m/2-n-1$ which leads to the definitions of the Yangian and the reflection algebra $\mathrm{B}(\mathfrak{osp}_{m|2n})$ associated with $\mathfrak{osp}_{m|2n}$ as in

Definition 3.1; see [1]. The evaluation homomorphism $\mathrm{B}(\mathfrak{osp}_{m|2n}) \to \mathrm{U}(\mathfrak{osp}_{m|2n})$ provided by Theorem 3.3 takes the same form (3.26), where N should be replaced by $m-2n$:

$$S(u) \mapsto \frac{u + F - (m-2n)/4}{u - F + (m-2n)/4}.$$

The embedding $\mathrm{U}(\mathfrak{osp}_{m|2n}) \hookrightarrow \mathrm{B}(\mathfrak{osp}_{m|2n})$ is defined by the same formula (3.43). Note also that the parameter ω of the Brauer algebra $\mathcal{B}_n(\omega)$ should be evaluated at $\omega = m-2n$ in the context of the super-version of the centralizer results involving $\mathfrak{osp}_{m|2n}$; see Sec. 4. $\square$

4 Tensor representations of reflection algebras

The action of the Lie algebra $\mathfrak{gl}_N$ in the vector representation $\mathbb{C}^N$ is given by the rule $E_{ij} \mapsto e_{ij}$ which corresponds to the natural action of the general linear group $\mathrm{GL}_N(\mathbb{C})$ on $\mathbb{C}^N$ by left multiplication. By restricting this action to the subgroups of orthogonal and symplectic matrices O_N and Sp_N (N is even for the latter) we make $\mathbb{C}^N$ and the tensor product space

$$\mathbb{C}^N \otimes \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N, \qquad n \ \text{ factors}, \tag{4.1}$$

into representations of O_N and Sp_N. To write the images of elements of the groups, introduce an extra copy of the vector space $\mathbb{C}^N$ and consider the endomorphism algebra

$$\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,(\mathbb{C}^N)^{\otimes n}, \tag{4.2}$$

labeling the tensor factors with the numbers $0, 1, \ldots, n$. Writing a group element $\mathbf{h}$ as $\mathbf{h} = \sum_{i,j=1}^N e_{ij} \otimes \mathbf{h}_{ij}$, for the image of $\mathbf{h}$ we have $\mathbf{h} \mapsto \mathbf{h}_1 \ldots \mathbf{h}_n$ with the notation similar to (3.11) and (3.19). Due to the work of Brauer [4], the centralizer of this action in the endomorphism algebra of (4.1) is the homomorphic image of the algebra $\mathcal{B}_n(\omega)$ with an appropriately specified value of the parameter ω. Namely, $\omega = N$ in the orthogonal case, and the action of the Brauer algebra $\mathcal{B}_n(N)$ in the vector space (4.1) is given by

$$s_{ij} \mapsto P_{ij}, \qquad \epsilon_{ij} \mapsto Q_{ij}, \qquad i < j. \tag{4.3}$$

In the symplectic case, $\omega = -N$ and the action of the Brauer algebra $\mathcal{B}_n(-N)$ is given by

$$s_{ij} \mapsto -P_{ij}, \qquad \epsilon_{ij} \mapsto -Q_{ij}, \qquad i < j; \tag{4.4}$$

see [4], [30], where the operators P_{ij} and Q_{ij} on the vector space (4.1) act as the respective operators P and Q defined in (3.7) and (3.9) on the tensor product on the i-th and j-th copies of $\mathbb{C}^N$ and act as the identity operators on each of the remaining copies. Furthermore, in the orthogonal case the vector space (4.1) is decomposed as

$$(\mathbb{C}^N)^{\otimes n} \cong \bigoplus_{f=0}^{\lfloor n/2 \rfloor} \bigoplus_{\substack{\lambda \vdash n-2f \\ \lambda_1' + \lambda_2' \leqslant N}} V_\lambda \otimes L(\lambda),$$

where V_λ and $L(\lambda)$ are the respective irreducible representations of $\mathcal{B}_n(N)$ and O_N associated with the diagram λ, and λ'_j denotes the number of boxes in the column j of λ; see [31]. Similarly, in the symplectic case

$$(\mathbb{C}^N)^{\otimes n} \cong \bigoplus_{f=0}^{\lfloor n/2 \rfloor} \bigoplus_{\substack{\lambda \vdash n-2f \\ 2\lambda'_1 \leqslant N}} V_{\lambda'} \otimes L(\lambda),$$

where $V_{\lambda'}$ and $L(\lambda)$ are the respective irreducible representations of $\mathcal{B}_n(-N)$ and Sp_N associated with λ' and λ; see *loc. cit.*

