QUIVER SCHUR ALGEBRAS FOR THE LINEAR QUIVER I

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ABSTRACT. We define a graded quasi-hereditary covering for the cyclotomic quiver Hecke algebras $\mathcal{R}_n^\Lambda$ of type A when $e=0$ (the linear quiver) or $e \geq n$. We show that these algebras are quasi-hereditary graded cellular algebras by giving explicit homogeneous bases for them. When $e=0$ we show that the KLR grading on the quiver Hecke algebras is compatible with the gradings on parabolic category $\mathcal{O}_n^\Lambda$ previously introduced in the works of Beilinson, Ginzburg and Soergel and Backelin. As a consequence, we show that when $e=0$ our graded Schur algebras are Koszul over field of characteristic zero. Finally, we give an LLT-like algorithm for computing the graded decomposition numbers of the quiver Schur algebras in characteristic zero when $e=0$.

CONTENTS

1. INTRODUCTION

Khovanov and Lauda [28, 29] and Rouquier [40] have introduced a remarkable family of $\mathbb{Z}$-graded algebras which are now known to categorify the canonical bases of Kac-Moody algebras [11, 15, 43]. Brundan and Kleshchev [10] initiated the study of 'cyclotomic' quotients of these algebras by showing that they are isomorphic to the degenerate and non-degenerate cyclotomic Hecke algebras of type $G(\ell,1,n)$; see also [40].

This is the first of two papers which define and study quasi-hereditary covers of the cyclotomic quiver Hecke algebras of the linear quiver and of 'large' cyclic quivers. These algebras are graded analogues of the cyclotomic Hecke algebras $\underline{\mathcal{S}}_n^{\mathrm{DJM}}$ of type $G(\ell,1,n)$ at non-roots of unity [9, 17]. Our first main result is the following.

Theorem A. *Suppose that $e=0$ or $e \geq n$ and let $\mathcal{Z}=K$ be an arbitrary field. The algebra $\mathcal{S}_n^\Lambda$ is a quasi-hereditary graded cellular algebra with graded standard*

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modules $\{\,\Delta^{\boldsymbol{\mu}} \mid \boldsymbol{\mu}\in\mathscr{P}^\Lambda_n\,\}$ *and irreducible modules* $\{\,L^{\boldsymbol{\mu}} \mid \boldsymbol{\mu}\in\mathscr{P}^\Lambda_n\,\}$*. Moreover, there is an equivalence of (ungraded) categories*

$$\mathsf{F}^\Lambda_n : \mathcal{S}^\Lambda_n\text{-Mod} \longrightarrow \underline{\mathcal{S}}^{DJM}_n\text{-Mod}$$

which sends standard modules to standard modules and simple modules to simple modules in the obvious way.

In fact, we define the quiver Schur algebras over more general rings. In particular, the quiver Schur algebra $\mathcal{S}^\Lambda_n$ is defined over $\mathbb{Z}$ when $e=0$ or when $e>n$ is prime.

Like the cyclotomic Schur algebras, the quiver Schur algebra $\mathcal{S}^\Lambda_n$ is defined to be the endomorphism algebra of a direct sum of "graded permutation modules"; see Definition 4.16. Our graded permutation modules turn out to be direct summands of modules used to define the cyclotomic Schur algebras. Therefore, $\mathcal{S}^\Lambda_n$ and $\underline{\mathcal{S}}^{\mathrm{DJM}}_n$ are Morita equivalent (Theorem 6.13). The algebras $\mathcal{S}^\Lambda_n$ and $\underline{\mathcal{S}}^{\mathrm{DJM}}_n$ are not isomorphic in general, however, because $\mathcal{S}^\Lambda_n$ usually has smaller dimension.

Quite surprisingly, given that their definitions are so different, if Λ is a dominant weight of level 2 then we show in [24] that $\mathcal{S}^\Lambda_n$ is isomorphic as a graded algebra to one of the quasi-hereditary covers of the Khovanov's diagram algebra defined and studied by Brundan and Stroppel [14]. In particular, in this case $\mathcal{S}^\Lambda_n$ is a positively graded basic algebra with a Koszul grading.

The key to both the definition and our understanding the algebras $\mathcal{S}^\Lambda_n$ is the graded cellular bases of the quiver Hecke algebras $\mathcal{R}^\Lambda_n$ that we constructed in [25]. The graded permutation modules have homogeneous bases which are a subset of the cellular bases of $\mathcal{R}^\Lambda_n$ (Theorem 4.10), and we show how to lift the bases of the permutation modules to give an explicit homogeneous cellular basis for the quiver Schur algebras over an arbitrary commutative ring when $e=0$ (Theorem 4.20). As a consequence, we show that the quiver Schur algebras are quasi-hereditary (Theorem 4.25), and that they naturally decompose into a direct sum of blocks $\mathcal{S}^\Lambda_n = \bigoplus_{\beta\in Q^+_n}\mathcal{S}^\Lambda_\beta$, with each block $\mathcal{S}^\Lambda_\beta$ being a quasi-hereditary graded cellular algebra (Theorem 4.36). We show that there is a graded Schur functor (Proposition 4.31), describe the graded Young modules (Proposition 5.6) and give an explicit description of the graded tilting modules (Corollary 5.15) which is similar in spirit to Donkin's construction of the tilting modules of an algebraic group [18].

If $e=0$ and we work over the field of complex numbers then Brundan and Kleshchev have shown that the degenerate cyclotomic Schur algebras are Morita equivalent to blocks of parabolic category $\mathcal{O}^\Lambda_n$ for the Lie algebra of the general linear group [9]. By results of Backelin [4], and Beilinson, Ginzburg and Soergel [5], parabolic category $\mathcal{O}^\Lambda_n$ admits a Koszul grading. By [9], the endomorphism algebra of a prinjective generator of $\mathcal{O}^\Lambda_n$ is Morita equivalent to the degenerate cyclotomic Hecke algebra $\mathcal{H}^\Lambda_n$ of type A. Therefore, the Koszul grading on $\mathcal{O}^\Lambda_n$ induces a grading on the module category of $\mathcal{H}^\Lambda_n$. This gives two ostensibly different gradings on the degenerate cyclotomic Hecke algebra $\mathcal{H}^\Lambda_n$: one coming from parabolic category $\mathcal{O}^\Lambda_n$ and the KLR grading given by the Brundan-Kleshchev isomorphism $\mathcal{H}^\Lambda_n\cong\mathcal{R}^\Lambda_n$ [10] when $e=0$.

Theorem B. *Suppose that* $e=0$ *and* $\mathcal{Z}=\mathbb{C}$ *is the field of complex numbers. Then graded category* $\mathcal{O}^\Lambda_n$ *and the quiver Hecke algebra* $\mathcal{R}^\Lambda_n$ *induce graded Morita equivalent gradings on* $\mathcal{H}^\Lambda_n$-Mod.

Our proof of Theorem B in section 7.2 is essentially a delicate counting argument (Proposition 7.17) using the facts that that indecomposable prinjective modules in $\mathcal{O}^\Lambda_n$ are rigid, because $\mathcal{O}^\Lambda_n$ is Koszul, and that the graded decomposition numbers of $\mathcal{R}^\Lambda_n$ have been computed by Brundan and Kleshchev [11]. Ultimately, however, our argument relies on Ariki's categorification theorem [2] and the Koszulity of parabolic category $\mathcal{O}_n^\Lambda$ [4, 5], both of which are proved using heavy geometric machinery.

Building on Theorem B, in section 7.3 we prove a graded analogue of [9, Theorem C], thus lifting Brundan and Kleshchev's "higher Schur-Weyl duality" to the graded setting.

Theorem C. *Suppose that $e = 0$ and $\mathcal{Z} = \mathbb{C}$. Then there are graded Schur functors $\mathsf{F}_n^{\mathcal{O}} : \mathcal{O}_n^\Lambda\text{-Mod} \longrightarrow \mathcal{R}_n^\Lambda\text{-Mod}$ and $\mathsf{F}_n^\Lambda : \mathcal{S}_n^\Lambda\text{-Mod} \longrightarrow \mathcal{R}_n^\Lambda\text{-Mod}$ and a graded equivalence $\mathsf{E}_n^{\mathcal{O}} : \mathcal{O}_n^\Lambda \longrightarrow \mathcal{S}_n^\Lambda\text{-Mod}$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{O}_n^\Lambda \arrow[r, "\mathsf{E}_n^{\mathcal{O}}"] \arrow[dr, "\mathsf{F}_n^{\mathcal{O}}"'] & \mathcal{S}_n^\Lambda\text{-Mod} \arrow[d, "\mathsf{F}_n^\Lambda"] \\
 & \mathcal{R}_n^\Lambda\text{-Mod}
\end{tikzcd}$$

In particular, $\mathcal{S}_n^\Lambda$-Mod is Koszul.

Webster [45, Corollary 5.7] has obtained a graded lift of Brundan and Kleshchev's functor $\underline{\mathsf{F}}_n^{\mathcal{O}}$ which is similar to our Theorem C. He uses a very different geometric construction which is based on his categorification of irreducible representations of Kac-Moody algebras.

Since the module category of $\mathcal{S}_n^\Lambda$ is Koszul when $e = 0$ the graded decomposition numbers of $\mathcal{S}_n^\Lambda$ are polynomials with non-negative coefficients which, as one might expect, are (known) parabolic Kazhdan–Lusztig polynomials. Using our graded cellular bases of $\mathcal{S}_n^\Lambda$, in section 7.5 we give a fast algorithm for computing these polynomials which is similar in spirit to the LLT algorithm for the Hecke algebras of type A [32]. This is interesting because our analogue of the LLT algorithm computes the graded decomposition numbers of the quiver Schur algebras when $e = 0$ whereas the extension of the LLT algorithm to the q-Schur algebras [33] is non-trivial because it requires first computing the action of the bar involution on the Fock space.

In the sequel to this paper [24] we show that the decomposition numbers of the quiver Schur algebras are independent of the characteristic of the field when $e = 0$. As a consequence, the formal characters of the irreducible modules of the quiver Hecke algebras are independent of the field when $e = 0$, thus proving a conjecture of Kleshchev and Ram [31, Conjecture 7.3]. Moreover, using Theorem C, this implies that the module category of the cyclotomic quiver Schur algebras is Koszul over an arbitrary field when $e = 0$.

As we were finishing this paper we received a preprint by Stroppel and Webster [42] which, building on [45], constructs a family of graded algebras as convolution algebras on the cohomology of quiver varieties. Over an algebraically closed field of characteristic zero they show that cyclotomic quotients of these algebras are isomorphic to the cyclotomic Schur algebras associated to arbitrary quivers of type A. Further, the graded decomposition numbers of the Stroppel-Webster cyclotomic quiver Schur algebras are polynomials with non-negative coefficients, so that the basic algebras of the these algebras are positively graded. Therefore, Theorem C and the uniqueness of Koszul gradings implies that the Stroppel-Webster quiver Schur algebras for the linear quiver are graded Morita equivalent to our quiver Schur algebras in characteristic zero.

Index of notation

$D^{\boldsymbol\mu}$	Simple $\mathcal{R}^\Lambda_n$-module
$d(\mathfrak{t}), d'(\mathfrak{t})$	Permutations: $\mathfrak{t} = \mathfrak{t}^{\boldsymbol\mu} d(\mathfrak{t}) = \mathfrak{t}_{\boldsymbol\mu} d'(\mathfrak{t})$
$d_{\boldsymbol\lambda\boldsymbol\mu}(q)$	$[\Delta^{\boldsymbol\lambda} : L^{\boldsymbol\mu}]_q$
$\deg \mathfrak{t}$	Tableau degree
$\operatorname{codeg} \mathfrak{t}$	Tableau codegree
$\operatorname{def} \beta$	Defect of $\beta \in Q^+$
$\operatorname{Dim} M$	Graded dimension of M
$\Delta^{\boldsymbol\lambda}, \nabla^{\boldsymbol\lambda}$	Weyl and costandard modules
$\Delta_{\boldsymbol\lambda}, \nabla_{\boldsymbol\lambda}$	Sign dual (co)standard modules
$e^{\boldsymbol\mu}, e_{\boldsymbol\mu}$	KLR idempotents $e(\mathbf{i}^{\boldsymbol\mu}), e(\mathbf{i}_{\boldsymbol\mu})$
$e_{\boldsymbol\lambda\boldsymbol\mu}(q)$	Inverse decomposition number
$E^{\boldsymbol\mu}, E_{\boldsymbol\mu}$	Graded exterior powers
$\mathsf{F}^\Lambda_n, \mathsf{F}^\Lambda_\beta$	Graded Schur functors
$G^{\boldsymbol\mu}, G_{\boldsymbol\mu}$	Graded permutation modules
G^Λ_n	$\bigoplus_{\boldsymbol\mu \in \mathscr{P}^\Lambda_n} G^{\boldsymbol\mu}$
End_A	End in A-Mod
END_A	All A-module endomorphisms
$\mathfrak{F}^\Lambda$	Combinatorial Fock space
$\mathcal{H}^\Lambda_n, \mathcal{H}^\Lambda_\beta$	Cyclotomimc Hecke algebras
Hom_A	Degree preserving maps in A-Mod
HOM_A	All A-module homomorphisms
$\mathbf{i}^{\boldsymbol\mu}, \mathbf{i}_{\boldsymbol\mu}$	$\operatorname{res}(\mathfrak{t}^{\boldsymbol\mu})$ and $\operatorname{res}(\mathfrak{t}_{\boldsymbol\mu})$
I^β	$\{\, \mathbf{i} \in I^n \mid \sum_{r=1}^{\ell} \alpha_{i_r} = \beta \,\}$
$\boldsymbol\kappa$	Multicharge determining $\Lambda_{\boldsymbol\kappa}$
$\mathcal{K}^\Lambda_n$	Restricted multipartitions for $\mathcal{R}^\Lambda_n$
$L^{\boldsymbol\mu}$	Simple $\mathcal{S}^\Lambda_n$-module
$\boldsymbol\mu'$	Conjugate multipartition
P^+	Positive weight lattice
$P^{\boldsymbol\mu}$	Projective cover of $L^{\boldsymbol\mu}$
$\mathscr{P}^\Lambda_n$	Multipartitions of n
$\mathscr{P}^\Lambda_\beta$	$\{\, \boldsymbol\mu \in \mathscr{P}^\Lambda_n \mid \mathbf{i}^{\boldsymbol\mu} \in I^\beta \,\}$
$\Psi^{\boldsymbol\mu\boldsymbol\lambda}_{\mathfrak{s}\mathfrak{t}}$	Basis elements of $\mathcal{S}^\Lambda_n$
$\psi_{\mathfrak{s}\mathfrak{t}}, \psi'_{\mathfrak{s}\mathfrak{t}}$	Basis elements of $\mathcal{R}^\Lambda_n$
$\Psi^{\boldsymbol\mu}$	Identity map on $G^{\boldsymbol\mu}$
Q^+	Positive root lattice
Q^+_n	$\{\, \beta \in Q^+ \mid \mathscr{P}^\Lambda_\beta \neq 0 \,\}$
res	Residue sequence for tableaux
$\mathcal{R}^\Lambda_n$	Quiver Hecke algebra
$\mathcal{R}^\Lambda_\beta$	A block of $\mathcal{R}^\Lambda_n$
$\mathtt{sgn}$	Sign automorphism
$\mathcal{S}^\Lambda_n$	Quiver Schur algebra
$\mathcal{S}^\Lambda_\beta$	A block of $\mathcal{S}^\Lambda_n$
$\mathcal{S}^{\beta'}_{\Lambda'}$	Sign-dual quiver Schur algebra
$S^{\boldsymbol\mu}, S_{\boldsymbol\mu}$	Graded Specht modules
$\operatorname{Std}(\mathscr{P}^\Lambda_n)$	Standard tableaux
$\operatorname{Std}^{\boldsymbol\mu}(\mathscr{P}^\Lambda_n)$	$\{\, \mathfrak{t} \mid \mathfrak{t} \trianglerighteq \mathfrak{t}^{\boldsymbol\mu}$ and $\operatorname{res}(\mathfrak{t}) = \mathbf{i}^{\boldsymbol\mu} \,\}$
$\operatorname{Std}_{\boldsymbol\mu}(\mathscr{P}^\Lambda_n)$	$\{\, \mathfrak{t} \mid \mathfrak{t}_{\boldsymbol\mu} \trianglerighteq \mathfrak{t}$ and $\operatorname{res}(\mathfrak{t}) = \mathbf{i}_{\boldsymbol\mu} \,\}$
$\mathcal{T}^{\boldsymbol\lambda}$	$\{\, (\boldsymbol\mu, \mathfrak{s}) \mid \mathfrak{s} \in \operatorname{Std}^{\boldsymbol\mu}(\boldsymbol\lambda) \,\}$
$\mathcal{T}_{\boldsymbol\lambda}$	$\{\, (\boldsymbol\mu, \mathfrak{s}) \mid \mathfrak{s} \in \operatorname{Std}_{\boldsymbol\mu}(\boldsymbol\lambda) \,\}$
$T^{\boldsymbol\mu}, T_{\boldsymbol\mu}$	Tilting modules
τ_β	Trace form on $\mathcal{R}^\Lambda_\beta$
$\mathfrak{t}^{\boldsymbol\mu}, \mathfrak{t}_{\boldsymbol\mu}$	Initial and final $\boldsymbol\mu$-tableaux
$y^{\boldsymbol\mu}, y_{\boldsymbol\mu}$	$\psi_{\mathfrak{t}^{\boldsymbol\mu}\mathfrak{t}^{\boldsymbol\mu}} = e^{\boldsymbol\mu} y^{\boldsymbol\mu}$, $\psi'_{\mathfrak{t}_{\boldsymbol\mu}\mathfrak{t}_{\boldsymbol\mu}} = e_{\boldsymbol\mu} y_{\boldsymbol\mu}$
$Y^{\boldsymbol\mu}, Y_{\boldsymbol\mu}$	Young modules
$\mathcal{Z}$	A commutative ring
$Z^{\boldsymbol\mu}$	Graded symmetric power
$\rhd, \blacktriangleright$	Dominance orderings
$[M{:}L^{\boldsymbol\mu}]_q$	Graded decomposition number
$[N{:}D^{\boldsymbol\mu}]_q$	Graded decomposition number
$\circledast$	Contragredient dual
$\#$	$\operatorname{HOM}_A(?, A)$-dual

2. Graded representation theory and combinatorics

In this chapter we set our notation and give the reader some quick reminders about graded modules and graded algebras, by which we mean $\mathbb{Z}$-graded modules and $\mathbb{Z}$-graded algebras. Expert readers may wish to skip this chapter.

2.1. **Modules and algebras.** Throughout this paper, $\mathcal{Z}$ will be an integral domain. In this paper a **graded $\mathcal{Z}$-module** is a $\mathbb{Z}$-graded $\mathcal{Z}$-module M. That is, as $\mathcal{Z}$-module, M has a direct sum decomposition

$$M = \bigoplus_{d \in \mathbb{Z}} M_d.$$

If $m \in M_d$, for $d \in \mathbb{Z}$, then m is **homogeneous** of **degree** d and we set $\deg m = d$. If M is a graded $\mathcal{Z}$-module and $s \in \mathbb{Z}$ let $M\langle s\rangle$ be the graded $\mathcal{Z}$-module obtained by shifting the grading on M up by s; that is, $M\langle s\rangle_d = M_{d-s}$, for $d \in \mathbb{Z}$. Let q be an indeterminate. If $\mathcal{Z} = K$ is a field then **graded dimension** of M is the Laurent polynomial

$$\operatorname{Dim} M = \sum_{d \in \mathbb{Z}} \dim_K M_d \in \mathbb{N}[q, q^{-1}]. \tag{2.1}$$

In particular, $\dim_K M = (\operatorname{Dim} M)\big|_{q=1}$. If M is a graded $\mathcal{Z}$-module let $\underline{M}$ be the ungraded $\mathcal{Z}$-module obtained by forgetting the grading on M. All modules in this paper will be graded unless otherwise mentioned.

If M is a graded module and if $f(q) = \sum_{d\in\mathbb{Z}} f_d q^d \in \mathbb{N}[q, q^{-1}]$ is a Laurent polynomial with non-negative coefficients $\{f_d\}_{d\in\mathbb{Z}}$ then define

$$f(q)M = \bigoplus_{d\in\mathbb{Z}} M\langle d\rangle^{\oplus f_d}.$$

Thus, $\operatorname{D\scriptsize{IM}}(f(q)M) = f(q) \operatorname{D\scriptsize{IM}} M$.

A **graded $\mathcal{Z}$-algebra** is a unital associative $\mathcal{Z}$-algebra $A = \bigoplus_{d\in\mathbb{Z}} A_d$ which is a graded $\mathcal{Z}$-module such that $A_d A_e \subseteq A_{d+e}$, for all $d, e \in \mathbb{Z}$. It follows that $1 \in A_0$ and that A_0 is a graded subalgebra of A. A graded (right) A-module is a graded $\mathcal{Z}$-module M such that $\underline{M}$ is an $\underline{A}$-module and $M_d A_e \subseteq M_{d+e}$, for all $d, e \in \mathbb{Z}$. Graded submodules, graded left A-modules and so on are all defined in the obvious way.

Let A-Mod be the category of finitely generated *graded* A-modules with degree preserving maps. Then

$$\operatorname{Hom}_A(M, N) = \{\, f \in \operatorname{Hom}_{\underline{A}}(\underline{M}, \underline{N}) \mid f(M_d) \subseteq N_d \text{ for all } d \in \mathbb{Z} \,\},$$

for all $M, N \in A$-Mod. The elements of $\operatorname{Hom}_A(M, N)$ are homogeneous maps of degree 0. More generally, for each $d \in \mathbb{Z}$ set

$$\operatorname{Hom}_A(M, N)_d = \operatorname{Hom}_A(M\langle d\rangle, N) \cong \operatorname{Hom}_A(M, N\langle -d\rangle).$$

Thus, $\operatorname{Hom}_A(M, N) = \operatorname{Hom}_A(M, N)_0$. If $f \in \operatorname{Hom}_A(M, N)_d$ then f is **homogeneous** of degree d and we set $\deg f = d$. Define

$$\operatorname{H\scriptsize{OM}}_A(M, N) = \bigoplus_{d\in\mathbb{Z}} \operatorname{Hom}_A(M, N)_d = \bigoplus_{d\in\mathbb{Z}} \operatorname{Hom}_A(M\langle d\rangle, N).$$

Then $\operatorname{Hom}_{\underline{A}}(\underline{M}, \underline{N}) \cong \operatorname{H\scriptsize{OM}}_A(M, N)$ as a $\mathcal{Z}$-module. Define

$$\operatorname{End}_A(M) = \operatorname{Hom}_A(M, M) \quad \text{and} \quad \operatorname{E\scriptsize{ND}}_A(M) = \operatorname{H\scriptsize{OM}}_A(M, M)$$

similarly.

If $r \geq 0$ and M and N are graded A-modules let $\operatorname{Ext}^r_A(M, N)$ be the space of r-fold extensions of M by N in the category A-Mod of (graded) A-modules and set

$$\operatorname{E\scriptsize{XT}}^r_A(M, N) = \bigoplus_{d\in\mathbb{Z}} \operatorname{Ext}^r_A(M\langle d\rangle, N).$$

Once again, $\operatorname{Ext}^r_{\underline{A}}(\underline{M}, \underline{N}) \cong \operatorname{E\scriptsize{XT}}^r_A(M, N)$, for all $r \geq 0$.

We emphasize that Hom_A and Ext_A are the spaces of homomorphisms and extensions in the category A-Mod of finitely generated (graded) A-modules. These should not be confused with $\operatorname{Hom}_{\underline{A}}$ and $\operatorname{Ext}_{\underline{A}}$ in the (ungraded) category $\underline{A}$-Mod.

Now suppose that A comes equipped with a homogeneous anti-isomorphism $\star$. Then the **contragredient dual** of the graded A-module M is the graded A-module

$$M^{\circledast} = \operatorname{H\scriptsize{OM}}_{\mathcal{Z}}(M, \mathcal{Z}) = \bigoplus_{d\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{Z}}(M\langle d\rangle, \mathcal{Z}) \tag{2.2}$$

where $\mathcal{Z}$ is concentrated in degree zero and where the action of A on $M^{\circledast}$ is given by $(fa)(m) = f(ma^{\star})$ for all $f \in M^{\circledast}$, $a \in A$ and $m \in M$. The module M is **self dual** if $M \cong M^{\circledast}$ as graded A-modules. If $\mathcal{Z} = K$ is a field then, as a vector space, $M^{\circledast}_d = \operatorname{Hom}_{\mathcal{Z}}(M_{-d}, K)$, so that $\operatorname{D\scriptsize{IM}} M^{\circledast} = \overline{\operatorname{D\scriptsize{IM}} M}$, where the bar involution $^{-} : \mathbb{Z}[q, q^{-1}] \longrightarrow \mathbb{Z}[q, q^{-1}]$ is the linear map determined by $q \mapsto q^{-1}$ and $q^{-1} \mapsto q$.

If m is an $\underline{A}$-module then a **graded lift** of m is an A-module M such that $\underline{M} \cong m$ as A-modules. In general, there is no guarantee that an $\underline{A}$-module will have a graded lift but it is easy to see that if an indecomposable $\underline{A}$-module has a graded lift then this lift is unique up to isomorphism and grading shift; see for example [5, Lemma 2.5.3]. The irreducible and projective indecomposable $\underline{A}$-modules always have graded lifts; see [21].

Suppose that M is a graded A-module and that $X = \{ X^\mu \mid \mu \in \mathscr{P} \}$ is a collection of A-modules such that $\{ \underline{X}^\mu \mid \mu \in \mathscr{P} \}$ are pairwise non-isomorphic $\underline{A}$-modules. Then M has a X-module **filtration** if there exists a filtration

$$M = M_0 \supset M_1 \supset \cdots \supset M_s = 0$$

such that there exist $\mu_r \in \mathscr{P}$ and $d_r \in \mathbb{Z}$ with $M_r/M_{r+1} \cong X^{\mu_r}\langle d_r\rangle$, for $0 \le r < s$. The **graded multiplicity** of X^μ in M is the Laurent polynomial

$$(M : X^\mu)_q = \sum_{r=0}^{s-1} q^{d_r} \in \mathbb{N}[q, q^{-1}]. \tag{2.3}$$

In general, this multiplicity will depend upon the choice of filtration but for many modules, such as irreducible modules and Weyl modules, the Laurent polynomial $(M : X^\mu)_q$ will be independent of this choice. We set $[M : X^\mu]_q = (M : X^\mu)_q$ when this multiplicity is independent of the choice of filtration.

2.2. **Cellular algebras.** All of the algebras considered in this paper are (graded) cellular algebras so we quickly recall the definition and some of the important properties of these algebras. Cellular algebras were defined by Graham and Lehrer [22] with their natural extension to the graded setting given in [25].

2.4. **Definition** (Graded cellular algebra [22, 25])**.** Suppose that A is a $\mathbb{Z}$-graded $\mathcal{Z}$-algebra which is free of finite rank over $\mathcal{Z}$. A **graded cell datum** for A is an ordered quadruple $(\mathscr{P}, T, B, \deg)$, where $(\mathscr{P}, \rhd)$ is the **weight poset**, $T(\lambda)$ is a finite set for $\lambda \in \mathscr{P}$, and

$$B\colon \coprod_{\lambda\in\mathscr{P}} T(\lambda)\times T(\lambda) \longrightarrow A; (\mathfrak{s},\mathfrak{t}) \mapsto b_{\mathfrak{s}\mathfrak{t}}, \quad \text{and} \quad \deg : \coprod_{\lambda\in\mathscr{P}} T(\lambda) \longrightarrow \mathbb{Z}$$

are two functions such that B is injective and

(GC$_d$) If $\lambda \in \mathscr{P}$ and $\mathfrak{s},\mathfrak{t} \in \mathsf{T}(\lambda)$ then $b_{\mathfrak{s}\mathfrak{t}}$ is homogeneous of degree $\deg b_{\mathfrak{s}\mathfrak{t}} = \deg \mathfrak{s} + \deg \mathfrak{t}$.

(GC$_1$) $\{ b_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in T(\lambda) \text{ for } \lambda \in \mathscr{P} \}$ is a $\mathcal{Z}$-basis of A.

(GC$_2$) If $\mathfrak{s},\mathfrak{t} \in T(\lambda)$, for some $\lambda \in \mathscr{P}$, and $a \in A$ then there exist scalars $r_{\mathfrak{t}\mathfrak{v}}(a)$, which do not depend on $\mathfrak{s}$, such that

$$b_{\mathfrak{s}\mathfrak{t}} a = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{t}\mathfrak{v}}(a) b_{\mathfrak{s}\mathfrak{v}} \pmod{A^{\rhd\lambda}},$$

where $A^{\rhd\lambda}$ is the $\mathcal{Z}$-submodule of A spanned by $\{ b^\mu_{\mathfrak{a}\mathfrak{b}} \mid \mu \rhd \lambda \text{ and } \mathfrak{a},\mathfrak{b} \in T(\mu) \}$.

(GC$_3$) The $\mathcal{Z}$-linear map $\star\colon A \longrightarrow A$ determined by $(b_{\mathfrak{s}\mathfrak{t}})^\star = b_{\mathfrak{t}\mathfrak{s}}$, for all $\lambda \in \mathscr{P}$ and all $\mathfrak{s},\mathfrak{t} \in T(\lambda)$, is a homogeneous anti-isomorphism of A.

A **graded cellular algebra** is a graded algebra which has a graded cell datum. The basis $\{ b_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \mathscr{P} \text{ and } \mathfrak{s},\mathfrak{t} \in T(\lambda \}$ is a **graded cellular basis** of A.

If we omit the degree assumption (GC$_d$) then we recover Graham and Lehrer's [22] definition of an (ungraded) cellular algebra.

Fix a graded cellular algebra A with graded cellular basis $\{b_{\mathfrak{s}\mathfrak{t}}\}$. If $\lambda \in \mathscr{P}$ then the graded **cell module** is the $\mathcal{Z}$-module Δ^λ with basis $\{ b_\mathfrak{t} \mid \mathfrak{t} \in T(\lambda) \}$ and with A-action

$$b_\mathfrak{t} a = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{t}\mathfrak{v}}(a) b_\mathfrak{v},$$

where the scalars $r_{\mathfrak{t}\mathfrak{v}}(a) \in \mathcal{Z}$ are the same scalars appearing in (GC$_2$). One of the key properties of the graded cell modules is that by [25, Lemma 2.7] they come equipped with a homogeneous bilinear form $\langle\ ,\ \rangle$ of degree zero which is determined by the equation

$$\langle b_\mathfrak{t}, b_\mathfrak{u}\rangle b_{\mathfrak{s}\mathfrak{v}} \equiv b_{\mathfrak{s}\mathfrak{t}} b_{\mathfrak{u}\mathfrak{v}} \pmod{A^{\rhd\lambda}},$$

for $\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v} \in T(\lambda)$. The radical of this form

$$\operatorname{rad} \Delta^\lambda = \{\, x \in \Delta^\lambda \mid \langle x, y \rangle = 0 \text{ for all } y \in \Delta^\lambda \,\}$$

is a graded A-submodule of Δ^λ so that $L^\lambda = \Delta^\lambda / \operatorname{rad} \Delta^\lambda$ is a graded A-module.

2.5. **Theorem** ([25, Theorem 2.10]). *Suppose that $\mathcal{Z}$ is a field and that A is a graded cellular algebra. Then:*

a) *If $L^\lambda \neq 0$, for $\lambda \in \mathscr{P}$, then L^λ is an absolutely irreducible graded A-module and $(L^\lambda)^\circledast \cong L^\lambda$.*
b) *$\{\, L^\lambda\langle k \rangle \mid \lambda \in \mathscr{P}, L^\lambda \neq 0$ and $k \in \mathbb{Z} \,\}$ is a complete set of pairwise non-isomorphic irreducible (graded) A-modules.*

Suppose that $\mathcal{Z} = K$ is a field and let M be a (graded) A-module and L^μ be a graded simple A-module, for $\mu \in \mathscr{P}$. We define

$$[M : L^\mu]_q = \sum_{d \in \mathbb{Z}} [M : L^\mu\langle d \rangle] q^d \tag{2.6}$$

to be the graded multiplicity of L^μ in M. By the Jordon-Hölder theorem, $[M : L^\mu]_q$ depends only on M and L^μ and not on the choice of composition series for M. Moreover, $[M : L^\mu]_q \in \mathbb{N}[q, q^{-1}]$ and $[M : L^\mu]_{q=1} = [\underline{M} : \underline{L}^\mu]$ is the usual decomposition multiplicity of $\underline{L}^\mu$ in $\underline{M}$.

2.7. **Corollary** ([25, Lemma 2.13]). *Suppose that $\mathcal{Z}$ is a field and that $\lambda, \mu \in \Lambda$ with $L^\mu \neq 0$. Then $[\Delta^\mu : L^\mu]_q = 1$ and $[\Delta^\lambda : L^\mu]_q \neq 0$ only if $\lambda \trianglerighteq \mu$.*

Let $\mathscr{P}_0 = \{\, \mu \in \mathscr{P} \mid L^\mu \neq 0 \,\}$. Then $\mathbf{D}_A(q) = ([\Delta^\lambda : L^\mu]_q)_{\lambda \in \mathscr{P}, \mu \in \mathscr{P}_0}$ is the **decomposition matrix** of A. For each $\mu \in \mathscr{P}_0$ let P^μ be the projective cover of L^μ in A-Mod. Then $\mathbf{C}_A(q) = ([P^\lambda : L^\mu]_q)_{\lambda, \mu \in \mathscr{P}_0}$ is the **Cartan matrix** of A.

If $M = (m_{ij})$ is a matrix let $M^{tr} = (m_{ji})$ be its transpose. We will need the following fact.

2.8. **Corollary** (Brauer-Humphreys reciprocity [25, Theorem 2.17]).
Suppose that $\mathcal{Z} = K$ is a field. Then $\mathbf{C}_A(q) = \mathbf{D}_A(q)^{tr} \mathbf{D}_A(q)$. In particular, $\mathbf{C}_A(q)$ is a symmetric matrix.

Finally, we note the following criterion for a cellular algebra to be quasi-hereditary. In particular, this implies that A-Mod is a highest weight category. The definitions of these objects can be found, for example, in [19, Appendix]. Alternatively, the reader can take the following result to be the definition of a (graded split) quasi-hereditary algebra (with a graded duality).

2.9. **Corollary** ([22, Remark 3.10]). *Suppose that A is a graded cellular algebra. Then A is a split quasi-hereditary algebra, with standard modules $\{\, \Delta^\mu \mid \mu \in \mathscr{P} \,\}$, if and only if $L^\mu \neq 0$ for all $\mu \in \mathscr{P}$.*

2.3. **Basic algebras and graded Morita equivalences.** Let $\mathcal{Z} = K$ be a field. Recall that a finite dimensional *ungraded* graded, K-algebra $\underline{B}_0$ is a **basic algebra** if every irreducible $\underline{B}_0$-module is one dimensional. It is well-known that every finite dimensional (ungraded) K-algebra $\underline{B}$ is Morita equivalent to a unique (up to isomorphism) basic algebra $\underline{B}_0$. In fact, if $\{\underline{P}_1, \dots, \underline{P}_z\}$ is a complete set of pairwise non-isomorphic projective indecomposable $\underline{B}$-modules then the basic algebra of $\underline{B}$ is isomorphic to $\operatorname{End}_{\underline{B}}(\underline{P}_1 \oplus \dots \oplus \underline{P}_r)$.

Now let A and B be two finite dimensional graded K-algebras. Following [21, §5], the algebras A and B are **graded Morita equivalent** if there is a equivalence of graded module categories A-Mod $\cong B$-Mod. Equivalently, by the results of [21, §5], A and B are graded Morita equivalent if and only if there is an (ungraded) Morita equivalence $\underline{E} : \underline{A}\text{-Mod} \cong \underline{B}\text{-Mod}$ and a functor of the graded module categories $G : A\text{-Mod} \to B\text{-Mod}$ such that the following diagram commutes:

$$\begin{tikzcd}
A\text{-Mod} \arrow[r, "G"] \arrow[d, "\text{Forget}"'] & B\text{-Mod} \arrow[d, "\text{Forget}"] \\
\underline{A}\text{-Mod} \arrow[r, "\underline{E}"'] & \underline{B}\text{-Mod}
\end{tikzcd}$$

where the vertical functors are the natural forgetful functors. Let $\{P_1, \dots, P_z\}$ be a complete set of pairwise non-isomorphic graded projective indecomposable B-modules such that $P_i \ncong P_j\langle k\rangle$ for any $i \neq j$ and $k \in \mathbb{Z}$. The **graded basic algebra** of A is the endomorphism algebra

$$^{\flat}A = \mathrm{E{\scriptstyle ND}}_A\big(P_1 \oplus \cdots \oplus P_r\big).$$

By construction, every irreducible $^{\flat}A$-module is one dimensional so $^{\flat}A$ is a basic algebra. Moreover, $^{\flat}A$ is naturally $\mathbb{Z}$-graded and, on forgetting the grading, $^{\flat}\underline{A}$ is the basic algebra of $\underline{A}$. Note, however, that unlike in the ungraded case, two graded Morita equivalent graded basic algebras need not be isomorphic as graded algebras; see the discussion following [21, Corollary 5.11].

2.4. **Schur functors.** Several places in this paper rely on Auslander's theory of "Schur functors" which we now briefly recall in the graded setting following [8, §3.1].

Let A be a finite dimensional graded algebra (with 1) over a field K and let A-Mod be the category of finite dimensional graded right A-modules. Suppose that $e \in A$ is a non-zero idempotent of degree zero and consider the subalgebra eAe of A. Then eAe is a graded algebra with identity element e. (In all of our applications, A will be a quasi-hereditary graded cellular algebra.)

Define functors $\mathsf{F}: A\text{-Mod} \longrightarrow eAe\text{-Mod}$ and $\mathsf{G}: eAe\text{-Mod} \longrightarrow A\text{-Mod}$ by

$$\mathsf{F}(M) = Me \cong \mathrm{Hom}_A(eA, M) \qquad \text{and} \qquad \mathsf{G}(N) = N \otimes_{eAe} eA, \tag{2.10}$$

for $M \in A$-Mod and $N \in eAe$-Mod, together with the obvious action on morphisms. By definition, these functors respect the gradings on both categories. In general, however, these functors do not define equivalences between the (graded) module categories of A and eAe.

To define an equivalence between eAe-Mod and a subcategory of A-Mod we need to work a little harder. Suppose that M is an A-module and define $\mathsf{O}_e(M)$ to be the largest submodule M' of M such that $\mathsf{F}(M') = 0$ and define $\mathsf{O}^e(M)$ to be the smallest submodule M'' of M such that $\mathsf{F}(M/M'') = 0$. Any A-module homomorphism $M \longrightarrow N$ sends $\mathsf{O}_e(M)$ to $\mathsf{O}_e(N)$ and $\mathsf{O}^e(M)$ to $\mathsf{O}^e(N)$, so O_e and O^e define functors on the category of A-modules.

2.11. **Lemma** ([8, Corollary 3.1c]). *Suppose that M and N are A-modules such that $\mathsf{O}^e(M) \cong M$ and $\mathsf{O}_e(N) = 0$. Then* $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_{eAe}(\mathsf{F}M, \mathsf{F}N)$.

Let $A(e)$-Mod be the full subcategory of A-Mod with objects all A-modules M such that $\mathsf{O}_e(M) = 0$ and $\mathsf{O}^e(M) = M$. It is easy to check that any A-module homomorphism $M \longrightarrow N$ induces a well-defined map $M/\mathsf{O}_e(M) \longrightarrow N/\mathsf{O}_e(N)$ so that there is an exact functor

$$\mathsf{H}: A\text{-Mod} \longrightarrow A\text{-Mod}; M \mapsto M/\mathsf{O}_e(M).$$

By [8, Lemma 3.1a], the functors $\mathsf{H}\circ\mathsf{G}\circ\mathsf{F}$ and $\mathsf{F}\circ\mathsf{H}\circ\mathsf{G}$ are isomorphic to the identity functors on $A(e)$-Mod and on eAe-Mod respectively. This implies the following.

2.12. **Theorem** ([8, Theorem 3.1d]). *The restrictions of the functors* F *and* $\mathsf{H}\circ\mathsf{G}$ *induce mutually inverse equivalences of categories between* $A(e)$*-*Mod *and* eAe*-*Mod*.*

In [8] this result is proved only for ungraded algebras, however, the proof there generalizes without change to graded module categories.

2.5. **Koszul algebras.** In this section we recall the definition of Koszul algebras and the properties of these algebras that we will need. Throughout this section we work over a field K.

Let $A = \bigoplus_{d\in\mathbb{Z}} A_d$ be a finite dimensional graded K-algebra. Then A is **positively graded** if $A_d = 0$ whenever $d < 0$. That is, all of the homogeneous elements of A have non-negative degree.

Suppose that A is positively graded and that $M = \bigoplus_{d\in\mathbb{Z}} M_d$ is a finite dimensional A-module. For each $d \in \mathbb{Z}$ let $\mathcal{G}r_d\, M = \bigoplus_{k\geq d} M_k$. Since A is positively graded $\mathcal{G}r_d\, M$ is an A-submodule of M. Let a be minimal and z be maximal such that $\mathcal{G}r_a\, M = M$ and $\mathcal{G}r_z\, M = 0$, respectively. Then the **grading filtration** of M is the filtration

$$M = \mathcal{G}r_a\, M \supseteq \mathcal{G}r_{a-1}\, M \supseteq \cdots \supseteq \mathcal{G}r_z\, M = 0.$$

If A_0 is semisimple then the quotients $\mathcal{G}r_d\, M/\,\mathcal{G}r_{d+1}\, M$ are semisimple for all $d \in \mathbb{Z}$.

Let $M \supset \operatorname{rad}^1 M \supset \operatorname{rad}^2 M \supset \cdots \supset \operatorname{rad}^r M \supset 0$ be the **radical filtration** of M so that $\operatorname{rad}^1 M = \operatorname{rad} M$ and $\operatorname{rad}^{i+1} M = \operatorname{rad}(\operatorname{rad}^i M)$ for each $i \geq 1$. Similarly, let $M \supset \operatorname{soc}^s M \supset \operatorname{soc}^{s-1} M \supset \cdots \supset \operatorname{soc}^1 M \supset 0$ be the **socle filtration** of M where $\operatorname{soc}^1 M = \operatorname{soc} M$ and $\operatorname{soc}^{i+1} M$ is the inverse image of $\operatorname{soc}(M/\operatorname{soc}^i M)$ under the natural projection $M \twoheadrightarrow M/\operatorname{soc}^i M$. It is a general fact that the radical and socle filtrations of finite dimensional modules have the same length $\ell\ell(M)$, which is the **Loewy length** of M.

An A-module M is **rigid** if its socle and radical filtrations coincide. That is,

$$\operatorname{rad}^r M/\operatorname{rad}^{r+1} M \cong \operatorname{soc}^{\ell\ell(M)-r+1}/\operatorname{soc}^{\ell\ell(M)-r} M,$$

for $0 \leq r \leq \ell\ell(M)$.

The following result follows easily from the definitions.