From now on we will impose the condition on the parameters,

$$N \geqslant 2n+1 \qquad \text{and} \qquad N \geqslant 2n \tag{4.5}$$

in the orthogonal and symplectic case, respectively. It implies, in particular, that the Brauer algebras $\mathcal{B}_n(N)$ and $\mathcal{B}_n(-N)$ are semisimple; see e.g. [28] where a criterion of semisimplicity of $\mathcal{B}_n(\omega)$ is given. Explicit projections of $(\mathbb{C}^N)^{\otimes n}$ on the irreducible representations of the orthogonal and symplectic groups are provided by the idempotents of the corresponding Brauer algebra. More precisely, suppose that $\mathcal{U} = (\Lambda_1, \dots, \Lambda_n)$ is an up-down tableau of shape $\lambda = \Lambda_n$. Consider the corresponding idempotent $E_{\mathcal{U}}$, the sequence of contents $(c_1, \dots, c_n)$ associated with $\mathcal{U}$ and the subspace

$$L_{\mathcal{U}} = E_{\mathcal{U}}(\mathbb{C}^N)^{\otimes n}. \tag{4.6}$$

The condition (4.5) ensures that $E_{\mathcal{U}}$ can be defined by (2.2) with non-vanishing denominators. Under the action (4.3) of the Brauer algebra $\mathcal{B}_n(N)$ the subspace is nonzero if the number of boxes in the first two columns of λ does not exceed N, which follows automatically from (4.5). In this case $L_{\mathcal{U}}$ is an irreducible representation of O_N isomorphic to $L(\lambda)$. Similarly, under the action (4.4) of the Brauer algebra $\mathcal{B}_n(-N)$ the subspace (4.6) is nonzero if the number of boxes in the first column of λ does not exceed $N/2$; this holds automatically by (4.5). In this case $L_{\mathcal{U}}$ is an irreducible representation of Sp_N isomorphic to $L(\lambda)$.

The vector space $L_{\mathcal{U}}$ carries a representation of the corresponding Lie algebra $\mathfrak{g}_N = \mathfrak{o}_N$ or $\mathfrak{g}_N = \mathfrak{sp}_N$. Employing the evaluation homomorphism (3.26), we can equip $L_{\mathcal{U}}$ with the action of the reflection algebra $\mathrm{B}(\mathfrak{g}_N)$ associated with the Lie algebra $\mathfrak{g}_N$.

On the other hand, the vector space $(\mathbb{C}^N)^{\otimes n}$ carries a representation of the extended Yangian $\mathrm{X}(\mathfrak{g}_N)$ defined by the assignment

$$T(u) \mapsto a(u)\, R_{01}(u - z_1) \dots R_{0n}(u - z_n), \tag{4.7}$$

where $z_1, \dots, z_n$ are fixed complex numbers and $a(u) \in 1 + \mathbb{C}[[u^{-1}]]\, u^{-1}$ is a fixed formal series. Here we regard the image of $T(u)$ as a formal series in u^{-1} with coefficients in the algebra (4.2). Indeed, verifying that (4.7) defines a representation of the Yangian amounts to checking that the image of $T(u)$ satisfies the defining relations (3.20), which is straightforward. Furthermore, using Proposition 3.2, we can equip the vector space $(\mathbb{C}^N)^{\otimes n}$ with the action of the reflection algebra $\mathrm{B}(\mathfrak{g}_N)$ by

$$S(u) \mapsto a(u)\, a(-u)^{-1}\, R_{0n}(-u-z_n)^{-1} \ldots R_{01}(-u-z_1)^{-1}\, R_{01}(u-z_1) \ldots R_{0n}(u-z_n). \quad (4.8)$$

We keep using the double sign convention as in Sec. 3 so that the upper sign is taken in the orthogonal case and the lower sign in the symplectic case.