2.13. **Lemma** ([5, Proposition 2.4.1]). *Suppose that* A *is positively graded and that* A_0 *is semisimple and that* A *is generated by* A_0 *and* A_1*. Let* M *be any finite dimensional* A*-module.*

a) *The radical filtration of* M *coincides with the grading filtration of* M*, up to shift, whenever* $M/\operatorname{rad} M$ *is irreducible.*
b) *The socle filtration of* M *coincides with the grading filtrations of* M*, up to shift, whenever* $\operatorname{soc} M$ *is irreducible.*

Consequently, M *is rigid whenever* $\operatorname{soc} M$ *and* $M/\operatorname{rad} M$ *are irreducible.*

2.14. **Definition** (Beilinson, Ginzburg and Soergel [5, Definition 1.2.1]).
A **Koszul algebra** is a positively graded algebra $A = \bigoplus_{d\geq 0} A_d$ such that A_0 is semisimple and, as a right A-module, A_0 has a (graded) projective resolution

$$\cdots \to P^2 \to P^1 \to P^0 \to A_0 \to 0$$

such that $P^d = P^d_d A$ is generated by its elements of degree d, for $d \geq 0$.

More generally, if A is a graded algebra then A-Mod is **Koszul** if it is graded Morita equivalent to the module category of a Koszul algebra.

We will need the following property of Koszul algebras.

2.15. **Proposition** ([5, Corollary 2.3.3]). *Suppose that* A *is a Koszul algebra. Then* A *is quadratic. That is,* A *is generated by* A_0 *and* A_1 *with homogeneous relations of degree two.*

3. Cyclotomic Quiver Hecke algebras and combinatorics

In this chapter we recall the facts about the cyclotomic quiver Hecke algebras of type Γ_e and the cyclotomic Hecke algebras of type $G(\ell,1,n)$ that we need in this paper.

3.1. **Cyclotomic quiver Hecke algebras.** Khovanov and Lauda [28, 29] and Rouquier [40] introduced (cyclotomic) quiver Hecke algebras for arbitrary oriented quivers. In this paper we consider mainly the linear quiver of type A_∞.

For the rest of this paper we fix a non-negative integer n and an integer $e \in \{0,2,3,4\dots\}$. Let Γ_e be oriented quiver with vertex set $I = \mathbb{Z}/e\mathbb{Z}$ and edges $i \longrightarrow i+1$, for all $i \in I$. To the quiver Γ_e we attach the standard Lie theoretic data of a Cartan matrix $(a_{ij})_{i,j\in I}$, fundamental weights $\{\,\Lambda_i \mid i \in I\,\}$, positive weights $P^+ = \sum_{i\in I}\mathbb{N}\Lambda_i$, positive roots $Q^+ = \bigoplus_{i\in I}\mathbb{N}\alpha_i$ and we let $(\cdot,\cdot)$ be the bilinear form determined by

$$(\alpha_i,\alpha_j) = a_{ij} \qquad \text{and} \qquad (\Lambda_i,\alpha_j) = \delta_{ij}, \qquad \text{for } i,j \in I.$$

More details can be found, for example, in [27, Chapt. 1].

Fix, once and for all, a **multicharge** $\boldsymbol{\kappa} = (\kappa_1,\dots,\kappa_\ell) \in \mathbb{Z}^\ell$ and define $\Lambda = \Lambda(\boldsymbol{\kappa}) = \Lambda_{\bar\kappa_1} + \dots + \Lambda_{\bar\kappa_\ell}$, where $\bar\kappa = \kappa \pmod e$. Equivalently, Λ is the unique element of P^+ such that

$$(\Lambda,\alpha_i) = \#\{\,1 \le l \le \ell \mid \kappa_l \equiv i \pmod e\,\}, \qquad \text{for all } i \in I. \tag{3.1}$$

All of the bases for the modules and algebras in this paper depend implicitly on κ even though the algebras themselves depend only on Λ.

Let $\mathfrak{S}_n$ is the symmetric group on n letters and let $s_r = (r, r+1)$. Then $\{s_1, s_2, \dots, s_{n-1}\}$ is the standard set of Coxeter generators for $\mathfrak{S}_n$. The group $\mathfrak{S}_n$ acts from the left on I^n by place permutations. More explicitly, if $1 \le r < n$ and $\mathbf{i} = (i_1,\dots,i_n) \in I^n$ then $s_r\mathbf{i} = (i_1,\dots,i_{r-1},i_{r+1},i_r,i_{r+2},\dots,i_n) \in I^n$.

3.2. **Definition.** Suppose that $n \ge 0$ and $e \in \{0,2,3,4,\dots\}$. The **cyclotomic quiver Hecke algebra**, or **cyclotomic Khovanov-Lauda–Rouquier algebra**, of weight Λ and type Γ_e is the unital associative $\mathcal{Z}$-algebra $\mathcal{R}_n^\Lambda = \mathcal{R}_{n,\mathcal{Z}}^\Lambda$ with generators

$$\{\psi_1,\dots,\psi_{n-1}\} \cup \{y_1,\dots,y_n\} \cup \{\, e(\mathbf{i}) \mid \mathbf{i} \in I^n\,\}$$

and relations

$$\begin{aligned} y_1^{(\Lambda,\alpha_{i_1})}e(\mathbf{i}) &= 0, & e(\mathbf{i})e(\mathbf{j}) &= \delta_{\mathbf{ij}}e(\mathbf{i}), & \textstyle\sum_{\mathbf{i}\in I^n} e(\mathbf{i}) &= 1,\\ y_r e(\mathbf{i}) &= e(\mathbf{i})y_r, & \psi_r e(\mathbf{i}) &= e(s_r{\cdot}\mathbf{i})\psi_r, & y_r y_s &= y_s y_r, \end{aligned}$$

$$\psi_r y_{r+1}e(\mathbf{i}) = (y_r\psi_r + \delta_{i_ri_{r+1}})e(\mathbf{i}), \qquad y_{r+1}\psi_r e(\mathbf{i}) = (\psi_r y_r + \delta_{i_ri_{r+1}})e(\mathbf{i}), \tag{3.3}$$

$$\begin{aligned} \psi_r y_s &= y_s\psi_r, && \text{if } s \ne r, r+1, \\ \psi_r\psi_s &= \psi_s\psi_r, && \text{if } |r-s| > 1, \end{aligned} \tag{3.4}$$

$$\psi_r^2 e(\mathbf{i}) = \begin{cases} 0, & \text{if } i_r = i_{r+1}, \\ (y_{r+1} - y_r)e(\mathbf{i}), & \text{if } i_r \to i_{r+1}, \\ (y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_r \leftarrow i_{r+1}, \\ (y_{r+1} - y_r)(y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_r \rightleftarrows i_{r+1} \\ e(\mathbf{i}), & \text{otherwise,} \end{cases}$$

$$\psi_r\psi_{r+1}\psi_r e(\mathbf{i}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} + 1)e(\mathbf{i}), & \text{if } i_r = i_{r+2} \to i_{r+1}, \\ (\psi_{r+1}\psi_r\psi_{r+1} - 1)e(\mathbf{i}), & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\ \big(\psi_{r+1}\psi_r\psi_{r+1} + y_r - 2y_{r+1} + y_{r+2}\big)e(\mathbf{i}), & \\ & \text{if } i_r = i_{r+2} \rightleftarrows i_{r+1}, \\ \psi_{r+1}\psi_r\psi_{r+1}e(\mathbf{i}), & \text{otherwise.} \end{cases}$$

for $\mathbf{i}, \mathbf{j} \in I^n$ and all admissible r and s. Moreover, $\mathcal{R}_n^\Lambda$ is naturally $\mathbb{Z}$-graded with degree function determined by

$$\deg e(\mathbf{i}) = 0, \qquad \deg y_r = 2 \qquad \text{and} \qquad \deg \psi_s e(\mathbf{i}) = -a_{i_s, i_{s+1}},$$

for $1 \le r \le n$, $1 \le s < n$ and $\mathbf{i} \in I^n$.

Inspecting the relations in Definition 3.2, there is a unique anti-isomorphism $\star$ of $\mathcal{R}_n^\Lambda$ which fixes each of the generators of $\mathcal{R}_n^\Lambda$. Thus $\star$ is homogeneous of order 2. Hence, by twisting with $\star$ we can define the contragredient dual $M^\circledast$ of an $\mathcal{R}_n^\Lambda$-module $M^\circledast = \mathrm{H{\small OM}}_{\mathcal{Z}}(M, \mathcal{Z})$ as in (2.2).

In this paper we will mainly be concerned with the special cases when either $e = 0$ or $e > n$. We note that if $e = 0$ then $I = \mathbb{Z}$ so that, at first sight, the set $\{\, e(\mathbf{i}) \mid \mathbf{i} \in \mathbb{Z}^n \,\}$ is infinite and the relation $\sum_{\mathbf{i} \in I^n} e(\mathbf{i}) = 1$ does not make sense. However, at least when $\mathcal{Z}$ is a field, it follows from Theorem 3.7 below that $e(\mathbf{i}) \ne 0$ for only finitely many $\mathbf{i} \in I^n$. The presentation of $\mathcal{R}_n^\Lambda$ depends on the orientation of Γ_e, however, it is easy to see that different orientations of Γ_e yield isomorphic algebras; see the last section of [28].

3.2. **Cyclotomic Hecke algebras.** Recall that $\Lambda \in P^+$ and that we have fixed an integer $e \in \{0, 2, 3, 4, \dots\}$. We now define the 'integral' cyclotomic Hecke algebras $\mathcal{H}_n^\Lambda$ of type $G(\ell, 1, n)$, where $\ell = \sum_{i \in I} (\Lambda, \alpha_i)$ is the **level** of Λ.

Fix a integral domain $\mathcal{Z}$ which contains an element $\xi = \xi(e)$ such that one of the following holds:

- $e > 0$ and ξ is a primitive eth root of unity in $\mathcal{Z}$.
- $e = 0$ and ξ is not a root of unity.
- $\xi = 1$ and e is the characteristic of $\mathcal{Z}$.

Define $\delta_{\xi 1} = 1$ if $\xi = 1$ and $\delta_{\xi 1} = 0$ otherwise. For $k \in \mathbb{Z}$ set

$$\xi^{(k)} = \begin{cases} \xi^k, & \text{if } \xi \ne 1, \\ k, & \text{if } \xi = 1. \end{cases} \tag{3.5}$$

The definition of $\xi = \xi(e)$ above ensures that $\xi^{(i)} = \xi^{(i+e)}$. Hence, $\xi^{(i)}$ is well-defined for all $i \in I = \mathbb{Z}/e\mathbb{Z}$.

3.6. **Definition.** The (integral) **cyclotomic Hecke algebra** $\mathcal{H}_n^\Lambda = \mathcal{H}_n^\Lambda(\mathcal{Z}, \xi)$ of type $G(\ell, 1, n)$ is the unital associative $\mathcal{Z}$-algebra with generators $L_1, \dots, L_n$,

$T_1, \dots, T_{n-1}$ and relations

$$\prod_{i\in I}(L_1 - \xi^{(i)})^{(\Lambda,\alpha_i)} = 0, \qquad L_rL_t = L_tL_r,$$
$$(T_r+1)(T_r-\xi) = 0, \qquad T_rL_r + \delta_{\xi 1} = L_{r+1}(T_r - \xi + 1),$$
$$T_sT_{s+1}T_s = T_{s+1}T_sT_{s+1},$$
$$T_rL_t = L_tT_r, \qquad \text{if } t \neq r, r+1,$$
$$T_rT_s = T_sT_r, \qquad \text{if } |r-s| > 1,$$

where $1 \le r < n$, $1 \le s < n-1$ and $1 \le t \le n$.

It is well-known that $\mathcal{H}_n^\Lambda$ decomposes into a direct sum of simultaneous generalized eigenspaces for the elements $L_1, \dots, L_n$ (cf. [23]). Moreover, the possible eigenvalues for $L_1, \dots, L_n$ belong to the set $\{\, \xi^{(i)} \mid i \in I \,\}$. Hence, the generalized eigenspaces for these elements are indexed by I^n. For each $\mathbf{i} \in I^n$ let $e(\mathbf{i})$ be the corresponding idempotent in $\mathcal{H}_n^\Lambda$ (or zero if the corresponding eigenspace is zero).

3.7. **Theorem** (Brundan-Kleshchev [10, Theorem 1.1])**.** *Suppose that $\mathcal{Z} = K$ is a field, $\xi \in K$ as above, and that $\Lambda = \Lambda(\boldsymbol{\kappa})$. Then there is an isomorphism of algebras $\underline{\mathcal{R}}_n^\Lambda \cong \mathcal{H}_n^\Lambda$ which sends $e(\mathbf{i}) \mapsto e(\mathbf{i})$, for all $\mathbf{i} \in I^n$ and*

$$y_r \mapsto \begin{cases} \sum_{\mathbf{i}\in I^n} (1 - \xi^{-i_r}L_r)e(\mathbf{i}), & \text{if } \xi \neq 1,\\ \sum_{\mathbf{i}\in I^n} (L_r - i_r)e(\mathbf{i}), & \text{if } \xi = 1.\end{cases}$$
$$\psi_s \mapsto \sum_{\mathbf{i}\in I^n} (T_s + P_s(\mathbf{i}))Q_s(\mathbf{i})^{-1}e(\mathbf{i}),$$

where $P_r(\mathbf{i}), Q_r(\mathbf{i}) \in R[y_r, y_{r+1}]$, for $1 \le r \le n$ and $1 \le s < n$.

By [10, Theorem 1.1], the inverse isomorphism $\mathcal{H}_n^\Lambda \xrightarrow{\sim} \underline{\mathcal{R}}_n^\Lambda$ is determined by

$$L_r \mapsto \begin{cases} \sum_{\mathbf{i}\in I^n} \xi^{i_r}(1 - y_r)e(\mathbf{i}), & \text{if } \xi \neq 1,\\ \sum_{\mathbf{i}\in I^n} (i_r + y_r)e(\mathbf{i}), & \text{if } \xi = 1.\end{cases} \tag{3.8}$$

$$T_s \mapsto \sum_{\mathbf{i}\in I^n} (\psi_sQ_s(\mathbf{i}) - P_s(\mathbf{i}))e(\mathbf{i}), \tag{3.9}$$

for $1 \le r \le n$ and $1 \le s < n$.

Henceforth, we identify the algebras $\underline{\mathcal{R}}_n^\Lambda$ and $\mathcal{H}_n^\Lambda$ under this isomorphism. In particular, we will not distinguish between the homogeneous generators of $\underline{\mathcal{R}}_n^\Lambda$ and their images in $\mathcal{H}_n^\Lambda$ under the isomorphism of Theorem 3.7.

Even though we will not distinguish between $\underline{\mathcal{R}}_n^\Lambda$ and $\mathcal{H}_n^\Lambda$ we will usually write $\mathcal{R}_n^\Lambda$ when we are working with graded representations and $\mathcal{H}_n^\Lambda$ for ungraded representations.

3.3. **Tableaux combinatorics.** This section sets up the tableaux combinatorics that will be used throughout this paper. Recall that a **partition** of n is a weakly decreasing sequence $\mu = (\mu_1 \ge \mu_2 \ge \dots)$ of non-negative integers which sum to n. Set $|\mu| = n$.

A **multipartition** of n is an ℓ-tuple $\boldsymbol{\mu} = (\mu^{(1)}|\dots|\mu^{(\ell)})$ of partitions such that $|\mu^{(1)}| + \dots + |\mu^{(\ell)}| = n$. We *identify* a multipartition with its diagram

$$\boldsymbol{\mu} = \{\, (r,c,l) \mid 1 \le c \le \mu_r^{(l)} \text{ for } r \ge 1 \text{ and } 1 \le l \le \ell \,\},$$

which we think of as an ℓ-tuple of boxes in the plane. For example,

$$(3,2|2,1^2|3,1) = \left(\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \end{array} \;\middle|\; \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \\ \cline{1-1} \end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1} \end{array} \right).$$

The partitions $\mu^{(1)}, \dots, \mu^{(\ell)}$ are the **components** of $\boldsymbol{\mu}$ and we identify $\mu^{(l)}$ with the subdiagram $\{\, (r,c,l) \mid 1 \le c \le \mu^{(l)}_r \text{ for } r \ge 1 \,\}$ of $\boldsymbol{\mu}$. A **node** is any triple $A = (r,c,l) \in \mathbb{N}^2 \times \{1,2,\dots,\ell\}$. In particular, the elements of (the diagram of) $\boldsymbol{\mu}$ are nodes.

Let $\mathscr{P}^\Lambda_n$ be the set of multipartitions of n. Then $\mathscr{P}^\Lambda_n$ is a poset under the **dominance order** $\trianglerighteq$ where $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$, for multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ of n, if

$$\sum_{k=1}^{l-1} |\lambda^{(k)}| + \sum_{j=1}^{i} \lambda^{(l)}_j \ge \sum_{k=1}^{l-1} |\mu^{(k)}| + \sum_{j=1}^{i} \mu^{(l)}_j,$$

for $1 \le l \le \ell$ and $i \ge 1$. If $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \ne \boldsymbol{\mu}$ then we write $\boldsymbol{\lambda} \triangleright \boldsymbol{\mu}$.

Suppose that $\boldsymbol{\mu} \in \mathscr{P}^\Lambda_n$ is a multipartition of n. Then a **$\boldsymbol{\mu}$-tableau** is a bijective map $\mathfrak{t} \colon \boldsymbol{\mu} \longrightarrow \{1,2,\dots,n\}$. We think of a $\boldsymbol{\mu}$-tableau $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(\ell)})$ as a labelling of (the diagram of) $\boldsymbol{\mu}$, where $\mathfrak{t}^{(r)}$ is the restriction of $\mathfrak{t}$ to $\mu^{(r)}$. In this way, we talk of the rows, columns and components of a tableau $\mathfrak{t}$. For example,

$$\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \;\middle|\; \begin{array}{|c|c|}\hline 6 & 7 \\ \hline 8 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline 10 & 11 & 12 \\ \hline 13 \\ \cline{1-1} \end{array} \right) \text{ and } \left(\begin{array}{|c|c|c|}\hline 9 & 11 & 13 \\ \hline 10 & 12 \\ \cline{1-2} \end{array} \;\middle|\; \begin{array}{|c|c|}\hline 5 & 8 \\ \hline 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \right)$$

are two $(3,2|2,1^2|3,1)$-tableaux. If $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(\ell)})$ is a $\boldsymbol{\mu}$-tableau then define $\operatorname{Shape}(\mathfrak{t}) = \boldsymbol{\mu}$, so that $\operatorname{Shape}(\mathfrak{t}^{(r)}) = \mu^{(r)}$, for $1 \le r \le \ell$. If $\mathfrak{t}^{-1}(k) = (r,c,l)$, then we set $\operatorname{comp}_{\mathfrak{t}}(k) = l$.

A $\boldsymbol{\mu}$-tableau $\mathfrak{t}$ is **standard** if its entries increase along the rows and down the columns of each component. For example, the two tableaux above are standard. If $\mathfrak{t}$ is a standard tableau let $\mathfrak{t}_{\downarrow k}$ be the subtableau of $\mathfrak{t}$ which contains $1,2,\dots,k$. Then a tableau $\mathfrak{t}$ is standard if and only if $\operatorname{Shape}(\mathfrak{t}_{\downarrow k})$ is a multipartition for $1 \le k \le n$. The dominance order induces a partial order on the set of tableaux where $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if

$$\operatorname{Shape}(\mathfrak{s}_{\downarrow k}) \trianglerighteq \operatorname{Shape}(\mathfrak{t}_{\downarrow k}), \qquad \text{for } 1 \le k \le n,$$

for $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ and $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\mu})$, where $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\Lambda_n$. Again we write $\mathfrak{s} \triangleright \mathfrak{t}$ if $\mathfrak{s} \trianglerighteq \mathfrak{t}$ and $\mathfrak{s} \ne \mathfrak{t}$. Let $\operatorname{Std}(\boldsymbol{\mu})$ be the poset of standard $\boldsymbol{\mu}$-tableau and set $\operatorname{Std}^2(\boldsymbol{\mu}) = \operatorname{Std}(\boldsymbol{\mu}) \times \operatorname{Std}(\boldsymbol{\mu})$, $\operatorname{Std}(\mathscr{P}^\Lambda_n) = \bigcup_{\boldsymbol{\mu} \in \mathscr{P}^\Lambda_n} \operatorname{Std}(\boldsymbol{\mu})$ and $\operatorname{Std}^2(\mathscr{P}^\Lambda_n) = \bigcup_{\boldsymbol{\mu} \in \mathscr{P}^\Lambda_n} \operatorname{Std}^2(\boldsymbol{\mu})$.

We extend the dominance order to $\operatorname{Std}^2(\mathscr{P}^\Lambda_n)$ by declaring that $(\mathfrak{s},\mathfrak{t}) \underline{\blacktriangleright} (\mathfrak{u},\mathfrak{v})$ if $\mathfrak{s} \trianglerighteq \mathfrak{u}$ and $\mathfrak{t} \trianglerighteq \mathfrak{v}$. We write $(\mathfrak{s},\mathfrak{t}) \blacktriangleright (\mathfrak{u},\mathfrak{v})$ if $(\mathfrak{s},\mathfrak{t}) \underline{\blacktriangleright} (\mathfrak{u},\mathfrak{v})$ and $(\mathfrak{s},\mathfrak{t}) \ne (\mathfrak{u},\mathfrak{v})$.

If $\boldsymbol{\mu} \in \mathscr{P}^\Lambda_n$ let $\boldsymbol{\mu}' = (\mu^{(\ell)'}, \dots, \mu^{(1)'})$ be the **conjugate** multipartition which is obtained from $\boldsymbol{\mu}$ by reversing the order of its components and then swapping the rows and columns in each component. Similarly, the **conjugate** of the $\boldsymbol{\mu}$-tableau $\mathfrak{t}$ is the $\boldsymbol{\mu}'$-tableau $\mathfrak{t}'$ which is obtained from $\mathfrak{t}$ by reversing its components and then swapping its rows and columns in each component. The reader is invited to check that $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if and only if $\boldsymbol{\mu}' \trianglerighteq \boldsymbol{\lambda}'$ and that $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if and only if $\mathfrak{t}' \trianglerighteq \mathfrak{s}'$, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\Lambda_n$ and for $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\mathscr{P}^\Lambda_n)$.

Fix a multipartition $\boldsymbol{\mu} \in \mathscr{P}^\Lambda_n$. Define $\mathfrak{t}^{\boldsymbol{\mu}}$ to be the unique standard $\boldsymbol{\mu}$-tableau such that $\mathfrak{t}^{\boldsymbol{\mu}} \trianglerighteq \mathfrak{t}$, for all $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\mu})$. More explicitly, $\mathfrak{t}^{\boldsymbol{\mu}}$ is the $\boldsymbol{\mu}$-tableau which has the numbers $1,2,\dots,n$ entered in order, from left to right, and then top to bottom, along the rows of the components $\mu^{(1)}, \dots, \mu^{(\ell)}$ of $\boldsymbol{\mu}$. Define $\mathfrak{t}_{\boldsymbol{\mu}} = (\mathfrak{t}^{\boldsymbol{\mu}'})'$. By the last paragraph $\mathfrak{t}_{\boldsymbol{\mu}}$ is the unique $\boldsymbol{\mu}$-tableau such that $\mathfrak{t} \trianglerighteq \mathfrak{t}_{\boldsymbol{\mu}}$, for all $\mathfrak{t}_{\boldsymbol{\mu}} \in \operatorname{Std}(\boldsymbol{\mu})$. The numbers $1,2,\dots,n$ are entered in order down the columns of the components

$\mu^{(\ell)}, \dots, \mu^{(1)}$ of $\boldsymbol{\mu}$. The two tableaux displayed above are $\mathfrak{t}^{\boldsymbol{\mu}}$ an $\mathfrak{t}_{\boldsymbol{\mu}}$, respectively, for $\boldsymbol{\mu} = (3,2|2,1^2|3,1)$.

Recall from Section 3.1 that we have fixed a multicharge $\kappa \in \mathbb{Z}^\ell$. The **residue** of the node $A = (r,c,l)$ is $\operatorname{res}(A) = \kappa_l + c - r \pmod e$ (where we adopt the convention that $i \equiv i \pmod 0$, for $i \in \mathbb{Z}$). Thus, $\operatorname{res}(A) \in I$. A node A is an i**-node** if $\operatorname{res}(A) = i$. If $\mathfrak{t}$ is a $\boldsymbol{\mu}$-tableaux and $1 \le k \le n$ then the **residue** of k in $\mathfrak{t}$ is $\operatorname{res}_{\mathfrak{t}}(k) = \operatorname{res}(A)$, where $A \in \boldsymbol{\mu}$ is the unique node such that $\mathfrak{t}(A) = k$. The **residue sequence** of $\mathfrak{t}$ is

$$\operatorname{res}(\mathfrak{t}) = (\operatorname{res}_{\mathfrak{t}}(1), \operatorname{res}_{\mathfrak{t}}(2), \dots, \operatorname{res}_{\mathfrak{t}}(n)) \in I^n.$$

As two important special cases we set $\mathbf{i}^\mu = \operatorname{res}(\mathfrak{t}^{\boldsymbol{\mu}})$ and $\mathbf{i}_{\boldsymbol{\mu}} = \operatorname{res}(\mathfrak{t}_{\boldsymbol{\mu}})$, for $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$.

Following Brundan, Kleshchev and Wang [13, Definition. 3.5] we now define the degree and codegree of a standard tableau. Suppose that $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$. A node A is an **addable node** of $\boldsymbol{\mu}$ if $A \notin \boldsymbol{\mu}$ and $\boldsymbol{\mu} \cup \{A\}$ is the (diagram of) a multipartition of $n+1$. Similarly, a node B is a **removable node** of $\boldsymbol{\mu}$ if $B \in \boldsymbol{\mu}$ and $\boldsymbol{\mu} \setminus \{B\}$ is a multipartition of $n-1$. Given any two nodes $A = (r,c,l), B = (r',c',l')$, say that B is strictly below A, or A is strictly above B, if either $l' > l$ or $l' = l$ and $r' > r$. Suppose that A is an i-node and define integers

$$d_A(\boldsymbol{\mu}) = \#\left\{\begin{array}{c}\text{addable } i\text{-nodes of } \boldsymbol{\mu}\\ \text{strictly below } A\end{array}\right\} - \#\left\{\begin{array}{c}\text{removable } i\text{-nodes of } \boldsymbol{\mu}\\ \text{strictly below } A\end{array}\right\},$$

and

$$d^A(\boldsymbol{\mu}) = \#\left\{\begin{array}{c}\text{addable } i\text{-nodes of } \boldsymbol{\mu}\\ \text{strictly above } A\end{array}\right\} - \#\left\{\begin{array}{c}\text{removable } i\text{-nodes of } \boldsymbol{\mu}\\ \text{strictly above } A\end{array}\right\}.$$

If $\mathfrak{t}$ is a standard $\boldsymbol{\mu}$-tableau then its **degree** and **codegree** are defined inductively by setting $\deg \mathfrak{t} = 0 = \operatorname{codeg} \mathfrak{t}$, if $n = 0$, and if $n > 0$ then

$$\deg \mathfrak{t} = \deg \mathfrak{t}_{\downarrow(n-1)} + d_A(\boldsymbol{\mu}) \quad \text{and} \quad \operatorname{codeg} \mathfrak{t} = \operatorname{codeg} \mathfrak{t}_{\downarrow(n-1)} + d^A(\boldsymbol{\mu}),$$

where $A = \mathfrak{t}^{-1}(n)$. The definitions of the residue, degree and codegree of a tableau all depend on the choice of multicharge $\boldsymbol{\kappa}$. We write $\operatorname{res}_{\mathfrak{t}}^{\boldsymbol{\kappa}}$, $\deg^{\boldsymbol{\kappa}} \mathfrak{t}$ and $\operatorname{codeg}^{\boldsymbol{\kappa}} \mathfrak{t}$ when we want to emphasize this choice.

Recall that $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$ is the positive root lattice. Fix $\beta \in Q^+$ with $\sum_{i \in I} (\Lambda_i, \beta) = n$ and let

$$I^\beta = \{\, \mathbf{i} \in I^n \mid \alpha_{i_1} + \dots + \alpha_{i_n} = \beta \,\}.$$

Then I^β is an $\mathfrak{S}_n$-orbit of I^n and it is straightforward to check that every $\mathfrak{S}_n$-orbit can be written uniquely in this way for some $\beta \in Q^+$.

Fix $\beta \in Q^+$ and set $\mathscr{P}_\beta^\Lambda = \{\, \boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda \mid \mathbf{i}^{\boldsymbol{\lambda}} \in I^\beta \,\}$. The **defect** of β is the integer

$$\operatorname{def} \beta = (\Lambda, \beta) - \frac{1}{2}(\beta, \beta).$$

The defect of a block is closely related to the degree and codegree of the corresponding tableaux.

3.10. **Lemma** ([13, Lemma 3.12]). *Suppose that* $\beta \in Q^+$ *and* $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\mu})$, *for* $\boldsymbol{\mu} \in \mathscr{P}_\beta^\Lambda$. *Then* $\deg \mathfrak{t} + \operatorname{codeg} \mathfrak{t} = \operatorname{def} \beta$.

3.4. **Standard homogeneous bases.** We are now ready to define some bases for the cyclotomic quiver Hecke algebra $\mathcal{R}_n^\Lambda$. Recall from the last section that $\mathfrak{S}_n$ is the symmetric group on n letters and that $\{s_1, s_2, \dots, s_{n-1}\}$ is the standard set of Coxeter generators for $\mathfrak{S}_n$. If $w \in \mathfrak{S}_n$ then the **length** of w is the integer

$$\ell(w) = \min\{\, k \mid w = s_{r_1} \dots s_{r_k} \text{ for some } 1 \le r_1, \dots, r_k < n \,\}.$$

A **reduced expression** for w is a word $w = s_{r_1} \dots s_{r_k}$ such that $k = \ell(w)$. It is a general fact from the theory of Coxeter groups that any reduced expression for w can be transformed into any reduced expression using just the braid relations $s_rs_t = s_ts_t$, if $|r-t| > 1$, and $s_rs_{r+1}s_r = s_{r+1}s_rs_{r+1}$, for $1 \le r < n-1$.

Hereafter, unless otherwise stated, we fix a reduced expression $w = s_{r_1} \dots s_{r_k}$ for each element $w \in \mathfrak{S}_n$, with $1 \le r_1, \dots, r_k < n$. We define $\psi_w = \psi_{r_1} \dots \psi_{r_k}$. By Definition 3.2, the generators ψ_r, for $1 \le r < n$, do not satisfy the braid relations. Therefore, the element $\psi_w \in \mathcal{R}_n^\Lambda$ depends upon our choice of reduced expression for w.

The symmetric group $\mathfrak{S}_n$ acts from the right on the set of tableaux by composition of maps. If $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\mu})$ define two permutations $d(\mathfrak{t})$ and $d'(\mathfrak{t})$ in $\mathfrak{S}_n$ by $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\mu}} d(\mathfrak{t})$ and $\mathfrak{t} = \mathfrak{t}_{\boldsymbol{\mu}} d'(\mathfrak{t})$. Conjugating either of the last two equations shows that $d'(\mathfrak{t}) = d(\mathfrak{t}')$. Let $w_{\boldsymbol{\mu}} = d(\mathfrak{t}_{\boldsymbol{\mu}})$. Then it is easy to check that $w_{\boldsymbol{\mu}} = d(\mathfrak{t})d'(\mathfrak{t})^{-1}$ and $\ell(w_{\boldsymbol{\mu}}) = \ell(d(\mathfrak{t})) + \ell(d'(\mathfrak{t}))$, for all $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\mu})$.

Recall from section 3.3 that $\mathbf{i}^{\boldsymbol{\mu}} = \operatorname{res}(\mathfrak{t}^{\boldsymbol{\mu}})$ and that $\mathbf{i}_{\boldsymbol{\mu}} = \operatorname{res}(\mathfrak{t}_{\boldsymbol{\mu}})$.

3.11. **Definition** ([25, Definitions 4.9, 5.1 and 6.9])**.** Suppose that $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$. Define non-negative integers $d_1^{\boldsymbol{\mu}}, \dots, d_n^{\boldsymbol{\mu}}$ and $d_{\boldsymbol{\mu}}^1, \dots, d_{\boldsymbol{\mu}}^n$ recursively by requiring that

$$d_1^{\boldsymbol{\mu}} + \dots + d_k^{\boldsymbol{\mu}} = \deg(\mathfrak{t}^{\boldsymbol{\mu}}_{\downarrow k}) \quad \text{and} \quad d_{\boldsymbol{\mu}}^1 + \dots + d_{\boldsymbol{\mu}}^k = \operatorname{codeg}(\mathfrak{t}_{\boldsymbol{\mu}\downarrow k}),$$

for $1 \le k \le n$. Now set $e^{\boldsymbol{\mu}} = e(\mathbf{i}^{\boldsymbol{\mu}})$, $e_{\boldsymbol{\mu}} = e(\mathbf{i}_{\boldsymbol{\mu}})$,

$$y^{\boldsymbol{\mu}} = y_1^{d_1^{\boldsymbol{\mu}}} \dots y_n^{d_n^{\boldsymbol{\mu}}} \quad \text{and} \quad y_{\boldsymbol{\mu}} = y_1^{d_{\boldsymbol{\mu}}^1} \dots y_n^{d_{\boldsymbol{\mu}}^n}.$$

For a pair of tableaux $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\boldsymbol{\mu})$ define

$$\psi_{\mathfrak{s}\mathfrak{t}} = \psi^\star_{d(\mathfrak{s})} e^{\boldsymbol{\mu}} y^{\boldsymbol{\mu}} \psi_{d(\mathfrak{t})} \quad \text{and} \quad \psi'_{\mathfrak{s}\mathfrak{t}} = \psi^\star_{d'(\mathfrak{s})} e_{\boldsymbol{\mu}} y_{\boldsymbol{\mu}} \psi_{d'(\mathfrak{t})}.$$

3.12. *Remark.* We warn the reader that the element $\psi'_{\mathfrak{s}\mathfrak{t}}$ is equal to the element $\psi'_{\mathfrak{s}'\mathfrak{t}'}$ in the notation of [25, 26] so care should be taken when comparing the results in this paper with those in [25, 26]. We have changed notation because Definition 3.11 makes several subsequent definitions and results more intuitive. For example, see Corollary 3.19 and Proposition 3.26 below.

In general, the elements $\psi_{\mathfrak{s}\mathfrak{t}}$ and $\psi'_{\mathfrak{s}\mathfrak{t}}$ depend upon the choice of reduced expression that we fixed in Definition 3.11 because $\psi_1, \dots, \psi_{n-1}$ do not satisfy the braid relations. Similarly, $\psi_{d(\mathfrak{s})^{-1}}$ and $\psi^\star_{\mathfrak{s}}$ will generally be different elements of $\mathcal{R}_n^\Lambda$.

It follows from Definition 3.11 and the relations that if $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathscr{P}_n^\Lambda)$ then

$$e(\mathbf{i})\psi_{\mathfrak{s}\mathfrak{t}}e(\mathbf{j}) = \delta_{\mathbf{i},\operatorname{res}(\mathfrak{s})}\delta_{\mathbf{j},\operatorname{res}(\mathfrak{t})}\psi_{\mathfrak{s}\mathfrak{t}} \quad \text{and} \quad e(\mathbf{i})\psi'_{\mathfrak{s}\mathfrak{t}}e(\mathbf{j}) = \delta_{\mathbf{i},\operatorname{res}(\mathfrak{s})}\delta_{\mathbf{j},\operatorname{res}(\mathfrak{t})}\psi'_{\mathfrak{s}\mathfrak{t}}, \tag{3.13}$$

for all $\mathbf{i}, \mathbf{j} \in I^n$. More importantly we have the following.

3.14. **Theorem** (Hu-Mathas [25, Theorems 5.8 and 6.11])**.** *Suppose that $\mathcal{Z}$ is an integral domain such that e is invertible in $\mathcal{Z}$ whenever $e \ne 0$ and e is not prime. Then:*

a) $\{\, \psi_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathscr{P}_n^\Lambda) \,\}$ *is a graded cellular basis of* $\mathcal{H}_n^\Lambda$ *with weight poset* $(\mathscr{P}_n^\Lambda, \trianglerighteq)$ *and degree function* $\deg \psi_{\mathfrak{s}\mathfrak{t}} = \deg \mathfrak{s} + \deg \mathfrak{t}$.
b) $\{\, \psi'_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathscr{P}_n^\Lambda) \,\}$ *is a graded cellular basis of* $\mathcal{H}_n^\Lambda$ *with weight poset* $(\mathscr{P}_n^\Lambda, \trianglelefteq)$ *and degree function* $\deg \psi'_{\mathfrak{s}\mathfrak{t}} = \operatorname{codeg} \mathfrak{s} + \operatorname{codeg} \mathfrak{t}$.

As we explain in Proposition 3.26 below, these two bases are essentially equivalent. The ψ-basis and the ψ'-basis are dual to each other in the following sense.

3.15. **Lemma** ([26, Corollary 3.10])**.** *Suppose that* $(\mathfrak{s}, \mathfrak{t}), (\mathfrak{u}, \mathfrak{v}) \in \operatorname{Std}^2(\mathscr{P}_n^\Lambda)$. *Then:*

a) $\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{u}\mathfrak{v}} \ne 0$ *only if* $\operatorname{res}(\mathfrak{t}) = \operatorname{res}(\mathfrak{u})$ *and* $\mathfrak{u} \trianglerighteq \mathfrak{t}$.
b) $\psi'_{\mathfrak{u}\mathfrak{v}}\psi_{\mathfrak{s}\mathfrak{t}} \ne 0$ *only if* $\operatorname{res}(\mathfrak{s}) = \operatorname{res}(\mathfrak{v})$ *and* $\mathfrak{v} \trianglerighteq \mathfrak{s}$

We need the following dominance results. Recall from §3.3 that $(\mathfrak{s}, \mathfrak{t}) \mathrel{\underline{\blacktriangleright}} (\mathfrak{u}, \mathfrak{v})$ if $\mathfrak{s} \trianglerighteq \mathfrak{u}$ and $\mathfrak{t} \trianglerighteq \mathfrak{v}$.

3.16. **Lemma** ([26, Corollary 3.11]). *Suppose that* $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}(\mathscr{P}^\Lambda_n)$ *and* $1\le r\le n$. *Then*

$$\psi_{\mathfrak{s}\mathfrak{t}}y_r=\sum_{\substack{(\mathfrak{u},\mathfrak{v})\in\mathscr{P}^\Lambda_n\\(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})}}a_{\mathfrak{u}\mathfrak{v}}\psi_{\mathfrak{u}\mathfrak{v}}\quad\textit{and}\quad\psi'_{\mathfrak{s}\mathfrak{t}}y_r=\sum_{\substack{(\mathfrak{u},\mathfrak{v})\in\mathscr{P}^\Lambda_n\\(\mathfrak{s},\mathfrak{t})\blacktriangleright(\mathfrak{u},\mathfrak{v})}}b_{\mathfrak{u}\mathfrak{v}}\psi_{\mathfrak{u}\mathfrak{v}},$$

for some scalars $a_{\mathfrak{u}\mathfrak{v}},b_{\mathfrak{u}\mathfrak{v}}\in\mathcal{Z}$.

The next result strengthens [25, Lemma 5.7].

3.17. **Lemma.** *Suppose that* $\psi_{\mathfrak{s}\mathfrak{t}}$ *and* $\hat\psi_{\mathfrak{s}\mathfrak{t}}$ *are defined using possibly different reduced expressions for* $d(\mathfrak{s})$ *and* $d(\mathfrak{t})$, *where* $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ *for some* $\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_n$. *Then*

$$\psi_{\mathfrak{s}\mathfrak{t}}-\hat\psi_{\mathfrak{s}\mathfrak{t}}=\sum_{(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})}s_{\mathfrak{u}\mathfrak{v}}\psi_{\mathfrak{u}\mathfrak{v}}\quad\textit{and}\quad\psi'_{\mathfrak{s}\mathfrak{t}}-\hat\psi'_{\mathfrak{s}\mathfrak{t}}=\sum_{(\mathfrak{s},\mathfrak{t})\blacktriangleright(\mathfrak{u},\mathfrak{v})}t_{\mathfrak{u}\mathfrak{v}}\psi'_{\mathfrak{u}\mathfrak{v}}$$

where $s_{\mathfrak{u}\mathfrak{v}}\ne0$ *only if* $\operatorname{res}(\mathfrak{u})=\operatorname{res}(\mathfrak{s})$, $\operatorname{res}(\mathfrak{v})=\operatorname{res}(\mathfrak{t})$ *and* $\deg\mathfrak{u}+\deg\mathfrak{v}=\deg\mathfrak{s}+\deg\mathfrak{t}$ *and* $t_{\mathfrak{u}\mathfrak{v}}\ne0$ *only if* $\operatorname{res}(\mathfrak{u})=\operatorname{res}(\mathfrak{s})$, $\operatorname{res}(\mathfrak{v})=\operatorname{res}(\mathfrak{t})$ *and* $\operatorname{codeg}\mathfrak{u}+\operatorname{codeg}\mathfrak{v}=\operatorname{codeg}\mathfrak{s}+\operatorname{codeg}\mathfrak{t}$.

Proof. By [26, Theorem 3.9] the transition matrices between the ψ-basis and the (non-homogeneous) standard basis of $\mathcal{H}^\Lambda_n$ from [17] is triangular with respect to strong dominance partial order $\trianglerighteq$. The same remark applies to the $\hat\psi$-basis, which is defined using possibly different choices of reduced expressions. Applying this result twice to rewrite $\hat\psi_{\mathfrak{s}\mathfrak{t}}$ in terms of the ψ-basis, via the standard basis, proves the first statement. The second statement can be proved similarly. □

3.5. **The blocks of $\mathcal{R}^\Lambda_n$.** We now show how Theorem 3.14 restricts to give a basis for the blocks, or the indecomposable two-sided ideals, of $\mathcal{R}^\Lambda_n$. Suppose that $\beta\in Q^+$ and define

$$\mathcal{R}^\Lambda_\beta=e_\beta\mathcal{R}^\Lambda_n=e_\beta\mathcal{R}^\Lambda_ne_\beta,\qquad\text{where }e_\beta=\sum_{\mathbf{i}\in I^\beta}e(\mathbf{i}).$$

Set $Q^+_n=\{\,\beta\in Q^+\mid e_\beta\ne0\text{ in }\mathcal{R}^\Lambda_n\,\}$. By [35, Theorem 2.11] and [7, Theorem 1], if $\mathcal{Z}=K$ is a field then $\mathcal{R}^\Lambda_\beta$ is a block of $\mathcal{R}^\Lambda_n$. That is,

$$\mathcal{R}^\Lambda_n=\bigoplus_{\beta\in Q^+_n}\mathcal{R}^\Lambda_\beta\tag{3.18}$$

is the decomposition of $\mathcal{R}^\Lambda_n$ into blocks. Theorem 3.7 implies that $\underline{\mathcal{R}}^\Lambda_\beta\cong\mathcal{H}^\Lambda_\beta$, where $\mathcal{H}^\Lambda_\beta=e_\beta\mathcal{H}^\Lambda_ne_\beta$.

Recall that $\mathscr{P}^\Lambda_\beta=\{\,\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_n\mid\mathbf{i}^{\boldsymbol{\lambda}}\in I^\beta\,\}$. Combining Theorem 3.14, (3.13) and (3.18) we obtain the following.

3.19. **Corollary** ([25]). *Suppose that* $\mathcal{Z}=K$ *is a field and that* $\beta\in Q^+_n$. *Then*

$$\{\,\psi_{\mathfrak{s}\mathfrak{t}}\mid\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})\textit{ for }\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_\beta\,\}\quad\textit{and}\quad\{\,\psi'_{\mathfrak{s}\mathfrak{t}}\mid\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})\textit{ for }\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_\beta\,\}$$

are graded cellular bases of $\mathcal{R}^\Lambda_\beta$. *In particular,* $\mathcal{R}^\Lambda_\beta$ *is a graded cellular algebra.*

3.6. **Trace forms and contragredient duality.** Recall that a **trace form** on A is a map $\operatorname{tr}:A\longrightarrow\mathcal{Z}$ such that $\operatorname{tr}(ab)=\operatorname{tr}(ba)$, for all $a,b\in A$. The trace form tr is **non-degenerate** if whenever $a\in A$ is non-zero then $\operatorname{tr}(ab)\ne0$ for some $b\in A$. An algebra A is a **symmetric** algebra if it has a non-degenerate trace form.

3.20. **Theorem** ([25, Theorem 6.17]). *Suppose that* $\beta\in Q^+_n$ *and that* $\mathcal{Z}=K$ *is a field. Then there is a non-degenerate homogeneous trace form* $\tau_\beta:\mathcal{R}^\Lambda_\beta\longrightarrow K$ *of degree* $-2\operatorname{def}\beta$ *such that* $\tau_\beta(\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{v}\mathfrak{u}})\ne0$ *only if* $(\mathfrak{u},\mathfrak{v})\trianglerighteq(\mathfrak{s},\mathfrak{t})$, *for* $(\mathfrak{s},\mathfrak{t}),(\mathfrak{u},\mathfrak{v})\in\operatorname{Std}^2(\mathscr{P}^\Lambda_n)$. *Moreover,* $\tau_\beta(\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{t}\mathfrak{s}})\ne0$, *for all* $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathscr{P}^\Lambda_n)$. *Consequently,* $\mathcal{R}^\Lambda_\beta$ *is a graded symmetric algebra.*

By the results in Section 2.2 the two cellular bases $\{\psi_{\mathfrak{st}}\}$ and $\{\psi'_{\mathfrak{uv}}\}$ can both determine cell modules for $\mathcal{R}^\Lambda_n$. Suppose that $\boldsymbol{\mu}\in\mathscr{P}^\Lambda_n$. The **Specht module** $S^{\boldsymbol{\mu}}$ is the cell module of $\mathcal{R}^\Lambda_n$ corresponding to $\boldsymbol{\mu}$ determined by the ψ-basis and the dual Specht module $S_{\boldsymbol{\mu}}$ is the cell module corresponding to $\boldsymbol{\mu}$ determined by the ψ'-basis. As their names suggest, the modules $S^{\boldsymbol{\mu}}$ and $S_{\boldsymbol{\mu}}$ are dual to each other.

3.21. **Proposition** ([25, Proposition 6.19]). *Suppose that $\boldsymbol{\mu}\in\mathscr{P}^\Lambda_\beta$, where $\beta\in Q^+_n$. Then $S^{\boldsymbol{\mu}}\cong S^{\circledast}_{\boldsymbol{\mu}}\langle\operatorname{def}\beta\rangle$ as graded $\mathcal{R}^\Lambda_n$-modules.*

We warn the reader that the module $S_{\boldsymbol{\mu}}$ is denoted $S_{\boldsymbol{\mu}'}$ in [25, §6]. This change in notation is a consequence of Remark 3.12. The notation for Specht modules and dual Specht modules in this paper is compatible with [30].

As in section 2.2, define $D^{\boldsymbol{\mu}}=S^{\boldsymbol{\mu}}/\operatorname{rad}S^{\boldsymbol{\mu}}$. A multipartition $\boldsymbol{\mu}\in\mathscr{P}^\Lambda_n$ is a **Kleshchev** multipartition if $\underline{D}^{\boldsymbol{\mu}}\ne 0$. Let

$$\mathcal{K}^\Lambda_n=\{\,\boldsymbol{\mu}\in\mathscr{P}^\Lambda_n\mid \underline{D}^{\boldsymbol{\mu}}\ne0\,\}$$

be the set of Kleshchev multipartitions. Ariki [1] has given a recursive description of the Kleshchev multipartitions. Observe that $\mathcal{K}^\Lambda_n$ depends on the choice of multicharge $\boldsymbol{\kappa}$ and not just on Λ. Building on Ariki's result, we classified the irreducible graded $\mathcal{R}^\Lambda_n$-modules.

3.22. **Proposition** ([25, Corollary 5.11]). *Suppose that $\mathcal{Z}=K$ is a field. Then*

$$\{\,D^{\boldsymbol{\mu}}\langle d\rangle\mid \boldsymbol{\mu}\in\mathcal{K}^\Lambda_n \text{ and } d\in\mathbb{Z}\,\}$$

is a complete set of pairwise non-isomorphic irreducible graded $\mathcal{R}^\Lambda_n$-modules.

In the case when $e=0$ the Kleshchev multipartition are sometimes called **restricted** and **FLOTW** multipartitions in the literature. In this case they have a particularly simple description.

3.23. **Corollary** ([44]). *Suppose that $e=0$, $\kappa_1\ge\kappa_2\ge\dots\ge\kappa_\ell$ and $\boldsymbol{\mu}\in\mathscr{P}^\Lambda_n$. Then $\boldsymbol{\mu}=(\mu^{(1)},\dots,\mu^{(\ell)})$ is a Kleshchev multipartition if and only $\mu^{(l)}_{r+\kappa_l-\kappa_{l+1}}\le\mu^{(l+1)}_r$ for $1\le l<\ell$ and $r\ge1$.*

3.7. **The sign isomorphism.** Following [30, §3.2] we now introduce an analogue of the sign involution of the symmetric groups for the quiver Hecke algebras. Unlike the case of the symmetric groups, this map is generally not an automorphism of $\mathcal{R}^\Lambda_n$.

In section 3.1 we fixed the multicharge $\boldsymbol{\kappa}=(\kappa_1,\dots,\kappa_\ell)\in\mathbb{Z}^\ell$ which determines $\Lambda=\Lambda(\boldsymbol{\kappa})$. Define $\boldsymbol{\kappa}'=(-\kappa_\ell,\dots,-\kappa_1)\in\mathbb{Z}^\ell$ and let $\Lambda'=\Lambda(\boldsymbol{\kappa}')$. Then $\Lambda'\in P^+$. More precisely, if $\Lambda=\sum_{i\in I}l_i\Lambda_i$, for $l_i\in\mathbb{N}$, then $\Lambda'=\sum_{i\in I}l_i\Lambda_{-i}$. Similarly, if $\beta=\sum_{i\in I}b_i\alpha_i$, for some $b_i\in\mathbb{N}$, define $\beta'=\sum_{i\in I}b_i\alpha_{-i}$. Then $\beta'\in Q^+$.

As noted in [30, §3.2], the relations easily imply that there is a unique degree preserving isomorphism of graded algebras $\mathtt{sgn}\colon\mathcal{R}^\Lambda_\beta\longrightarrow\mathcal{R}^{\Lambda'}_{\beta'}$ such that

$$e(\mathbf{i})\mapsto e(-\mathbf{i}),\qquad y_r\mapsto -y_r\quad\text{and}\quad \psi_s\mapsto-\psi_s \tag{3.24}$$

for $\mathbf{i}\in I^{\beta'}$, $1\le r\le n$, and $1\le s<n$. The map $\mathtt{sgn}$ induces a graded Morita equivalence

$$\mathcal{R}^{\Lambda'}_{\beta'}\text{-Mod}\xrightarrow{\sim}\mathcal{R}^\Lambda_\beta\text{-Mod}$$

in the sense of Section 2.3. This equivalence sends an $\mathcal{R}^{\Lambda'}_{\beta'}$-module M to the $\mathcal{R}^\Lambda_\beta$-module $M^{\mathtt{sgn}}$ where $M^{\mathtt{sgn}}$ is equal to M as a graded vector space and where the $\mathcal{R}^\Lambda_n$-action on $M^{\mathtt{sgn}}$ is given by $m\cdot a=m\,\mathtt{sgn}(a)$, for $a\in\mathcal{R}^\Lambda_n$ and $m\in M^{\mathtt{sgn}}$.

In section 3.3 we defined the residue sequence $\operatorname{res}(\mathfrak{t})=\operatorname{res}^{\boldsymbol{\kappa}}(\mathfrak{t})$, degree $\deg\mathfrak{t}=\deg^{\boldsymbol{\kappa}}\mathfrak{t}$ and codegree $\operatorname{codeg}\mathfrak{t}=\operatorname{codeg}^{\boldsymbol{\kappa}}\mathfrak{t}$ of a standard tableau $\mathfrak{t}$, all of which depend on $\boldsymbol{\kappa}$. Similarly, we set $\operatorname{res}'(\mathfrak{t})=\operatorname{res}^{\boldsymbol{\kappa}'}(\mathfrak{t})$, $\deg'(\mathfrak{t})=\deg^{\boldsymbol{\kappa}'}(\mathfrak{t})$ and $\operatorname{codeg}'(\mathfrak{t})=$

$\operatorname{codeg}^{\boldsymbol{\kappa}'}(\mathfrak{t})$. Recall also that $d'(\mathfrak{t}) \in \mathfrak{S}_n$ is the permutation determined by $\mathfrak{t} = \mathfrak{t}_{\boldsymbol{\mu}} d'(\mathfrak{t})$ and that $\mathfrak{t}'$ is the tableau which is conjugate to $\mathfrak{t}$.

The following result is easily checked using the definitions.

3.25. **Lemma.** *Suppose that $\beta \in Q_n^+$. Then $\mathscr{P}_{\beta'}^{\Lambda'} = \{\, \boldsymbol{\mu}' \mid \boldsymbol{\mu} \in \mathscr{P}_\beta^\Lambda \,\}$. Moreover, if $\mathfrak{t} \in \operatorname{Std}(\mathscr{P}_\beta^\Lambda)$ then $\mathfrak{t} \in \operatorname{Std}(\mathscr{P}_{\beta'}^{\Lambda'})$, $\operatorname{res}'(\mathfrak{t}) = -\operatorname{res}(\mathfrak{t}')$, $\deg' \mathfrak{t} = \operatorname{codeg} \mathfrak{t}'$, $\operatorname{codeg}' \mathfrak{t} = \deg \mathfrak{t}'$, and $d'(\mathfrak{t}) = d(\mathfrak{t}')$.*

Deploying this notation we obtain a ψ-basis and a ψ'-basis for $\mathcal{R}_{\beta'}^{\Lambda'}$. An easy exercise in the definitions shows that these bases are closely related to the corresponding bases of $\mathcal{R}_\beta^\Lambda$. More precisely, Definition 3.11 and Lemma 3.25 quickly give the following:

3.26. **Proposition.** *Suppose that $\beta \in Q_n^+$, that $\mathtt{sgn}\colon \mathcal{R}_\beta^\Lambda \longrightarrow \mathcal{R}_{\beta'}^{\Lambda'}$ is the sign isomorphism and $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathscr{P}_n^\Lambda)$. Then*

$$\mathtt{sgn}(\psi_{\mathfrak{s}\mathfrak{t}}) = \varepsilon_{\mathfrak{s}\mathfrak{t}} \psi'_{\mathfrak{s}'\mathfrak{t}'} \quad \textit{and} \quad \mathtt{sgn}(\psi'_{\mathfrak{s}\mathfrak{t}}) = \varepsilon'_{\mathfrak{s}\mathfrak{t}} \psi_{\mathfrak{s}'\mathfrak{t}'},$$

where $\varepsilon_{\mathfrak{s}\mathfrak{t}} = (-1)^{\deg \mathfrak{t}^{\boldsymbol{\mu}} + \ell(d(\mathfrak{s})) + \ell(d(\mathfrak{t}))}$ and $\varepsilon'_{\mathfrak{s}\mathfrak{t}} = (-1)^{\operatorname{codeg} \mathfrak{t}_{\boldsymbol{\mu}} + \ell(d'(\mathfrak{s})) + \ell(d'(\mathfrak{t}))}$.

Applying $\mathtt{sgn}$ to the construction of the Specht modules of $\mathcal{R}_{\beta'}^{\Lambda'}$, from section 2.2, we obtain:

3.27. **Corollary.** *Suppose that $\mathcal{Z}$ is an integral domain such that e is invertible in $\mathcal{Z}$ whenever $e \neq 0$ and e is not prime. Then*

$$S^{\boldsymbol{\mu}} \cong (S_{\boldsymbol{\mu}'})^{\mathtt{sgn}} \quad \textit{and} \quad S_{\boldsymbol{\mu}} \cong (S^{\boldsymbol{\mu}'})^{\mathtt{sgn}}.$$

This is in agreement with [30, Theorem 8.5]. See also Proposition 3.21.

4. Graded Schur algebras

In this chapter we introduce the quiver Schur algebras, or graded cyclotomic Schur algebras. We will show that they are quasi-hereditary graded cellular algebras. Unless otherwise stated, the following assumption will be in force for the rest of this paper.

4.1. **Assumption.** *We assume that $e = 0$ or $e \geq n$ and that $\mathcal{Z}$ is an integral domain in which e is invertible if $e \neq 0$ and e is not prime.*

By [25, Lemma 5.13], this assumption assures that $\mathcal{R}_n^\Lambda$ is free as a $\mathcal{Z}$-module. We expect that this is true for any commutative ring, in which case our quiver Schur algebras are free over any commutative domain.

4.1. Permutation modules.

Following the recipe in [17] we will define the graded cyclotomic Schur algebra to be the algebra of graded $\mathcal{R}_n^\Lambda$-endomorphisms of a particular $\mathcal{R}_n^\Lambda$-module G_n^Λ. In this section we introduce and investigate the summands of G_n^Λ.

4.2. **Definition.** Suppose that $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$. Define $G^{\boldsymbol{\mu}}$ and $G_{\boldsymbol{\mu}}$ to be the $\mathcal{R}_n^\Lambda$-modules:

$$G^{\boldsymbol{\mu}} = \psi_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}^{\boldsymbol{\mu}}} \mathcal{R}_n^\Lambda \langle -\deg \mathfrak{t}^{\boldsymbol{\mu}} \rangle \quad \text{and} \quad G_{\boldsymbol{\mu}} = \psi'_{\mathfrak{t}_{\boldsymbol{\mu}}\mathfrak{t}_{\boldsymbol{\mu}}} \mathcal{R}_n^\Lambda \langle -\operatorname{codeg} \mathfrak{t}_{\boldsymbol{\mu}} \rangle.$$

The degree shifts appear in Definition 4.2 because we want $G^{\boldsymbol{\mu}}$ to have a graded Specht filtration in which $S^{\boldsymbol{\mu}}$ has graded multiplicity one and we want $G_{\boldsymbol{\mu}}$ to have a graded dual Specht filtration in which $S_{\boldsymbol{\mu}}$ appears with multiplicity one.

The modules $G^{\boldsymbol{\mu}}$ and $G_{\boldsymbol{\mu}}$ are closely related. To explain this recall the isomorphism $\mathtt{sgn}\colon \mathcal{R}_n^\Lambda \longrightarrow \mathcal{R}_n^{\Lambda'}$ from (3.24).

4.3. **Lemma.** *Suppose that $\boldsymbol{\mu} \in \mathscr{P}^\Lambda_\beta$, for $\beta \in Q^+_n$. Then*

$$G^{\boldsymbol{\mu}} \cong (G_{\boldsymbol{\mu}'})^{\mathsf{sgn}} \quad \text{and} \quad G_{\boldsymbol{\mu}} \cong (G^{\boldsymbol{\mu}'})^{\mathsf{sgn}}$$

as graded $\mathcal{R}^\Lambda_n$-modules.

Proof. This is immediate from Definition 4.2 and Proposition 3.26. □

As a consequence, any result which we prove for $G^{\boldsymbol{\mu}}$ immediately translates into an equivalent "sign dual" result for $G_{\boldsymbol{\mu}}$. Our first aim is to give a basis for these modules. If $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$ define

$$\begin{aligned}
\operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda}) &= \{\, \mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}) \mid \mathfrak{s} \trianglerighteq \mathfrak{t}^{\boldsymbol{\mu}} \text{ and } \operatorname{res}(\mathfrak{s}) = \mathbf{i}^{\boldsymbol{\mu}} \,\},\\
\operatorname{Std}_{\boldsymbol{\mu}}(\boldsymbol{\lambda}) &= \{\, \mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}) \mid \mathfrak{t}_{\boldsymbol{\mu}} \trianglerighteq \mathfrak{s} \text{ and } \operatorname{res}(\mathfrak{s}) = \mathbf{i}_{\boldsymbol{\mu}} \,\},
\end{aligned}$$

and set $\operatorname{Std}^{\boldsymbol{\mu}}(\mathscr{P}^\Lambda_n) = \bigcup_{\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_n} \operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda})$ and $\operatorname{Std}_{\boldsymbol{\mu}}(\mathscr{P}^\Lambda_n) = \bigcup_{\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_n} \operatorname{Std}_{\boldsymbol{\mu}}(\boldsymbol{\lambda})$.

4.4. **Lemma.** *Suppose that $\boldsymbol{\mu} \in \mathscr{P}^\Lambda_n$. Then*

a) *$G^{\boldsymbol{\mu}}$ is spanned by $\{\, \psi_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}^{\boldsymbol{\mu}}}\psi'_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u} \in \operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda}), \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n \,\}$.*
b) *$G_{\boldsymbol{\mu}}$ is spanned by $\{\, \psi'_{\mathfrak{t}_{\boldsymbol{\mu}}\mathfrak{t}_{\boldsymbol{\mu}}}\psi_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s} \in \operatorname{Std}_{\boldsymbol{\mu}}(\boldsymbol{\lambda}), \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n \,\}$.*

Proof. By definition, $e^{\boldsymbol{\mu}}y^{\boldsymbol{\mu}} = \psi_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}^{\boldsymbol{\mu}}}$ so, by Theorem 3.14, $G^{\boldsymbol{\mu}}$ is spanned by the elements of the form $\psi_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}^{\boldsymbol{\mu}}}\psi'_{\mathfrak{u}\mathfrak{v}}$, for $(\mathfrak{u},\mathfrak{v}) \in \operatorname{Std}^2(\mathscr{P}^\Lambda_n)$. Hence, part (a) follows from Lemma 3.15(a). Part (b) follows by a similar argument or by applying Lemma 4.3. □

For any positive integer $m \le n$ set $s_{m,m} = 1$ and $\psi_{m,m} = 1$. If $1 \le r < m$ let

$$s_{r,m} = s_r \dots s_{m-1} \quad \text{and} \quad \psi_{r,m} = \psi_r \dots \psi_{m-1},$$

and set $s_{m,r} = s^{-1}_{r,m}$ and $\psi_{m,r} = \psi^\star_{r,m}$. To show that the elements in Lemma 4.4 give bases of $G^{\boldsymbol{\mu}}$ and $G_{\boldsymbol{\mu}}$ we need the following technical lemma. This result does not require the assumption that $e = 0$ or $e \ge n$.

4.5. **Lemma.** *Suppose that $e \in \{0, 2, 3, 4 \dots\}$, $\mathbf{i} \in I^n$ and that there exists an integer r, with $1 \le r \le n$, and non-negative integers $d_r, \dots, d_n, d_{n+1}$ such that*

$$d_r \ge d_s \ge d_t \ge d_{n+1} \text{ whenever } r \le s \le t < n \text{ and } i_r = i_s = i_t.$$

Then $\psi_{n,r}y_r^{d_r} \dots y_n^{d_n}e(\mathbf{i}) \in y_r^{d_{r+1}} \dots y_n^{d_{n+1}}e(s_{n,r}\mathbf{i})\mathcal{R}^\Lambda_n$.

Proof. We argue by downwards induction on r. If $r = n$ then $\psi_{n,r} = 1$ and there is nothing to prove since, by assumption, $d_r \ge d_{n+1}$. Suppose then that $r < n$. We divide the proof into two cases.

First suppose that $i_r \ne i_{r+1}$. Then, using (3.3), we have that

$$\begin{aligned}
\psi_{n,r}y_r^{d_r}y_{r+1}^{d_{r+1}} \dots y_n^{d_n}e(\mathbf{i}) &= \psi_{n,r+1}\psi_r y_r^{d_r}y_{r+1}^{d_{r+1}} \dots y_n^{d_n}e(\mathbf{i})\\
&= \psi_{n,r+1}y_{r+1}^{d_r}y_r^{d_{r+1}}\psi_r y_{r+2}^{d_{r+2}} \dots y_n^{d_n}e(\mathbf{i})\\
&= y_r^{d_{r+1}}\psi_{n,r+1}y_{r+1}^{d_r}y_{r+2}^{d_{r+2}} \dots y_n^{d_n}e(s_r\mathbf{i})\psi_r.
\end{aligned}$$

Therefore, $\psi_{n,r}y_r^{d_r}y_{r+1}^{d_{r+1}} \dots y_n^{d_n}e(\mathbf{i}) \in y_r^{d_{r+1}}y_{r+1}^{d_{r+2}} \dots y_n^{d_{n+1}}e(s_{n,r}\mathbf{i})\mathcal{R}^\Lambda_n$ by induction because the sequence $s_r\mathbf{i}$ and the non-negative integers $d_r, d_{r+2}, \dots, d_{n+1}$ satisfy the assumptions of the Lemma.

Now consider the remaining case when $i_r = i_{r+1}$. A quick calculation using (3.3) shows that ψ_r commutes with any symmetric polynomial in y_r and y_{r+1}, so that $\psi_r(y_ry_{r+1}) = (y_ry_{r+1})\psi_r$. By assumption, $d_r \ge d_{r+1} \ge d_{n+1}$, so

$$\begin{aligned}
\psi_{n,r}y_r^{d_r} \dots y_n^{d_n}e(\mathbf{i}) &= \psi_{n,r+1}\psi_r(y_ry_{r+1})^{d_{r+1}}y_{r+2}^{d_{r+2}} \dots y_n^{d_n}e(\mathbf{i})y_r^{d_r-d_{r+1}}\\
&= \psi_{n,r+1}(y_ry_{r+1})^{d_{r+1}}y_{r+2}^{d_{r+2}} \dots y_n^{d_n}e(s_r\mathbf{i})\psi_r y_r^{d_r-d_{r+1}}\\
&= y_r^{d_{r+1}}\psi_{n,r+1}y_{r+1}^{d_{r+1}}y_{r+2}^{d_{r+2}} \dots y_n^{d_n}e(s_r\mathbf{i})\psi_r y_r^{d_r-d_{r+1}}.
\end{aligned}$$

Since $i_r = i_{r+1}$ the sequence $s_r\mathbf{i}$ and the integers $d_{r+1}, \dots, d_n$ again satisfy the assumptions of the Lemma. Hence, the result follows by induction. $\square$

Before we can give bases for $G^{\boldsymbol{\mu}}$ and $G_{\boldsymbol{\mu}}$ we need to introduce a special choice of reduced expression. Recall our definition of $\psi_{r,n}$ and $\psi_{n,r}$ (for any $1 \le r \le n$) above Lemma 4.5. It is well-known and easy to prove that

$$\mathfrak{S}_n = \bigsqcup_{r=1}^{n} s_{r,n}\mathfrak{S}_{n-1} \qquad \text{(disjoint union)},$$

and that $\ell(s_{r,n}w) = \ell(s_{r,n}) + \ell(w) = \ell(w) + n - r$, for all $w \in \mathfrak{S}_{n-1}$. Hence, we have the following:

4.6. **Lemma.** *Suppose that $w \in \mathfrak{S}_n$. Then there exist unique integers $r_2, \dots, r_n$, with $1 \le r_k \le k$, such that $w = s_{r_n,n} \dots s_{r_2,2}$ and $\ell(w) = \ell(s_{r_n,n}) + \dots + \ell(s_{r_2,2})$.*

The factorization $w = s_{r_n,n} \dots s_{r_2,2}$ in Lemma 4.6 gives a reduced word for w. As a temporary notation, define $\hat\psi_w = \psi_{r_n,n} \dots \psi_{r_2,2} \in \mathcal{R}_n^\Lambda$ and if $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\boldsymbol{\lambda})$ let $\hat\psi_{\mathfrak{s}\mathfrak{t}} = \hat\psi^\star_{d(\mathfrak{s})} e^{\boldsymbol{\lambda}} y^{\boldsymbol{\lambda}} \hat\psi_{d(\mathfrak{t})}$. Define $\hat\psi'_{\mathfrak{s}\mathfrak{t}}$ similarly.

Observe that the choice of reduced expression used to define $\hat\psi_{\mathfrak{s}\mathfrak{t}}$ is compatible with the natural embeddings $\mathfrak{S}_m \hookrightarrow \mathfrak{S}_n$, for $1 \le m \le n$. More precisely, if n appears in $\mathfrak{t}$ in the same position as r appears in $\mathfrak{t}^{\boldsymbol{\mu}}$ then $d(\mathfrak{t}) = s_{r,n} d(\mathfrak{t}_{\downarrow_{n-1}})$ and $\ell(d(\mathfrak{t})) = n - r + \ell(d(\mathfrak{t}_{\downarrow_{n-1}}))$. Consequently, $\hat\psi_{d(\mathfrak{t})} = \psi_{r,n}\hat\psi_{\mathfrak{t}_{\downarrow_{n-1}}}$. Similarly, if n appears in $\mathfrak{t}$ in the same position as r appears in $\mathfrak{t}_{\boldsymbol{\mu}}$ then $d'(\mathfrak{t}) = s_{r,n} d'(\mathfrak{t}_{\downarrow_{n-1}})$ and $\ell(d'(\mathfrak{t})) = n - r + \ell(d'(\mathfrak{t}_{\downarrow_{n-1}}))$ so that $\hat\psi'_{d'(\mathfrak{t})} = \psi_{r,n}\hat\psi'_{d'(\mathfrak{t}_{\downarrow_{n-1}})}$.

The next Lemma makes heavy use of Assumption 4.1.

4.7. **Lemma.** *Suppose that $e = 0$ or $e \ge n$ and $\mathfrak{s} \in \operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda})$ and $\mathfrak{u} \in \operatorname{Std}_{\boldsymbol{\mu}}(\boldsymbol{\lambda})$, for some $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$. Then $\hat\psi_{\mathfrak{s}\mathfrak{t}^{\boldsymbol{\lambda}}} \in G^{\boldsymbol{\mu}}$ and $\hat\psi'_{\mathfrak{u}\mathfrak{t}_{\boldsymbol{\lambda}}} \in G_{\boldsymbol{\mu}}$.*

Proof. We prove only that $\hat\psi_{\mathfrak{s}\mathfrak{t}^{\boldsymbol{\lambda}}} \in G^{\boldsymbol{\mu}}$, the second statement being equivalent by Proposition 3.26 and Lemma 4.3.

We argue by induction on n. If $n = 1$ then $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\mu}}$ and there is nothing to prove, so assume that $n > 0$. Let $\mathfrak{s}_\downarrow = \mathfrak{s}_{\downarrow(n-1)}$, $\boldsymbol{\lambda}^{\mathfrak{s}_\downarrow} = \operatorname{Shape}(\mathfrak{s}_\downarrow)$, $\boldsymbol{\mu}_\downarrow = \operatorname{Shape}(\mathfrak{t}^{\boldsymbol{\mu}}_{\downarrow(n-1)})$. Then $\mathfrak{s}_\downarrow \in \operatorname{Std}^{\boldsymbol{\mu}_\downarrow}(\boldsymbol{\lambda}^{\mathfrak{s}_\downarrow})$. Suppose that n appears in the same position in $\mathfrak{s}$ as r does in $\mathfrak{t}^{\boldsymbol{\lambda}}$. Let $\boldsymbol{\nu} = \operatorname{Shape}(\mathfrak{s}_{\downarrow(r-1)})$. By definition, $\hat\psi_{d(\mathfrak{s})} = \psi_{r,n}\hat\psi_{\mathfrak{s}_\downarrow}$ so, recalling the definition of the integers $d_1^{\boldsymbol{\lambda}}, \dots, d_n^{\boldsymbol{\lambda}}$ from Definition 3.11, we have

$$\hat\psi_{\mathfrak{s}\mathfrak{t}^{\boldsymbol{\lambda}}} = \hat\psi^\star_{d(\mathfrak{s})} y^{\boldsymbol{\lambda}} e^{\boldsymbol{\lambda}} = \hat\psi^\star_{\mathfrak{s}_\downarrow}\psi_{n,r} y^{\boldsymbol{\nu}} y_r^{d_r^{\boldsymbol{\lambda}}} \dots y_n^{d_n^{\boldsymbol{\lambda}}} e^{\boldsymbol{\lambda}} = \hat\psi^\star_{\mathfrak{s}_\downarrow} y^{\boldsymbol{\nu}} \psi_{n,r} y_r^{d_r^{\boldsymbol{\lambda}}} \dots y_n^{d_n^{\boldsymbol{\lambda}}} e^{\boldsymbol{\lambda}},$$

where the last equality follows because if $r \le j < n$ then ψ_j commutes with $y^{\boldsymbol{\nu}}$ by (3.3). In order to apply Lemma 4.5 to the sequence $d_r = d_r^{\boldsymbol{\lambda}}, \dots, d_n = d_n^{\boldsymbol{\lambda}}, d_{n+1} = d_n^{\boldsymbol{\mu}}$ we have to check that $d_r^{\boldsymbol{\lambda}} \ge d_s^{\boldsymbol{\lambda}} \ge d_t^{\boldsymbol{\lambda}} \ge d_n^{\boldsymbol{\mu}}$ whenever $r \le s \le t < n$ and $i_r^{\boldsymbol{\lambda}} = i_s^{\boldsymbol{\lambda}} = i_t^{\boldsymbol{\lambda}}$. To this end, observe because $e = 0$ or $e > n$ each component contains at most one addable or removable node of each residue. Therefore, if $\boldsymbol{\rho} \in \mathscr{P}_n^\Lambda$ and $1 \le m \le n$ then

$$d_m^{\boldsymbol{\rho}} = \#\{\, 1 \le l \le \ell \mid l > \operatorname{comp}_{\mathfrak{t}^{\boldsymbol{\rho}}}(m) \text{ and } \kappa_l \equiv i_m^{\boldsymbol{\rho}} \pmod e \,\}.$$

Now if $r \le s \le t < n$ then, by definition, $\operatorname{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(r) \le \operatorname{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(s) \le \operatorname{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(t)$ so that $d_r^{\boldsymbol{\lambda}} \ge d_s^{\boldsymbol{\lambda}} \ge d_t^{\boldsymbol{\lambda}}$ whenever $i_r^{\boldsymbol{\lambda}} = i_s^{\boldsymbol{\lambda}} = i_t^{\boldsymbol{\lambda}}$. Further, $\operatorname{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(r) \le \operatorname{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(s) \le \operatorname{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(t) \le \operatorname{comp}_{\mathfrak{t}^{\boldsymbol{\mu}}}(n)$, since $\mathfrak{s} \trianglerighteq \mathfrak{t}^{\boldsymbol{\mu}}$, so that $d_r^{\boldsymbol{\lambda}} \ge d_s^{\boldsymbol{\lambda}} \ge d_t^{\boldsymbol{\lambda}} \ge d_n^{\boldsymbol{\mu}}$ because $i_t^{\boldsymbol{\lambda}} = i_r^{\boldsymbol{\lambda}} = i_n^{\mathfrak{s}} = i_n^{\boldsymbol{\mu}}$. Therefore, by Lemma 4.5, there exists $h \in \mathcal{R}_n^\Lambda$ such that

$$\hat\psi_{\mathfrak{s}\mathfrak{t}^{\boldsymbol{\lambda}}} = \hat\psi^\star_{d(\mathfrak{s})} y^{\boldsymbol{\lambda}} e^{\boldsymbol{\lambda}} = \hat\psi^\star_{\mathfrak{s}_\downarrow} y^{\boldsymbol{\nu}} y_r^{d_{r+1}^{\boldsymbol{\lambda}}} \dots y_{n-1}^{d_n^{\boldsymbol{\lambda}}} y_n^{d_n^{\boldsymbol{\mu}}} e(s_{n,r}\mathbf{i}^{\boldsymbol{\lambda}})h.$$

Note that by definition $y^{\boldsymbol{\lambda}^{\mathfrak{s}\downarrow}} = y^{\boldsymbol{\nu}} y_r^{d_{r+1}^{\boldsymbol{\lambda}}} \dots y_{n-1}^{d_n^{\boldsymbol{\lambda}}}$ and that y_n commutes with $\hat{\psi}_{\mathfrak{s}\downarrow}$ by (3.4). Therefore, it follows by induction that there exists $h' \in \mathcal{R}_n^\Lambda$ such that

$$\hat{\psi}_{\mathfrak{s}\mathfrak{t}^{\boldsymbol{\lambda}}} = y_n^{d_n^{\boldsymbol{\mu}}} \hat{\psi}^\star_{\mathfrak{s}_\downarrow} y^{\boldsymbol{\lambda}^{\mathfrak{s}\downarrow}} e(\mathbf{i}^{\boldsymbol{\lambda}^{\mathfrak{s}\downarrow}} \vee i_n^{\boldsymbol{\mu}})h = y_n^{d_n^{\boldsymbol{\mu}}} e^{\boldsymbol{\mu}} y_1^{d_1^{\boldsymbol{\mu}}} \dots y_{n-1}^{d_{n-1}^{\boldsymbol{\mu}}} h'h \in e^{\boldsymbol{\mu}} y^{\boldsymbol{\mu}} \mathcal{R}_n^\Lambda.$$

This completes the proof of the Lemma. □

4.8. *Remark.* If we drop Assumption 4.1 then it is easy to construct examples where the argument of Lemma 4.9 fails if $0 < e \le n$.

4.9. **Corollary.** *Suppose that $e = 0$ or $e \ge n$ and $\mathfrak{s} \in \mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda})$ and $\mathfrak{u} \in \mathrm{Std}_{\boldsymbol{\mu}}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$. Then for any $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, $\psi_{\mathfrak{s}\mathfrak{t}} \in G^{\boldsymbol{\mu}}$ and $\psi'_{\mathfrak{u}\mathfrak{t}} \in G_{\boldsymbol{\mu}}$.*

Proof. We show only that $\psi_{\mathfrak{s}\mathfrak{t}} \in G^{\boldsymbol{\mu}}$. If $\psi_{\mathfrak{s}\mathfrak{t}} = \hat{\psi}_{\mathfrak{s}\mathfrak{t}}$ then the result follows by Lemma 4.7. Otherwise, by Lemma 3.17, there exist $s_{\mathfrak{u}\mathfrak{v}} \in \mathcal{Z}$ such that

$$\psi_{\mathfrak{s}\mathfrak{t}} = \hat{\psi}_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{(\mathfrak{u},\mathfrak{v}) \in \mathrm{Std}^2(\mathscr{P}_n^\Lambda) \\ (\mathfrak{u},\mathfrak{v}) \blacktriangleright (\mathfrak{s},\mathfrak{t})}} s_{\mathfrak{u}\mathfrak{v}} \psi_{\mathfrak{u}\mathfrak{v}},$$

where $s_{\mathfrak{u}\mathfrak{v}} \ne 0$ only if $\mathrm{res}(\mathfrak{u}) = \mathrm{res}(\mathfrak{s})$. Consequently, if $s_{\mathfrak{u}\mathfrak{v}} \ne 0$ then $\mathfrak{u} \trianglerighteq \mathfrak{s} \trianglerighteq \mathfrak{t}^{\boldsymbol{\mu}}$ and $\mathfrak{v} \trianglerighteq \mathfrak{t}$ so that $\mathfrak{u} \in \mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu})$, for some $\boldsymbol{\nu} \trianglerighteq \boldsymbol{\lambda}$. By induction on dominance, $\psi_{\mathfrak{u}\mathfrak{v}} = \psi_{\mathfrak{u}\mathfrak{t}^{\boldsymbol{\nu}}} \psi_{d(\mathfrak{v})}$ belongs to $G^{\boldsymbol{\mu}}$ whenever $(\mathfrak{u},\mathfrak{u}) \blacktriangleright (\mathfrak{s},\mathfrak{t})$. Moreover, $\hat{\psi}_{\mathfrak{s}\mathfrak{t}} \in G^{\boldsymbol{\mu}}$ by Lemma 4.7. Hence, $\psi_{\mathfrak{s}\mathfrak{t}} \in G^{\boldsymbol{\mu}}$ as we wanted to show. □

We can now give bases for $G^{\boldsymbol{\mu}}$ and $G_{\boldsymbol{\mu}}$. Almost everything in this paper relies on the next result.

4.10. **Theorem.** *Suppose that $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$. Then*

a) $\{\, \psi_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s} \in \mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ *and* $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\nu})$, *for* $\boldsymbol{\nu} \in \mathscr{P}_n^\Lambda \,\}$ *is a basis of* $G^{\boldsymbol{\mu}}$.

b) $\{\, \psi'_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u} \in \mathrm{Std}_{\boldsymbol{\mu}}(\boldsymbol{\nu})$ *and* $\mathfrak{v} \in \mathrm{Std}(\boldsymbol{\nu})$, *for* $\boldsymbol{\nu} \in \mathscr{P}_n^\Lambda \,\}$ *is a basis of* $G_{\boldsymbol{\mu}}$.

Proof. Parts (a) and (b) are equivalent by Lemma 4.3 and Proposition 3.26, so it is enough to prove (a). Suppose first that $\mathcal{Z} = K$ is a field. By Corollary 4.9, $\psi_{\mathfrak{s}\mathfrak{t}} \in G^{\boldsymbol{\mu}}$ whenever $\mathfrak{s} \in \mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ and $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\nu})$, for some multipartition $\boldsymbol{\nu} \in \mathscr{P}_n^\Lambda$. Therefore, by Theorem 3.14,

$$\dim_K G^{\boldsymbol{\mu}} \ge \sum_{\mathfrak{u} \in \mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu})} \# \mathrm{Std}(\boldsymbol{\nu}).$$

On the other hand, by Lemma 4.4 the dimension of $G^{\boldsymbol{\mu}}$ is at most the number on the right hand side. Hence, the set in the statement of the theorem is a basis of $G^{\boldsymbol{\mu}}$, so that the Lemma holds over any field K.

To prove the proposition when $\mathcal{Z}$ is not a field by Assumption 4.1 it suffices to consider the cases where $\mathcal{Z} = \mathbb{Z}$ if $e = 0$ or e is a prime; or $\mathcal{Z} = \mathbb{Z}[e^{-1}]$ if $e > 0$ and e is not a prime. In these cases, $\mathcal{Z}$ is always a principal ideal domain. Let G be the $\mathcal{Z}$-module of $G^{\boldsymbol{\mu}}$ with basis $\{\, \psi_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s} \in \mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu}), \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\nu}),$ for $\boldsymbol{\nu} \in \mathscr{P}_n^\Lambda \,\}$. We have a short exact sequence of $\mathcal{Z}$-modules

$$0 \longrightarrow G \longrightarrow G^{\boldsymbol{\mu}} \longrightarrow G^{\boldsymbol{\mu}}/G \longrightarrow 0.$$

Therefore, for every field K which is a $\mathcal{Z}$-algebra there is an exact sequence

$$G \otimes_{\mathcal{Z}} K \longrightarrow G^{\boldsymbol{\mu}} \otimes_{\mathcal{Z}} K \to G^{\boldsymbol{\mu}}/G \otimes_{\mathcal{Z}} K \to 0.$$

By the first paragraph of the proof, the first homomorphism in the last exact sequence is an isomorphism. It follows that $G^{\boldsymbol{\mu}}/G \otimes_{\mathcal{Z}} K = 0$ for any field K which is an $\mathcal{Z}$-algebra. Applying Nakayama's Lemma (see, for example, [3, Proposition 3.8]), $G^{\boldsymbol{\mu}}/G = 0$. That is, $G^{\boldsymbol{\mu}} = G$. Hence, elements in the statement of the theorem are a basis for $G^{\boldsymbol{\mu}}$ as required. □

Theorem 4.10 has several useful corollaries. We first note that it gives explicit formulae for the graded dimensions of these two modules:

$$\operatorname{DIM} G^{\boldsymbol{\mu}} = \sum_{\mathfrak{s}\in\operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu})} \sum_{\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\nu})} q^{\deg\mathfrak{s}+\deg\mathfrak{t}-\deg\mathfrak{t}^{\boldsymbol{\mu}}},$$

$$\operatorname{DIM} G_{\boldsymbol{\mu}} = \sum_{\mathfrak{u}\in\operatorname{Std}_{\boldsymbol{\mu}}(\boldsymbol{\nu})} \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\nu})} q^{\operatorname{codeg}\mathfrak{u}+\operatorname{codeg}\mathfrak{v}-\operatorname{codeg}\mathfrak{t}_{\boldsymbol{\mu}}}.$$

4.11. **Corollary.** *Suppose that $\boldsymbol{\mu},\boldsymbol{\lambda}\in\mathscr{P}_n^\Lambda$. Then*

$$\{\,\psi_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}\in\operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu}) \text{ and } \mathfrak{t}\in\operatorname{Std}^{\boldsymbol{\lambda}}(\boldsymbol{\nu}), \text{for } \boldsymbol{\nu}\in\mathscr{P}_n^\Lambda\,\}$$

is a basis of $G^{\boldsymbol{\mu}}\cap(G^{\boldsymbol{\lambda}})^\star$ and

$$\{\,\psi'_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}\in\operatorname{Std}_{\boldsymbol{\mu}}(\boldsymbol{\nu}) \text{ and } \mathfrak{t}\in\operatorname{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\nu}), \text{for } \boldsymbol{\nu}\in\mathscr{P}_n^\Lambda\,\}$$

is a basis of $G_{\boldsymbol{\mu}}\cap(G_{\boldsymbol{\lambda}})^\star$.

Proof. Suppose that $a\in G^{\boldsymbol{\mu}}\cap(G^{\boldsymbol{\lambda}})^\star$ and write $a=\sum_{(\mathfrak{s},\mathfrak{t})\in\mathscr{P}_n^\Lambda} r_{\mathfrak{s}\mathfrak{t}}\psi_{\mathfrak{s}\mathfrak{t}}$, for $r_{\mathfrak{s}\mathfrak{t}}\in\mathcal{Z}$ and $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}(\mathscr{P}_n^\Lambda)$. Then $r_{\mathfrak{s}\mathfrak{t}}\ne0$ only if $\mathfrak{s}\in\operatorname{Std}^{\boldsymbol{\mu}}(\mathscr{P}_n^\Lambda)$ by Theorem 4.10. Similarly, since $a^\star\in G^{\boldsymbol{\lambda}}$ we see that $r_{\mathfrak{s}\mathfrak{t}}\ne0$ only if $\mathfrak{t}\in\operatorname{Std}^{\boldsymbol{\lambda}}(\mathscr{P}_n^\Lambda)$. Moreover, if $\mathfrak{s}\in\operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ and $\mathfrak{t}\in\operatorname{Std}^{\boldsymbol{\lambda}}(\boldsymbol{\nu})$ then $\psi_{\mathfrak{s}\mathfrak{t}}\in G^{\boldsymbol{\mu}}\cap(G^{\boldsymbol{\lambda}})^\star$ by two more applications of Theorem 4.10. This proves the first claim. The second statement follows similarly. □

4.12. **Corollary.** *Suppose that $\boldsymbol{\mu}\in\mathscr{P}_n^\Lambda$.*

a) *Write $\operatorname{Std}^{\boldsymbol{\mu}}(\mathscr{P}_n^\Lambda)=\{\mathfrak{s}_1,\dots,\mathfrak{s}_m\}$, ordered so that $i\le j$ whenever $\mathfrak{s}_i\trianglerighteq\mathfrak{s}_j$ and set $\boldsymbol{\nu}^i=\operatorname{Shape}(\mathfrak{s}_i)$, for $1\le i,j\le m$. Then $G^{\boldsymbol{\mu}}$ has a (graded) Specht filtration*
$$G^{\boldsymbol{\mu}}=G^m\ge G^{m-1}\ge\dots\ge G^1\ge G^0=0$$
such that $G^i/G^{i-1}\cong S^{\boldsymbol{\nu}^i}\langle\deg\mathfrak{s}_i\rangle$, for $1\le i\le m$.

b) *Write $\operatorname{Std}_{\boldsymbol{\mu}}(\mathscr{P}_n^\Lambda)=\{\mathfrak{u}_1,\dots,\mathfrak{u}_l\}$, ordered so that $i\ge j$ whenever $\mathfrak{u}_i\trianglerighteq\mathfrak{u}_j$ and set $\boldsymbol{\nu}_i=\operatorname{Shape}(\mathfrak{u}_i)$, for $1\le i,j\le l$. Then $G_{\boldsymbol{\mu}}$ has a (graded) dual Specht filtration*
$$G_{\boldsymbol{\mu}}=G_l\ge G_{l-1}\ge\dots\ge G_1\ge G_0=0$$
such that $G_i/G_{i-1}\cong S_{\boldsymbol{\nu}_i}\langle\operatorname{codeg}\mathfrak{u}_i\rangle$, for $1\le i\le l$.