Proposition 4.1. *The subspace* $L_{\mathcal{U}}$ *of the vector space* (4.1) *is invariant under the action of* $\mathrm{B}(\mathfrak{g}_N)$ *given by*

$$S(u) \mapsto \frac{u - N/4}{u + N/4}\, R_{0n}(-u \mp c_n + \kappa/2)^{-1} \ldots R_{01}(-u \mp c_1 + \kappa/2)^{-1} \\ \times R_{01}(u \mp c_1 + \kappa/2) \ldots R_{0n}(u \mp c_n + \kappa/2).$$

Moreover, the representation of $\mathrm{B}(\mathfrak{g}_N)$ *on* $L_{\mathcal{U}}$ *is isomorphic to the evaluation module* $L_{\mathcal{U}}$ *obtained via the homomorphism* (3.26).

Proof. Consider the anti-automorphism of the Brauer algebra $\mathcal{B}_n(\omega)$ identical on the generators. Apply this anti-automorphism to the identity (2.10), then replace u by $u + \varkappa/2$. Together with (1.8) this leads to the following relation in the orthogonal case:

$$R_{0n}(-u - c_n + \kappa/2)^{-1} \ldots R_{01}(-u - c_1 + \kappa/2)^{-1} \\ \times R_{01}(u - c_1 + \kappa/2) \ldots R_{0n}(u - c_n + \kappa/2)\, E_{\mathcal{U}} = \frac{u + N/4}{u - N/4}\, E_{\mathcal{U}}\, \frac{u + X_0 - N/4}{u - X_0 + N/4},$$

where $X_0 = Q_{01} + \dots + Q_{0,n} - P_{01} - \dots - P_{0,n}$ so that the parameter n in (2.10) is replaced by $n+1$ and m is replaced by 0. We have used the fact that the image of $\rho_{ij}(u)$ coincides with $R_{ij}(u)$ under the specialization (4.3) with $\omega = N$.

To get the symplectic counterpart of the relation, note that under the specialization (4.4) with $\omega = -N$ the image of $\rho_{ij}(u)$ coincides with $R_{ij}(-u)$. Before applying the above argument, we invert all factors of E_U which occur in (2.10) so that in the symplectic case we get

$$R_{0n}(-u + c_n + \kappa/2)^{-1} \ldots R_{01}(-u + c_1 + \kappa/2)^{-1} \\ \times R_{01}(u + c_1 + \kappa/2) \ldots R_{0n}(u + c_n + \kappa/2)\, E_{\mathcal{U}} = \frac{u + N/4}{u - N/4}\, E_{\mathcal{U}}\, \frac{u + X_0 - N/4}{u - X_0 + N/4}.$$

This implies the first part of the proposition. The second part follows from the observation that X_0 coincides with the image of the element F in the representation (4.1) so that the two actions of the reflection algebra $\mathrm{B}(\mathfrak{g}_N)$ on $L_{\mathcal{U}}$ coincide. □

We conclude with an interpretation of the new fusion procedure for the symmetric group provided by Corollary 2.7, from the viewpoint of the representation theory of the Yangians. Recall that the *Yangian for* $\mathfrak{gl}_N$ is an associative algebra $\mathrm{Y}(\mathfrak{gl}_N)$ with generators $t_{ij}^{(r)}$, where $1 \leqslant i, j \leqslant N$ and $r = 1, 2, \dots$, satisfying certain quadratic relations. They are written with the use of the generating functions (3.17) and have the form of the *RTT* relation (3.20), where instead of the R-matrix (3.15) we take the *Yang R-matrix*

$$R(u) = 1 - \frac{P}{u}; \tag{4.9}$$

see e.g. [19, Ch. 1] for a detailed description of the algebraic structure of $\mathrm{Y}(\mathfrak{gl}_N)$. The *reflection algebra* $\mathrm{B}(\mathfrak{gl}_N)$ (see [29]) is isomorphic to the subalgebra of $\mathrm{Y}(\mathfrak{gl}_N)$ generated by the coefficients of the series $s_{ij}(u)$ where the matrix $S(u) = [s_{ij}(u)]$ is given by $S(u) = T(-u)^{-1}\, T(u)$; see [20].