Proof. Suppose that $1\le i\le m$. Define G^i to be the $\mathcal{Z}$-submodule of $G^{\boldsymbol{\mu}}$ spanned by $\{\,\psi_{\mathfrak{s}_j\mathfrak{t}} \mid 1\le j\le i \text{ and } \mathfrak{t}\in\operatorname{Std}(\boldsymbol{\nu}^j)\,\}$. Then G^i is a submodule of $G^{\boldsymbol{\mu}}$ by Theorem 4.10 and Definition 2.4(GC$_2$). Finally, $G^i/G^{i-1}\cong S^{\boldsymbol{\nu}^i}\langle\deg\mathfrak{s}_i\rangle$ by the construction of the cell modules given in section 2.2. More precisely, if $\mathfrak{s}=\mathfrak{s}_i$ then the isomorphism is given by $\psi_{\mathfrak{t}}\mapsto\psi_{\mathfrak{s}\mathfrak{t}}+G^{i-1}$, for all $\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\nu}^i)$. This proves (a). The proof of (b) is almost identical. □

In particular, note that $S^{\boldsymbol{\mu}}$ is a quotient of $G^{\boldsymbol{\mu}}$ and that $S_{\boldsymbol{\mu}}$ is a quotient of $G_{\boldsymbol{\mu}}$.

4.13. **Corollary.** *Suppose that $\boldsymbol{\mu}\in\mathscr{P}_n^\Lambda$. Then:*

a) $\{\,\psi_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}^{\boldsymbol{\mu}}}\psi'_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u}\in\operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu}) \text{ and } \mathfrak{t}\in\operatorname{Std}(\boldsymbol{\nu}), \text{for } \boldsymbol{\nu}\in\mathscr{P}_n^\Lambda\,\}$ *is a basis of $G^{\boldsymbol{\mu}}$.*
b) $\{\,\psi'_{\mathfrak{t}_{\boldsymbol{\mu}}\mathfrak{t}_{\boldsymbol{\mu}}}\psi_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}\in\operatorname{Std}_{\boldsymbol{\mu}}(\boldsymbol{\nu}) \text{ and } \mathfrak{t}\in\operatorname{Std}(\boldsymbol{\nu}), \text{for } \boldsymbol{\nu}\in\mathscr{P}_n^\Lambda\,\}$ *is a basis of $G_{\boldsymbol{\mu}}$.*

Proof. By Lemma 4.4 and Theorem 3.14(b) the elements in (a) span $G^{\boldsymbol{\mu}}$, so it remains to show that they are linearly independent. This is a direct consequence of Theorem 4.10. The proof of (b) is similar. □

4.14. **Corollary.** *Suppose that $\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda$. Using the notation of Corollary 4.12:*

a) *$G^{\boldsymbol{\mu}}$ has a dual Specht filtration $G^{\boldsymbol{\mu}}=H_m\ge H_{m-1}\ge\dots\ge H_1\ge H_0=0$ such that $H_i/H_{i-1}\cong S_{\boldsymbol{\nu}^i}\langle\deg\mathfrak{t}^{\boldsymbol{\mu}}+\operatorname{codeg}\mathfrak{s}_i\rangle$, for $1\le i\le m$.*

b) *$G_{\boldsymbol{\mu}}$ has a Specht filtration $G_{\boldsymbol{\mu}} = H^l \geq H^{l-1} \geq \cdots \geq H^1 \geq H^0 = 0$ such that $H^i/H^{i-1} \cong S^{\boldsymbol{\nu}_i}\langle \operatorname{codeg} \mathfrak{t}_{\boldsymbol{\mu}} + \deg \mathfrak{u}_i\rangle$, for $1 \leq i \leq l$.*

Proof. We prove only (b). Part (a) can be proved in a similar way. Mirroring the proof of Corollary 4.12, define H^i to be the $\mathcal{Z}$-submodule of $G_{\boldsymbol{\mu}}$ spanned by the elements

$$\{\, \psi'_{\mathfrak{t}_{\boldsymbol{\mu}}\mathfrak{t}_{\boldsymbol{\mu}}}\psi_{\mathfrak{u}_j\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\nu}_j) \text{ and } l+1-i \leq j \leq l \,\}.$$

This is an $\mathcal{R}^\Lambda_n$-submodule of $G_{\boldsymbol{\mu}}$ by Theorem 3.14 and (GC$_2$) of Definition 2.4. As in the proof of Corollary 4.12 it is easy to verify that $H_i/H_{i-1} \cong S^{\boldsymbol{\nu}^i}\langle \operatorname{codeg} \mathfrak{t}_{\boldsymbol{\mu}} + \deg \mathfrak{u}_i\rangle$; compare with [26, Corollaries 3.11, 3.12]. The degree shift is just the difference of the degrees of the basis elements of $S^{\boldsymbol{\nu}^i}$ and the degrees of the elements $\psi'_{\mathfrak{t}_{\boldsymbol{\mu}}\mathfrak{t}_{\boldsymbol{\mu}}}\psi_{\mathfrak{u}_i\mathfrak{t}}$. □

Recall from (3.18) that $\mathcal{R}^\Lambda_n = \bigoplus_\beta \mathcal{R}^\Lambda_\beta$ and that $\mathcal{R}^\Lambda_\beta$ carries a non-degenerate homogeneous trace form τ_β of degree $-2\operatorname{def}\beta$ by Theorem 3.20. The following argument is lifted from [36, Proposition 5.13].

4.15. **Theorem.** *Suppose that $\mathcal{Z} = K$ is a field and that $\boldsymbol{\mu} \in \mathscr{P}^\Lambda_\beta$, for $\beta \in Q^+_n$. Then, as $\mathcal{R}^\Lambda_n$-modules,*

$$(G^{\boldsymbol{\mu}})^{\circledast} \cong G^{\boldsymbol{\mu}}\langle -2\operatorname{def}\beta\rangle \quad \textit{and} \quad (G_{\boldsymbol{\mu}})^{\circledast} \cong G_{\boldsymbol{\mu}}\langle -2\operatorname{def}\beta\rangle.$$

Proof. Both isomorphisms can be proved similarly, so we consider only the first one. Using Theorem 4.10 and Corollary 4.13, define a pairing $G^{\boldsymbol{\mu}} \times G^{\boldsymbol{\mu}} \longrightarrow \mathcal{Z}$ by

$$\langle \psi_{\mathfrak{s}\mathfrak{t}}, \psi_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}^{\boldsymbol{\mu}}}\psi'_{\mathfrak{u}\mathfrak{v}}\rangle_{\boldsymbol{\mu}} = \tau_\beta(\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{v}\mathfrak{u}}),$$

for all $\mathfrak{s} \in \operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda})$, $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, $\mathfrak{u} \in \operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu})$, $\mathfrak{v} \in \operatorname{Std}(\boldsymbol{\nu})$, for some $\boldsymbol{\lambda}, \boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta$. By Theorem 3.20, $\tau_\beta(\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{t}\mathfrak{s}}) \neq 0$ and $\tau_\beta(\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{v}\mathfrak{u}}) \neq 0$ only if $(\mathfrak{u}, \mathfrak{v}) \blacktriangleright (\mathfrak{s}, \mathfrak{t})$. Therefore, the Gram matrix of $\langle\ ,\ \rangle_{\boldsymbol{\mu}}$ is upper triangular with non-zero elements on the diagonal so that $\langle\ ,\ \rangle_{\boldsymbol{\mu}}$ is non-degenerate. Recalling the degree shift in the definition of $G^{\boldsymbol{\mu}}$ from Definition 4.2, it is easy to check that $\langle\ ,\ \rangle_{\boldsymbol{\mu}}$ is a homogeneous bilinear map of degree $-2\operatorname{def}\beta$. Therefore, to complete the proof it is enough to show that $\langle\ ,\ \rangle_{\boldsymbol{\mu}}$ is associative in the sense that

$$\langle \psi_{\mathfrak{s}\mathfrak{t}}h, \psi_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}^{\boldsymbol{\mu}}}\psi'_{\mathfrak{u}\mathfrak{v}}\rangle_{\boldsymbol{\mu}} = \langle \psi_{\mathfrak{s}\mathfrak{t}}, \psi_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}^{\boldsymbol{\mu}}}\psi'_{\mathfrak{u}\mathfrak{v}}h^\star\rangle_{\boldsymbol{\mu}},$$

for all $h \in \mathcal{R}^\Lambda_n$ and all $(\mathfrak{s}, \mathfrak{t})$ and $(\mathfrak{u}, \mathfrak{v})$ as above. Write $\psi'_{\mathfrak{u}\mathfrak{v}}h^\star = \sum r_{\mathfrak{a}\mathfrak{b}}\psi'_{\mathfrak{a}\mathfrak{b}}$, where in the sum $(\mathfrak{a}, \mathfrak{b}) \in \operatorname{Std}^2(\mathscr{P}^\Lambda_\beta)$ and $r_{\mathfrak{a}\mathfrak{b}} \in \mathcal{Z}$. Then the left hand side is equal to

$$\langle \psi_{\mathfrak{s}\mathfrak{t}}h, \psi_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}^{\boldsymbol{\mu}}}\psi'_{\mathfrak{u}\mathfrak{v}}\rangle_{\boldsymbol{\mu}} = \tau_\beta(\psi_{\mathfrak{s}\mathfrak{t}}h\psi'_{\mathfrak{v}\mathfrak{u}}) = \sum_{(\mathfrak{a},\mathfrak{b})\in\operatorname{Std}^2(\mathscr{P}^\Lambda_\beta)} r_{\mathfrak{a}\mathfrak{b}}\tau_\beta(\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{b}\mathfrak{a}})$$

Now τ_β is a trace form, so $\tau_\beta(\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{b}\mathfrak{a}}) = \tau_\beta(\psi'_{\mathfrak{b}\mathfrak{a}}\psi_{\mathfrak{s}\mathfrak{t}})$ is non-zero only if $\mathfrak{a} \trianglerighteq \mathfrak{s}$ and $\operatorname{res}(\mathfrak{a}) = \operatorname{res}(\mathfrak{s})$ by Lemma 3.15, so that $\mathfrak{a} \in \operatorname{Std}^{\boldsymbol{\mu}}(\mathscr{P}^\Lambda_\beta)$. Consequently,

$$\begin{aligned}
\langle \psi_{\mathfrak{s}\mathfrak{t}}h, \psi_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}^{\boldsymbol{\mu}}}\psi'_{\mathfrak{u}\mathfrak{v}}\rangle_{\boldsymbol{\mu}} &= \sum_{\substack{\mathfrak{a}\in\operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu}),\mathfrak{b}\in\operatorname{Std}(\boldsymbol{\nu})\\ \boldsymbol{\nu}\in\mathscr{P}^\Lambda_\beta}} r_{\mathfrak{a}\mathfrak{b}}\tau_\beta(\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{b}\mathfrak{a}})\\
&= \sum_{\substack{\mathfrak{a}\in\operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu}),\mathfrak{b}\in\operatorname{Std}(\boldsymbol{\nu})\\ \boldsymbol{\nu}\in\mathscr{P}^\Lambda_\beta}} r_{\mathfrak{a}\mathfrak{b}}\langle \psi_{\mathfrak{s}\mathfrak{t}}, \psi_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}^{\boldsymbol{\mu}}}\psi'_{\mathfrak{a}\mathfrak{b}}\rangle_{\boldsymbol{\mu}}\\
&= \langle \psi_{\mathfrak{s}\mathfrak{t}}, \psi_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}^{\boldsymbol{\mu}}}\psi'_{\mathfrak{u}\mathfrak{v}}h^\star\rangle_{\boldsymbol{\mu}},
\end{aligned}$$

where the last equality follows using Lemma 3.15 and Corollary 4.13. Hence, the form $\langle\ ,\ \rangle_{\boldsymbol{\mu}}$ is associative. Since taking duals reverses the grading, the map $x \mapsto \langle x, ?\rangle_{\boldsymbol{\mu}}$, for $x \in G^{\boldsymbol{\mu}}$, gives the required isomorphism. □

4.2. **Quiver Schur algebras.** We are now ready to define the quiver Schur algebras of type Γ_e, which are the main objects of study in this paper.

4.16. **Definition.** Suppose that $\Lambda \in P^+$ and let $G^\Lambda_n = \bigoplus_{\boldsymbol{\mu}\in\mathscr{P}^\Lambda_n} G^{\boldsymbol{\mu}}$. The **quiver Schur algebra** of type (Γ_e, Λ) is the endomorphism algebra

$$\mathcal{S}^\Lambda_n = \mathcal{S}^\Lambda_n(\Gamma_e) = \mathrm{End}_{\mathcal{R}^\Lambda_n}(G^\Lambda_n).$$

By definition $\mathcal{S}^\Lambda_n$ is a graded algebra. As a $\mathcal{Z}$-module, $\mathcal{S}^\Lambda_n$ admits a decomposition

$$\mathcal{S}^\Lambda_n = \bigoplus_{\boldsymbol{\nu},\boldsymbol{\mu}\in\mathscr{P}^\Lambda_n} \mathrm{Hom}_{\mathcal{R}^\Lambda_n}(G^{\boldsymbol{\nu}}, G^{\boldsymbol{\mu}}).$$

By Theorem 3.20, $\mathcal{R}^\Lambda_n$ is a graded symmetric algebra, so by [16, 61.2]

$$\mathrm{Hom}_{\mathcal{R}^\Lambda_n}(G^{\boldsymbol{\nu}}, G^{\boldsymbol{\mu}}) \cong G^{\boldsymbol{\mu}} \cap (G^{\boldsymbol{\nu}})^\star \tag{4.17}$$

as graded $\mathcal{Z}$-modules, where the isomorphism is given by $\Psi \mapsto \Psi(e^{\boldsymbol{\nu}}y^{\boldsymbol{\nu}})$. By Corollary 4.11, if $\mathfrak{s} \in \mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda})$ and $\mathfrak{t} \in \mathrm{Std}^{\boldsymbol{\nu}}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_\beta$, then $\psi_{\mathfrak{s}\mathfrak{t}} \in G^{\boldsymbol{\mu}} \cap (G^{\boldsymbol{\nu}})^\star$ so we can define a homomorphism $\Psi^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{s}\mathfrak{t}} \in \mathrm{Hom}_{\mathcal{R}^\Lambda_n}(G^{\boldsymbol{\nu}}, G^{\boldsymbol{\mu}})$ by

$$\Psi^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{s}\mathfrak{t}}(e^{\boldsymbol{\nu}}y^{\boldsymbol{\nu}}h) = \psi_{\mathfrak{s}\mathfrak{t}}h, \qquad \text{for all } h \in \mathcal{R}^\Lambda_n. \tag{4.18}$$

We think of $\Psi^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{s}\mathfrak{t}}$ as an element of $\mathcal{S}^\Lambda_n$ in the obvious way.

4.19. **Example** It is necessary to include $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in the notation $\Psi^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{s}\mathfrak{t}}$ because a given tableau can belong to $\mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ for many different $\boldsymbol{\mu}$. The simplest example of this phenomenon occurs when $\mathfrak{t} = (\boxed{1}|\emptyset)$ and $\boldsymbol{\kappa} = (0,0)$, so that $\Lambda = 2\Lambda_0$. Let $\boldsymbol{\mu} = (1|-)$ and $\boldsymbol{\nu} = (-|1)$. Then $\mathfrak{t} \in \mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\mu}) \cap \mathrm{Std}^{\boldsymbol{\nu}}(\boldsymbol{\mu})$ and $\psi_{\mathfrak{t}\mathfrak{t}} = e^{\boldsymbol{\mu}}y^{\boldsymbol{\mu}} \in G^{\boldsymbol{\mu}} \cap G^{\boldsymbol{\nu}} \cap (G^{\boldsymbol{\mu}})^\star \cap (G^{\boldsymbol{\nu}})^\star$ by Corollary 4.11. Therefore, the tableau $\mathfrak{t}$ determines four different maps in $\mathcal{S}^\Lambda_n$:

$$\begin{aligned}
&\Psi^{\boldsymbol{\mu}\boldsymbol{\mu}}_{\mathfrak{t}\mathfrak{t}} : G^{\boldsymbol{\mu}} \longrightarrow G^{\boldsymbol{\mu}}; e^{\boldsymbol{\mu}}y^{\boldsymbol{\mu}}h \mapsto \psi_{\mathfrak{t}\mathfrak{t}}h, \qquad &&\Psi^{\boldsymbol{\nu}\boldsymbol{\mu}}_{\mathfrak{t}\mathfrak{t}} : G^{\boldsymbol{\mu}} \longrightarrow G^{\boldsymbol{\nu}}; e^{\boldsymbol{\mu}}y^{\boldsymbol{\mu}}h \mapsto \psi_{\mathfrak{t}\mathfrak{t}}h,\\
&\Psi^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{t}\mathfrak{t}} : G^{\boldsymbol{\nu}} \longrightarrow G^{\boldsymbol{\mu}}; e^{\boldsymbol{\nu}}y^{\boldsymbol{\nu}}h \mapsto \psi_{\mathfrak{t}\mathfrak{t}}h, \qquad &&\Psi^{\boldsymbol{\nu}\boldsymbol{\nu}}_{\mathfrak{t}\mathfrak{t}} : G^{\boldsymbol{\nu}} \longrightarrow G^{\boldsymbol{\nu}}; e^{\boldsymbol{\nu}}y^{\boldsymbol{\nu}}h \mapsto \psi_{\mathfrak{t}\mathfrak{t}}h.
\end{aligned}$$

We have $\deg \Psi^{\boldsymbol{\mu}\boldsymbol{\mu}}_{\mathfrak{t}\mathfrak{t}} = 0$, $\deg \Psi^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{t}\mathfrak{t}} = 1 = \deg \Psi^{\boldsymbol{\nu}\boldsymbol{\mu}}_{\mathfrak{t}\mathfrak{t}}$ and $\deg \Psi^{\boldsymbol{\nu}\boldsymbol{\nu}}_{\mathfrak{t}\mathfrak{t}} = 2$. ◊

For $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$ let $\mathcal{T}^{\boldsymbol{\lambda}} = \{\, (\boldsymbol{\mu}, \mathfrak{s}) \mid \mathfrak{s} \in \mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\mu} \in \mathscr{P}^\Lambda_n \,\}$.

4.20. **Theorem.** *Suppose that $e = 0$ or $e \geq n$ and let $\mathcal{Z}$ be an integral domain such that e is invertible in $\mathcal{Z}$ whenever $e \neq 0$ and e is not prime. Then $\mathcal{S}^\Lambda_n$ is a graded cellular algebra with cellular basis $\{\, \Psi^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{s}\mathfrak{t}} \mid (\boldsymbol{\mu}, \mathfrak{s}), (\boldsymbol{\nu}, \mathfrak{t}) \in \mathcal{T}^{\boldsymbol{\lambda}}$ and $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n \,\}$, weight poset $(\mathscr{P}^\Lambda_n, \trianglerighteq)$ and degree function $\deg \Psi^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{s}\mathfrak{t}} = \deg \mathfrak{s} - \deg \mathfrak{t}^{\boldsymbol{\mu}} + \deg \mathfrak{t} - \deg \mathfrak{t}^{\boldsymbol{\nu}}$.*

Proof. By Corollary 4.11 and (4.17) the maps in the statement of the Theorem are a basis of $\mathcal{S}^\Lambda_n$. As in [17, §6], it is now a purely formal argument to show that this basis is a cellular basis of $\mathcal{S}^\Lambda_n$. We have already verified axioms (GC$_d$) and (GC$_1$) from section 2.2. Axiom (GC$_3$) is a straightforward calculation using the fact that $\psi^\star_{\mathfrak{s}\mathfrak{t}} = \psi_{\mathfrak{t}\mathfrak{s}}$ by Theorem 3.14; see [17, Proposition 6.9]. It remains to check (GC$_2$) but this follows by repeating the argument from [17, Theorem 6.6(ii)], essentially without change, using Corollary 4.11 and Theorem 3.14. □

4.21. *Remark.* In [17, Theorem 6.6] the cellular basis of the cyclotomic q-Schur algebras is labelled by *semistandard tableaux* of type $\boldsymbol{\nu}$. The tableaux in $\mathcal{T}^{\boldsymbol{\lambda}}$ are, in fact, closely related to semistandard tableaux. Using the notation of [17, Definition 4.2], if $(\boldsymbol{\nu}, \mathfrak{t}) \in \mathcal{T}^{\boldsymbol{\lambda}}$ then $\boldsymbol{\nu}(\mathfrak{t})$ is a semistandard $\boldsymbol{\lambda}$-tableau of type $\boldsymbol{\nu}$.

4.22. **Example** If $\ell = 2$ then it is an interesting combinatorial exercise to show that $\mathcal{S}^\Lambda_\beta$ is positively graded; see [24]. If $\ell > 2$ then $\mathcal{S}^\Lambda_\beta$ is in general only $\mathbb{Z}$-graded.

For example, suppose that $\Lambda = 3\Lambda_0$, $\boldsymbol{\mu} = (1|2, 1|2^2)$ and

$$\mathfrak{t} = \left(\begin{array}{|c|c|}\hline 1 & 6 \\ \hline 7 \\ \cline{1-1} \end{array} \;\middle|\; \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 8 \\ \hline \end{array} \;\middle|\; \begin{array}{|c|}\hline 5 \\ \hline \end{array} \right).$$

Then it is easy to check that $\mathfrak{t} \in \operatorname{Std}^{\boldsymbol{\mu}}(2, 1|2^2|1)$ and that $\deg \mathfrak{t} = 2 < \deg \mathfrak{t}^{\boldsymbol{\mu}} = 3$. Therefore, $\deg \Psi^{\boldsymbol{\mu}\boldsymbol{\mu}}_{\mathfrak{t}\mathfrak{t}} = -2$. $\Diamond$

Now that we know that $\mathcal{S}^\Lambda_n$ is a graded cellular algebra we can use the general theory from section 2.2 to construct cell modules and irreducible $\mathcal{S}^\Lambda_n$-modules.

Suppose that $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$. The **graded Weyl module** $\Delta^{\boldsymbol{\lambda}}$ is the cell module for $\mathcal{S}^\Lambda_n$ corresponding to $\boldsymbol{\lambda}$. More explicitly, $\Delta^{\boldsymbol{\lambda}}$ is the $\mathcal{S}^\Lambda_n$-module with basis

$$\{\, \Psi^{\boldsymbol{\nu}}_{\mathfrak{t}} \mid (\boldsymbol{\nu}, \mathfrak{t}) \in \mathcal{T}^{\boldsymbol{\lambda}} \,\} \tag{4.23}$$

such that $\big(\Phi^{\boldsymbol{\lambda}\boldsymbol{\lambda}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}\mathcal{S}^\Lambda_n + (\mathcal{S}^\Lambda_n)^{\rhd\boldsymbol{\lambda}}\big)/(\mathcal{S}^\Lambda_n)^{\unrhd\boldsymbol{\lambda}} \cong \Delta^{\boldsymbol{\lambda}}$ under the map which sends $\Psi^{\boldsymbol{\lambda}\boldsymbol{\nu}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + (\mathcal{S}^\Lambda_n)^{\rhd\boldsymbol{\lambda}}$ to $\Psi^{\boldsymbol{\nu}}_{\mathfrak{t}}$, for $(\boldsymbol{\nu}, \mathfrak{t}) \in \mathcal{T}^{\boldsymbol{\lambda}}$.

As in section 2.2, the graded Weyl module $\Delta^{\boldsymbol{\lambda}}$ comes equipped with a homogeneous bilinear form $\langle\ ,\ \rangle$ of degree zero such that

$$\langle \Psi^{\boldsymbol{\mu}}_{\mathfrak{s}}, \Psi^{\boldsymbol{\nu}}_{\mathfrak{t}} \rangle \Psi^{\boldsymbol{\lambda}\boldsymbol{\lambda}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \equiv \Psi^{\boldsymbol{\lambda}\boldsymbol{\mu}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}} \Psi^{\boldsymbol{\nu}\boldsymbol{\lambda}}_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\lambda}}} \pmod{(\mathcal{S}^\Lambda_n)^{\rhd\boldsymbol{\lambda}}}, \tag{4.24}$$

for $(\boldsymbol{\mu}, \mathfrak{s}), (\boldsymbol{\nu}, \mathfrak{t}) \in \mathcal{T}^{\boldsymbol{\lambda}}$. Define $L^{\boldsymbol{\lambda}} = \Delta^{\boldsymbol{\lambda}}/\operatorname{rad}\Delta^{\boldsymbol{\lambda}}$, where $\operatorname{rad}\Delta^{\boldsymbol{\lambda}}$ is the radical of this form. Set $\nabla^{\boldsymbol{\lambda}} = (\Delta^{\boldsymbol{\lambda}})^{\circledast}$.

4.25. **Theorem.** *Suppose that $e = 0$ or $e \geq n$. Then $\mathcal{S}^\Lambda_n$ is quasi-hereditary graded cellular algebra with:*

- *weight poset $(\mathscr{P}^\Lambda_n, \unrhd)$,*
- *graded standard modules $\{\, \Delta^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n \,\}$,*
- *graded costandard modules $\{\, \nabla^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n \,\}$, and,*
- *graded simple modules $\{\, L^{\boldsymbol{\lambda}}\langle k\rangle \mid \boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$ and $k \in \mathbb{Z} \,\}$.*

Moreover, $L^{\boldsymbol{\lambda}} \cong (L^{\boldsymbol{\lambda}})^{\circledast}$ for all $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$.

Proof. By definition, $\Psi^{\boldsymbol{\lambda}\boldsymbol{\lambda}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ is the identity map on $G^{\boldsymbol{\lambda}}$, so $\langle \Psi^{\boldsymbol{\lambda}}_{\mathfrak{t}^{\boldsymbol{\lambda}}}, \Psi^{\boldsymbol{\lambda}}_{\mathfrak{t}^{\boldsymbol{\lambda}}} \rangle = 1$ by (4.24). Consequently, $L^{\boldsymbol{\lambda}} \neq 0$ for all $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$. Therefore, $L^{\boldsymbol{\lambda}} \cong (L^{\boldsymbol{\lambda}})^{\circledast}$, for $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$, and

$$\{\, L^{\boldsymbol{\lambda}}\langle k\rangle \mid \boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n \text{ and } k \in \mathbb{Z} \,\}$$

is a complete set of pairwise non-isomorphic irreducible $\mathcal{S}^\Lambda_n$-modules by Theorem 2.5. In turn, this implies that $\mathcal{S}^\Lambda_n$ is a quasi-hereditary algebra by Corollary 2.9, with standard and costandard modules as stated. $\square$

For each $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$ set $\Psi^{\boldsymbol{\lambda}} = \Psi^{\boldsymbol{\lambda}\boldsymbol{\lambda}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$. Then $\Psi^{\boldsymbol{\lambda}}$ (restricts to) the identity map on $G^{\boldsymbol{\lambda}}$ and $\sum_{\boldsymbol{\lambda}} \Psi^{\boldsymbol{\lambda}}$ is the identity element of $\mathcal{S}^\Lambda_n$. As an $\mathcal{Z}$-module, every $\mathcal{S}^\Lambda_n$-module M has a **weight space** decomposition

$$M = \bigoplus_{\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n} M_{\boldsymbol{\lambda}}, \qquad \text{where } M_{\boldsymbol{\lambda}} = M\Psi^{\boldsymbol{\lambda}}. \tag{4.26}$$

In particular, if $\boldsymbol{\lambda}, \boldsymbol{\nu} \in \mathscr{P}^\Lambda_n$ then $\{\, \Psi^{\boldsymbol{\nu}}_{\mathfrak{t}} \mid (\boldsymbol{\nu}, \mathfrak{t}) \in \mathcal{T}^{\boldsymbol{\lambda}} \,\}$ is a basis of $\Delta^{\boldsymbol{\lambda}}_{\boldsymbol{\nu}}$ by (4.23).

4.27. *Remark.* Although we will not need this, the reader can check that if $(\boldsymbol{\nu}, \mathfrak{t}) \in \mathcal{T}^{\boldsymbol{\lambda}}$ then we can identify $\Psi^{\boldsymbol{\nu}}_{\mathfrak{t}}$ with the homomorphism $G^{\boldsymbol{\nu}} \to S^{\boldsymbol{\lambda}}$ which sends $\psi_{\mathfrak{t}^{\boldsymbol{\nu}}\mathfrak{t}^{\boldsymbol{\nu}}}h$ to $\psi_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}}h$, for $h \in \mathcal{R}^\Lambda_n$. In this way, $\Delta^{\boldsymbol{\lambda}}$ can be identified with a $\mathcal{S}^\Lambda_n$-submodule of $\operatorname{Hom}_{\mathcal{R}^\Lambda_n}(G^\Lambda_n, S^{\boldsymbol{\lambda}})$. By Corollary 4.12 there is a projection map $\pi^{\boldsymbol{\lambda}} : G^{\boldsymbol{\lambda}} \twoheadrightarrow S^{\boldsymbol{\lambda}}$ such that $\pi^{\boldsymbol{\lambda}}(\psi_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}h) = \psi_{\mathfrak{t}^{\boldsymbol{\lambda}}}h$, for all $h \in \mathcal{R}^\Lambda_n$. So, by Theorem 4.20 and the remarks after (4.23), the weight space $\Delta^{\boldsymbol\lambda}_{\boldsymbol\nu}$ of the Weyl module $\Delta^{\boldsymbol\lambda}$ can be identified with the set of maps in $\operatorname{Hom}_{\mathcal{R}^\Lambda_n}(G^{\boldsymbol\nu}, S^{\boldsymbol\lambda})$ which factor through $\pi^{\boldsymbol\lambda}$.

$$\begin{tikzcd} G^{\boldsymbol\nu} \arrow[r, dashed, "\exists\Psi'"] \arrow[dr, "\Psi"'] & G^{\boldsymbol\lambda} \arrow[d, "\pi^{\boldsymbol\lambda}"] \\ & S^{\boldsymbol\lambda} \end{tikzcd}$$

4.3. **Graded Schur functors.** We now define an exact functor from the category of graded $\mathcal{S}^\Lambda_n$-modules to the category of graded $\mathcal{R}^\Lambda_n$-modules and use this to relate the graded decomposition numbers of the two algebras. To do this it is useful to introduce a slightly larger version of the quiver Schur algebra $\mathcal{S}^\Lambda_n$.

To this end let $\dot{\mathscr{P}}^\Lambda_n = \mathscr{P}^\Lambda_n \cup \{\omega\}$, where ω is a dummy symbol, and set $G^\omega = \mathcal{R}^\Lambda_n$ and $\dot{G}^\Lambda_n = G^\Lambda_n \oplus G^\omega$. The **extended quiver Schur algebra** is the algebra

$$\dot{\mathcal{S}}^\Lambda_n = \operatorname{End}_{\mathcal{R}^\Lambda_n}(\dot{G}^\Lambda_n).$$

Suppose that $\boldsymbol\nu \in \mathscr{P}^\Lambda_n$. For convenience of notation, set $\operatorname{Std}^\omega(\boldsymbol\nu) = \operatorname{Std}(\boldsymbol\nu)$ and define $e^\omega = 1 = y^\omega \in \mathcal{R}^\Lambda_n$ so that $G^\omega = e^\omega y^\omega \mathcal{R}^\Lambda_n$. Let $\mathfrak{t}^\omega = 1$ and set $\psi_{\mathfrak{t}^\omega\mathfrak{t}^\omega} = e^\omega y^\omega = 1$ and define $\deg \mathfrak{t}^\omega = 0$. Extending (4.18), if $\boldsymbol\nu, \boldsymbol\mu \in \dot{\mathscr{P}}^\Lambda_n$ and $\mathfrak{s} \in \operatorname{Std}^{\boldsymbol\mu}(\boldsymbol\nu)$ and $\mathfrak{t} \in \operatorname{Std}^{\boldsymbol\nu}(\boldsymbol\nu)$ then define

$$\Psi^{\boldsymbol\mu\boldsymbol\nu}_{\mathfrak{s}\mathfrak{t}}(e^{\boldsymbol\nu} y^{\boldsymbol\nu} h) = \psi_{\mathfrak{s}\mathfrak{t}} h, \qquad \text{for all } h \in \mathcal{R}^\Lambda_n.$$

Then $\Psi^{\boldsymbol\mu\boldsymbol\nu}_{\mathfrak{s}\mathfrak{t}} \in \dot{\mathcal{S}}^\Lambda_n$ and $\deg \Psi^{\boldsymbol\mu\boldsymbol\nu}_{\mathfrak{s}\mathfrak{t}} = \deg \mathfrak{s} - \deg \mathfrak{t}^{\boldsymbol\mu} + \deg \mathfrak{t} - \deg \mathfrak{t}^{\boldsymbol\nu}$. For each multipartition $\boldsymbol\lambda \in \mathscr{P}^\Lambda_n$ set $\dot{\mathcal{T}}^{\boldsymbol\lambda} = \{\, (\boldsymbol\nu, \mathfrak{t}) \mid \mathfrak{t} \in \operatorname{Std}^{\boldsymbol\nu}(\boldsymbol\lambda) \text{ for } \boldsymbol\nu \in \dot{\mathscr{P}}^\Lambda_n \,\} = \mathcal{T}^{\boldsymbol\lambda} \cup \{\omega\} \times \operatorname{Std}(\boldsymbol\lambda)$.

4.28. **Proposition.** *The algebra $\dot{\mathcal{S}}^\Lambda_n$ is a graded cellular algebra with cellular basis*

$$\{\, \Psi^{\boldsymbol\mu\boldsymbol\nu}_{\mathfrak{s}\mathfrak{t}} \mid (\boldsymbol\mu, \mathfrak{s}), (\boldsymbol\nu, \mathfrak{t}) \in \dot{\mathcal{T}}^{\boldsymbol\lambda} \text{ for } \boldsymbol\lambda \in \mathscr{P}^\Lambda_n \,\},$$

weight poset $(\mathscr{P}^\Lambda_n, \trianglerighteq)$ and degree function

$$\deg \Psi^{\boldsymbol\mu\boldsymbol\nu}_{\mathfrak{s}\mathfrak{t}} = \deg \mathfrak{s} - \deg \mathfrak{t}^{\boldsymbol\mu} + \deg \mathfrak{t} - \deg \mathfrak{t}^{\boldsymbol\nu}.$$

Moreover, $\dot{\mathcal{S}}^\Lambda_n$ is a quasi-hereditary algebra with standard modules $\{\, \dot\Delta^{\boldsymbol\lambda} \mid \boldsymbol\lambda \in \mathscr{P}^\Lambda_n \,\}$ and simple modules $\{\, \dot L^{\boldsymbol\lambda}\langle k\rangle \mid \boldsymbol\lambda \in \mathscr{P}^\Lambda_n \text{ and } k \in \mathbb{Z} \,\}$.

Proof. By definition, $\mathcal{S}^\Lambda_n$ is a subalgebra of $\dot{\mathcal{S}}^\Lambda_n$ and, as a $\mathcal{Z}$-module,

$$\dot{\mathcal{S}}^\Lambda_n = \mathcal{S}^\Lambda_n \oplus \operatorname{Hom}_{\mathcal{R}^\Lambda_n}(G^\omega, G^\Lambda_n) \oplus \operatorname{Hom}_{\mathcal{R}^\Lambda_n}(G^\Lambda_n, G^\omega) \oplus \operatorname{End}_{\mathcal{R}^\Lambda_n}(G^\omega).$$

For $\boldsymbol\mu \in \dot{\mathscr{P}}^\Lambda_n$ there are isomorphisms of graded $\mathcal{Z}$-modules $G^{\boldsymbol\mu} \cong \operatorname{Hom}_{\mathcal{R}^\Lambda_n}(G^\omega, G^{\boldsymbol\mu})$ given by $\psi_{\mathfrak{s}\mathfrak{t}} \mapsto \Psi^{\boldsymbol\mu\omega}_{\mathfrak{s}\mathfrak{t}}$, for $\mathfrak{s} \in \operatorname{Std}^{\boldsymbol\mu}(\boldsymbol\nu)$ and $\mathfrak{t} \in \operatorname{Std}^\omega(\boldsymbol\nu)$ and $\boldsymbol\nu \in \mathscr{P}^\Lambda_n$. Therefore, the elements in the statement of the Proposition give a basis of $\dot{\mathcal{S}}^\Lambda_n$ by Theorem 4.20 and Theorem 4.10. Repeating the arguments from Theorem 4.20 and Theorem 4.25 shows that $\dot{\mathcal{S}}^\Lambda_n$ is a quasi-hereditary graded cellular algebra. $\square$

By Proposition 4.28, there exist Weyl modules $\dot\Delta^{\boldsymbol\lambda}$ and a simple modules $\dot L^{\boldsymbol\lambda} = \dot\Delta^{\boldsymbol\lambda} / \operatorname{rad} \dot\Delta^{\boldsymbol\lambda}$ for $\dot{\mathcal{S}}^\Lambda_n$, for each $\boldsymbol\lambda \in \mathscr{P}^\Lambda_n$. As in (4.23), let $\{\, \Psi^{\boldsymbol\nu}_{\mathfrak{t}} \mid (\boldsymbol\nu, \mathfrak{t}) \in \dot{\mathcal{T}}^{\boldsymbol\lambda} \,\}$ be the basis of $\dot\Delta^{\boldsymbol\lambda}$.

Set $\Psi^\Lambda_n = \sum_{\boldsymbol\mu \in \mathscr{P}^\Lambda_n} \Psi^{\boldsymbol\mu}$ and let Ψ^ω be the identity map on $G^\omega = \mathcal{R}^\Lambda_n$. Then Ψ^Λ_n is the identity element of $\mathcal{S}^\Lambda_n$ and $\Psi^\Lambda_n + \Psi^\omega$ is the identity element of $\dot{\mathcal{S}}^\Lambda_n$. By definition, Ψ^Λ_n and Ψ^ω are both idempotents in $\dot{\mathcal{S}}^\Lambda_n$ and $\Psi^\Lambda_n \dot{\mathcal{S}}^\Lambda_n \Psi^\Lambda_n \cong \mathcal{S}^\Lambda_n$. Therefore, by (2.10), there are exact functors

$$\dot{\mathsf{F}}^\omega_n : \dot{\mathcal{S}}^\Lambda_n\text{-Mod} \longrightarrow \mathcal{S}^\Lambda_n\text{-Mod} \quad \text{ and } \quad \dot{\mathsf{G}}^n_\omega : \mathcal{S}^\Lambda_n\text{-Mod} \longrightarrow \dot{\mathcal{S}}^\Lambda_n\text{-Mod}$$

given by $\dot{\mathsf{F}}^\omega_n(M) = M\Psi^\Lambda_n$ and $\dot{\mathsf{G}}^n_\omega(N) = N \otimes_{\mathcal{S}^\Lambda_n} \Psi^\Lambda_n\dot{\mathcal{S}}^\Lambda_n$. By section 2.4 we also have functors $\mathsf{H}_\omega, \mathsf{O}_\omega, \mathsf{O}^\omega : \dot{\mathcal{S}}^\Lambda_n\text{-Mod} \longrightarrow \dot{\mathcal{S}}^\Lambda_n\text{-Mod}$ such that $\mathsf{H}_\omega(M) = M/\mathsf{O}_\omega(M)$.

4.29. **Lemma.** *The functors $\dot{\mathsf{F}}^\omega_n$ and $\dot{\mathsf{G}}^n_\omega$ induce mutually inverse equivalences of categories between $\dot{\mathcal{S}}^\Lambda_n$-Mod and $\mathcal{S}^\Lambda_n$-Mod. Moreover,*

$$\dot{\mathsf{F}}^\omega_n(\dot{\Delta}^{\boldsymbol{\lambda}}) \cong \Delta^{\boldsymbol{\lambda}} \qquad \textit{and} \qquad \dot{\mathsf{F}}^\omega_n(\dot{L}^{\boldsymbol{\lambda}}) \cong L^{\boldsymbol{\lambda}},$$

for all $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$.

Proof. Let M be an $\dot{\mathcal{S}}^\Lambda_n$-module. Then, extending (4.26), M has a weight space decomposition

$$M = \bigoplus_{\boldsymbol{\mu}\in\dot{\mathscr{P}}^\Lambda_n} M_{\boldsymbol{\mu}}, \qquad \text{where } M_{\boldsymbol{\mu}} = M\Psi^{\boldsymbol{\mu}}.$$

Then, essentially by definition, $\dot{\mathsf{F}}^\omega_n(M) = \bigoplus_{\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_n} M_{\boldsymbol{\lambda}}$. That is, $\dot{\mathsf{F}}^\omega_n$ removes the ω-weight space of M. In particular, $\dot{\mathsf{F}}^\omega_n(\dot{\Delta}^{\boldsymbol{\mu}}) = \Delta^{\boldsymbol{\mu}}$ and $\dot{\mathsf{F}}^\omega_n(\dot{L}^{\boldsymbol{\mu}}) = L^{\boldsymbol{\mu}}$, for all $\boldsymbol{\mu} \in \mathscr{P}^\Lambda_n$. The fact that $\dot{\mathsf{F}}^\omega_n(\dot{L}^{\boldsymbol{\mu}}) = L^{\boldsymbol{\mu}}$ for all $\boldsymbol{\mu} \in \mathscr{P}^\Lambda_n$ implies that $\mathsf{O}^\omega(M) = M$, $\mathsf{O}_\omega(M) = 0$, for all $M \in \dot{\mathcal{S}}^\Lambda_n$-Mod. Therefore, H_ω is the identity functor and $\dot{\mathsf{G}}^n_\omega \cong \mathsf{H}_\omega \circ \dot{\mathsf{G}}^n_\omega$. Hence, the Lemma is an immediate consequence of Theorem 2.12. □

The identity map Ψ^ω on $\mathcal{R}^\Lambda_n = G^\omega$ is idempotent in $\dot{\mathcal{S}}^\Lambda_n$ and there is a graded isomorphism of $\mathcal{Z}$-algebras $\Psi^\omega\dot{\mathcal{S}}^\Lambda_n\Psi^\omega \cong \mathcal{R}^\Lambda_n$. Therefore, by (2.10), there are functors

$$\dot{\mathsf{F}}^\Lambda_n : \dot{\mathcal{S}}^\Lambda_n\text{-Mod} \longrightarrow \mathcal{R}^\Lambda_n\text{-Mod} \quad \text{and} \quad \dot{\mathsf{G}}^\Lambda_n : \mathcal{R}^\Lambda_n\text{-Mod} \longrightarrow \dot{\mathcal{S}}^\Lambda_n\text{-Mod} \tag{4.30}$$

given by $\dot{\mathsf{F}}^\Lambda_n(M) = M\Psi^\omega = M_\omega$ and $\dot{\mathsf{G}}^\Lambda_n(N) = N \otimes_{\mathcal{R}^\Lambda_n} \Psi^\omega\dot{\mathcal{S}}^\Lambda_n$.

4.31. **Proposition.** *Suppose that $\mathcal{Z} = K$ is a field. Then there is an exact functor $\mathsf{F}^\Lambda_n : \mathcal{S}^\Lambda_n\text{-Mod} \longrightarrow \mathcal{R}^\Lambda_n\text{-Mod}$ given by*

$$\mathsf{F}^\Lambda_n(M) = (M \otimes_{\mathcal{S}^\Lambda_n} \Psi^\Lambda_n\dot{\mathcal{S}}^\Lambda_n)\Psi^\omega, \qquad \textit{for } M \in \mathcal{S}^\Lambda_n\text{-Mod},$$

such that if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\Lambda_n$ then $\mathsf{F}^\Lambda_n(\Delta^{\boldsymbol{\lambda}}) \cong S^{\boldsymbol{\lambda}}$ and

$$\mathsf{F}^\Lambda_n(L^{\boldsymbol{\mu}}) \cong \begin{cases} D^{\boldsymbol{\mu}}, & \textit{if } \boldsymbol{\mu} \in \mathcal{K}^\Lambda_n,\\ 0, & \textit{if } \boldsymbol{\mu} \notin \mathcal{K}^\Lambda_n.\end{cases}$$

Proof. By definition, $\mathsf{F}^\Lambda_n = \dot{\mathsf{F}}^\Lambda_n \circ \dot{\mathsf{G}}^n_\omega$, so F^Λ_n is an exact functor from $\mathcal{S}^\Lambda_n$-Mod to $\mathcal{R}^\Lambda_n$-Mod. The functor $\dot{\mathsf{F}}^\Lambda_n$ is nothing more than projection onto the ω-weight space. Hence, if $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$ then $\dot{\mathsf{F}}^\Lambda_n(\dot{\Delta}^{\boldsymbol{\lambda}})$ is spanned by the maps $\{\, \Psi^\omega_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$, since $\operatorname{Std}^\omega(\boldsymbol{\lambda}) = \operatorname{Std}(\boldsymbol{\lambda})$. The map $\Phi^\omega_{\mathfrak{t}} \mapsto \psi_{\mathfrak{t}}$, for $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, shows that $\dot{\mathsf{F}}^\Lambda_n(\dot{\Delta}^{\boldsymbol{\lambda}}) \cong S^{\boldsymbol{\lambda}}$. Hence, $\mathsf{F}^\Lambda_n(\Delta^{\boldsymbol{\lambda}}) \cong S^{\boldsymbol{\lambda}}$ by Lemma 4.29. By Theorem 2.12, $\mathsf{F}^\Lambda_n(L^{\boldsymbol{\mu}})$ is an irreducible $\mathcal{R}^\Lambda_n$-module whenever it is non-zero. A straightforward argument by induction on the dominance ordering using $\mathsf{F}^\Lambda_n(\Delta^{\boldsymbol{\lambda}}) \cong S^{\boldsymbol{\lambda}}$, Corollary 2.7 and Corollary 3.22 now shows that $\mathsf{F}^\Lambda_n(L^{\boldsymbol{\mu}}) \cong D^{\boldsymbol{\mu}}$ if $\boldsymbol{\mu} \in \mathcal{K}^\Lambda_n$ and that $\mathsf{F}^\Lambda_n(L^{\boldsymbol{\mu}}) = 0$ otherwise. □

Since F^Λ_n is an exact functor, we obtain the promised relationship between the graded decomposition numbers of $\mathcal{S}^\Lambda_n$ and $\mathcal{R}^\Lambda_n$.

4.32. **Corollary.** *Suppose that $\mathcal{Z} = K$ is a field and that $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$ and $\boldsymbol{\mu} \in \mathcal{K}^\Lambda_n$. Then $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]_q = [\Delta^{\boldsymbol{\lambda}} : L^{\boldsymbol{\mu}}]_q$.*

The graded decomposition multiplicities $[\Delta^{\boldsymbol{\lambda}} : L^{\boldsymbol{\mu}}]_q$ are one of the main objects of interest in this paper so we give them a special name.

4.33. **Definition.** Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\Lambda_n$. Set

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = [\Delta^{\boldsymbol{\lambda}} : L^{\boldsymbol{\mu}}]_q = \sum_{d\in\mathbb{Z}} [\Delta^{\boldsymbol{\lambda}} : L^{\boldsymbol{\mu}}\langle d\rangle]\, q^d.$$

Let $\mathbf{D}_{\mathcal{S}_n^\Lambda}(q) = (d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q))_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_n^\Lambda}$ and $\mathbf{D}_{\mathcal{R}_n^\Lambda}(q) = (d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q))_{\boldsymbol{\lambda}\in\mathscr{P}_n^\Lambda,\boldsymbol{\mu}\in\mathcal{K}_n^\Lambda}$ be the graded decomposition matrix of $\mathcal{S}_n^\Lambda$ and $\mathcal{R}_n^\Lambda$, respectively.

By Corollary 4.32, $\mathbf{D}_{\mathcal{R}_n^\Lambda}(q)$ can be considered as a submatrix of $\mathbf{D}_{\mathcal{S}_n^\Lambda}(q)$. For future use we note the following important property of these (Laurent) polynomials. This is a special case of Corollary 2.7.

4.34. **Corollary.** *Suppose that $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_n^\Lambda$. Then $d_{\boldsymbol{\mu}\boldsymbol{\mu}}(q)=1$ and $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\neq 0$ only if $\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}$ and $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda$ for some $\beta\in Q_n^+$.*

4.4. **Blocks of quiver Schur algebras.** We now give the block decomposition of the graded Schur algebra $\mathcal{S}_n^\Lambda$. The key observation is the following double centralizer result.

4.35. **Lemma** (A double centralizer property)**.** *There are canonical isomorphisms of graded algebras such that*

$$\dot{\mathcal{S}}_n^\Lambda \cong \mathrm{END}_{\mathcal{R}_n^\Lambda}(\dot{G}_n^\Lambda) \qquad \text{and} \qquad \mathcal{R}_n^\Lambda \cong \mathrm{END}_{\dot{\mathcal{S}}_n^\Lambda}(\dot{G}_n^\Lambda).$$

Proof. The first isomorphism is the definition of $\dot{\mathcal{S}}_n^\Lambda$ whereas the second follows directly from the definition of $\dot{\mathcal{S}}_n^\Lambda$ because

$$\mathcal{R}_n^\Lambda \cong \mathrm{HOM}_{\mathcal{R}_n^\Lambda}(\mathcal{R}_n^\Lambda,\mathcal{R}_n^\Lambda) \cong \Psi^\omega\dot{\mathcal{S}}_n^\Lambda\Psi^\omega \cong \mathrm{END}_{\dot{\mathcal{S}}_n^\Lambda}(\Psi^\omega\dot{\mathcal{S}}_n^\Lambda),$$

and $\Psi^\omega\dot{\mathcal{S}}_n^\Lambda \cong \dot{G}_n^\Lambda$ as right $\dot{\mathcal{S}}_n^\Lambda$-modules. □

In order to describe the block decomposition of $\mathcal{S}_n^\Lambda$ we set $G_\beta^\Lambda = \bigoplus_{\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda} G^{\boldsymbol{\mu}}$ and define $\mathcal{S}_\beta^\Lambda = \mathrm{END}_{\mathcal{R}_n^\Lambda}(G_\beta^\Lambda)$ if $\beta\in Q_n^+$. Equivalently, $\mathcal{S}_\beta^\Lambda = \Psi^\beta\mathcal{S}_n^\Lambda\Psi^\beta$, where $\Psi^\beta = \sum_{\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda}\Psi^{\boldsymbol{\mu}}$.

The subalgebras $\mathcal{S}_\beta^\Lambda$ of $\mathcal{S}_n^\Lambda$ are the blocks of $\mathcal{S}_n^\Lambda$. More precisely, we have the following.

4.36. **Theorem.** *Suppose that $\mathcal{Z}=K$ is a field. Then*

$$\mathcal{S}_n^\Lambda = \bigoplus_{\beta\in Q_n^+}\mathcal{S}_\beta^\Lambda,$$

is the block decomposition of $\mathcal{S}_n^\Lambda$ into a direct sum of indecomposable two-sided ideals. Moreover, if $\beta\in Q_n^+$ then the cellular basis of $\mathcal{S}_n^\Lambda$ in Theorem 4.20 restricts to give a graded cellular basis of $\mathcal{S}_\beta^\Lambda$. In particular, $\mathcal{S}_\beta^\Lambda$ is a quasi-hereditary graded cellular algebra, for each $\beta\in Q_n^+$.

Proof. First observe that if $\boldsymbol{\lambda}\in\mathscr{P}_\alpha^\Lambda$ and $\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda$, for $\alpha\neq\beta\in Q^+$, then all of the composition factors of $G^{\boldsymbol{\lambda}}$ and $G^{\boldsymbol{\mu}}$ belong to different blocks by 3.18 and Corollary 4.12. Therefore, $\mathrm{HOM}_{\mathcal{R}_n^\Lambda}(G^{\boldsymbol{\lambda}},G^{\boldsymbol{\mu}})=0$ so that, as $\mathcal{Z}$-modules,

$$\begin{aligned}\mathcal{S}_n^\Lambda = \mathrm{END}_{\mathcal{R}_n^\Lambda}(G_n^\Lambda) &= \bigoplus_{\alpha,\beta\in Q_n^+}\mathrm{HOM}_{\mathcal{R}_n^\Lambda}(G_\alpha^\Lambda,G_\beta^\Lambda)\\ &= \bigoplus_{\beta\in Q_n^+}\mathrm{END}_{\mathcal{R}_\beta^\Lambda}(G_\beta^\Lambda) = \bigoplus_{\beta\in Q_n^+}\mathcal{S}_\beta^\Lambda.\end{aligned}$$

It follows that the cellular basis of Theorem 4.20 restricts to give cellular bases for the algebras $\mathcal{S}_\beta^\Lambda$, for $\beta\in Q_n^+$. Therefore, $\mathcal{S}_\beta^\Lambda$ is a quasi-hereditary graded cellular algebra for each $\beta\in Q_n^+$.

It remains to show that each of the algebras $\mathcal{S}_\beta^\Lambda$ is indecomposable. By the double centralizer property, Lemma 4.35, the algebras $\mathcal{R}_n^\Lambda$ and $\dot{\mathcal{S}}_n^\Lambda$ have the same number of blocks and $\mathcal{S}_n^\Lambda$ and $\dot{\mathcal{S}}_n^\Lambda$ have the same number of indecomposable two-sided ideals by Lemma 4.29. By (3.18) the blocks of $\mathcal{R}_n^\Lambda$ are indexed by Q_n^+. As the elements of Q_n^+ also index the subalgebras $\mathcal{S}_\beta^\Lambda$, the non-zero algebras $\mathcal{S}_\beta^\Lambda$ must be indecomposable giving the result. □

For each $\beta \in Q_n^+$ define $\mathsf{F}_\beta^\Lambda(M) = \mathsf{F}_n^\Lambda(M\Psi^\beta)$, for an $\mathcal{S}_n^\Lambda$-module M. Tracing through the constructions of section 2.4 we obtain the following.

4.37. **Corollary.** *Suppose that $\beta \in Q_n^+$. Then F_n^Λ restricts to give an exact functor*

$$\mathsf{F}_\beta^\Lambda : \mathcal{S}_\beta^\Lambda\text{-Mod} \longrightarrow \mathcal{R}_\beta^\Lambda\text{-Mod}.$$

Moreover, there is a decomposition of functors $\mathsf{F}_n^\Lambda \cong \bigoplus_{\beta\in Q_n^+} \mathsf{F}_\beta^\Lambda$.

4.38. **Corollary.** *Suppose that $\beta \in Q_n^+$. Then $\mathcal{S}_\beta^\Lambda$ is a quasi-hereditary cover of $\mathcal{R}_\beta^\Lambda$ in the sense of Rouquier* [41, Definition 4.34]*.*

Proof. This follows because, by definition, F_β^Λ is the composition of $\dot{\mathsf{F}}_n^\Lambda$ with an equivalence of categories and $\dot{\mathsf{F}}_n^\Lambda$ is fully faithful on projectives by Lemma 2.11. □

4.5. **Sign-dual quiver Schur algebras.** Suppose that $\beta \in Q_n^+$ and recall the sign isomorphism $\mathtt{sgn} : \mathcal{R}_\beta^\Lambda \longrightarrow \mathcal{R}_{\beta'}^{\Lambda'}$ from (3.24). Consider the $\mathcal{R}_{\beta'}^{\Lambda'}$-module

$$G_{\Lambda'}^{\beta'} = \bigoplus_{\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda} G_{\boldsymbol{\mu}'}.$$

The **sign-dual quiver Schur algebra** of type $(\Gamma_e, \Lambda')_{\beta'}$ is the algebra

$$\mathcal{S}_{\Lambda'}^{\beta'} = \mathcal{S}_{\Lambda'}^{\beta'}(\Gamma_e) = \operatorname{End}_{\mathcal{R}_{\beta'}^{\Lambda'}}(G_{\Lambda'}^{\beta'}).$$

By (3.24) and Lemma 4.3 we have

$$\mathcal{S}_{\Lambda'}^{\beta'} = \operatorname{End}_{\mathcal{R}_{\beta'}^{\Lambda'}}\Big(\bigoplus_{\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda} G_{\boldsymbol{\mu}'}\Big) \cong \operatorname{End}_{\mathcal{R}_\beta^\Lambda}\Big(\bigoplus_{\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda} G^{\boldsymbol{\mu}}\Big) = \mathcal{S}_\beta^\Lambda.$$

That is, $\mathcal{S}_{\Lambda'}^{\beta'} \cong \mathcal{S}_\beta^\Lambda$ as graded algebras. For $\boldsymbol{\lambda}' \in \mathscr{P}_\beta^\Lambda$ let

$$\mathcal{T}_{\boldsymbol{\lambda}} = \{\, (\boldsymbol{\nu}, \mathfrak{t}) \mid (\boldsymbol{\nu}', \mathfrak{t}') \in \mathcal{T}^{\boldsymbol{\lambda}'} \,\} = \{\, (\boldsymbol{\nu}, \mathfrak{t}) \mid \mathfrak{t} \in \operatorname{Std}_{\boldsymbol{\nu}}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\nu} \in \mathscr{P}_{\beta'}^{\Lambda'} \,\}.$$

Chasing the isomorphism $\mathcal{S}_{\Lambda'}^{\beta'} \cong \mathcal{S}_\beta^\Lambda$ through Theorem 4.20, using Proposition 3.26, shows that $\mathcal{S}_{\Lambda'}^{\beta'}$ is a graded cellular algebra with weight poset $(\mathscr{P}_{\beta'}^{\Lambda'}, \trianglelefteq)$ and basis

$$\{\, \Psi_{\boldsymbol{\mu}\boldsymbol{\nu}}^{\mathfrak{s}\mathfrak{t}} \mid (\boldsymbol{\mu}, \mathfrak{s}), (\boldsymbol{\nu}, \mathfrak{t}) \in \mathcal{T}_{\boldsymbol{\lambda}} \text{ for } \boldsymbol{\lambda} \in \mathscr{P}_{\beta'}^{\Lambda'} \,\},$$

where $\Psi_{\boldsymbol{\mu}\boldsymbol{\nu}}^{\mathfrak{s}\mathfrak{t}}$ is the $\mathcal{R}_{\beta'}^{\Lambda'}$-endomorphism of $G_{\Lambda'}^{\beta'}$ given by

$$\Psi_{\boldsymbol{\mu}\boldsymbol{\nu}}^{\mathfrak{s}\mathfrak{t}}(e_{\boldsymbol{\rho}} y_{\boldsymbol{\rho}} h) = \delta_{\boldsymbol{\rho}\boldsymbol{\nu}} \psi'_{\mathfrak{s}\mathfrak{t}} h,$$

for $(\boldsymbol{\mu}, \mathfrak{s})$ and $(\boldsymbol{\nu}, \mathfrak{t})$ as above and $\boldsymbol{\rho} \in \mathscr{P}_{\beta'}^{\Lambda'}$. Alternatively, this can be proved by applying $\mathtt{sgn}$ to Theorem 4.25.

If $\boldsymbol{\lambda} \in \mathscr{P}_{\beta'}^{\Lambda'}$ let $\Delta_{\boldsymbol{\lambda}}$ be the corresponding Weyl module of $\mathcal{S}_{\Lambda'}^{\beta'}$ determined by this basis and let $L_{\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}} / \operatorname{rad} \Delta_{\boldsymbol{\lambda}}$ be its simple head.

Following the development of section 4.2 it is easy to show that $\mathcal{S}_{\Lambda'}^{\beta'}$ is a quasi-hereditary graded cellular algebra with weight poset $(\mathscr{P}_{\beta'}^{\Lambda'}, \trianglelefteq)$. Alternatively, this can be proved by applying $\mathtt{sgn}$ to Theorem 4.25. Applying $\mathtt{sgn}$ to the graded Weyl modules and graded simple modules for $\mathcal{S}_{\Lambda'}^{\beta'}$-modules we deduce the following result which should be compared with Corollary 3.27.

4.39. **Theorem.** *Suppose that $\beta \in Q_n^+$. The sign isomorphism $\mathtt{sgn}:\mathcal{R}_\beta^\Lambda \longrightarrow \mathcal{R}_{\beta'}^{\Lambda'}$ induces a canonical degree persevering, poset reversing, isomorphism of quasi-hereditary graded cellular algebras $\mathtt{sgn}:\mathcal{S}_\beta^\Lambda \longrightarrow \mathcal{S}_{\Lambda'}^{\beta'}$. Moreover, we have isomorphisms*

$$\Delta^{\boldsymbol{\mu}} \cong \Delta_{\boldsymbol{\mu}'}^{\mathtt{sgn}} \quad \textit{and} \quad L^{\boldsymbol{\mu}} \cong L_{\boldsymbol{\mu}'}^{\mathtt{sgn}}$$

of $\mathcal{S}_\beta^\Lambda$-modules, for $\boldsymbol{\mu} \in \mathscr{P}_\beta^\Lambda$. Consequently,

$$[\Delta^{\boldsymbol{\lambda}} : L^{\boldsymbol{\mu}}]_q = [\Delta_{\boldsymbol{\lambda}'} : L_{\boldsymbol{\mu}'}]_q,$$

for all $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \mathscr{P}_\beta^\Lambda$.

5. Tilting modules

In this chapter we introduce the tilting modules for $\mathcal{S}_n^\Lambda$, and the closely related Young modules for $\mathcal{R}_n^\Lambda$, which play an important role in the following chapters. Throughout this chapter we maintain our standing assumption 4.1.

5.1. **Young modules.** In this section we show that there exists a family of indecomposable $\mathcal{R}_n^\Lambda$-modules indexed by $\mathscr{P}_n^\Lambda$ and that $G^{\boldsymbol{\mu}}$ is a direct sum of these modules, for each $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$.

Fix $\beta \in Q_n^+$ and recall from (4.26) that every $\mathcal{S}_\beta^\Lambda$-module has a weight space decomposition. Analogously, as a right $\mathcal{S}_\beta^\Lambda$-module, the regular representation of $\mathcal{S}_\beta^\Lambda$ has a decomposition into a direct sum of left weight spaces:

$$\mathcal{S}_\beta^\Lambda = \bigoplus_{\boldsymbol{\mu} \in \mathscr{P}_\beta^\Lambda} Z^{\boldsymbol{\mu}}, \quad \text{where } Z^{\boldsymbol{\mu}} = \Psi^{\boldsymbol{\mu}} \mathcal{S}_\beta^\Lambda \text{ for } \boldsymbol{\mu} \in \mathscr{P}_\beta^\Lambda. \tag{5.1}$$

Since $\Psi^{\boldsymbol{\mu}}$ is an idempotent in $\mathcal{S}_\beta^\Lambda$, each weight space $Z^{\boldsymbol{\mu}}$ is a projective $\mathcal{S}_\beta^\Lambda$-module.

Let $P^{\boldsymbol{\mu}}$ be the projective cover of $L^{\boldsymbol{\mu}}$ (in the category of graded $\mathcal{S}_\beta^\Lambda$-modules). By the theory of (graded) cellular algebras, $P^{\boldsymbol{\mu}}$ has a filtration by Weyl modules such that $\Delta^{\boldsymbol{\lambda}}$ appears with graded multiplicity $[\Delta^{\boldsymbol{\lambda}} : L^{\boldsymbol{\mu}}]_q$. On the other hand, $Z^{\boldsymbol{\mu}}$ has basis $\{\, \Psi_{\mathfrak{s}\mathfrak{t}}^{\boldsymbol{\mu}\boldsymbol{\nu}} \mid (\boldsymbol{\mu}, \mathfrak{s}), (\boldsymbol{\nu}, \mathfrak{t}) \in \mathcal{T}^{\boldsymbol{\lambda}} \text{ and } \boldsymbol{\lambda} \in \mathscr{P}_\beta^\Lambda \,\}$.

Write $\operatorname{Std}^{\boldsymbol{\mu}}(\mathscr{P}_\beta^\Lambda) = \{\mathfrak{s}_1, \dots, \mathfrak{s}_z\}$, ordered so that $a > b$ whenever $\boldsymbol{\lambda}_a \rhd \boldsymbol{\lambda}_b$, where $\boldsymbol{\lambda}_c = \operatorname{Shape}(\mathfrak{s}_c)$. In particular, $\mathfrak{s}_1 = \mathfrak{t}^{\boldsymbol{\mu}}$. If $a \geq 1$ let M_a be the $\mathcal{Z}$-submodule of $Z^{\boldsymbol{\mu}}$ spanned by the elements $\{\, \Psi_{\mathfrak{s}_b\mathfrak{t}}^{\boldsymbol{\mu}\boldsymbol{\nu}} \mid \mathfrak{t} \in \operatorname{Std}^{\boldsymbol{\nu}}(\boldsymbol{\lambda}_b) \text{ for } \boldsymbol{\nu} \in \mathscr{P}_\beta^\Lambda \text{ and } b \geq a \,\}$. By Theorem 4.20, and Definition 2.4(GC$_2$),

$$Z^{\boldsymbol{\mu}} = M_1 \supset M_2 \supset \cdots \supset M_z \supset 0 \tag{5.2}$$

is an $\mathcal{S}_\beta^\Lambda$-module filtration of $Z^{\boldsymbol{\mu}}$ with $M_a/M_{a+1} \cong \Delta^{\boldsymbol{\lambda}_a}\langle \deg \mathfrak{s}_a - \deg \mathfrak{t}^{\boldsymbol{\mu}} \rangle$, for $1 \leq a \leq z$. Thus, in the notation of section 2.1, $Z^{\boldsymbol{\mu}}$ has a Δ-filtration in which $\Delta^{\boldsymbol{\lambda}}$ appears with graded multiplicity

$$[Z^{\boldsymbol{\mu}} : \Delta^{\boldsymbol{\lambda}}]_q := \sum_{\mathfrak{s} \in \operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda})} q^{\deg \mathfrak{s} - \deg \mathfrak{t}^{\boldsymbol{\mu}}}. \tag{5.3}$$

Since $\mathcal{S}_\beta^\Lambda$ is quasi-hereditary $[Z^{\boldsymbol{\mu}} : \Delta^{\boldsymbol{\lambda}}]_q$ is independent of the choice of Δ-filtration.

By the last paragraph $[Z^{\boldsymbol{\mu}} : \Delta^{\boldsymbol{\mu}}]_q = 1$ and there is a surjection $Z^{\boldsymbol{\mu}} \twoheadrightarrow \Delta^{\boldsymbol{\mu}}$. Moreover, $\Delta^{\boldsymbol{\lambda}}$ appears in $Z^{\boldsymbol{\mu}}$ only if $\boldsymbol{\lambda} \unrhd \boldsymbol{\mu}$. Therefore, since $Z^{\boldsymbol{\mu}}$ is projective, it follows that

$$Z^{\boldsymbol{\mu}} = P^{\boldsymbol{\mu}} \oplus \bigoplus_{\boldsymbol{\lambda} \rhd \boldsymbol{\mu}} p_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) P^{\boldsymbol{\lambda}}, \tag{5.4}$$

for some Laurent polynomials $p_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in \mathbb{N}[q, q^{-1}]$.

5.5. **Definition.** Suppose that $\boldsymbol{\mu} \in \mathscr{P}_\beta^\Lambda$. The **graded Young modules** are the $\mathcal{R}_\beta^\Lambda$-modules $Y^{\boldsymbol{\mu}} = \mathsf{F}_\beta^\Lambda(P^{\boldsymbol{\mu}})$ and $Y_{\boldsymbol{\mu}} = (Y^{\boldsymbol{\mu}'})^{\mathtt{sgn}}$, where $Y^{\boldsymbol{\mu}'}$ is a Young module for $\mathcal{S}_{\Lambda'}^{\beta'}$.

The next result gives some justification for this terminology. In Lemma 6.11 below we will show that the graded Young modules are graded lifts of the Young modules for $\mathcal{H}_n^\Lambda$ introduced in [36].

5.6. **Proposition.** *Suppose that $\beta \in Q_n^+$ and that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_\beta^\Lambda$. Then:*

a) *$Y^{\boldsymbol{\mu}}$ and $Y_{\boldsymbol{\mu}}$ are indecomposable $\mathcal{R}_\beta^\Lambda$-modules.*
b) *If $d \in \mathbb{Z}$ then $Y^{\boldsymbol{\mu}} \cong Y^{\boldsymbol{\nu}}\langle d\rangle$ if and only if $\boldsymbol{\lambda} = \boldsymbol{\mu}$ and $d = 0$. Similarly, $Y_{\boldsymbol{\mu}} \cong Y_{\boldsymbol{\lambda}}\langle d\rangle$ if and only if $\boldsymbol{\lambda} = \boldsymbol{\mu}$ and $d = 0$.*
c) *$G^{\boldsymbol{\mu}} \cong Y^{\boldsymbol{\mu}} \oplus \bigoplus_{\boldsymbol{\lambda} \rhd \boldsymbol{\mu}} p_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) Y^{\boldsymbol{\lambda}}$ and $G_{\boldsymbol{\mu}} \cong Y_{\boldsymbol{\mu}} \oplus \bigoplus_{\boldsymbol{\lambda} \unlhd \boldsymbol{\mu}} p_{\boldsymbol{\lambda}'\boldsymbol{\mu}'}(q) Y_{\boldsymbol{\lambda}}$.*
d) *$Y^{\boldsymbol{\mu}}$ has a graded Specht filtration in which $S^{\boldsymbol{\lambda}}$ appears with graded multiplicity*
$$(Y^{\boldsymbol{\mu}} : S^{\boldsymbol{\lambda}})_q = [\Delta^{\boldsymbol{\lambda}} : L^{\boldsymbol{\mu}}]_q$$
and $Y_{\boldsymbol{\mu}}$ has a dual graded Specht filtration in which $S_{\boldsymbol{\lambda}}$ appears with graded multiplicity
$$(Y_{\boldsymbol{\mu}} : S_{\boldsymbol{\lambda}})_q = [\Delta_{\boldsymbol{\lambda}} : L_{\boldsymbol{\mu}}]_q.$$
e) *if $\boldsymbol{\mu} \in \mathcal{K}_\beta^\Lambda$ then $Y^{\boldsymbol{\mu}}$ is the projective cover of $D^{\boldsymbol{\mu}}$.*

Proof. By Corollary 4.38, the functor F_β^Λ is fully faithful on projective modules, so $\operatorname{END}_{\mathcal{R}_\beta^\Lambda}(Y^{\boldsymbol{\mu}}) \cong \operatorname{END}_{\mathcal{S}_\beta^\Lambda}(P^{\boldsymbol{\mu}})$ is a local ring since $P^{\boldsymbol{\mu}}$ is indecomposable. Hence, $Y^{\boldsymbol{\mu}}$ and $Y_{\boldsymbol{\mu}}$ are indecomposable $\mathcal{R}_\beta^\Lambda$-modules. Moreover, the fact that F_β^Λ is fully faithful on projectives also implies (b) since the $P^{\boldsymbol{\mu}}\langle d\rangle$ are pairwise non-isomorphic.

Applying the Schur functor from Proposition 4.31,
$$\mathsf{F}_\beta^\Lambda(Z^{\boldsymbol{\mu}}) = \Psi^{\boldsymbol{\mu}} \dot{\mathcal{S}}_\beta^\Lambda \Psi^\omega \cong \operatorname{HOM}_{\mathcal{R}_\beta^\Lambda}(\mathcal{R}_\beta^\Lambda, G^{\boldsymbol{\mu}}) \cong G^{\boldsymbol{\mu}}.$$
Hence, part (c) follows from (5.4).

Now consider (d). If $\boldsymbol{\mu} \in \mathscr{P}_\beta^\Lambda$ then $L^{\boldsymbol{\mu}} \neq 0$ by Theorem 4.25. Therefore,
$$[P^{\boldsymbol{\mu}} : \Delta^{\boldsymbol{\lambda}}]_q = [\Delta^{\boldsymbol{\lambda}} : L^{\boldsymbol{\mu}}]_q$$
by Corollary 2.8, for $\boldsymbol{\lambda} \in \mathscr{P}_\beta^\Lambda$. In particular, the multiplicity $[P^{\boldsymbol{\mu}} : \Delta^{\boldsymbol{\lambda}}]_q$ depends only on $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. Since F_β^Λ is exact, and using Corollary 3.27,
$$(Y_{\boldsymbol{\mu}} : S_{\boldsymbol{\lambda}})_q = (Y^{\boldsymbol{\mu}'} : S^{\boldsymbol{\lambda}'})_q = [\Delta^{\boldsymbol{\lambda}'} : L^{\boldsymbol{\mu}'}]_q = [\Delta_{\boldsymbol{\lambda}} : L_{\boldsymbol{\mu}}]_q,$$
where the last equality comes from Theorem 4.39. Hence, (d) holds. Note that we are not claiming that the graded Specht filtration multiplicities for $Y^{\boldsymbol{\mu}}$ are independent of the choice of filtration.

Finally, part (e) follows by the exactness of F_β^Λ and Proposition 4.31 because $P^{\boldsymbol{\mu}}$ is the projective cover of $L^{\boldsymbol{\mu}}$. □

5.7. *Remark.* The Laurent polynomials $p_{\boldsymbol{\lambda}'\boldsymbol{\mu}'}(q)$ and $[\Delta_{\boldsymbol{\lambda}} : L_{\boldsymbol{\mu}}]_q$ in parts (c) of Proposition 5.6 must be computed using the algebra $\mathcal{S}_{\Lambda'}^{\beta'}$.

Using Theorem 4.15 to argue by induction on the dominance ordering we obtain the following.

5.8. **Corollary.** *Suppose that $\boldsymbol{\mu} \in \mathscr{P}_\beta^\Lambda$, for $\beta \in Q_n^+$. Then, as $\mathcal{R}_\beta^\Lambda$-modules*
$$(Y^{\boldsymbol{\mu}})^{\circledast} \cong Y^{\boldsymbol{\mu}}\langle -2\operatorname{def}\beta\rangle \quad \text{and} \quad (Y_{\boldsymbol{\mu}})^{\circledast} \cong Y_{\boldsymbol{\mu}}\langle -2\operatorname{def}\beta\rangle.$$

Proof. If $\boldsymbol{\mu}$ is maximal in $\mathscr{P}_\beta^\Lambda$ then $Y^{\boldsymbol{\mu}} = G^{\boldsymbol{\mu}}$ so in this case the result is a special case of Theorem 4.15. If $\boldsymbol{\mu}$ is not maximal then the result follows by induction on dominance using Proposition 5.6(b) and Theorem 4.15. □

Finally, we note that because $P^{\boldsymbol{\mu}}$ is the projective cover of $L^{\boldsymbol{\mu}}$, and because F_β^Λ is an exact functor, that we have the following.

5.9. **Corollary.** *Suppose that $\boldsymbol{\mu} \in \mathcal{K}_\beta^\Lambda$, for $\beta \in Q_n^+$. Then $Y^{\boldsymbol{\mu}}$ is the projective cover of $D^{\boldsymbol{\mu}}$.*

5.2. **Tilting modules.** By Theorem 4.25, $\mathcal{S}^\Lambda_\beta$ is a quasi-hereditary algebra. An $\mathcal{S}^\Lambda_\beta$-module T is a (graded) **tilting module** if it has both a filtration by shifted Weyl modules $\Delta^{\boldsymbol{\lambda}}\langle k\rangle$, for $\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_\beta$ and $k\in\mathbb{Z}$, and a filtration by the contragredient duals of shifted Weyl modules.

On forgetting the grading, Theorem 4.25 says that the ungraded algebra $\underline{\mathcal{S}}^\Lambda_n$ is quasi-hereditary. Therefore, by a famous theorem of Ringel [39], for each $\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_\beta$ there exists a unique $\underline{\mathcal{S}}^\Lambda_n$-module $\underline{T}^{\boldsymbol{\lambda}}$ such that

a) $\underline{T}^{\boldsymbol{\lambda}}$ is indecomposable.
b) $\underline{T}^{\boldsymbol{\lambda}}$ has both a $\underline{\Delta}$-filtration and a $\underline{\nabla}$-filtration.
c) $[\underline{T}^{\boldsymbol{\lambda}}:\underline{\Delta}^{\boldsymbol{\lambda}}]=1$ and $[\underline{T}^{\boldsymbol{\lambda}}:\underline{\Delta}^{\boldsymbol{\mu}}]\neq 0$ only if $\boldsymbol{\lambda}\geq\boldsymbol{\mu}$.

Ringel's construction (see the proof of [39, Lemma 3]), extends to the graded case to show that every tilting module for $\underline{\mathcal{S}}^\Lambda_n$ has a graded lift. Since $\underline{T}^{\boldsymbol{\mu}}$ is indecomposable it follows that there is a unique graded lift $T^{\boldsymbol{\mu}}$ of $\underline{T}^{\boldsymbol{\mu}}$ for which has a degree zero homomorphism $\Delta^{\boldsymbol{\mu}}\hookrightarrow T^{\boldsymbol{\mu}}$. The aim of this section is to show that $T^{\boldsymbol{\mu}}\cong(T^{\boldsymbol{\mu}})^{\circledast}$ is graded self-dual. To prove this we need another description of these modules.

Fix $\boldsymbol{\mu}\in\mathscr{P}^\Lambda_\beta$ and let $\theta_{\boldsymbol{\mu}}\in\operatorname{Hom}_{\mathcal{R}^\Lambda_\beta}(\mathcal{R}^\Lambda_\beta,G_{\boldsymbol{\mu}})$ be the map in $\dot{\mathcal{S}}^\Lambda_n$ given by

$$\theta_{\boldsymbol{\mu}}(h)=\psi'_{\mathfrak{t}_{\boldsymbol{\mu}}\mathfrak{t}_{\boldsymbol{\mu}}}h,\qquad\text{for all } h\in\mathcal{R}^\Lambda_\beta.$$

We define analogues of the exterior powers for $\mathcal{S}^\Lambda_\beta$ using the functor $\dot{\mathsf{F}}^\omega_n$ from (4.30).

5.10. **Definition.** Suppose that $\boldsymbol{\mu}\in\mathscr{P}^\Lambda_\beta$. Define $E^{\boldsymbol{\mu}}=\dot{\mathsf{F}}^\omega_n(\theta_{\boldsymbol{\mu}}\dot{\mathcal{S}}^\Lambda_n)\langle-\operatorname{def}\beta\rangle$.

Observe that $E^{\boldsymbol{\mu}}$ is a right $\mathcal{S}^\Lambda_\beta$-module under composition of maps because, by definition, $E^{\boldsymbol{\mu}}$ is the set of maps from G^Λ_n to $G_{\boldsymbol{\mu}}$ which factor through $\theta_{\boldsymbol{\mu}}$:

$$\begin{tikzcd} G^\Lambda_n \arrow[r, dashed, "\exists\theta'"] \arrow[dr, "\theta"'] & \mathcal{R}^\Lambda_\beta \arrow[d, "\theta_{\boldsymbol{\mu}}"] \\ & G_{\boldsymbol{\mu}} \end{tikzcd}$$

This is similar to the description of the Weyl module $\Delta^{\boldsymbol{\mu}}$ given in Remark 4.27.

Our first aim is to give a basis for $E^{\boldsymbol{\mu}}$. Notice that if $\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_\beta$, $\mathfrak{s}\in\operatorname{Std}_{\boldsymbol{\mu}}(\boldsymbol{\lambda})$ and $\mathfrak{t}\in\operatorname{Std}^{\boldsymbol{\nu}}(\boldsymbol{\lambda})$ then $\psi'_{\mathfrak{t}_{\boldsymbol{\mu}}\mathfrak{t}_{\boldsymbol{\mu}}}\psi_{\mathfrak{s}\mathfrak{t}}\in G_{\boldsymbol{\mu}}\cap(G^{\boldsymbol{\nu}})^\star$ by Corollary 4.13. Therefore, we can define $\theta^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{s}\mathfrak{t}}\in\operatorname{Hom}_{\mathcal{R}^\Lambda_\beta}(G^{\boldsymbol{\nu}},G_{\boldsymbol{\mu}})$ by

$$\theta^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{s}\mathfrak{t}}(e^{\boldsymbol{\nu}}y^{\boldsymbol{\nu}}h)=\psi'_{\mathfrak{t}_{\boldsymbol{\mu}}\mathfrak{t}_{\boldsymbol{\mu}}}\psi_{\mathfrak{s}\mathfrak{t}}h,$$

for all $h\in\mathcal{R}^\Lambda_\beta$.

5.11. **Theorem.** *Suppose that $\boldsymbol{\mu}\in\mathscr{P}^\Lambda_\beta$, for $\beta\in Q^+_n$. Then*

$$\{\,\theta^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{s}\mathfrak{t}}\mid(\boldsymbol{\mu},\mathfrak{s})\in\mathcal{T}_{\boldsymbol{\lambda}}\text{ and }(\boldsymbol{\nu},\mathfrak{t})\in\mathcal{T}^{\boldsymbol{\lambda}}\text{ for some }\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_\beta\,\}$$

is a basis of $E^{\boldsymbol{\mu}}$. Moreover, considered as an element of $E^{\boldsymbol{\mu}}$,

$$\deg\theta^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{s}\mathfrak{t}}=\deg\mathfrak{s}-\deg\mathfrak{t}_{\boldsymbol{\mu}}+\deg\mathfrak{t}-\deg\mathfrak{t}^{\boldsymbol{\nu}}.$$

Proof. Let $\dot{E}^{\boldsymbol{\mu}}=\theta_{\boldsymbol{\mu}}\dot{\mathcal{S}}^\Lambda_n$. Then $\dot{E}^{\boldsymbol{\mu}}$ is a right $\dot{\mathcal{S}}^\Lambda_n$-module under composition of maps and $E^{\boldsymbol{\mu}}=\dot{\mathsf{F}}^\omega_n(\dot{E}^{\boldsymbol{\mu}})$. By Proposition 4.28, $\dot{E}^{\boldsymbol{\mu}}$ is spanned by the maps $\theta_{\boldsymbol{\mu}}\Psi^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{s}\mathfrak{t}}$, for $\boldsymbol{\mu},\boldsymbol{\nu}\in\dot{\mathscr{P}}^\Lambda_n$, $\mathfrak{s}\in\operatorname{Std}_{\boldsymbol{\mu}}(\boldsymbol{\lambda})$, $\mathfrak{t}\in\operatorname{Std}^{\boldsymbol{\nu}}(\boldsymbol{\lambda})$, and $\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_\beta$. By definition, we have that

$$\theta_{\boldsymbol{\mu}}\Psi^{\boldsymbol{\mu}\boldsymbol{\nu}}_{\mathfrak{s}\mathfrak{t}}(e^{\boldsymbol{\nu}}y^{\boldsymbol{\nu}}h)=\delta_{\boldsymbol{\mu}\omega}\psi'_{\mathfrak{t}_{\boldsymbol{\mu}}\mathfrak{t}_{\boldsymbol{\mu}}}\psi_{\mathfrak{s}\mathfrak{t}}h.$$

Therefore, applying Lemma 3.15, $\theta_{\boldsymbol{\mu}}\Psi^{\boldsymbol{\mu\nu}}_{\mathfrak{st}}$ is non-zero only if $\boldsymbol{\mu}=\omega$, $\operatorname{res}(\mathfrak{s})=\operatorname{res}(\mathfrak{t}_{\boldsymbol{\mu}})$ and $\mathfrak{t}_{\boldsymbol{\mu}}\trianglerighteq\mathfrak{s}$. Moreover, in this case, $\theta_{\boldsymbol{\mu}}\Psi^{\boldsymbol{\mu\nu}}_{\mathfrak{st}}=\theta^{\boldsymbol{\mu\nu}}_{\mathfrak{st}}$. Therefore, the elements

$$\{\,\theta^{\boldsymbol{\mu\nu}}_{\mathfrak{st}}\mid \mathfrak{s}\in\operatorname{Std}_{\boldsymbol{\mu}}(\boldsymbol{\lambda})\text{ and }\mathfrak{t}\in\operatorname{Std}^{\boldsymbol{\nu}}(\boldsymbol{\lambda})\text{ for some }\boldsymbol{\nu}\in\dot{\mathscr{P}}^{\Lambda}_{n}\text{ and }\boldsymbol{\lambda}\in\mathscr{P}^{\Lambda}_{\beta}\,\}$$

span $\dot{E}^{\boldsymbol{\mu}}$. On the other hand, these elements are linearly independent because $\{\theta^{\boldsymbol{\mu\nu}}_{\mathfrak{st}}(e^{\boldsymbol{\nu}}y^{\boldsymbol{\nu}})\}$ is a linearly independent subset of $G_{\boldsymbol{\mu}}$ by Corollary 4.13(a). Hence, we have found a basis for $\dot{E}^{\boldsymbol{\mu}}$. Applying the functor $\dot{\mathsf{F}}^{\omega}_{n}$ kills the ω-weight space of $\dot{E}^{\boldsymbol{\mu}}$. So $\dot{\mathsf{F}}^{\omega}_{n}$ maps the basis we have found for $E^{\boldsymbol{\mu}}$ to the elements in the statement of the Theorem.