Now consider the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$ and let $E = [E_{ij}]$ be the $N \times N$ matrix whose (i, j) entry is the generator E_{ij} of $\mathrm{U}(\mathfrak{gl}_N)$. For any complex parameter $\omega \in \mathbb{C}$ the mapping

$$T(u) \to \frac{u - E^t - \omega/4}{u - \omega/4}$$

defines an *evaluation homomorphism* $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$; see e.g. [19, Secs. 1.1-1.3]. Its restriction to the reflection algebra is a homomorphism $\mathrm{B}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$ given by

$$S(u) \mapsto \frac{u + \omega/4}{u - \omega/4} \cdot \frac{u - E^t - \omega/4}{u + E^t + \omega/4}. \tag{4.10}$$

Now, let λ be a partition of n and suppose that $\mathcal{U}$ is a standard tableau of shape λ. Consider the corresponding idempotent $E_{\mathcal{U}}$ and the sequence of contents $(\sigma_1, \dots, \sigma_n)$ as defined in Corollary 2.7. The symmetric group $\mathfrak{S}_n$ acts naturally on the space (4.1). By the Schur–Weyl duality the subspace

$$L_{\mathcal{U}} = E_{\mathcal{U}} (\mathbb{C}^N)^{\otimes n} \tag{4.11}$$

is nonzero if the number of boxes in the first column of λ does not exceed N; in this case $L_{\mathcal{U}}$ is an irreducible representation of $\mathfrak{gl}_N$ with the highest weight λ.

Proposition 4.2. *The subspace $L_{\mathcal{U}}$ is invariant under the action of $\mathrm{B}(\mathfrak{gl}_N)$ given by*

$$\begin{aligned} S(u) \mapsto \frac{u - \omega/4}{u + \omega/4}\, R_{0n}(-u - \sigma_n - \omega/4)^{-1} \dots R_{01}(-u - \sigma_1 - \omega/4)^{-1} \\ \times R_{01}(u - \sigma_1 - \omega/4) \dots R_{0n}(u - \sigma_n - \omega/4). \end{aligned}$$

Moreover, the representation of $\mathrm{B}(\mathfrak{gl}_N)$ on $L_{\mathcal{U}}$ is isomorphic to the evaluation module $L_{\mathcal{U}}$ obtained via the homomorphism (4.10).

Proof. Using the matrix notation and regarding the matrix E as the element

$$E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij} \in \operatorname{End} \mathbb{C}^N \otimes \mathrm{U}(\mathfrak{gl}_N)$$

we find that the image of the transposed matrix E^t in (4.2) coincides with the element X_0 given by

$$X_0 = P_{01} + P_{02} + \cdots + P_{0n}.$$

The argument is now competed in the same way as the proof of Proposition 4.1 with the use of the image of the identity (2.10) in the group algebra of the symmetric group. □

References

[1] D. Arnaudon, J. Avan, N. Crampé, L. Frappat and E. Ragoucy, *R-matrix presentation for super-Yangians* $Y(\mathrm{osp}(m|2n))$, J. Math. Phys. **44** (2003), 302–308.

[2] D. Arnaudon, A. Molev and E. Ragoucy, *On the R-matrix realization of Yangians and their representations*, Annales Henri Poincaré **7** (2006), 1269–1325.

[3] J. Birman and H. Wenzl, *Braids, link polynomials and a new algebra*, Trans. AMS **313** (1989), 249–273.

[4] R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. Math. **38** (1937), 854–872.

[5] I. V. Cherednik, *On special bases of irreducible finite-dimensional representations of the degenerate affine Hecke algebra*, Funct. Analysis Appl. **20** (1986), 87–89.

[6] I. V. Cherednik, *A new interpretation of Gelfand–Tzetlin bases*, Duke Math. J. **54** (1987), 563–577.