Finally, if $(\boldsymbol{\mu},\mathfrak{s}),(\boldsymbol{\nu},\mathfrak{t})\in\mathcal{T}^{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda}\in\mathscr{P}^{\Lambda}_{\beta}$, it remains to compute $\deg\theta^{\boldsymbol{\mu\nu}}_{\mathfrak{st}}$ when $\theta^{\boldsymbol{\mu\nu}}_{\mathfrak{st}}$ is considered as an element of $E^{\boldsymbol{\mu}}$. Recalling the degree shifts in the definition of the three modules $G^{\boldsymbol{\nu}}$, $G_{\boldsymbol{\mu}}$ and $E^{\boldsymbol{\mu}}$, we find that

$$\deg\theta^{\boldsymbol{\mu\nu}}_{\mathfrak{st}}=\operatorname{codeg}\mathfrak{t}_{\boldsymbol{\mu}}+\deg\mathfrak{s}+\deg\mathfrak{t}-\deg\mathfrak{t}^{\boldsymbol{\nu}}-\operatorname{def}\beta.$$

Applying Lemma 3.10 this is equal to the expression in the statement of the theorem. □

Let $\operatorname{Std}_{\boldsymbol{\mu}}(\mathscr{P}^{\Lambda}_{\beta})=\{\mathfrak{s}_1,\dots,\mathfrak{s}_y\}$ ordered so that $a>b$ whenever $\boldsymbol{\lambda}_{\mathfrak{a}}\vartriangleright\boldsymbol{\lambda}_b$, where we set $\boldsymbol{\lambda}_c=\operatorname{Shape}(\mathfrak{s}_c)$ for $1\le c\le y$. (Thus, $\mathfrak{s}_y=\mathfrak{t}_{\boldsymbol{\mu}}$.) The proof of Theorem 5.11 shows that $\theta^{\boldsymbol{\mu\nu}}_{\mathfrak{st}}=\theta_{\boldsymbol{\mu}}\Psi^{\omega\boldsymbol{\nu}}_{\mathfrak{st}}$, so arguing as in (5.2) we obtain the following.

5.12. **Corollary.** *Suppose that $\boldsymbol{\mu}\in\mathscr{P}^{\Lambda}_{\beta}$. Then $E^{\boldsymbol{\mu}}$ has a Δ-filtration*

$$E^{\boldsymbol{\mu}}=E_1>E_2>\dots>E_y>0,$$

such that $E_r/E_{r+1}\cong\Delta^{\boldsymbol{\lambda}_r}\langle\deg\mathfrak{s}_r-\deg\mathfrak{t}_{\boldsymbol{\mu}}\rangle$, where $\boldsymbol{\nu}_r=\operatorname{Shape}(\mathfrak{s}_r)$, for $1\le r\le y$. In particular, $\Delta^{\boldsymbol{\mu}}$ is a submodule of $E^{\boldsymbol{\mu}}$, $[E^{\boldsymbol{\mu}}:\Delta^{\boldsymbol{\mu}}]_q=1$ and $[E^{\boldsymbol{\mu}}:\Delta^{\boldsymbol{\lambda}}]_q\ne0$ only if $\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}$.

We now give a second basis of $E^{\boldsymbol{\mu}}$ and use it show that $E^{\boldsymbol{\mu}}$ is a tilting module. Suppose that $\mathfrak{u}\in\operatorname{Std}_{\boldsymbol{\mu}}(\boldsymbol{\nu})$ and $\mathfrak{v}\in\operatorname{Std}^{\boldsymbol{\lambda}}(\boldsymbol{\nu})$, for $\boldsymbol{\lambda},\boldsymbol{\nu}\in\mathscr{P}^{\Lambda}_{\beta}$. Then $\psi'_{\mathfrak{uv}}\psi_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}\in G_{\boldsymbol{\mu}}\cap(G^{\boldsymbol{\lambda}})^{\star}$ by Corollary 4.12. Therefore, we can define $\theta^{\mathfrak{uv}}_{\boldsymbol{\mu\lambda}}\in\operatorname{HOM}_{\mathcal{R}^{\Lambda}_{\beta}}(G^{\boldsymbol{\lambda}},G_{\boldsymbol{\mu}})$ by

$$\theta^{\mathfrak{uv}}_{\boldsymbol{\mu\lambda}}(e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}h)=\psi'_{\mathfrak{uv}}\psi_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}h,$$

for $h\in\mathcal{R}^{\Lambda}_{\beta}$.

5.13. **Lemma.** *Suppose that $\mathcal{Z}=K$ is a field and that $\boldsymbol{\mu}\in\mathscr{P}^{\Lambda}_{\beta}$. Then*

$$\{\,\theta^{\mathfrak{uv}}_{\boldsymbol{\mu\nu}}\mid(\boldsymbol{\mu},\mathfrak{u})\in\mathcal{T}_{\boldsymbol{\lambda}},(\boldsymbol{\nu},\mathfrak{v})\in\mathcal{T}^{\boldsymbol{\lambda}}\text{ for some }\boldsymbol{\lambda}\in\mathscr{P}^{\Lambda}_{\beta}\,\}$$

is a basis of $E^{\boldsymbol{\mu}}$. Moreover, considered as an element of $E^{\boldsymbol{\mu}}$,

$$\deg\theta^{\mathfrak{uv}}_{\boldsymbol{\mu\nu}}=\operatorname{codeg}\mathfrak{u}-\operatorname{codeg}\mathfrak{t}_{\boldsymbol{\mu}}+\operatorname{codeg}\mathfrak{v}-\deg\mathfrak{t}^{\boldsymbol{\nu}}.$$

Proof. We first show that $\theta^{\mathfrak{uv}}_{\boldsymbol{\mu\nu}}\in E^{\boldsymbol{\mu}}$ whenever $\mathfrak{u}\in\operatorname{Std}_{\boldsymbol{\mu}}(\boldsymbol{\lambda})$ and $\mathfrak{v}\in\operatorname{Std}^{\boldsymbol{\nu}}(\boldsymbol{\lambda})$, for some $\boldsymbol{\lambda}\in\mathscr{P}^{\Lambda}_{\beta}$. By Theorem 4.10, $\psi'_{\mathfrak{uv}}=\psi'_{\mathfrak{t}_{\boldsymbol{\mu}}\mathfrak{t}_{\boldsymbol{\mu}}}x$, for some $x\in\mathcal{R}^{\Lambda}_{\beta}$. Therefore,

$$\theta^{\mathfrak{uv}}_{\boldsymbol{\mu\nu}}(e^{\boldsymbol{\nu}}y^{\boldsymbol{\nu}}h)=\psi'_{\mathfrak{uv}}\psi_{\mathfrak{t}^{\boldsymbol{\nu}}\mathfrak{t}^{\boldsymbol{\nu}}}h=\psi'_{\mathfrak{t}_{\boldsymbol{\mu}}\mathfrak{t}_{\boldsymbol{\mu}}}x\psi_{\mathfrak{t}^{\boldsymbol{\nu}}\mathfrak{t}^{\boldsymbol{\nu}}}h=\theta_{\boldsymbol{\mu}}(x\psi_{\mathfrak{t}^{\boldsymbol{\nu}}\mathfrak{t}^{\boldsymbol{\nu}}}h).$$

That is, $\theta^{\mathfrak{uv}}_{\boldsymbol{\mu\nu}}$ factors through $\theta_{\boldsymbol{\mu}}$ so that $\theta^{\mathfrak{uv}}_{\boldsymbol{\mu\nu}}\in E^{\boldsymbol{\mu}}$ as claimed. The elements in the statement of the theorem are linearly independent because $\{\theta^{\mathfrak{uv}}_{\boldsymbol{\mu\nu}}(e^{\boldsymbol{\nu}}y^{\boldsymbol{\nu}})\}$ is a linearly independent subset of $G_{\boldsymbol{\mu}}$ by applying $\star$ to Theorem 4.13(a). Therefore, since we are working over a field, we see that we have a basis by counting dimensions using Theorem 5.11.

Finally, the degree of $\theta^{\mathfrak{uv}}_{\boldsymbol{\mu\nu}}$ is easy to compute using Lemma 3.10 as in the last paragraph of the proof of Theorem 5.11. □

Notice that unlike Theorem 5.11, the basis of Lemma 5.13 does not obviously yield a Δ-filtration of $E^{\boldsymbol\mu}$ because it is not clear how to write the basis elements $\theta^{\mathfrak{u}\mathfrak{v}}_{\boldsymbol\mu\boldsymbol\nu}(e^{\boldsymbol\nu}y^{\boldsymbol\nu})$ in terms of the cellular basis of $\mathcal{S}^\Lambda_\beta$. By appealing to Theorem 4.39 it is possible to construct a ∇-filtration of $E^{\boldsymbol\mu}$ using the basis of Lemma 5.13. The existence of a ∇-filtration is also implied by the next result.

5.14. **Theorem.** *Suppose that $\mathcal{Z} = K$ is a field and that $\boldsymbol\mu \in \mathscr{P}^\Lambda_\beta$, for $\beta \in Q^+_n$. Then $E^{\boldsymbol\mu} \cong (E^{\boldsymbol\mu})^{\circledast}$.*

Proof. Using the two bases of $E^{\boldsymbol\mu}$ given by Theorem 5.11 and Lemma 5.13, define $\langle\!\langle\ ,\ \rangle\!\rangle_{\boldsymbol\mu} : E^{\boldsymbol\mu} \times E^{\boldsymbol\mu} \longrightarrow K$ to be the unique bilinear map such that

$$\langle\!\langle\theta^{\boldsymbol\mu\boldsymbol\nu}_{\mathfrak{s}\mathfrak{t}}, \theta^{\mathfrak{u}\mathfrak{v}}_{\boldsymbol\mu\boldsymbol\tau}\rangle\!\rangle_{\boldsymbol\mu} = \tau_\beta(\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{v}\mathfrak{u}}),$$

for $(\boldsymbol\mu,\mathfrak{s}) \in \mathcal{T}_{\boldsymbol\lambda}$, $(\boldsymbol\nu,\mathfrak{t}) \in \mathcal{T}^{\boldsymbol\lambda}$, $(\boldsymbol\mu,\mathfrak{u}) \in \mathcal{T}_{\boldsymbol\rho}$ and $(\boldsymbol\tau,\mathfrak{v}) \in \mathcal{T}^{\boldsymbol\rho}$ for some $\boldsymbol\lambda,\boldsymbol\rho \in \mathscr{P}^\Lambda_\beta$. By Theorem 3.20, $\langle\!\langle\theta^{\boldsymbol\mu\boldsymbol\nu}_{\mathfrak{s}\mathfrak{t}}, \theta^{\mathfrak{s}\mathfrak{t}}_{\boldsymbol\mu\boldsymbol\tau}\rangle\!\rangle_{\boldsymbol\mu} \ne 0$ and $\langle\!\langle\theta^{\boldsymbol\mu\boldsymbol\nu}_{\mathfrak{s}\mathfrak{t}}, \theta^{\mathfrak{u}\mathfrak{v}}_{\boldsymbol\mu\boldsymbol\tau}\rangle\!\rangle_{\boldsymbol\mu} \ne 0$ only if $(\mathfrak{u},\mathfrak{v}) \mathrel{\underline{\blacktriangleright}} (\mathfrak{s},\mathfrak{t})$ and $\deg(\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{v}\mathfrak{u}}) = 2\operatorname{def}\beta$. Therefore, $\langle\!\langle\ ,\ \rangle\!\rangle_{\boldsymbol\mu}$ is a non-degenerate bilinear form.

We claim that the bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle_{\boldsymbol\mu}$ is homogeneous. To see this suppose that $\langle\!\langle\theta^{\boldsymbol\mu\boldsymbol\lambda}_{\mathfrak{s}\mathfrak{t}}, \theta^{\mathfrak{u}\mathfrak{v}}_{\boldsymbol\mu\boldsymbol\tau}\rangle\!\rangle_{\boldsymbol\mu} \ne 0$, for basis elements as above. Then $\deg(\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{v}\mathfrak{u}}) = 2\operatorname{def}\beta$ since τ_β is homogeneous of degree $-2\operatorname{def}\beta$. Using the degree formulae in Theorem 5.11 and Lemma 5.13, together with Lemma 3.10, this implies that $\deg\theta^{\boldsymbol\mu\boldsymbol\lambda}_{\mathfrak{s}\mathfrak{t}} + \deg\theta^{\mathfrak{u}\mathfrak{v}}_{\boldsymbol\mu\boldsymbol\tau} = 0$. Hence, $\langle\!\langle\ ,\ \rangle\!\rangle_{\boldsymbol\mu}$ is a homogeneous bilinear form of degree zero.

To complete the proof it is enough to show that the form $\langle\!\langle\ ,\ \rangle\!\rangle_{\boldsymbol\mu}$ is associative because then the map which sends $\theta^{\boldsymbol\mu\boldsymbol\lambda}_{\mathfrak{s}\mathfrak{t}}$ to the function $x \mapsto \langle\!\langle\theta^{\boldsymbol\mu\boldsymbol\lambda}_{\mathfrak{s}\mathfrak{t}}, x\rangle\!\rangle_{\boldsymbol\mu}$ is an $\mathcal{S}^\Lambda_\beta$-module homomorphism. This can be proved by essentially repeating the argument from the proof of Theorem 4.15. ∎

5.15. **Corollary.** *Suppose that $\mathcal{Z} = K$ is a field and that $\boldsymbol\mu \in \mathscr{P}^\Lambda_\beta$. Then $E^{\boldsymbol\mu}$ is a tilting module. Moreover,*

$$E^{\boldsymbol\mu} = T^{\boldsymbol\mu} \oplus \bigoplus_{\boldsymbol\lambda \rhd \boldsymbol\mu} t_{\boldsymbol\lambda\boldsymbol\mu}(q) T^{\boldsymbol\lambda},$$

for some Laurent polynomials $t_{\boldsymbol\lambda\boldsymbol\mu}(q) \in \mathbb{N}[q,q^{-1}]$ such that $t_{\boldsymbol\lambda\boldsymbol\mu}(q) = t_{\boldsymbol\lambda\boldsymbol\mu}(q^{-1})$.

Proof. By Corollary 5.12 $E^{\boldsymbol\mu}$ has a Δ-filtration. Therefore, since $E^{\boldsymbol\mu} \cong (E^{\boldsymbol\mu})^{\circledast}$ it also as a ∇-filtration. Hence, $E^{\boldsymbol\mu}$ is a tilting module so that $E^{\boldsymbol\mu}$ has a unique decomposition into a direct sum of tilting modules. By Corollary 5.12 $T^{\boldsymbol\lambda}$ is a summand of $E^{\boldsymbol\mu}$ only if $\boldsymbol\lambda \trianglerighteq \boldsymbol\mu$ and, moreover, $T^{\boldsymbol\mu}$ appears with multiplicity one. Therefore, $E^{\boldsymbol\mu} = T^{\boldsymbol\mu} \oplus \bigoplus_{\boldsymbol\lambda \rhd \boldsymbol\mu} t_{\boldsymbol\mu\boldsymbol\lambda}(q) T^{\boldsymbol\lambda}$ for some polynomials $t_{\boldsymbol\lambda\boldsymbol\mu}(q) \in \mathbb{N}[q,q^{-1}]$. Finally, $t_{\boldsymbol\lambda\boldsymbol\mu}(q) = t_{\boldsymbol\lambda\boldsymbol\mu}(q^{-1})$ since $E^{\boldsymbol\mu}$ is (graded) self-dual and because $T^{\boldsymbol\lambda} \cong T^{\boldsymbol\nu}\langle d\rangle$ only if $\boldsymbol\lambda = \boldsymbol\nu$ and $d = 0$. ∎

Arguing by induction on dominance we obtain the main result of this section.

5.16. **Corollary.** *Suppose that $\boldsymbol\mu \in \mathscr{P}^\Lambda_\beta$. Then $(T^{\boldsymbol\mu})^{\circledast} \cong T^{\boldsymbol\mu}$.*

5.3. **Twisted tilting modules.** This section introduces the twisted tilting modules of $\mathcal{S}^\Lambda_\beta$ which will later play the role of the canonical bases in the Fock space. We start by discussing Ringel duality in the graded setting.

A **full tilting module** for $\mathcal{S}^\Lambda_\beta$ is a tilting module which contains every indecomposable tilting module, up to shift, as a direct summand. Hence,

$$E^\Lambda_\beta = \bigoplus_{\boldsymbol\mu \in \mathscr{P}^\Lambda_\beta} E^{\boldsymbol\mu}$$

is a full tilting module for $\mathcal{S}^\Lambda_\beta$. Define the **Ringel dual** of $\mathcal{S}^\Lambda_\beta$ to be the graded algebra $\textsc{End}_{\mathcal{S}^\Lambda_\beta}(E^\Lambda_\beta)$. (Strictly speaking, this is *a* Ringel dual of $\mathcal{S}^\Lambda_\beta$.)

Recall the graded Schur functor $\mathsf{F}^\Lambda_\beta : \mathcal{S}^\Lambda_\beta\text{-Mod} \longrightarrow \mathcal{R}^\Lambda_\beta\text{-Mod}$ from Proposition 4.31.

5.17. **Lemma.** *Suppose that* $\boldsymbol{\mu} \in \mathscr{P}^{\Lambda}_{\beta}$. *Then* $\mathsf{F}^{\Lambda}_{\beta}(E^{\boldsymbol{\mu}}) \cong G_{\boldsymbol{\mu}}\langle -\operatorname{def}\beta\rangle$ *as an* $\mathcal{R}^{\Lambda}_{\beta}$-*module.*

Proof. By Proposition 4.31 and Lemma 4.29 , and Definition 5.10,

$$\begin{aligned}\mathsf{F}^{\Lambda}_{\beta}(E^{\boldsymbol{\mu}}) = \mathsf{F}^{\Lambda}_{n}\big(\dot{\mathsf{F}}^{\omega}_{n}(\theta_{\boldsymbol{\mu}}\dot{\mathcal{S}}^{\Lambda}_{n})\langle -\operatorname{def}\beta\rangle\big) &= \dot{\mathsf{F}}^{\Lambda}_{n}(\theta_{\boldsymbol{\mu}}\dot{\mathcal{S}}^{\Lambda}_{n}\langle -\operatorname{def}\beta\rangle) = \theta_{\boldsymbol{\mu}}\dot{\mathcal{S}}^{\Lambda}_{n}\Psi^{\omega}\langle -\operatorname{def}\beta\rangle \\ &\cong \operatorname{Hom}_{\mathcal{R}^{\Lambda}_{n}}(\mathcal{R}^{\Lambda}_{n}, G_{\boldsymbol{\mu}})\langle -\operatorname{def}\beta\rangle \cong G_{\boldsymbol{\mu}}\langle -\operatorname{def}\beta\rangle,\end{aligned}$$

as required. □

5.18. **Corollary.** *Suppose that* $\boldsymbol{\mu} \in \mathscr{P}^{\Lambda}_{\beta}$. *Then* $\mathsf{F}^{\Lambda}_{\beta}(T^{\boldsymbol{\mu}}) \cong Y_{\boldsymbol{\mu}}\langle -\operatorname{def}\beta\rangle$ *as an* $\mathcal{R}^{\Lambda}_{\beta}$-*module.*

Proof. If $\boldsymbol{\mu}$ is maximal with respect to dominance in $\mathscr{P}^{\Lambda}_{\beta}$ then $E^{\boldsymbol{\mu}} = T^{\boldsymbol{\mu}}$ by Corollary 5.15 and $G_{\boldsymbol{\mu}} = Y_{\boldsymbol{\mu}}$ by Proposition 5.6(b), so the result is just Lemma 5.17 in this case. If $\boldsymbol{\mu}$ is not maximal in $\mathscr{P}^{\Lambda}_{\beta}$ the result follows by downwards induction on the dominance order using Lemma 5.17, Corollary 5.15 and Proposition 5.6(b). □

If A is an algebra let A^{op} be the opposite algebra which is obtained by reversing the order of multiplication.

5.19. **Theorem.** *Suppose that* $\beta \in Q^{+}_{n}$. *Then the Ringel dual of* $\mathcal{S}^{\Lambda}_{\beta}$ *is isomorphic to* $(\mathcal{S}^{\beta}_{\Lambda})^{op}$. *In particular,* $\operatorname{End}_{\mathcal{S}^{\Lambda}_{\beta}}(E^{\Lambda}_{\beta})$ *is a quasi-hereditary graded cellular algebra.*

Proof. There is an injection $\operatorname{Hom}_{\mathcal{R}^{\Lambda}_{\beta}}(G_{\boldsymbol{\mu}}, G_{\boldsymbol{\nu}}) \hookrightarrow \operatorname{Hom}_{\mathcal{S}^{\Lambda}_{\beta}}(E^{\boldsymbol{\mu}}, E^{\boldsymbol{\nu}})$ given by composition of maps, for $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}^{\Lambda}_{\beta}$. By Lemma 5.17 this map is surjective. Therefore, the Ringel dual of $\mathcal{S}^{\Lambda}_{\beta}$ is isomorphic, as a graded algebra, to $\operatorname{End}_{\mathcal{S}^{\Lambda}_{\beta}}(G^{\beta}_{\Lambda})^{op} = (\mathcal{S}^{\beta}_{\Lambda})^{op}$. □

By Theorems 4.39, $\mathcal{S}^{\beta}_{\Lambda} \cong \mathcal{S}^{\Lambda'}_{\beta'}$ as graded algebras. Note, however, that this is not an isomorphism of quasi-hereditary algebras because the isomorphism reverses the partial ordering on the standard and irreducible modules of these algebras.

By standard arguments (see, for example, [19, Lemma A4.6]), we have:

5.20. **Corollary.** *Suppose that* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^{\Lambda}_{\beta}$, *for* $\beta \in Q^{+}_{n}$. *Then*

$$[T^{\boldsymbol{\lambda}} : \nabla^{\boldsymbol{\mu}}]_q = [\Delta_{\boldsymbol{\mu}} : L_{\boldsymbol{\lambda}}]_q,$$

where $[\Delta_{\boldsymbol{\mu}} : L_{\boldsymbol{\lambda}}]_q$ *is a graded decomposition number for the sign-dual quiver Schur algebra* $\mathcal{S}^{\beta}_{\Lambda}$.

Even though we have been working with the tilting modules $T^{\boldsymbol{\mu}}$ throughout this chapter, it is actually the twisted tilting modules that we will need later. In analogy with Theorem 4.39 we make the following definition.

5.21. **Definition.** Suppose that $\boldsymbol{\lambda} \in \mathscr{P}^{\Lambda}_{\beta}$ and let $T^{\beta'}_{\boldsymbol{\lambda}'}$ be the self-dual tilting module for the sign dual quiver Schur algebra $\mathcal{S}^{\beta'}_{\Lambda'}$. The **twisted tilting module** is the $\mathcal{S}^{\Lambda}_{\beta}$-module $T_{\boldsymbol{\lambda}} = (T^{\beta'}_{\boldsymbol{\lambda}'})^{\mathtt{sgn}}$.

By Theorem 5.19, $T^{\beta'}_{\boldsymbol{\lambda}'}$ is the tilting module of the Ringel dual of $\mathcal{S}^{\Lambda'}_{\beta'}$.

5.22. **Proposition.** *Suppose that* $\beta \in Q^{+}_{n}$ *and* $\boldsymbol{\lambda} \in \mathscr{P}^{\Lambda}_{\beta}$, *for* $\beta \in Q^{+}_{n}$. *Then* $T_{\boldsymbol{\lambda}} \cong T^{\circledast}_{\boldsymbol{\lambda}}$ *is a self-dual tilting module and* $[T_{\boldsymbol{\lambda}} : \Delta^{\boldsymbol{\mu}}]_q = \overline{[\Delta^{\boldsymbol{\mu}'} : L^{\boldsymbol{\lambda}'}]_q}$, *for all* $\boldsymbol{\mu} \in \mathscr{P}^{\Lambda}_{\beta}$.

Proof. Using the definitions it is straightforward to check that there is an isomorphism of functors $\mathtt{sgn} \circ \circledast \cong \circledast \circ \mathtt{sgn}$ from $\mathcal{S}^{\beta'}_{\Lambda'}$-Mod to $\mathcal{S}^{\Lambda}_{\beta}$-Mod. Therefore,

$$(T_{\boldsymbol{\lambda}})^{\circledast} \cong (T^{\beta'}_{\boldsymbol{\lambda}'})^{\mathtt{sgn}\circ\circledast} \cong (T^{\beta'}_{\boldsymbol{\lambda}'})^{\circledast\circ\mathtt{sgn}} \cong (T^{\beta'}_{\boldsymbol{\lambda}'})^{\mathtt{sgn}} = T_{\boldsymbol{\lambda}},$$

by Corollary 5.16. For the second statement, using Theorem 4.39, if $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}^{\Lambda}_{\beta}$ then

$$[T_{\boldsymbol{\lambda}}:\nabla^{\boldsymbol{\mu}}]_q=[T^{\beta'}_{\boldsymbol{\lambda}'}:\nabla_{\boldsymbol{\mu}'}]_q=[\Delta^{\boldsymbol{\mu}'}:L^{\boldsymbol{\lambda}'}]_q,$$

where the last equality uses the analogue of Corollary 5.20 for $\mathcal{S}^{\beta'}_{\Lambda'}$. Taking duals, $[T_{\boldsymbol{\lambda}}:\Delta^{\boldsymbol{\mu}}]_q=\overline{[T^{\circledast}_{\boldsymbol{\lambda}}:\nabla^{\boldsymbol{\mu}}]_q}=\overline{[\Delta^{\boldsymbol{\mu}'}:L^{\boldsymbol{\lambda}'}]_q}$ as required. □

5.23. *Remark.* The uniqueness of self-dual tilting modules implies that there is an involution $\downarrow$ on the set of multipartitions such that $T_{\boldsymbol{\lambda}}\cong T^{\boldsymbol{\lambda}\downarrow}$, for all $\boldsymbol{\lambda}\in\mathscr{P}^{\Lambda}_{\beta}$. If $\boldsymbol{\lambda}\in\mathcal{K}^{\Lambda}_{\beta}$ then it follows from [12, Theorem 2.7] that $\boldsymbol{\lambda}\downarrow$ is given by the inverse *rectification map* of [12, (2.33)].

6. Cyclotomic Schur algebras

We are now ready to connect the quiver Schur algebras with the (ungraded) cyclotomic Hecke algebras introduced in [17] and [9, Theorem C].

6.1. **Cyclotomic permutation modules.** Throughout this section we work with the ungraded Hecke algebra $\mathcal{H}^{\Lambda}_n$. Consequently, as in Theorem 3.7, we assume that $\mathcal{Z}=K$ is a suitable field. If $w\in\mathfrak{S}_n$ define $T_w=T_{i_1}\dots T_{i_k}$, where $w=s_{i_1}\dots s_{i_k}$ is a reduced expression for w. Unlike the element $\psi_w\in\mathcal{R}^{\Lambda}_n$, T_w is independent of the choice of reduced expression for w.

Suppose that $\boldsymbol{\mu}\in\mathscr{P}^{\Lambda}_n$. Recall that if $1\le k\le n$ and $\mathfrak{t}=(\mathfrak{t}^{(1)},\dots,\mathfrak{t}^{(\ell)})$ is a tableau then $\operatorname{comp}_{\mathfrak{t}}(k)=s$ if k appears in $\mathfrak{t}^{(s)}$. Define $m^{\boldsymbol{\mu}}=u^{\boldsymbol{\mu}}x^{\boldsymbol{\mu}}$ where

$$u^{\boldsymbol{\mu}}=\prod_{k=1}^{n}\prod_{s=\operatorname{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(k)+1}^{\ell}(L_k-\xi^{(\kappa_s)})\quad\text{and}\quad x^{\boldsymbol{\mu}}=\sum_{w\in\mathfrak{S}_{\boldsymbol{\mu}}}T_w,$$

where $\xi^{(k)}$ is as defined in (3.5). These definitions reduce to [17, Definition 3.5] when $\xi\ne 1$ and to [9, (6.12)–(6.13)] when $\xi=1$.

6.1. **Definition** ([9,17]). Suppose that $\boldsymbol{\mu}\in\mathscr{P}^{\Lambda}_n$ and define $\underline{M}^{\boldsymbol{\mu}}=m^{\boldsymbol{\mu}}\mathcal{H}^{\Lambda}_n$.

We write $\underline{M}^{\boldsymbol{\mu}}$ rather than $M^{\boldsymbol{\mu}}$ to emphasize that $\underline{M}^{\boldsymbol{\mu}}$ is not (naturally) $\mathbb{Z}$-graded. We will not define a graded lift of $\underline{M}^{\boldsymbol{\mu}}$, instead, the aim of this section is to show that $\underline{G}^{\boldsymbol{\mu}}$ is a direct summand of $\underline{M}^{\boldsymbol{\mu}}$.

Remind the reader of our standing assumption that $e=0$ or $e\ge n$. This assumption is crucial for the next two results – and consequently for all of the results in this section.

6.2. **Lemma.** *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}^{\Lambda}_n$ and $1\ne w\in\mathfrak{S}_{\boldsymbol{\lambda}}$. Then $e^{\boldsymbol{\lambda}}\psi_we^{\boldsymbol{\lambda}}=0$.*

Proof. By Definition 3.2, $\psi_we^{\boldsymbol{\lambda}}=e(\mathbf{j})\psi_w$ where $\mathbf{j}=w\cdot\mathbf{i}^{\boldsymbol{\lambda}}$. Now, the assumption that $e=0$ or $e\ge n$ implies that all of the nodes in row a of $\lambda^{(l)}$ have pairwise distinct residues whenever $\lambda^{(l)}_a\ne 0$, for $a\ge 0$ and $1\le l\le\ell$. Consequently, $\mathbf{j}\ne\mathbf{i}^{\boldsymbol{\lambda}}$ since $1\ne w\in\mathfrak{S}_{\boldsymbol{\lambda}}$. Therefore, $e^{\boldsymbol{\lambda}}\psi_we^{\boldsymbol{\lambda}}=e^{\boldsymbol{\lambda}}e(\mathbf{j})\psi_w=0$. □

6.3. **Lemma.** *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}^{\Lambda}_n$. Then $e^{\boldsymbol{\lambda}}u^{\boldsymbol{\lambda}}=g^{\boldsymbol{\lambda}}(y)e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}$, where $g^{\boldsymbol{\lambda}}(y)$ is an invertible element of $K[y_1,\dots,y_n]$.*

Proof. We prove the Lemma only when $\xi\ne 1$ and leave the case when $\xi=1$, which is similar, to the reader. Write $\mathbf{i}^{\boldsymbol{\lambda}}=(i_1,\dots,i_n)$ and let $d^{\boldsymbol{\lambda}}_1,\dots,d^{\boldsymbol{\lambda}}_n$ be as defined in Definition 3.11, so that $d^{\boldsymbol{\lambda}}_r=\{\operatorname{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(r)<t\le\ell\mid i_r\equiv\kappa_t\pmod e\}$,

for $1 \le r \le n$. Then, using (3.8),

$$\begin{aligned}
e^{\boldsymbol{\lambda}} u^{\boldsymbol{\lambda}} &= \prod_{r=1}^{n} \prod_{t=\mathrm{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(r)+1}^{\ell} e^{\boldsymbol{\lambda}}(L_r - \xi^{\kappa_t}) \\
&= \prod_{r=1}^{n} \prod_{t=\mathrm{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(r)+1}^{\ell} e^{\boldsymbol{\lambda}}(\xi^{i_r} - \xi^{\kappa_t} - \xi^{i_r} y_r) \\
&= \prod_{r=1}^{n} (-\xi^{i_r})^{d_r^{\boldsymbol{\lambda}}} y_r^{d_r^{\boldsymbol{\lambda}}} \prod_{\substack{\mathrm{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(r) < t \le \ell \\ i_r \not\equiv \kappa_t \pmod{e}}} e^{\boldsymbol{\lambda}}(\xi^{i_r} - \xi^{\kappa_t} - \xi^{i_r} y_r) \\
&= e^{\boldsymbol{\lambda}} y^{\boldsymbol{\lambda}} \cdot \prod_{r=1}^{n} (-\xi^{i_r})^{d_r^{\boldsymbol{\lambda}}} \prod_{\substack{\mathrm{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(r) < t \le \ell \\ i_r \not\equiv \kappa_t \pmod{e}}} (\xi^{i_r} - \xi^{\kappa_t} - \xi^{i_r} y_r).
\end{aligned}$$

The factor to the right of $e^{\boldsymbol{\lambda}} y^{\boldsymbol{\lambda}}$ in the last equation is a polynomial in $K[y_1, \dots, y_n]$ with non-zero constant term. Since each y_r is nilpotent (it has positive degree), it follows that $g(y)$ is invertible. All of the terms in the last equation commute, so the lemma follows. □

6.4. **Theorem.** *Suppose that $e = 0$ or $e \ge n$ and let $\boldsymbol{\lambda} \in \mathscr{P}_n^{\Lambda}$. Then there exists an invertible element $f^{\boldsymbol{\lambda}}(y) \in K[y_1, \dots, y_n]$ such that*

$$e^{\boldsymbol{\lambda}} m^{\boldsymbol{\lambda}} e^{\boldsymbol{\lambda}} = f^{\boldsymbol{\lambda}}(y) e^{\boldsymbol{\lambda}} y^{\boldsymbol{\lambda}}.$$

Proof. By Lemma 6.3, there exists an invertible element $g^{\boldsymbol{\lambda}}(y) \in K[y_1, \dots, y_n]$ such that

$$e^{\boldsymbol{\lambda}} m^{\boldsymbol{\lambda}} e^{\boldsymbol{\lambda}} = e^{\boldsymbol{\lambda}} u^{\boldsymbol{\lambda}} x^{\boldsymbol{\lambda}} e^{\boldsymbol{\lambda}} = g^{\boldsymbol{\lambda}}(y) y^{\boldsymbol{\lambda}} \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} e^{\boldsymbol{\lambda}} T_w e^{\boldsymbol{\lambda}}.$$

By (3.9), if $w \in \mathfrak{S}_n$ and $\mathbf{j} \in I^n$ then $T_r e(\mathbf{j}) = (\psi_r Q_r(\mathbf{j}) - P_r(\mathbf{j})) e(\mathbf{j})$ so the last equation can be rewritten as

$$e^{\boldsymbol{\lambda}} m^{\boldsymbol{\lambda}} e^{\boldsymbol{\lambda}} = g^{\boldsymbol{\lambda}}(y) y^{\boldsymbol{\lambda}} \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} r_w(y) e^{\boldsymbol{\lambda}} \psi_w e^{\boldsymbol{\lambda}},$$

for some $r_w(y) \in K[y_1, \dots, y_n]$. Applying Lemma 6.2, this sum collapses to give

$$e^{\boldsymbol{\lambda}} m^{\boldsymbol{\lambda}} e^{\boldsymbol{\lambda}} = g^{\boldsymbol{\lambda}}(y) e^{\boldsymbol{\lambda}} y^{\boldsymbol{\lambda}} r_1(y) = f^{\boldsymbol{\lambda}}(y) e^{\boldsymbol{\lambda}} y^{\boldsymbol{\lambda}},$$

for some polynomial $f^{\boldsymbol{\lambda}}(y) \in K[y_1, \dots, y_n]$. It remains to show that $f^{\boldsymbol{\lambda}}(y)$ is invertible or, equivalently, that it has non-zero constant term. By [26, Corollary 3.11], if $1 \le r \le n$ and $(\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\mathscr{P}_n^{\Lambda})$ then $y_r \psi_{\mathfrak{s}\mathfrak{t}}$ is a linear combination of terms $\psi_{\mathfrak{u}\mathfrak{v}}$, where $(\mathfrak{u}, \mathfrak{v}) \blacktriangleright (\mathfrak{s}, \mathfrak{t})$. Therefore, since $e^{\boldsymbol{\lambda}} y^{\boldsymbol{\lambda}} = \psi_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$, there exist scalars $b_{\mathfrak{u}\mathfrak{v}} \in K$ such that

$$f^{\boldsymbol{\lambda}}(y) e^{\boldsymbol{\lambda}} y^{\boldsymbol{\lambda}} = b_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \psi_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} + \sum_{\substack{(\mathfrak{u}, \mathfrak{v}) \blacktriangleright (\mathfrak{t}^{\boldsymbol{\lambda}}, \mathfrak{t}^{\boldsymbol{\lambda}}) \\ \mathfrak{u}, \mathfrak{v} \in \mathrm{Std}^{\boldsymbol{\lambda}}(\mathscr{P}_n^{\Lambda})}} b_{\mathfrak{u}\mathfrak{v}} \psi_{\mathfrak{u}\mathfrak{v}},$$

where $b_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = f^{\boldsymbol{\lambda}}(0)$ is the constant term of $f^{\boldsymbol{\lambda}}(y)$. On the other hand, by [26, Theorem 3.9] there exist scalars $c_{\mathfrak{u}\mathfrak{v}} \in K$ such that $c_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \ne 0$ and

$$e^{\boldsymbol{\lambda}} m^{\boldsymbol{\lambda}} e^{\boldsymbol{\lambda}} = e^{\boldsymbol{\lambda}} \Big(\sum_{\mathfrak{u}, \mathfrak{v} \,\underline{\blacktriangleright}\, \mathfrak{t}^{\boldsymbol{\lambda}}} c_{\mathfrak{u}\mathfrak{v}} \psi_{\mathfrak{u}\mathfrak{v}} \Big) e^{\boldsymbol{\lambda}} = c_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \psi_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} + \sum_{\substack{\mathfrak{u}, \mathfrak{v} \ne \mathfrak{t}^{\boldsymbol{\lambda}} \\ \mathfrak{u}, \mathfrak{v} \in \mathrm{Std}^{\boldsymbol{\lambda}}(\mathscr{P}_n^{\Lambda})}} c_{\mathfrak{u}\mathfrak{v}} \psi_{\mathfrak{u}\mathfrak{v}},$$

where the second equality follows from (3.13). Hence, $f^{\boldsymbol{\lambda}}(0) = c_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \ne 0$ by Theorem 3.14, and the proof is complete. □

6.5. *Remark.* Using Theorem 6.4 it is possible to show that $e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}} = f^{\boldsymbol{\lambda}}(y)e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}} + \epsilon$ where ϵ is a linear combination of homogeneous terms of degree strictly greater than $2\deg \mathfrak{t}^{\boldsymbol{\lambda}} = \deg(e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}})$. To see this first show that $e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}}$ is a linear combination of terms of the form $e^{\boldsymbol{\lambda}}m_{\boldsymbol{\lambda}}e(\mathbf{j})$, where $\mathbf{j} \in I^{\boldsymbol{\lambda}} = \{\, \mathbf{i} \in I^n \mid \mathbf{i} = \sigma \cdot \mathbf{i}^{\boldsymbol{\lambda}} \text{ for some } \sigma \in \mathfrak{S}_{\boldsymbol{\lambda}} \,\}$. The key observation is then that $\deg \psi_w e(\mathbf{j}) > 0$ whenever $1 \neq w \in \mathfrak{S}_{\boldsymbol{\lambda}}$ and $\mathbf{j} \in I^{\boldsymbol{\lambda}}$, which can be proved by adapting the argument of Lemma 6.2. Consequently, $e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}$ is the homogeneous component of $e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}}$ of minimal degree. Examples show that this does not always happen if we drop the assumption that $e = 0$ or $e \geq n$.

6.6. **Corollary.** *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$. Then*

$$e^{\boldsymbol{\lambda}}\underline{M}^{\boldsymbol{\lambda}} = e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda = e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}}e^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda = e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda = \underline{G}^{\boldsymbol{\lambda}}.$$

Proof. By definition, $e^{\boldsymbol{\lambda}}\underline{M}^{\boldsymbol{\lambda}} = e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda$ and $\underline{G}^{\boldsymbol{\lambda}} = e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda$ so we only need to check the two middle equalities. By Theorem 6.4 there exists an invertible element $f^{\boldsymbol{\lambda}}(y)$ such that $e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}}e^{\boldsymbol{\lambda}} = f^{\boldsymbol{\lambda}}(y)e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}$. Consequently, $e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}}e^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda = e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda$. To complete the proof it is enough to show that $e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}} \in e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda$. This is immediate because $e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}} = e^{\boldsymbol{\lambda}}u^{\boldsymbol{\lambda}}x^{\boldsymbol{\lambda}} \in e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda$ by Lemma 6.3. □

6.7. **Definition.** Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$. Let $\pi^{\boldsymbol{\lambda}} : \underline{M}^{\boldsymbol{\lambda}} \longrightarrow e^{\boldsymbol{\lambda}}\underline{M}^{\boldsymbol{\lambda}}$ be the surjective $\mathcal{H}_n^\Lambda$-module homomorphism given by $\pi^{\boldsymbol{\lambda}}(h) = e^{\boldsymbol{\lambda}}h$, for $h \in \underline{M}^{\boldsymbol{\lambda}}$.

6.8. **Proposition.** *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$. Then the epimorphism $\pi^{\boldsymbol{\lambda}}$ splits. That is, $\pi^{\boldsymbol{\lambda}}$ has a right inverse $\phi^{\boldsymbol{\lambda}}$ and $\underline{M}^{\boldsymbol{\lambda}} \cong e^{\boldsymbol{\lambda}}\underline{M}^{\boldsymbol{\lambda}} \oplus \operatorname{Ker}\pi^{\boldsymbol{\lambda}}$.*

Proof. By Theorem 6.4, $e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}}e^{\boldsymbol{\lambda}} = f^{\boldsymbol{\lambda}}(y)e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}$ where is an invertible element of $\mathcal{H}_n^\Lambda$. Define $\phi^{\boldsymbol{\lambda}}$ to be the map

$$\phi^{\boldsymbol{\lambda}} : e^{\boldsymbol{\lambda}}\underline{M}^{\boldsymbol{\lambda}} \longrightarrow \underline{M}^{\boldsymbol{\lambda}}; e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}h \mapsto m^{\boldsymbol{\lambda}}e^{\boldsymbol{\lambda}}f^{\boldsymbol{\lambda}}(y)^{-1}h,$$

for $h \in \mathcal{H}_n^\Lambda$. To prove that $\phi^{\boldsymbol{\lambda}}$ is well-defined suppose that $e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}h = 0$ for some $h \in \mathcal{H}_n^\Lambda$. By Corollary 6.6, there exists $h^{\boldsymbol{\lambda}} \in \mathcal{H}_n^\Lambda$ such that $e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}} = e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}h^{\boldsymbol{\lambda}}$. Let $*$ be the non-homogeneous anti-isomorphism of $\mathcal{H}_n^\Lambda$ which fixes each of the non-homogeneous generators T_r and L_s, for $1 \leq r < n$ and $1 \leq s \leq n$.. Then $(e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}h^{\boldsymbol{\lambda}})^* = (h^{\boldsymbol{\lambda}})^*e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}$ because $e^{\boldsymbol{\lambda}}$ and $y^{\boldsymbol{\lambda}}$ are polynomials in $L_1, \cdots, L_n$ by [25, Proposition 4.8] and Theorem 3.7, respectively. Therefore,

$$m^{\boldsymbol{\lambda}}e^{\boldsymbol{\lambda}}f^{\boldsymbol{\lambda}}(y)^{-1}h = (e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}h^{\boldsymbol{\lambda}})^*f^{\boldsymbol{\lambda}}(y)^{-1}h = (h^{\boldsymbol{\lambda}})^*f^{\boldsymbol{\lambda}}(y)^{-1}e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}h = 0.$$

That is, $\phi^{\boldsymbol{\lambda}}(e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}h) = 0$. Hence, $\phi^{\boldsymbol{\lambda}}$ is a well-defined $\mathcal{H}_n^\Lambda$-module homomorphism. Moreover, if $h \in \mathcal{H}_n^\Lambda$ then

$$(\pi^{\boldsymbol{\lambda}} \circ \phi^{\boldsymbol{\lambda}})(e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}h) = e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}}e^{\boldsymbol{\lambda}}f^{\boldsymbol{\lambda}}(y)^{-1}h = e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}f^{\boldsymbol{\lambda}}(y)f^{\boldsymbol{\lambda}}(y)^{-1}h = e^{\boldsymbol{\lambda}}y^{\boldsymbol{\lambda}}h.$$

That is, $\pi^{\boldsymbol{\lambda}} \circ \phi^{\boldsymbol{\lambda}}$ is the identity map on $e^{\boldsymbol{\lambda}}\underline{M}^{\boldsymbol{\lambda}}$. Hence, $\pi^{\boldsymbol{\lambda}}$ splits as claimed. □

6.9. **Corollary.** *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$. Then $\phi^{\boldsymbol{\lambda}}$ induces an $\mathcal{H}_n^\Lambda$-module isomorphism $e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda \cong m^{\boldsymbol{\lambda}}e^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda$.*

Proof. This follows directly from the proof of Proposition 6.8. In fact, we have that $\phi^{\boldsymbol{\lambda}}(e^{\boldsymbol{\lambda}}m^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda) = m^{\boldsymbol{\lambda}}e^{\boldsymbol{\lambda}}\mathcal{H}_n^\Lambda$. □

6.2. **Cyclotomic Schur algebras.** We are now ready to show that $\mathcal{S}_n^\Lambda$ is Morita equivalent to one of the cyclotomic Schur algebras introduced in [9, 17].