[7] V. G. Drinfeld, *Hopf algebras and the quantum Yang–Baxter equation*, Soviet Math. Dokl. **32** (1985), 254–258.

[8] V. G. Drinfeld, *Quantum Groups*, in "International Congress of Mathematicians (Berkeley, 1986)", Amer. Math. Soc., Providence RI, 1987, pp. 798–820.

[9] A. P. Isaev and A. I. Molev, *Fusion procedure for the Brauer algebra*, Algebra i Analiz **22** (2010), 142–154.

[10] A. P. Isaev, A. I. Molev and O. V. Ogievetsky, *Fusion procedure for the Birman–Murakami–Wenzl algebra and evaluation homomorphisms for quantum algebras*, in preparation.

[11] A. P. Isaev, A. I. Molev and A. F. Os'kin, *On the idempotents of Hecke algebras*, Lett. Math. Phys. **85** (2008), 79–90.

[12] A. P. Isaev and O. V. Ogievetsky, *On Baxterized solutions of reflection equation and integrable chain models*, Nuclear Phys. B **760** (2007), 167–183.

[13] A. P. Isaev and O. V. Ogievetsky, *Jucys-Murphy elements for Birman–Murakami–Wenzl algebras*, in: "Proc. of Int. Workshop Supersymmetries and Quantum Symmetries", Dubna 2009; `arXiv:0912.4010`.

[14] A. Jucys, *On the Young operators of the symmetric group*, Lietuvos Fizikos Rinkinys **6** (1966), 163–180.

[15] A. Jucys, *Factorization of Young projection operators for the symmetric group*, Lietuvos Fizikos Rinkinys **11** (1971), 5–10.

[16] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, *Yang–Baxter equation and representation theory*, Lett. Math. Phys. **5** (1981), 393–403.

[17] R. Leduc and A. Ram, *A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: The Brauer, Birman-Wenzl and type A Iwahori-Hecke algebras*, Adv. Math. **125** (1997), 1–94.

[18] A. I. Molev, *On the fusion procedure for the symmetric group*, Reports Math. Phys. **61** (2008), 181–188.

[19] A. Molev, *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, 143. American Mathematical Society, Providence, RI, 2007.

[20] A. I. Molev and E. Ragoucy, *Representations of reflection algebras*, Rev. Math. Phys. **14** (2002), 317–342.

[21] G. E. Murphy, *A new construction of Young's seminormal representation of the symmetric groups* , J. Algebra **69** (1981), 287–297.

[22] M. Nazarov, *Yangians and Capelli identities*, in: "Kirillov's Seminar on Representation Theory" (G. I. Olshanski, Ed.), Amer. Math. Soc. Transl. **181**, Amer. Math. Soc., Providence, RI, pp. 139–163 (1998).

[23] M. Nazarov, *Young's orthogonal form for Brauer's centralizer algebra*, J. Algebra **182** (1996), 664–693.

[24] M. Nazarov, *Representations of twisted Yangians associated with skew Young diagrams*, Selecta Math. (N.S.) **10** (2004), 71-129.

[25] M. Nazarov, *A mixed hook-length formula for affine Hecke algebras*, European J. Combin. **25** (2004), 1345–1376.

[26] G. Olshanski, *Twisted Yangians and infinite-dimensional classical Lie algebras*, in: "Quantum Groups (Leningrad, 1990)", Lecture Notes in Math. **1510**, Springer, Berlin, 1992, pp. 103–120.

[27] N. Yu. Reshetikhin, L. A. Takhtajan and L. D. Faddeev, *Quantization of Lie Groups and Lie algebras*, Leningrad Math. J. **1** (1990), 193–225.

[28] H. Rui, *A criterion on the semisimple Brauer algebras*, J. Comb. Theor., A **111** (2005), 78–88.

[29] E. K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. **A21** (1988), 2375–2389.

[30] H. Wenzl, *On the structure of Brauer's centralizer algebras*, Ann. Math. (2) **128** (1988), 173–193.

[31] H. Weyl, *Classical groups, their invariants and representations*, Princeton Univ. Press, Princeton NJ, 1946.

[32] A. B. Zamolodchikov and Al. B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, Ann. Phys. **120** (1979), 253–291.