6.10. **Definition** ([9, 17]). The **cyclotomic Schur algebra** is the algebra

$$\underline{\mathcal{S}}_n^{\mathrm{DJM}} = \operatorname{End}_{\mathcal{H}_n^\Lambda}\Big(\bigoplus_{\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda} \underline{M}^{\boldsymbol{\mu}}\Big).$$

Again, we write $\underline{\mathcal{S}}_n^{\mathrm{DJM}}$ to emphasize that $\underline{\mathcal{S}}_n^{\mathrm{DJM}}$ is not $\mathbb{Z}$-graded. Note that the algebra $\underline{\mathcal{S}}_n^{\mathrm{DJM}}$ depends implicitly on the dominant weight Λ.

By [17, Corollary 6.18], $\underline{\mathcal{S}}_n^{\mathrm{DJM}}$ is a quasi-hereditary cellular algebra with Weyl modules $\underline{\Delta}^{\boldsymbol{\lambda}}_{\mathrm{DJM}}$ and irreducible modules $\underline{L}^{\boldsymbol{\mu}}_{\mathrm{DJM}}$, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$. By [35, Theorem 2.11] and [7] the blocks of $\underline{\mathcal{S}}_n^{\mathrm{DJM}}$ are again labelled by Q_n^+, however, the direct summands of $\underline{M}^{\boldsymbol{\mu}}$ do not necessarily belong to the same block so it is difficult to describe the blocks of $\underline{\mathcal{S}}_n^{\mathrm{DJM}}$ explicitly; however, see [37, Theorem 4.5].

Recall the graded Young modules $Y^{\boldsymbol{\mu}}$, for $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$, from Definition 5.5.

6.11. **Lemma.** *Suppose that $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$. Then $\underline{M}^{\boldsymbol{\mu}} \cong \underline{Y}^{\boldsymbol{\mu}} \oplus \bigoplus_{\boldsymbol{\lambda} \rhd \boldsymbol{\mu}} (\underline{Y}^{\boldsymbol{\lambda}})^{m_{\boldsymbol{\lambda}\boldsymbol{\mu}}}$ for some integers $m_{\boldsymbol{\lambda}\boldsymbol{\mu}} \in \mathbb{N}$.*

Proof. By [36, (3.5)] there is a family of (ungraded) indecomposable $\mathcal{H}_n^\Lambda$-modules $\{\, \underline{y}^{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_n^\Lambda \,\}$ which are uniquely determined, up to isomorphism, by the property that

$$\underline{M}^{\boldsymbol{\mu}} \cong \underline{y}^{\boldsymbol{\mu}} \oplus \bigoplus_{\boldsymbol{\lambda} \rhd \boldsymbol{\mu}} (\underline{y}^{\boldsymbol{\lambda}})^{\oplus m_{\boldsymbol{\lambda}\boldsymbol{\mu}}} \tag{6.12}$$

for some integers $m_{\boldsymbol{\lambda}\boldsymbol{\mu}} \in \mathbb{N}$. We show by induction on the dominance ordering that $\underline{Y}^{\boldsymbol{\nu}} \cong \underline{y}^{\boldsymbol{\nu}}$, for all $\boldsymbol{\nu} \in \mathscr{P}_n^\Lambda$.

First suppose that $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$ is maximal in the dominance ordering. Then $\underline{M}^{\boldsymbol{\mu}} = \underline{y}^{\boldsymbol{\mu}}$ by (6.12) and $G^{\boldsymbol{\mu}} = Y^{\boldsymbol{\mu}}$ by Proposition 5.6(c). Therefore, $\underline{Y}^{\boldsymbol{\mu}} \cong \underline{y}^{\boldsymbol{\mu}}$ since $\underline{G}^{\boldsymbol{\mu}}$ is a summand of $\underline{M}^{\boldsymbol{\mu}}$ by Proposition 6.8.

Now suppose that $\boldsymbol{\mu}$ is not maximal in the dominance ordering. Then $Y^{\boldsymbol{\mu}}$ is isomorphic to a direct summand of $G^{\boldsymbol{\mu}}$ by Proposition 5.6(c). Therefore, there exists a multipartition $\boldsymbol{\lambda} \unrhd \boldsymbol{\mu}$ such that that $\underline{Y}^{\boldsymbol{\mu}} \cong \underline{y}^{\boldsymbol{\lambda}}$ by Proposition 6.8 and (6.12). By induction, if $\boldsymbol{\nu} \rhd \boldsymbol{\mu}$ then $\underline{y}^{\boldsymbol{\nu}} \cong \underline{Y}^{\boldsymbol{\nu}}$, so this forces $\boldsymbol{\lambda} = \boldsymbol{\mu}$ by Proposition 5.6(b). That is, $\underline{Y}^{\boldsymbol{\mu}} \cong \underline{y}^{\boldsymbol{\mu}}$ as claimed. This completes the proof. □

6.13. **Theorem.** *Suppose that $\mathcal{Z}$ is a field and that $e = 0$ or $e \geq n$. Then there is an equivalence of highest weight categories*

$$\mathsf{E}^\Lambda_{DJM} : \underline{\mathcal{S}}_n^\Lambda\text{-Mod} \xrightarrow{\sim} \underline{\mathcal{S}}_n^{DJM}\text{-Mod}$$

such that $\mathsf{E}^\Lambda_{DJM}(\underline{\Delta}^{\boldsymbol{\lambda}}) \cong \underline{\Delta}^{\boldsymbol{\lambda}}_{DJM}$ and $\mathsf{E}^\Lambda_{DJM}(\underline{L}^{\boldsymbol{\mu}}) \cong \underline{L}^{\boldsymbol{\mu}}_{DJM}$, for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$.

Proof. By Lemma 6.11, the algebra

$$\mathrm{End}_{\mathcal{H}_n^\Lambda}\Big(\bigoplus_{\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda} \underline{Y}^{\boldsymbol{\mu}}\Big)$$

is the basic algebra of $\underline{\mathcal{S}}_n^\Lambda$ and it is also the basic algebra of $\underline{\mathcal{S}}_n^{\mathrm{DJM}}$. Hence the result follows by the discussion in section 2.3. □

Using the combinatorics of the cellular bases of the algebras $\underline{\mathcal{S}}_n^{\mathrm{DJM}}$ and $\mathcal{S}_n^\Lambda$ it is easy to see that over field $\dim \underline{\mathcal{S}}_n^\Lambda \leq \dim \underline{\mathcal{S}}_n^{\mathrm{DJM}}$ and that this equality is strict except for small n; see Remark 4.21. In particular, the algebras $\underline{\mathcal{S}}_n^\Lambda$ and $\underline{\mathcal{S}}_n^{\mathrm{DJM}}$ are not isomorphic in general.

6.14. **Corollary.** *Suppose that $\mathcal{Z}$ is a field and that $e = 0$ or $e \geq n$. Then, up to Morita equivalence, $\underline{\mathcal{S}}_n^{DJM}$ depends only on e, Λ and the characteristic of $\mathcal{Z}$.*

In particular, if $e = 0$ or $e \geq n$ then the decomposition numbers of the degenerate and non-degenerate cyclotomic Schur algebras depend only e, Λ and the characteristic of the field. This generalizes [10, Corollary 6.3] which is the analogous result for the cyclotomic Hecke algebras (without any restriction on e).

Using Lemma 6.11 it is not hard to show that the degenerate and non-degenerate cyclotomic Schur algebras are isomorphic over any field when $e = 0$. Gordon and

Losev [20, Proposition 6.6] have constructed an explicit isomorphism between these algebras over $\mathbb{C}$, extending Brundan and Kleshchev's isomorphism theorem 3.7.

7. Positivity, decomposition numbers and the Fock space

Theorem 6.13 shows that $\mathcal{S}_n^\Lambda$-Mod induces a grading on the category of finite dimensional modules for the cyclotomic Schur algebras $\underline{\mathcal{S}}_n^{\mathrm{DJM}}$. On the other hand, in the degenerate case Brundan and Kleshchev [9] have constructed an equivalence of categories $\mathcal{O}_n^\Lambda \xrightarrow{\sim} \underline{\mathcal{S}}_n^{\mathrm{DJM}}$-Mod, where $\mathcal{O}_n^\Lambda$ is a sum of certain integral blocks of the BGG parabolic category $\mathcal{O}$ for $\mathfrak{gl}_N(\mathbb{C})$. Deep results of Beilinson, Ginzburg and Soergel [5, Theorem 1.1.3] and Backelin [4, Theorem 1.1] show that $\mathcal{O}_n^\Lambda$ admits a Koszul grading and hence that $\underline{\mathcal{S}}_n^{\mathrm{DJM}}$ can be endowed with a Koszul grading as well. This chapter matches up the gradings on $\mathcal{R}_n^\Lambda$ and $\mathcal{S}_n^\Lambda$ with the gradings coming from category $\mathcal{O}_n^\Lambda$ and, as a consequence, shows that the graded module category $\mathcal{S}_n^\Lambda$-Mod is Koszul.

Throughout this chapter we assume that $e = 0$ and that $\mathcal{Z} = \mathbb{C}$, considerably strengthening our standing assumption 4.1.

7.1. **Parabolic category $\mathcal{O}$.** Recall from section 3.1 that the dominant weight $\Lambda = \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_\ell}$ is determined by our fixed choice of multicharge $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell)$. For the rest of this chapter we assume that

$$\kappa_1 = 0 \geq \kappa_2 \geq \cdots \geq \kappa_\ell.$$

There is no loss of generality in making this assumption because we can permute the numbers in the multicharge, and shift all of the generators $L_1, \dots, L_n$ of $\mathcal{H}_n^\Lambda$ in Definition 3.6 by the same scalar, without changing the isomorphism type of $\mathcal{H}_n^\Lambda$ or the graded isomorphism type of $\mathcal{R}_n^\Lambda$.

Fix an integer $m \geq n - \kappa_\ell$ and define $\mu_l = m + \kappa_l$, for $1 \leq l \leq \ell$. Then $\mu = (\mu_1, \dots, \mu_\ell)$ is a partition of $N = \mu_1 + \cdots + \mu_\ell$ and, by assumption, $\mu_1 \geq \cdots \geq \mu_\ell \geq n$. Let $\mu' = (\mu'_1, \dots, \mu'_m)$ be the partition which is conjugate to μ.

Let $J = \{1-m, 2-m, \dots, n-1\}$ and $J_+ := J \cup (J+1)$. The **diagram** of Λ with respect to μ, which should not be confused with the Young diagram defined in section 3.3, is the array of boxes with rows indexed by J_+ in increasing order from bottom to top and with μ'_i boxes in row i and μ_j boxes in column j. A **Λ-tableau** is a labelling of the boxes of the Λ-diagram with the integers in J_+, possibly with repeats. The **ground state** Λ-tableau is the Λ-tableau with an j in all of the boxes in row j, for all $j \in J$.

For example, if $\ell = 4$, $\boldsymbol{\kappa} = (0, 0, -2, -3)$, so that $\Lambda = 2\Lambda_0 + \Lambda_{-2} + \Lambda_{-3}$, and $n = 3$, $m = 6$ then $\mu = (6, 6, 4, 3)$, $\mu' = (4, 4, 4, 3, 2, 2)$, $N = 19$ and the ground state Λ-tableau is

0	0		
-1	-1		
-2	-2	-2	
-3	-3	-3	-3
-4	-4	-4	-4
-5	-5	-5	-5

Let $\mathfrak{gl}_N(\mathbb{C})$ be the general linear Lie algebra of $N \times N$ matrices with its standard Cartan subalgebra $\mathfrak{h}$. Let $\mathfrak{p}$ be the standard parabolic subalgebra of $\mathfrak{gl}_N(\mathbb{C})$ which has Levi subalgebra $\mathfrak{gl}_{\mu_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}_{\mu_\ell}(\mathbb{C})$.

Let $\mathcal{O}^\Lambda$ be the category of all finitely generated $\mathfrak{gl}_N(\mathbb{C})$-modules which are locally finite over $\mathfrak{p}$ and integrable over $\mathfrak{h}$. This is the usual parabolic analogue of the BGG category $\mathcal{O}$, except that we are only allowing modules with integral weights or, equivalently, integral central characters. The irreducible modules in category $\mathcal{O}^\Lambda_n$ are naturally parametrised by highest weights. Following [9, 12] we give a non-standard labelling of the irreducible modules in $\mathcal{O}^\Lambda_n$ by the multiplications $\bigcup_\beta \mathscr{P}^\Lambda_\beta$, for suitable β. To this end, let $\varepsilon_1, \dots, \varepsilon_N \in \mathfrak{h}^*$ be the standard coordinate functions on $\mathfrak{h}$ so that if $M = (m_{ab})$ is a matrix in $\mathfrak{gl}_N(\mathbb{C})$ then $\varepsilon_i(M) = m_{ii}$ picks out the i^{th} diagonal entry of M.

Following [12, (2.50)], if $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$ then the **Λ-tableau** of $\boldsymbol{\lambda}$ is the Λ-tableau $\mathsf{T}^{\boldsymbol{\lambda}}$ which has $\lambda^{(k)}_r + r$ in row $r \in J_+ \cap \mathbb{Z}^{\le 0}$ and column $k \in \{1, 2, \dots, \ell\}$. For example, the Λ-diagram of the empty multipartition is the ground state Λ-tableau. The **column reading** of $\mathsf{T}^{\boldsymbol{\lambda}}$ is the sequence $(t_1, \dots, t_n)$ where $t_1, t_2, \dots, t_N$ are the entries of $\mathsf{T}^{\boldsymbol{\lambda}}$, read from top to bottom and then from left to right down the columns of $\mathsf{T}^{\boldsymbol{\lambda}}$. Set $\operatorname{wt}(\boldsymbol{\lambda}) = t_1\varepsilon_1 + (t_2+1)\varepsilon_2 + \dots + (t_N + N - 1)\varepsilon_N$, a dominant weight for $\mathfrak{gl}_N(\mathbb{C})$. Now define $\underline{L}^{\boldsymbol{\lambda}}_{\mathcal{O}}$ to be the irreducible highest weight module in $\mathcal{O}^\Lambda$ of highest weight $\operatorname{wt}(\boldsymbol{\lambda})$.

Now suppose that $\beta \in Q^+_n$. Then $\boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta$ if and only if the Λ-tableau of $\boldsymbol{\nu}$ has the weight $\Lambda - \beta$, see [12, section 2.1] for the definition of the weight of a Λ-tableau. Let $\mathcal{O}^\Lambda_\beta$ be the Serre subcategory of $\mathcal{O}^\Lambda$ generated by the irreducible $\mathfrak{gl}_N(\mathbb{C})$-modules $\{\,\underline{L}^{\boldsymbol{\nu}}_{\mathcal{O}} \mid \boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta\,\}$. That is, $\mathcal{O}^\Lambda_\beta$ is the full subcategory of $\mathcal{O}^\Lambda$ consisting of modules which have all their composition factors in $\{\,\underline{L}^{\boldsymbol{\nu}}_{\mathcal{O}} \mid \boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta\,\}$. Brundan [7, Theorem 2] shows that $\mathcal{O}^\Lambda_\beta$ is a single block of parabolic category $\mathcal{O}^\Lambda$ and, moreover, that

$$\mathcal{O}^\Lambda = \bigoplus_{\beta \in Q_+} \mathcal{O}^\Lambda_\beta \tag{7.1}$$

is the block decomposition of $\mathcal{O}^\Lambda$. Set $\mathcal{O}^\Lambda_n = \bigoplus_{\beta \in Q^+_n} \mathcal{O}^\Lambda_\beta$.

We are almost ready to state a deep result of Backelin's, that builds on fundamental work of Beilinson, Ginzburg and Soergel [5, Theorem 1.1.3], which says that $\mathcal{O}^\Lambda_\beta$ admits a Koszul grading. In fact, Backelin proved this result more generally for the blocks of parabolic category $\mathcal{O}$ for an arbitrary semisimple complex Lie algebra.

To state Backelin's theorem we need to know that there is a 'dual' category $\mathcal{O}^\beta_\Lambda$ to $\mathcal{O}^\Lambda_\beta$ which has irreducible modules $\{\,\underline{L}^{\mathcal{O}}_{\boldsymbol{\nu}} \mid \boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta\,\}$. The category $\mathcal{O}^\beta_\Lambda$ is again a subcategory of $\mathcal{O}$ and it is defined in completely analogous way to $\mathcal{O}^\Lambda_\beta$. As we do not need to know the precise description of $\mathcal{O}^\beta_\Lambda$ we omit further details and refer the interested reader to [4]. Now define

$$\mathcal{S}^{\mathcal{O}}_\beta = \operatorname{Ext}^\bullet_{\mathcal{O}^\beta_\Lambda}\big(\underline{L}^{\mathcal{O}}_\beta, \underline{L}^{\mathcal{O}}_\beta\big),$$

where $\underline{L}^{\mathcal{O}}_\beta = \bigoplus_{\boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta} \underline{L}^{\mathcal{O}}_{\boldsymbol{\nu}}$ is the direct sum of the irreducible $\mathcal{O}^\beta_\Lambda$-modules. We consider $\mathcal{S}^{\mathcal{O}}_\beta$ as a positively graded algebra under the Yoneda product.

For any module M let ι_M be the identity map on M. For example, $\{\,\iota_{\underline{L}^{\mathcal{O}}_{\boldsymbol{\nu}}} \mid \boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta\,\}$ is a basis of $(\mathcal{S}^{\mathcal{O}}_\beta)_0$.

7.2. **Theorem** (Backelin [4]). *Suppose that $\beta \in Q^+_n$ and $\mathcal{Z} = \mathbb{C}$. Then $\mathcal{S}^{\mathcal{O}}_\beta$ is a Koszul algebra, and there is an algebra isomorphism*

$$\mathcal{S}^{\mathcal{O}}_\beta \cong \operatorname{End}_{\mathcal{O}^\Lambda_\beta}\big(\underline{P}^\beta_{\mathcal{O}}, \underline{P}^\beta_{\mathcal{O}}\big)$$

where $\underline{P}^\beta_{\mathcal{O}} = \bigoplus_{\boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta} \underline{P}^{\boldsymbol{\nu}}_{\mathcal{O}}$ is a minimal projective generator for $\mathcal{O}^\Lambda_\beta$. Moreover, this isomorphism sends $\iota_{\underline{L}^{\mathcal{O}}_{\boldsymbol{\nu}}}$ to $\iota_{\underline{P}^{\boldsymbol{\nu}}_{\mathcal{O}}}$, for $\boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta$.

Proof. The existence of such an isomorphism is proved by Backelin in [4, Theorem 1.1]. The isomorphism identifies $\iota_{\underline{L}^{\mathcal{O}}_{\boldsymbol{\nu}}}$ and $\iota_{\underline{P}^{\boldsymbol{\nu}}_{\mathcal{O}}}$ by [4, Remark 3.8]. $\square$

We remark that Mazorchuk [38, Theorem 7.1] has given an algebraic proof of Backelin's result assuming that the Kazhdan-Lusztig conjecture holds for $\mathcal{O}^\Lambda_\beta$.

The algebra on the right hand side of Theorem 7.2 is a finite dimensional algebra which, by definition, is the basic algebra for category $\mathcal{O}^\Lambda_\beta$. Thus, we have an equivalence of categories $\mathsf{E}^{\mathcal{O}}_\beta : \mathcal{O}^\Lambda_\beta \xrightarrow{\sim} \underline{\mathcal{S}}^{\mathcal{O}}_\beta$-Mod. For each $\boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta$ let $L^{\boldsymbol{\nu}}_{\mathcal{O}}$ be the unique irreducible $\mathcal{S}^{\mathcal{O}}_\beta$-module which is concentrated in degree zero and such that $L^{\boldsymbol{\nu}}_{\mathcal{O}} \cong \mathsf{E}^{\mathcal{O}}_\beta(\underline{L}^{\boldsymbol{\nu}}_{\mathcal{O}})$. We are abusing notation here because $L^{\boldsymbol{\nu}}_{\mathcal{O}}$ is one dimensional so that $\underline{L}^{\boldsymbol{\nu}}_{\mathcal{O}}$ is *not* the module obtained from $L^{\boldsymbol{\nu}}_{\mathcal{O}}$ by forgetting the grading. Similarly, let $P^{\boldsymbol{\nu}}_{\mathcal{O}}$ be the projective cover of $L^{\boldsymbol{\nu}}_{\mathcal{O}}$ in $\mathcal{S}^{\mathcal{O}}_\beta$-Mod, so that $P^{\boldsymbol{\nu}}_{\mathcal{O}} \cong \mathsf{E}^{\mathcal{O}}_\beta(\underline{P}^{\boldsymbol{\nu}}_{\mathcal{O}})$.

Category $\mathcal{O}^\Lambda_n$ has a duality $\underline{\diamond}$ which is induced by the anti-isomorphism of $\mathfrak{gl}_N(\mathbb{C})$ which maps a matrix to its transpose. Since $(\underline{L}^{\boldsymbol{\nu}}_{\mathcal{O}})^{\underline{\diamond}} \cong \underline{L}^{\boldsymbol{\nu}}_{\mathcal{O}}$ taking duals induces natural isomorphisms

$$\operatorname{Ext}^{\bullet}_{\mathcal{O}^\beta_\Lambda}\big(\underline{L}^{\mathcal{O}}_{\boldsymbol{\lambda}}, \underline{L}^{\mathcal{O}}_{\boldsymbol{\nu}}\big) \cong \operatorname{Ext}^{\bullet}_{\mathcal{O}^\beta_\Lambda}\big(\underline{L}^{\mathcal{O}}_{\boldsymbol{\nu}}, \underline{L}^{\mathcal{O}}_{\boldsymbol{\lambda}}\big),$$

for $\boldsymbol{\lambda}, \boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta$. Therefore, $\underline{\diamond}$ lifts to a homogeneous duality $\diamond$ on $\mathcal{S}^{\mathcal{O}}_\beta$. Each simple $\mathcal{S}^{\mathcal{O}}_\beta$-module $L^{\boldsymbol{\nu}}_{\mathcal{O}}$ is one dimensional and concentrated in degree zero, so $\big(L^{\boldsymbol{\nu}}_{\mathcal{O}}\big)^{\diamond} \cong L^{\boldsymbol{\nu}}_{\mathcal{O}}$, for all $\boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta$.

7.3. **Corollary.** *Suppose that $\beta \in Q^+_n$ and $\boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta$. Then the radical and grading filtrations of the projective indecomposable module $P^{\boldsymbol{\nu}}_{\mathcal{O}}$ coincide. Moreover, if $\boldsymbol{\nu} \in \mathcal{K}^\Lambda_\beta$ then $P^{\boldsymbol{\nu}}_{\mathcal{O}}$ is rigid with both its radical and socle filtrations being equal to its grading filtration.*

Proof. By construction, $P^{\boldsymbol{\nu}}_{\mathcal{O}}/\operatorname{rad} P^{\boldsymbol{\nu}}_{\mathcal{O}} \cong L^{\boldsymbol{\nu}}_{\mathcal{O}}$. Moreover, if $\boldsymbol{\nu} \in \mathcal{K}^\Lambda_\beta$ then $(P^{\boldsymbol{\nu}}_{\mathcal{O}})^{\diamond} \cong P^{\boldsymbol{\nu}}_{\mathcal{O}}\langle k\rangle$, for some $k \in \mathbb{Z}$, by [12, (2.52), Lemma 3.2]. Consequently, $P^{\boldsymbol{\nu}}_{\mathcal{O}}$ has an irreducible socle. Since $\mathcal{S}^{\mathcal{O}}_\beta$ is Koszul by Theorem 7.2 the Corollary now follows by Proposition 2.15 and Lemma 2.13. □

Let $\underline{\Delta}^{\boldsymbol{\lambda}}_{\mathcal{O}}$ be the Verma module of highest weight $\operatorname{wt}(\boldsymbol{\lambda})$ in $\mathcal{O}^\Lambda_\beta$. Then there is a graded $\mathcal{S}^{\mathcal{O}}_\beta$–module $\Delta^{\boldsymbol{\lambda}}_{\mathcal{O}}$ such that $\Delta^{\boldsymbol{\lambda}}_{\mathcal{O}} \cong \mathsf{E}^{\mathcal{O}}_\beta(\underline{\Delta}^{\boldsymbol{\lambda}}_{\mathcal{O}})$ by [5, Proposition 3.5.7] and the proof of [4, Proposition 3.2]. Since $\underline{\Delta}^{\boldsymbol{\lambda}}_{\mathcal{O}}$ is indecomposable, we fix the grading on $\Delta^{\boldsymbol{\lambda}}_{\mathcal{O}}$ by requiring that the surjection $\Delta^{\boldsymbol{\lambda}}_{\mathcal{O}} \twoheadrightarrow L^{\boldsymbol{\lambda}}_{\mathcal{O}}$ is a homogeneous map of degree zero in $\mathcal{S}^{\mathcal{O}}_\beta$-Mod.

We are now ready to make the link between parabolic category $\mathcal{O}$ and the quiver Schur algebras. The following result is a reformulation of some of Brundan and Kleshchev's main results from [9, 12]. Our Theorem C from the introduction is a graded analogue of this result.

7.4. **Theorem** (Brundan and Kleshchev)**.** *Suppose that $\beta \in Q^+_n$, $e = 0$ and $\mathcal{Z} = \mathbb{C}$. Then there is an equivalence of categories $\underline{\mathsf{E}}^\Lambda_{\mathcal{O}} : \mathcal{O}^\Lambda_\beta \longrightarrow \underline{\mathcal{S}}^\Lambda_\beta$-Mod and an exact functor $\underline{\mathsf{F}}^{\mathcal{O}}_\beta : \mathcal{O}^\Lambda_\beta \longrightarrow \underline{\mathcal{R}}^\Lambda_\beta$-Mod such that the following diagram of functors commutes*

$$\begin{tikzcd}
\mathcal{O}^\Lambda_\beta \arrow[r, "\underline{\mathsf{E}}^\Lambda_{\mathcal{O}}"] \arrow[dr, "\underline{\mathsf{F}}^{\mathcal{O}}_\beta"'] & \underline{\mathcal{S}}^\Lambda_\beta\text{-Mod} \arrow[d, "\underline{\mathsf{F}}^\Lambda_\beta"] \\
 & \underline{\mathcal{R}}^\Lambda_\beta\text{-Mod}
\end{tikzcd}$$

Moreover, $\underline{\mathsf{E}}^\Lambda_{\mathcal{O}}(\underline{\Delta}^{\boldsymbol{\lambda}}_{\mathcal{O}}) \cong \underline{\Delta}^{\boldsymbol{\lambda}}$ and $\underline{\mathsf{E}}^\Lambda_{\mathcal{O}}(\underline{L}^{\boldsymbol{\mu}}_{\mathcal{O}}) \cong \underline{L}^{\boldsymbol{\mu}}$, for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\Lambda_n$.

Proof. By [9, Theorem C] there is an equivalence of categories from $\mathcal{O}^\Lambda_\beta$ to $\underline{\mathcal{S}}^{\mathrm{DJM}}_\beta$-Mod which sends $\underline{\Delta}^{\boldsymbol{\lambda}}_{\mathcal{O}}$ to $\underline{\Delta}^{\boldsymbol{\lambda}}_{\mathrm{DJM}}$ and $\underline{L}^{\boldsymbol{\mu}}_{\mathcal{O}}$ to $\underline{L}^{\boldsymbol{\mu}}_{\mathrm{DJM}}$, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\Lambda_\beta$. Hence, by Theorem 6.13

and the remarks above, there is an equivalence of categories $\underline{\mathsf{E}}^\Lambda_{\mathcal{O}}:\mathcal{O}^\Lambda_\beta\longrightarrow\underline{\mathcal{S}}^\Lambda_\beta$-Mod such that $\underline{\mathsf{E}}^\Lambda_{\mathcal{O}}(\underline{\Delta}^{\boldsymbol{\lambda}}_{\mathcal{O}})\cong\underline{\Delta}^{\boldsymbol{\lambda}}$ and $\underline{\mathsf{E}}^\Lambda_{\mathcal{O}}(\underline{L}^{\boldsymbol{\mu}}_{\mathcal{O}})\cong\underline{L}^{\boldsymbol{\mu}}$. If $\xi=1$ then $\mathcal{H}^\Lambda_n$ is isomorphic to a degenerate cyclotomic Hecke algebra. Hence, Brundan and Kleshchev [12, Theorem 3.6] have shown that there exists an exact functor $\underline{\mathsf{F}}^{\mathcal{O}}_\beta:\mathcal{O}^\Lambda_\beta\longrightarrow\mathcal{H}^\Lambda_\beta$-Mod with the required properties. By Theorem 3.7, $\mathcal{H}^\Lambda_\beta\cong\underline{\mathcal{R}}^\Lambda_\beta$, so this completes the proof. □

Consequently, there is an equivalence of categories

$$\mathcal{O}^\Lambda_n\cong\bigoplus_{\beta\in Q^+_n}\underline{\mathcal{S}}^{\mathcal{O}}_\beta\text{-Mod}\cong\underline{\mathcal{S}}^\Lambda_n\text{-Mod}\,.$$

Henceforth we identify parabolic category $\mathcal{O}^\Lambda_n$ and $\underline{\mathcal{S}}^\Lambda_n$-Mod.

7.2. **Comparing the KLR and category $\mathcal{O}$ gradings.** We want to lift Theorem 7.4 to the graded setting. As a first step we show that the KLR and category $\mathcal{O}^\Lambda_n$ gradings induce the same grading on $\mathcal{H}^\Lambda_n$.

We need a graded analogue of the Hecke algebra $\mathcal{H}^\Lambda_n$ with the grading coming from category $\mathcal{O}^\Lambda_n$. Define $e^{\mathcal{O}}_\beta=\sum_{\boldsymbol{\mu}\in\mathcal{K}^\Lambda_\beta}\iota_{\underline{L}^{\mathcal{O}}_{\boldsymbol{\mu}}}\in\mathcal{S}^{\mathcal{O}}_\beta$. Then $e^{\mathcal{O}}_\beta$ is a homogeneous idempotent of degree zero. Now define

$$\mathcal{R}^{\mathcal{O}}_\beta=e^{\mathcal{O}}_\beta\mathcal{S}^{\mathcal{O}}_\beta e^{\mathcal{O}}_\beta\cong\operatorname{Ext}^\bullet_{\mathcal{O}}\Big(\bigoplus_{\boldsymbol{\mu}\in\mathcal{K}^\Lambda_\beta}\underline{L}^{\mathcal{O}}_{\boldsymbol{\mu}},\bigoplus_{\boldsymbol{\mu}\in\mathcal{K}^\Lambda_\beta}\underline{L}^{\mathcal{O}}_{\boldsymbol{\mu}}\Big).$$

By (2.10) there is an exact functor

$$\mathsf{F}^{\mathcal{O}}_\beta:\mathcal{S}^{\mathcal{O}}_\beta\text{-Mod}\longrightarrow\mathcal{R}^{\mathcal{O}}_\beta\text{-Mod};M\mapsto Me^{\mathcal{O}}_\beta,\qquad\text{for }M\in\mathcal{S}^{\mathcal{O}}_\beta\text{-Mod}\,.\tag{7.5}$$

Observe that $(e^{\mathcal{O}}_\beta)^\diamond=e^{\mathcal{O}}_\beta$ so that the involution $\diamond$ restricts to a graded anti-involution of $\mathcal{R}^{\mathcal{O}}_\beta$.

Let $\mathcal{R}^{\mathcal{O}}_\beta=\bigoplus_{d\in\mathbb{Z}}(\mathcal{R}^{\mathcal{O}}_\beta)_d$ be the decomposition of $\mathcal{R}^{\mathcal{O}}_\beta$ into its homogeneous components.

7.6. **Proposition.** *Suppose that $\beta\in Q^+_n$. Then:*

a) *$\mathcal{R}^{\mathcal{O}}_\beta$ is a positively graded basic algebra;*
b) *$(\mathcal{R}^{\mathcal{O}}_\beta)_0$ is semisimple;*
c) *The ungraded algebras $\underline{\mathcal{R}}^{\mathcal{O}}_\beta$, $\underline{\mathcal{R}}^\Lambda_\beta$ and $\mathcal{H}^\Lambda_\beta$ are Morita equivalent.*

Proof. By Theorem 7.2 $\mathcal{S}^{\mathcal{O}}_\beta$ is a Koszul algebra so it is positively graded and its degree zero component is semisimple by the definition in section 2.5. Hence, parts (a) and (b) follow because $\mathcal{R}^{\mathcal{O}}_\beta=e^{\mathcal{O}}_\beta\mathcal{S}^{\mathcal{O}}_\beta e^{\mathcal{O}}_\beta$. Finally, the algebras $\underline{\mathcal{R}}^{\mathcal{O}}_\beta$ and $\underline{\mathcal{R}}^\Lambda_\beta$ are Morita equivalent by Theorem 7.4. Hence, (c) follows since $\underline{\mathcal{R}}^\Lambda_\beta\cong\mathcal{H}^\Lambda_\beta$ by Theorem 3.7 and the remarks after (3.18). □

We need analogues of Specht modules, Young modules and simple modules for the algebra $\mathcal{R}^{\mathcal{O}}_\beta$. Recalling the functor $\mathsf{F}^{\mathcal{O}}_\beta$ from (7.5), for $\boldsymbol{\mu}\in\mathcal{K}^\Lambda_\beta$ and $\boldsymbol{\lambda}\in\mathscr{P}^\Lambda_\beta$ define

$$D^{\boldsymbol{\mu}}_{\mathcal{O}}=\mathsf{F}^{\mathcal{O}}_\beta(L^{\boldsymbol{\mu}}_{\mathcal{O}}),\quad S^{\boldsymbol{\lambda}}_{\mathcal{O}}=\mathsf{F}^{\mathcal{O}}_\beta(\Delta^{\boldsymbol{\lambda}}_{\mathcal{O}})\quad\text{and}\quad Y^{\boldsymbol{\lambda}}_{\mathcal{O}}=\mathsf{F}^{\mathcal{O}}_\beta(P^{\boldsymbol{\lambda}}_{\mathcal{O}}).\tag{7.7}$$

If $\boldsymbol{\mu}\in\mathcal{K}^\Lambda_\beta$ then $D^{\boldsymbol{\mu}}_{\mathcal{O}}\cong\mathbb{C}\iota_{\underline{L}^{\boldsymbol{\mu}}_{\mathcal{O}}}$ is an irreducible $\mathcal{R}^{\mathcal{O}}_\beta$-module and $Y^{\boldsymbol{\mu}}_{\mathcal{O}}$ is the projective cover of $D^{\boldsymbol{\mu}}_{\mathcal{O}}$ in $\mathcal{R}^{\mathcal{O}}_\beta$-Mod by Theorem 7.4, Theorem 7.2 and Theorem 2.12. Moreover, $\{\,D^{\boldsymbol{\mu}}_{\mathcal{O}}\langle k\rangle\mid\boldsymbol{\mu}\in\mathcal{K}^\Lambda_\beta\text{ and }k\in\mathbb{Z}\,\}$ is a complete set of irreducible $\mathcal{R}^{\mathcal{O}}_\beta$-modules. By construction, $D^{\boldsymbol{\mu}}_{\mathcal{O}}$ is one dimensional and concentrated in degree zero, for all $\boldsymbol{\mu}\in\mathcal{K}^\Lambda_\beta$.

The next result, which is again a reformulation of results of Brundan and Kleshchev, will allow us to compare the gradings on $\mathcal{R}^{\mathcal{O}}_\beta$ and $\mathcal{R}^\Lambda_n$.

7.8. **Theorem** (Brundan and Kleshchev [11,12]). *Suppose that $e=0$, $\beta\in Q_n^+$ and $\mathcal{Z}=\mathbb{C}$ and let $\boldsymbol{\lambda}\in\mathscr{P}_\beta^\Lambda$ and $\boldsymbol{\mu}\in\mathcal{K}_\beta^\Lambda$. Then*

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)=[S^{\boldsymbol{\lambda}}:D^{\boldsymbol{\mu}}]_q=[S_{\mathcal{O}}^{\boldsymbol{\mu}}:D_{\mathcal{O}}^{\boldsymbol{\mu}}]_q=[\Delta_{\mathcal{O}}^{\boldsymbol{\lambda}}:L_{\mathcal{O}}^{\boldsymbol{\mu}}]_q$$

is a parabolic Kazhdan-Lusztig polynomial. In particular, $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\in\delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}+q\mathbb{N}[q]$.

Proof. Since $\mathsf{F}_\beta^{\mathcal{O}}$ is exact, $[S_{\mathcal{O}}^{\boldsymbol{\mu}}:D_{\mathcal{O}}^{\boldsymbol{\mu}}]_q=[\Delta_{\mathcal{O}}^{\boldsymbol{\lambda}}:L_{\mathcal{O}}^{\boldsymbol{\mu}}]_q$, for all $\boldsymbol{\lambda}\in\mathscr{P}_\beta^\Lambda$ and $\boldsymbol{\mu}\in\mathcal{K}_\beta^\Lambda$.

By [11, Theorem 5.14, Corollary 5.15], the projective indecomposable $\mathcal{R}_n^\Lambda$-modules categorify the canonical basis of the integral highest weight module $L(\Lambda)$ for $U_q(\widehat{\mathfrak{sl}}_e)$ and the graded decomposition numbers give the transition matrix between the standard basis and the canonical basis of $L(\Lambda)$. On the other hand, [12, Theorem 3.1 and (2.18)] computes this transition matrix explicitly and shows that $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$ is equal to the parabolic Kazhdan-Lusztig polynomial $[\Delta_{\mathcal{O}}^{\boldsymbol{\lambda}}:L_{\mathcal{O}}^{\boldsymbol{\mu}}]_q$. Note that when $e=0$ the papers [11,12] use the same bar involution so that the transition matrices appearing in both papers coincide. See [12, §2.5] and the remarks after [11, (3.60)]. $\square$

7.9. *Remark.* The graded decomposition numbers $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)=[\Delta_{\mathcal{O}}^{\boldsymbol{\lambda}}:L_{\mathcal{O}}^{\boldsymbol{\mu}}]_q$ are described explicitly as parabolic Kazhdan-Lusztig polynomials in [6, Remark 14]. See also [12, (2.16) and Theorem 3.1].

By Proposition 5.6(d), the basic algebra of $\mathcal{R}_\beta^\Lambda$ is (isomorphic to) the algebra

$${}^\flat\mathcal{R}_\beta^\Lambda=\mathrm{E{\scriptstyle ND}}_{\mathcal{R}_n^\Lambda}\Big(\bigoplus_{\boldsymbol{\mu}\in\mathcal{K}_\beta^\Lambda}Y^{\boldsymbol{\mu}}\Big).$$

This is a $\mathbb{Z}$-graded algebra which is graded Morita equivalent to $\mathcal{R}_\beta^\Lambda$. Let

$$\mathsf{E}_{\mathcal{R}_\beta^\Lambda}:\mathcal{R}_\beta^\Lambda\text{-Mod}\xrightarrow{\sim}{}^\flat\mathcal{R}_\beta^\Lambda\text{-Mod};M\mapsto\mathrm{H{\scriptstyle OM}}_{\mathcal{R}_\beta^\Lambda}\Big(\bigoplus_{\boldsymbol{\lambda}\in\mathcal{K}_\beta^\Lambda}Y^{\boldsymbol{\lambda}},M\Big),\tag{7.10}$$

be the corresponding equivalence of graded module categories.

Suppose that $\boldsymbol{\mu}\in\mathcal{K}_\beta^\Lambda$ and let ${}^\flat Y^{\boldsymbol{\mu}}=\mathsf{E}_{\mathcal{R}_\beta^\Lambda}(Y^{\boldsymbol{\mu}})$ and ${}^\flat D^{\boldsymbol{\mu}}=\mathsf{E}_{\mathcal{R}_\beta^\Lambda}(D^{\boldsymbol{\mu}})$. Then ${}^\flat D^{\boldsymbol{\mu}}$ is an irreducible ${}^\flat\mathcal{R}_\beta^\Lambda$-module and ${}^\flat Y^{\boldsymbol{\mu}}$ is the projective cover of ${}^\flat D^{\boldsymbol{\mu}}$ in ${}^\flat\mathcal{R}_\beta^\Lambda$-Mod. Moreover, directly from the definitions, ${}^\flat\mathcal{R}_\beta^\Lambda=\bigoplus_{\boldsymbol{\mu}\in\mathcal{K}_\beta^\Lambda}{}^\flat Y^{\boldsymbol{\mu}}$ as a $\mathbb{C}$-vector space.

Recall from Corollary 2.8, and section 2.2, that $\mathbf{C}_{\mathcal{R}_\beta^\Lambda}(q)=(c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q))_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{K}_\beta^\Lambda}$ is the Cartan matrix of $\mathcal{R}_\beta^\Lambda$, and $\mathbf{D}_{\mathcal{R}_\beta^\Lambda}(q)=(d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q))_{\boldsymbol{\lambda}\in\mathscr{P}_\beta^\Lambda,\boldsymbol{\mu}\in\mathcal{K}_\beta^\Lambda}$ is the decomposition matrix, where $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)=\mathrm{D{\scriptstyle IM}}\,\mathrm{H{\scriptstyle OM}}_{\mathcal{R}_\beta^\Lambda}(Y^{\boldsymbol{\mu}},Y^{\boldsymbol{\lambda}})$ and

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)=[S^{\boldsymbol{\lambda}}:D^{\boldsymbol{\mu}}]_q=\mathrm{D{\scriptstyle IM}}\,\mathrm{H{\scriptstyle OM}}_{\mathcal{R}_\beta^\Lambda}(Y^{\boldsymbol{\mu}},S^{\boldsymbol{\lambda}}).$$

Moreover, $\mathbf{C}_{\mathcal{R}_\beta^\Lambda}(q)=\mathbf{D}_{\mathcal{R}_\beta^\Lambda}(q)^{tr}\mathbf{D}_{\mathcal{R}_\beta^\Lambda}(q)$.

The next result should be compared with Proposition 7.6.

7.11. **Proposition.** *Suppose that $\beta\in Q_n^+$. Then:*

a) *As ungraded algebras, ${}^\flat\underline{\mathcal{R}}_\beta^\Lambda\cong\underline{\mathcal{R}}_\beta^{\mathcal{O}}$.*
b) $\mathrm{D{\scriptstyle IM}}\,{}^\flat\mathcal{R}_\beta^\Lambda=\mathrm{D{\scriptstyle IM}}\,\mathcal{R}_\beta^{\mathcal{O}}$, *so ${}^\flat\mathcal{R}_\beta^\Lambda$ is a positively graded algebra.*
c) *The degree zero component of ${}^\flat\mathcal{R}_\beta^\Lambda$ is semisimple and isomorphic to*

$$(\mathcal{R}_\beta^{\mathcal{O}})_0=\bigoplus_{\boldsymbol{\mu}\in\mathcal{K}_\beta^\Lambda}{}^\flat D^{\boldsymbol{\mu}}.$$

Proof. The algebras $\mathcal{R}_\beta^{\mathcal{O}}$ and ${}^\flat\mathcal{R}_\beta^\Lambda$ are both basic algebras, so $\underline{\mathcal{R}}_\beta^{\mathcal{O}}\cong{}^\flat\underline{\mathcal{R}}_\beta^\Lambda$ by Proposition 7.6(c). (In particular, this implies that $\dim\mathcal{R}_\beta^{\mathcal{O}}=\dim{}^\flat\mathcal{R}_\beta^\Lambda$.)

For part (b), calculating directly from the definitions

$$\operatorname{DIM}{}^{\flat}\mathcal{R}^{\Lambda}_{\beta} = \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{K}^{\Lambda}_{\beta}} \operatorname{DIM}\operatorname{HOM}_{\mathcal{R}^{\Lambda}_{\beta}}(Y^{\boldsymbol{\mu}}, Y^{\boldsymbol{\lambda}}) = \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{K}^{\Lambda}_{\beta}} c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q).$$

By Corollary 2.8, $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = \sum_{\boldsymbol{\nu}} d_{\boldsymbol{\nu}\boldsymbol{\lambda}}(q) d_{\boldsymbol{\nu}\boldsymbol{\mu}}(q)$. Similarly,

$$\operatorname{DIM}\mathcal{R}^{\mathcal{O}}_{\beta} = \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{K}^{\Lambda}_{\beta}} \operatorname{DIM}\operatorname{HOM}_{\mathcal{R}^{\mathcal{O}}_{\beta}}(Y^{\boldsymbol{\mu}}_{\mathcal{O}}, Y^{\boldsymbol{\lambda}}_{\mathcal{O}}).$$

Therefore, by BGG reciprocity and Theorem 7.8, $\operatorname{DIM}{}^{\flat}\mathcal{R}^{\Lambda}_{\beta} = \operatorname{DIM}\mathcal{R}^{\mathcal{O}}_{\beta}$. In particular, ${}^{\flat}\mathcal{R}^{\Lambda}_{\beta}$ is positively graded.

It remains to show that the degree zero component of ${}^{\flat}\mathcal{R}^{\Lambda}_{\beta}$ is semisimple. By Theorem 7.8, $d_{\boldsymbol{\nu}\boldsymbol{\mu}}(q) \in \delta_{\boldsymbol{\nu}\boldsymbol{\mu}} + q\mathbb{N}[q]$ so $c_{\boldsymbol{\nu}\boldsymbol{\mu}}(q) \in \delta_{\boldsymbol{\nu}\boldsymbol{\mu}} + q\mathbb{N}[q]$ and $\operatorname{DIM}{}^{\flat}\mathcal{R}^{\Lambda}_{\beta}$ is a polynomial in $\mathbb{N}[q]$ with constant term $|\mathcal{K}^{\Lambda}_{\beta}|$. It follows that the degree zero component of ${}^{\flat}\mathcal{R}^{\Lambda}_{\beta}$ is exactly the span of the identity maps on $Y^{\boldsymbol{\mu}}$ for $\boldsymbol{\mu} \in \mathcal{K}^{\Lambda}_{\beta}$, which is a semisimple algebra. In particular, $({}^{\flat}\mathcal{R}^{\Lambda}_{\beta})_0$ is equal to $\oplus_{\boldsymbol{\mu}\in\mathcal{K}^{\Lambda}_{\beta}} {}^{\flat}D^{\boldsymbol{\mu}}$ as claimed. $\square$

7.12. **Corollary.** *Suppose that $\beta \in Q^+_n$ and $\boldsymbol{\mu} \in \mathcal{K}^{\Lambda}_{\beta}$. Then $\operatorname{soc} Y^{\boldsymbol{\mu}}_{\mathcal{O}} = (Y^{\boldsymbol{\mu}}_{\mathcal{O}})_{2\operatorname{def}\beta} \cong D^{\boldsymbol{\mu}}_{\mathcal{O}}\langle 2\operatorname{def}\beta\rangle$ and $\operatorname{soc}{}^{\flat}Y^{\boldsymbol{\mu}} = ({}^{\flat}Y^{\boldsymbol{\mu}})_{2\operatorname{def}\beta} \cong {}^{\flat}D^{\boldsymbol{\mu}}\langle 2\operatorname{def}\beta\rangle$.*

Proof. If $\boldsymbol{\mu} \in \mathcal{K}^{\Lambda}_{\beta}$ then $(Y^{\boldsymbol{\mu}})^{\circledast} \cong Y^{\boldsymbol{\mu}}\langle -2\operatorname{def}\beta\rangle$ by Corollary 5.8. In view of the Proposition, this implies the result. $\square$

Since $\mathsf{F}^{\mathcal{O}}_{\beta}$ is fully faithful on projectives by Lemma 2.11, there is an isomorphism of graded algebras

$$\mathcal{R}^{\mathcal{O}}_{\beta} \cong \operatorname{END}_{\mathcal{S}^{\mathcal{O}}_{\beta}}\Big(\bigoplus_{\boldsymbol{\mu}\in\mathcal{K}^{\Lambda}_{\beta}} P^{\boldsymbol{\mu}}_{\mathcal{O}}\Big). \tag{7.13}$$

Henceforth, we identify these two algebras. The advantage of working with $P^{\boldsymbol{\mu}}_{\mathcal{O}}$ is that $P^{\boldsymbol{\mu}}_{\mathcal{O}}$ is rigid by Corollary 7.3, for $\boldsymbol{\mu} \in \mathcal{K}^{\Lambda}_{\beta}$. Using this fact we now construct a basis of $\mathcal{S}^{\mathcal{O}}_{\beta}$ using radical filtrations of $P^{\boldsymbol{\mu}}_{\mathcal{O}}$, for $\boldsymbol{\mu} \in \mathscr{P}^{\Lambda}_{\beta}$:

$$P^{\boldsymbol{\mu}}_{\mathcal{O}} = \operatorname{rad}^0 P^{\boldsymbol{\mu}}_{\mathcal{O}} \supset \operatorname{rad}^1 P^{\boldsymbol{\mu}}_{\mathcal{O}} \supset \cdots \supset \operatorname{rad}^z P^{\boldsymbol{\mu}}_{\mathcal{O}} \supset 0,$$

for some $z \geq 0$. By Corollary 7.3, $P^{\boldsymbol{\mu}}_{\mathcal{O}}$ is rigid with its radical filtrating being equal to its grading filtration. Therefore,

$$\operatorname{rad}^d P^{\boldsymbol{\mu}}_{\mathcal{O}} / \operatorname{rad}^{d+1} P^{\boldsymbol{\mu}}_{\mathcal{O}} \cong \bigoplus_{\boldsymbol{\lambda}\in\mathscr{P}^{\Lambda}_{\beta}} (L^{\boldsymbol{\lambda}}_{\mathcal{O}}\langle d\rangle)^{\oplus c^{(d)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}},$$

for $0 \leq d \leq z$. Here we write $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = \sum_{d\geq 0} c^{(d)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} q^d$ for $c^{(d)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} \in \mathbb{N}$. Fix $\boldsymbol{\lambda} \in \mathscr{P}^{\Lambda}_{\beta}$ and $d \geq 0$ with $c^{(d)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} \neq 0$. Since $P^{\boldsymbol{\lambda}}_{\mathcal{O}}\langle d\rangle$ is the projective cover of $L^{\boldsymbol{\lambda}}_{\mathcal{O}}\langle d\rangle$ there exist homogeneous maps $\theta^{(d,s)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} \in \operatorname{HOM}_{\mathcal{S}^{\mathcal{O}}_{\beta}}(P^{\boldsymbol{\lambda}}_{\mathcal{O}}, \operatorname{rad}^d P^{\boldsymbol{\mu}}_{\mathcal{O}})$ such that the diagrams commute

$$\begin{tikzcd}
 & & & P^{\boldsymbol{\lambda}}_{\mathcal{O}} \arrow[d, "\langle d\rangle"] \arrow[ddl, "\theta^{(d,s)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}"'] & \\
 & & & P^{\boldsymbol{\lambda}}_{\mathcal{O}}\langle d\rangle \arrow[d, two heads] \arrow[dl, dashed] & \\
0 \arrow[r] & \operatorname{rad}^{d+1}(P^{\boldsymbol{\mu}}_{\mathcal{O}}) \arrow[r] & \operatorname{rad}^{d}(P^{\boldsymbol{\mu}}_{\mathcal{O}}) \arrow[r] & L^{\boldsymbol{\lambda}}_{\mathcal{O}}\langle d\rangle \arrow[r] & 0
\end{tikzcd} \tag{7.14}$$

for $1 \le s \le c^{(d)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$. By embedding $\operatorname{rad}^d P^{\boldsymbol{\mu}}_{\mathcal{O}}$ into $P^{\boldsymbol{\mu}}_{\mathcal{O}}$ we consider $\theta^{(d,s)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ as a homogeneous element of $\mathcal{S}^{\mathcal{O}}_\beta$ of degree d.

7.15. **Lemma.** *Suppose that $\beta \in Q^+_n$. Then*

$$\Theta^{\mathcal{O}}_\beta = \{\, \theta^{(d,s)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} \mid 1 \le s \le c^{(d)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}, 0 \le d \le 2\operatorname{def}\beta \text{ and } \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\Lambda_\beta \,\}$$

is a homogeneous basis of $\mathcal{R}^{\mathcal{O}}_\beta \cong \operatorname{END}_{\mathcal{S}^{\mathcal{O}}_\beta}(\bigoplus_{\boldsymbol{\mu} \in \mathcal{K}^\Lambda_\beta} P^{\boldsymbol{\mu}}_{\mathcal{O}})$. Moreover, $\deg \theta^{(d,s)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} = d$ for all $\theta^{(d,s)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} \in \Theta^{\mathcal{O}}_\beta$.

Proof. Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\Lambda_\beta$. The set $\{\, \theta^{(d,s)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} \mid 1 \le s \le c^{(d)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} \text{ and } d \ge 0 \,\}$ is linearly independent by construction, so it is a homogeneous basis of $\operatorname{Hom}_{\mathcal{S}^{\mathcal{O}}_\beta}(P^{\boldsymbol{\lambda}}_{\mathcal{O}}, P^{\boldsymbol{\mu}}_{\mathcal{O}})$ by counting (graded) dimensions. If $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}^\Lambda_\beta$ then $c^{(d)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} = 0$ if $d > 2\operatorname{def}\beta$ by Corollary 7.12 (moreover, $c^{(2\operatorname{def}\beta)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} = 0$ if $\boldsymbol{\lambda} \ne \boldsymbol{\mu}$), so the lemma follows. □

We want to use the same construction to give a basis of $\operatorname{HOM}_{\mathcal{S}^\Lambda_\beta}(P^{\boldsymbol{\lambda}}, P^{\boldsymbol{\mu}})$, when $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}^\Lambda_\beta$. Unfortunately, we are not yet able to identify the radical filtration of $P^{\boldsymbol{\mu}}$ with a grading filtration because $\mathcal{S}^\Lambda_\beta$ is not positively graded in general. To remedy this define a filtration of $P^{\boldsymbol{\mu}}$ by setting

$$P^{\boldsymbol{\mu}}(f) = \sum_{\substack{\theta \in \operatorname{Hom}_{\mathcal{S}^\Lambda_\beta}(P^{\boldsymbol{\lambda}}, P^{\boldsymbol{\mu}}),\\ \deg\theta \ge f, \boldsymbol{\lambda} \in \mathcal{K}^\Lambda_\beta}} \operatorname{im}\theta,$$

for $f \ge 0$. We remark that it can happen that $P^{\boldsymbol{\mu}}(f') = P^{\boldsymbol{\mu}}(f)$ for $f' \ne f$. For example, direct calculations show that if $\Lambda = \ell\Lambda_0$ and $\beta = \alpha_0$ then $\mathcal{K}^\Lambda_\beta = \{(0|\dots|0|1)\}$ and if $\boldsymbol{\mu} = (0|\dots|0|1)$ then $P^{\boldsymbol{\mu}}$ is evenly graded (that is, all of the homogeneous elements of $P^{\boldsymbol{\mu}}$ have even degree), so that $P^{\boldsymbol{\mu}}(2f-1) = P^{\boldsymbol{\mu}}(2f)$ for all $f \ge 1$.

7.16. **Lemma.** *Suppose that $\boldsymbol{\mu} \in \mathcal{K}^\Lambda_\beta$. Then*

$$P^{\boldsymbol{\mu}} = P^{\boldsymbol{\mu}}(0) \supseteq P^{\boldsymbol{\mu}}(1) \supseteq \cdots \supseteq P^{\boldsymbol{\mu}}(2\operatorname{def}\beta) \supset 0$$

is a filtration of $P^{\boldsymbol{\mu}}$. Moreover, if $\boldsymbol{\lambda} \in \mathcal{K}^\Lambda_\beta$ then $[P^{\boldsymbol{\mu}}(f) : L^{\boldsymbol{\lambda}}\langle f\rangle] = c^{(f)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ and $L^{\boldsymbol{\lambda}}\langle s\rangle$ is a composition factor of $P^{\boldsymbol{\mu}}(f)$ only if $s \ge f$.

Proof. It is immediate that the construction above defines a filtration of $P^{\boldsymbol{\mu}}$. Moreover, $P^{\boldsymbol{\mu}}(f) = 0$ if $f > 2\operatorname{def}\beta$ by Corollary 7.12. By Theorem 7.8, if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}^\Lambda_\beta$ then $[P^{\boldsymbol{\mu}} : L^{\boldsymbol{\lambda}}]_q \in \delta_{\boldsymbol{\nu},\boldsymbol{\lambda}} + q\mathbb{N}[q]$. Therefore, $L^{\boldsymbol{\lambda}}\langle s\rangle$ is a composition factor of $P^{\boldsymbol{\mu}}(f)$ only if $s \ge f$ and, moreover, if $k \ge f$ then $L^{\boldsymbol{\lambda}}\langle k\rangle$ is not a composition factor of $P^{\boldsymbol{\mu}}/P^{\boldsymbol{\mu}}(f)$. Hence, the remaining statement in the lemma follows from Theorem 7.8. □

7.17. **Proposition.** *Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}^\Lambda_\beta$ and $d \ge 0$. Then*

$$[\operatorname{rad}^d P^{\boldsymbol{\mu}} / \operatorname{rad}^{d+1} P^{\boldsymbol{\mu}} : L^{\boldsymbol{\lambda}}]_q = c^{(d)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} q^d = [\operatorname{rad}^d P^{\boldsymbol{\mu}}_{\mathcal{O}} / \operatorname{rad}^{d+1} P^{\boldsymbol{\mu}}_{\mathcal{O}} : L^{\boldsymbol{\lambda}}_{\mathcal{O}}]_q.$$

Proof. First consider $P^{\boldsymbol{\mu}}_{\mathcal{O}}$. By Corollary 7.3, $P^{\boldsymbol{\mu}}_{\mathcal{O}}$ is rigid with both its radical and socle filtrations being equal to its grading filtration. Since $[P^{\boldsymbol{\mu}}_{\mathcal{O}} : L^{\boldsymbol{\lambda}}_{\mathcal{O}}]_q = c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$ by Theorem 7.8 (and Corollary 4.34), this proves the right hand equality for $P^{\boldsymbol{\mu}}_{\mathcal{O}}$.

To prove the result for $P^{\boldsymbol{\mu}}$ we argue by induction on d. If $d = 0$ the result is automatic since $P^{\boldsymbol{\mu}}$ is the projective cover of $L^{\boldsymbol{\mu}}$ in $\mathcal{S}^\Lambda_\beta$-Mod. Suppose then that $d > 0$. By Theorem 7.4, $\underline{\mathsf{E}}^\Lambda_{\mathcal{O}}(\underline{P}^{\boldsymbol{\mu}}_{\mathcal{O}}) \cong \underline{P}^{\boldsymbol{\mu}}$, so $\underline{\mathsf{E}}^\Lambda_{\mathcal{O}}(\operatorname{rad}^d \underline{P}^{\boldsymbol{\mu}}_{\mathcal{O}}) \cong \operatorname{rad}^d \underline{P}^{\boldsymbol{\mu}}$. Therefore, $[\operatorname{rad}^d \underline{P}^{\boldsymbol{\mu}} / \operatorname{rad}^{d+1} \underline{P}^{\boldsymbol{\mu}} : \underline{L}^{\boldsymbol{\lambda}}] = c^{(d)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$. Therefore, to complete the proof it is enough to show that $L^{\boldsymbol{\mu}}\langle s\rangle$ is a composition factor of $\operatorname{rad}^d P^{\boldsymbol{\mu}} / \operatorname{rad}^{d+1} P^{\boldsymbol{\mu}}$ only if $s = d$ since

$[P^{\boldsymbol{\mu}}:L^{\boldsymbol{\lambda}}]_q = c_{\boldsymbol{\lambda\mu}}(q)$ by Theorem 7.8. Refining the filtration from Lemma 7.16, if necessary, shows that $P^{\boldsymbol{\mu}}$ has a filtration of the form

$$P^{\boldsymbol{\mu}} = P_0 \supset P_1 \supset \cdots \supset P_{l-1} \supset P_l = 0,$$

such that P_k/P_{k+1} is semisimple for $0 \le k < l$ and $P^{\boldsymbol{\mu}}(f) = P_{p_f}$ for some integers $0 = p_0 \le p_1 \le \dots$. Now $P^{\boldsymbol{\mu}}_{\mathcal{O}}$ is rigid by Corollary 7.3, so $P^{\boldsymbol{\mu}}$ is rigid by Theorem 7.4. Therefore, the radical filtration is the unique filtration of $P^{\boldsymbol{\mu}}$ which has semisimple subquotients and length $\ell\ell(P^{\boldsymbol{\mu}})$, the Loewy length of $P^{\boldsymbol{\mu}}$. So, if $l = \ell\ell(P^{\boldsymbol{\mu}})$ then $\operatorname{rad}^k P^{\boldsymbol{\mu}} = P_k$ for $k \ge 0$. Otherwise, $l > \ell\ell(P^{\boldsymbol{\mu}})$ and by omitting some of the modules in the displayed equation above we can construct a filtration of $P^{\boldsymbol{\mu}}$ of length $\ell\ell(P^{\boldsymbol{\mu}})$ which has semisimple quotients. Therefore, by rigidity there exist integers $0 = r_0 > r_1 > \dots$ such that $\operatorname{rad}^k P^{\boldsymbol{\mu}} = P_{r_k}$ for $k \ge 0$. Consequently, for any two non-negative integers d and f either $\operatorname{rad}^d P^{\boldsymbol{\mu}} \subseteq P^{\boldsymbol{\mu}}(f)$ or $P^{\boldsymbol{\mu}}(f) \subseteq \operatorname{rad}^d P^{\boldsymbol{\mu}}$.

We are now ready to complete the proof. First observe that, since $\mathsf{E}^{\Lambda}_{\mathcal{O}}(\underline{P}^{\boldsymbol{\mu}}_{\mathcal{O}}) \cong \underline{P}^{\boldsymbol{\mu}}$,

$$[\operatorname{rad}^d \underline{P}^{\boldsymbol{\mu}}/\operatorname{rad}^{d+1}\underline{P}^{\boldsymbol{\mu}} : \underline{L}^{\boldsymbol{\lambda}}] = [\operatorname{rad}^d \underline{P}^{\boldsymbol{\mu}}_{\mathcal{O}}/\operatorname{rad}^{d+1}\underline{P}^{\boldsymbol{\mu}} : \underline{L}^{\boldsymbol{\lambda}}_{\mathcal{O}}] = c^{(d)}_{\boldsymbol{\lambda\mu}}. \tag{7.18}$$

Therefore, if $c^{(d)}_{\boldsymbol{\lambda\mu}} = 0$ then $L^{\boldsymbol{\mu}}\langle s\rangle$ is not a composition factor of $\operatorname{rad}^d P^{\boldsymbol{\mu}}/\operatorname{rad}^{d+1} P^{\boldsymbol{\mu}}$, for any $s \in \mathbb{Z}$, so the result holds. Suppose then that $c^{(d)}_{\boldsymbol{\lambda\mu}} \ne 0$ and let $L^{\boldsymbol{\lambda}}\langle s\rangle$ be a composition factor of $\operatorname{rad}^d P^{\boldsymbol{\mu}}/\operatorname{rad}^{d+1} P^{\boldsymbol{\mu}}$. Using induction for the second equality,

$$[\operatorname{rad}^d P^{\boldsymbol{\mu}} : L^{\boldsymbol{\lambda}}]_q = [P^{\boldsymbol{\mu}} : L^{\boldsymbol{\lambda}}]_q - [P^{\boldsymbol{\mu}}/\operatorname{rad}^d P^{\boldsymbol{\mu}} : L^{\boldsymbol{\lambda}}]_q = \sum_{k\ge d} c^{(k)}_{\boldsymbol{\lambda\mu}} q^k,$$

so $s \ge d$. Using the construction of (7.14) this implies that $P^{\boldsymbol{\mu}}(d) \subseteq \operatorname{rad}^d P^{\boldsymbol{\mu}}$. If $P^{\boldsymbol{\mu}}(d) \subseteq \operatorname{rad}^{d+1} P^{\boldsymbol{\mu}}$ then $[\operatorname{rad}^d P^{\boldsymbol{\mu}}/\operatorname{rad}^{d+1} P^{\boldsymbol{\mu}} : L^{\boldsymbol{\lambda}}\langle s\rangle] \le [\operatorname{rad}^d P^{\boldsymbol{\mu}}/P^{\boldsymbol{\mu}}(d) : L^{\boldsymbol{\lambda}}\langle s\rangle]$ which is non-zero only if $s < d$ using the definition of $P^{\boldsymbol{\mu}}(d)$. This is a contradiction, so $\operatorname{rad}^{d+1} P^{\boldsymbol{\mu}} \subsetneq P^{\boldsymbol{\mu}}(d)$ by the last paragraph. Let f be maximal such that $\operatorname{rad}^{d+1} P^{\boldsymbol{\mu}} \subseteq P^{\boldsymbol{\mu}}(f)$. Then $f \ge d+1$ and

$$\begin{aligned}[\operatorname{rad}^d P^{\boldsymbol{\mu}}/\operatorname{rad}^{d+1} P^{\boldsymbol{\mu}} : L^{\boldsymbol{\lambda}}]_q &\ge [\operatorname{rad}^d P^{\boldsymbol{\mu}}/P^{\boldsymbol{\mu}}(f) : L^{\boldsymbol{\lambda}}]_q \\ &= [P^{\boldsymbol{\mu}}(d)/P^{\boldsymbol{\mu}}(f) : L^{\boldsymbol{\lambda}}]_q = \sum_{k=d}^{f-1} c^{(k)}_{\boldsymbol{\lambda\mu}} q^k,\end{aligned}$$

where last two equalities come from Lemma 7.16. Therefore, by (7.18), either $f = d+1$ or $f > d+1$ and $c^{(d+1)}_{\boldsymbol{\lambda\mu}} = \cdots = c^{(f-1)}_{\boldsymbol{\lambda\mu}} = 0$. Consequently, $L^{\boldsymbol{\lambda}}$ is a composition factor of $\operatorname{rad}^d P^{\boldsymbol{\mu}}/\operatorname{rad}^{d+1} P^{\boldsymbol{\mu}}$ with graded multiplicity $c^{(d)}_{\boldsymbol{\lambda\mu}} q^d$. This completes the proof of the inductive step and, hence, the proposition. □

Just as in (7.13), in order to apply this result we now identify ${}^{\flat}\mathcal{R}^{\Lambda}_{\beta}$ with a subalgebra of the basic algebra of $\mathcal{S}^{\Lambda}_{\beta}$, which is the algebra

$${}^{\flat}\mathcal{S}^{\Lambda}_{\beta} = \operatorname{END}_{\mathcal{S}^{\Lambda}_{\beta}}\Big(\bigoplus_{\boldsymbol{\mu}\in\mathscr{P}^{\Lambda}_{\beta}} P^{\boldsymbol{\lambda}}\Big).$$

Mimicking (7.10), let $\mathsf{E}_{\mathcal{S}^{\Lambda}_{\beta}} : \mathcal{S}^{\Lambda}_{\beta}\text{-Mod} \longrightarrow {}^{\flat}\mathcal{S}^{\Lambda}_{\beta}\text{-Mod}; M \mapsto \operatorname{HOM}_{\mathcal{S}^{\Lambda}_{\beta}}(\bigoplus_{\boldsymbol{\mu}\in\mathscr{P}^{\Lambda}_{\beta}} P^{\boldsymbol{\mu}}, M)$ be corresponding equivalence of graded module categories and let $\mathsf{E}^*_{\mathcal{S}^{\Lambda}_{\beta}}$ be the inverse equivalence. If $\boldsymbol{\lambda} \in \mathscr{P}^{\Lambda}_{\beta}$ let ${}^{\flat}P^{\boldsymbol{\lambda}} = \mathsf{E}_{\mathcal{S}^{\Lambda}_{\beta}}(P^{\boldsymbol{\lambda}})$ be projective indecomposable ${}^{\flat}\mathcal{S}^{\Lambda}_{\beta}$-module which corresponds to $P^{\boldsymbol{\mu}}$ under this equivalence. As in (7.13), there is an isomorphism

$${}^{\flat}\mathcal{R}^{\Lambda}_{\beta} \cong \operatorname{END}_{\mathcal{S}^{\Lambda}_{\beta}}\Big(\bigoplus_{\boldsymbol{\mu}\in\mathcal{K}^{\Lambda}_{\beta}} P^{\boldsymbol{\mu}}\Big) \tag{7.19}$$

of graded algebras. We now identify these two algebras.

7.20. **Theorem.** *Suppose that $e = 0$, $\mathcal{Z} = \mathbb{C}$ and $\beta \in Q_n^+$. Then there is an isomorphism $\Xi : \mathcal{R}_\beta^{\mathcal{O}} \xrightarrow{\sim} {}^\flat\mathcal{R}_\beta^\Lambda$ of graded algebras.*

Proof. By Theorem 7.4 the algebras $\underline{\mathcal{S}}_\beta^{\mathcal{O}}$ and $\underline{\mathcal{S}}_\beta^\Lambda$ are Morita equivalent, so there is an isomorphism of (ungraded) basic algebras $\Xi : \underline{\mathcal{S}}_\beta^{\mathcal{O}} \longrightarrow {}^\flat\underline{\mathcal{S}}_\beta^\Lambda$ such that $\Xi(\underline{P}_{\mathcal{O}}^{\boldsymbol\lambda}) \cong {}^\flat\underline{P}^{\boldsymbol\lambda}$ as ${}^\flat\underline{\mathcal{S}}_\beta^\Lambda$-modules, for all $\boldsymbol\lambda \in \mathscr{P}_\beta^\Lambda$. Using the identifications in (7.13) and (7.19), Ξ restricts to an isomorphism $\underline{\mathcal{R}}_\beta^{\mathcal{O}} \xrightarrow{\sim} {}^\flat\underline{\mathcal{R}}_\beta^\Lambda$ which, by abuse of notation, we also denote by Ξ.

Recall from Lemma 7.15 that $\Theta_\beta^{\mathcal{O}} = \{\theta_{\boldsymbol{\lambda\mu}}^{(d,s)}\}$ is a basis of $\mathrm{H}\textsc{om}_{\mathcal{S}_\beta^{\mathcal{O}}}(P_{\mathcal{O}}^{\boldsymbol\lambda}, P_{\mathcal{O}}^{\boldsymbol\mu})$. For each $\theta_{\boldsymbol{\lambda\mu}}^{(d,s)} \in \Theta_\beta^{\mathcal{O}}$ set $\vartheta_{\boldsymbol{\lambda\mu}}^{(d,s)} = \mathsf{E}_{\mathcal{S}_\beta^\Lambda}^* \circ \Xi \circ \theta_{\boldsymbol{\lambda\mu}}^{(d,s)} \circ \Xi^{-1} \circ \mathsf{E}_{\mathcal{S}_\beta^\Lambda}$. Forgetting the gradings for the moment, this implies that $\{\vartheta_{\boldsymbol{\lambda\mu}}^{(d,s)}\}$ is a basis of $\mathrm{Hom}_{\underline{\mathcal{S}}_\beta^\Lambda}(\underline{P}^{\boldsymbol\lambda}, \underline{P}^{\boldsymbol\mu})$. Moreover, by (7.14) there is a commutative diagram

$$\begin{tikzcd}
 & & & \underline{P}^{\boldsymbol\lambda} \arrow[dl, "\vartheta_{\boldsymbol{\lambda\mu}}^{(d,s)}"'] \arrow[d] & \\
0 \arrow[r] & \mathrm{rad}^{d+1}(\underline{P}^{\boldsymbol\mu}) \arrow[r] & \mathrm{rad}^{d}(\underline{P}^{\boldsymbol\mu}) \arrow[r] & \underline{L}^{\boldsymbol\lambda} \arrow[r] & 0
\end{tikzcd}$$

By Proposition 7.17, $\vartheta_{\boldsymbol{\lambda\mu}}^{(d,s)}$ is a non-zero homogeneous element of degree d, possibly modulo terms of higher degree. By replacing $\vartheta_{\boldsymbol{\lambda\mu}}^{(d,s)}$ with the projection onto its degree d component, if necessary, we may assume that $\deg \vartheta_{\boldsymbol{\lambda\mu}}^{(d,s)} = d$. Since multiplication in $\mathcal{S}_\beta^{\mathcal{O}}$ and in ${}^\flat\mathcal{S}_\beta^\Lambda$ respects degrees, it is straightforward to check that the map $\theta_{\boldsymbol{\lambda\mu}}^{(d,s)} \mapsto \vartheta_{\boldsymbol{\lambda\mu}}^{(d,s)}$ is an isomorphism of graded algebras. $\square$

The proof of Theorem 7.20 is quite subtle in that we have to work with the projective indecomposable modules $P^{\boldsymbol\mu}$ for the quiver Schur algebra $\mathcal{S}_\beta^\Lambda$ and use the fact that these modules are rigid. In many ways it would be more natural to prove this result using the graded Young modules $Y^{\boldsymbol\mu}$ but as these modules are not known to be rigid we cannot argue this way. In all of the examples that we have computed it turns out that the Young modules are rigid. We do not know whether or not this is true in general.

7.3. **Graded decomposition numbers when $e = 0$.** Now that we have shown that the KLR and category $\mathcal{O}_n^\Lambda$ gradings coincide at the level of the Hecke algebras the next step is to show that the gradings on the Schur algebras $\mathcal{S}_\beta^{\mathcal{O}}$ and $\mathcal{S}_\beta^\Lambda$ agree.

Recall from the last section that $Y_{\mathcal{O}}^{\boldsymbol\mu} = \mathsf{F}_\beta^{\mathcal{O}}(P_{\mathcal{O}}^{\boldsymbol\mu})$, for $\boldsymbol\mu \in \mathscr{P}_\beta^\Lambda$, is a Young module for $\mathcal{R}_\beta^{\mathcal{O}}$. Similarly, let ${}^\flat Y^{\boldsymbol\mu} = \mathsf{E}_{\mathcal{R}_\beta^\Lambda}(Y^{\boldsymbol\mu})$ be a Young module for ${}^\flat\mathcal{R}_\beta^\Lambda$. Using the isomorphism $\Xi : \mathcal{R}_\beta^{\mathcal{O}} \xrightarrow{\sim} {}^\flat\mathcal{R}_\beta^\Lambda$ from Theorem 7.20 we can consider $Y_{\mathcal{O}}^{\boldsymbol\mu}$ as a ${}^\flat\mathcal{R}_\beta^\Lambda$-module.

7.21. **Lemma.** *Suppose that $\boldsymbol\mu \in \mathscr{P}_\beta^\Lambda$. Then ${}^\flat Y^{\boldsymbol\mu} \cong Y_{\mathcal{O}}^{\boldsymbol\mu}\langle a_{\boldsymbol\mu}\rangle$ as ${}^\flat\mathcal{R}_\beta^\Lambda$-modules, for some $a_{\boldsymbol\mu} \in \mathbb{Z}$.*

Proof. By [9, Lemma 6.11] (and Lemma 6.11), $Y_{\mathcal{O}}^{\boldsymbol\mu}$ is a graded lift of $\underline{Y}^{\boldsymbol\mu}$. By Proposition 5.6, $\underline{Y}^{\boldsymbol\mu}$ is indecomposable so, up to shift, it has a unique graded lift as an ${}^\flat\mathcal{R}_\beta^\Lambda$-module; see section 2.1. That is, ${}^\flat Y^{\boldsymbol\mu} \cong Y_{\mathcal{O}}^{\boldsymbol\mu}\langle a_{\boldsymbol\mu}\rangle$, for some $a_{\boldsymbol\mu} \in \mathbb{Z}$. $\square$

7.22. **Theorem.** *Suppose that $e=0$, $\mathcal{Z}=\mathbb{C}$ and $\beta\in Q_n^+$. Then*

$$\operatorname{DIM}{}^\flat\mathcal{S}_\beta^\Lambda=\operatorname{DIM}\mathcal{S}_\beta^\mathcal{O}.$$

In particular, ${}^\flat\mathcal{S}_\beta^\Lambda$ is positively graded.

Proof. For any $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda$, let $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$ and $c^\mathcal{O}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$ be the graded Cartan numbers of ${}^\flat\mathcal{S}_\beta^\Lambda$ and $\mathcal{S}_\beta^\mathcal{O}$, respectively. (Then $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)=c^\mathcal{O}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$ by Theorem 7.8, for $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{K}_\beta^\Lambda$.) Then $\operatorname{DIM}{}^\flat\mathcal{S}_\beta^\Lambda=\sum_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda}c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$. By definition,

$$\begin{aligned}
c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)&=\operatorname{DIM}\operatorname{HOM}_{\mathcal{S}_\beta^\Lambda}(P^{\boldsymbol{\mu}},P^{\boldsymbol{\lambda}})\\
&=\operatorname{DIM}\operatorname{HOM}_{\mathcal{R}_\beta^\Lambda}(Y^{\boldsymbol{\mu}},Y^{\boldsymbol{\lambda}}), && \text{by Lemma 2.11,}\\
&=\operatorname{DIM}\operatorname{HOM}_{{}^\flat\mathcal{R}_\beta^\Lambda}({}^\flat Y^{\boldsymbol{\mu}},{}^\flat Y^{\boldsymbol{\lambda}}) && \text{applying (7.10),}\\
&=\operatorname{DIM}\operatorname{HOM}_{\mathcal{R}_\beta^\mathcal{O}}(Y_\mathcal{O}^{\boldsymbol{\mu}}\langle a_{\boldsymbol{\mu}}\rangle,Y_\mathcal{O}^{\boldsymbol{\lambda}}\langle a_{\boldsymbol{\lambda}}\rangle), && \text{by Lemma 7.21,}\\
&=q^{a_{\boldsymbol{\lambda}}-a_{\boldsymbol{\mu}}}\operatorname{DIM}\operatorname{HOM}_{\mathcal{R}_\beta^\mathcal{O}}(Y_\mathcal{O}^{\boldsymbol{\mu}},Y_\mathcal{O}^{\boldsymbol{\lambda}})\\
&=q^{a_{\boldsymbol{\lambda}}-a_{\boldsymbol{\mu}}}c^\mathcal{O}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q).
\end{aligned}$$

By graded BGG reciprocity, $c^\mathcal{O}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)=c^\mathcal{O}_{\boldsymbol{\mu}\boldsymbol{\lambda}}(q)$, so that $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)=q^{2(a_{\boldsymbol{\lambda}}-a_{\boldsymbol{\mu}})}c_{\boldsymbol{\mu}\boldsymbol{\lambda}}(q)$. However, the Cartan matrix of $\mathcal{S}_\beta^\Lambda$ is symmetric by Corollary 2.8. Therefore, $a_{\boldsymbol{\lambda}}=a_{\boldsymbol{\mu}}$ for all $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda$ since $\mathcal{S}_\beta^\Lambda$ is indecomposable by Theorem 4.36. (In fact, $a_{\boldsymbol{\mu}}=0$ for all $\boldsymbol{\mu}\in\mathcal{K}_\beta^\Lambda$ because if $\boldsymbol{\mu}\in\mathcal{K}_\beta^\Lambda$ then ${}^\flat D^{\boldsymbol{\mu}}\cong\Xi(D_\mathcal{O}^{\boldsymbol{\mu}})$ by Corollary 5.9 since both modules are concentrated in degree zero.) Therefore, $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)=c^\mathcal{O}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\in\mathbb{N}[q]$. Hence, $\operatorname{DIM}{}^\flat\mathcal{S}_\beta^\Lambda=\operatorname{DIM}\mathcal{S}_\beta^\mathcal{O}$ so that ${}^\flat\mathcal{S}_\beta^\Lambda$ is positively graded as claimed. □

7.23. **Corollary.** *Suppose that $e=0$, $\mathcal{Z}=\mathbb{C}$ and $\beta\in Q_n^+$. Then ${}^\flat\mathcal{S}_\beta^\Lambda\cong\mathcal{S}_\beta^\mathcal{O}$ as graded algebras. In particular, ${}^\flat\mathcal{S}_\beta^\Lambda$ is Koszul.*

Proof. Since $\mathcal{S}_\beta^\mathcal{O}$ is Koszul by Theorem 7.2, and ${}^\flat\mathcal{S}_\beta^\Lambda$ is positively graded by the Theorem, this follows immediately from [5, Corollary 2.5.2]. More directly, using Lemma 2.11 twice, there are homogeneous isomorphisms

$$\begin{aligned}
\mathcal{S}_\beta^\mathcal{O}&\cong\operatorname{END}_{\mathcal{S}_\beta^\mathcal{O}}\Big(\bigoplus_{\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda}P_\mathcal{O}^{\boldsymbol{\mu}}\Big)\cong\operatorname{END}_{\mathcal{R}_\beta^\mathcal{O}}\Big(\bigoplus_{\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda}Y_\mathcal{O}^{\boldsymbol{\mu}}\Big)\\
&\cong\operatorname{END}_{{}^\flat\mathcal{R}_\beta^\Lambda}\Big(\bigoplus_{\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda}{}^\flat Y^{\boldsymbol{\lambda}}\Big)\cong\operatorname{END}_{\mathcal{S}_\beta^\Lambda}\Big(\bigoplus_{\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda}P^{\boldsymbol{\mu}}\Big)\cong{}^\flat\mathcal{S}_\beta^\Lambda,
\end{aligned}$$

where the third isomorphism follows from Lemma 7.21 using the fact that $a_{\boldsymbol{\mu}}=0$ for all $\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda$, as was noted in the proof of Theorem 7.22. Hence, ${}^\flat\mathcal{S}_\beta^\Lambda\cong\mathcal{S}_\beta^\mathcal{O}$ is Koszul by Theorem 7.2. □

Define non-negative integers $d^{(s)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ by $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)=\sum_{s\ge0}d^{(s)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}q^s$, for $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda$.

7.24. **Corollary.** *Suppose that $e=0$, $\mathcal{Z}=\mathbb{C}$ and $\beta\in Q_n^+$ and let $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_\beta^\Lambda$. Then $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\in\delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}+q\mathbb{N}[q]$ and $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\in\delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}+q\mathbb{N}[q]$ and if $s\ge0$ then*

$$[\operatorname{rad}^s\Delta^{\boldsymbol{\lambda}}/\operatorname{rad}^{s+1}\Delta^{\boldsymbol{\lambda}}:L^{\boldsymbol{\mu}}\langle s\rangle]_q=d^{(s)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}.$$

Proof. Since ${}^\flat\mathcal{S}_\beta^\Lambda$ is positively graded $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\in\delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}+q\mathbb{N}[q]$ by Corollary 4.34. Consequently, $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\in\delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}+q\mathbb{N}[q]$ by Proposition 2.8. Finally, since $\Delta^{\boldsymbol{\mu}}/\operatorname{rad}\Delta^{\boldsymbol{\mu}}\cong L^{\boldsymbol{\mu}}$ is irreducible the last statement follows from Corollary 7.23 and Lemma 2.13. □

Combining the results in this section we obtain a more precise version of Theorem C from the introduction.

7.25. **Theorem.** *Suppose that $\beta \in Q_n^+$, $e = 0$ and $\mathcal{Z} = \mathbb{C}$. Then there is an equivalence of categories $\mathsf{E}_\beta^{\mathcal{O}} : \mathcal{O}_\beta^\Lambda \longrightarrow \mathcal{S}_\beta^\Lambda\text{-Mod}$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{O}_\beta^\Lambda \arrow[r, "\mathsf{E}_\beta^{\mathcal{O}}"] \arrow[dr, "\mathsf{F}_\beta^{\mathcal{O}}"'] & \mathcal{S}_\beta^\Lambda\text{-Mod} \arrow[d, "\mathsf{F}_\beta^\Lambda"] \\
 & \mathcal{R}_\beta^\Lambda\text{-Mod}
\end{tikzcd}$$

Moreover, $\mathsf{E}_\beta^{\mathcal{O}}(\Delta_{\mathcal{O}}^{\boldsymbol{\lambda}}) \cong \Delta^{\boldsymbol{\lambda}}$ and $\mathsf{E}_\beta^{\mathcal{O}}(L_{\mathcal{O}}^{\boldsymbol{\mu}}) \cong L^{\boldsymbol{\mu}}$, for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$.

7.4. **The Fock Space.** The aim of this subsection is to realize the tilting and irreducible modules for $\mathcal{S}_n^\Lambda$ as canonical and dual canonical bases of the higher level Fock space. Throughout this subsection, we work over $\mathbb{C}$.

Let $\operatorname{Rep}(\mathcal{S}_n^\Lambda)$ be the Grothendieck group of finitely generated $\mathcal{S}_n^\Lambda$-modules. If M is an $\mathcal{S}_n^\Lambda$-module let $[M]$ be its image in $\operatorname{Rep}(\mathcal{S}_n^\Lambda)$. Observe that $\operatorname{Rep}(\mathcal{S}_n^\Lambda)$ is naturally a $\mathbb{Z}[q, q^{-1}]$-module where q acts by grading shift: $q[M] = [M\langle 1\rangle]$, for $M \in \mathcal{S}_n^\Lambda\text{-Mod}$. Similarly, let $\operatorname{Proj}(\mathcal{S}_n^\Lambda)$ be the Grothendieck group for the category of finitely generated projective $\mathcal{S}_n^\Lambda$-modules. The Cartan pairing is the sesquilinear map (anti-linear in the first argument, linear in the second)

$$(\ ,\ ) : \operatorname{Proj}(\mathcal{S}_n^\Lambda) \times \operatorname{Rep}(\mathcal{S}_n^\Lambda) \longrightarrow \mathbb{Z}[q, q^{-1}], \quad ([P], [M]) = \operatorname{Dim} \operatorname{Hom}_{\mathcal{S}_n^\Lambda}(P, M),$$

for $[P] \in \operatorname{Proj}(\mathcal{S}_n^\Lambda)$ and $[M] \in \operatorname{Rep}(\mathcal{S}_n^\Lambda)$. There is a natural embedding $\operatorname{Proj}(\mathcal{S}_n^\Lambda) \hookrightarrow \operatorname{Rep}(\mathcal{S}_n^\Lambda)$.

Define the **combinatorial Fock space** of weight Λ to be

$$\mathfrak{F}^\Lambda = \bigoplus_{n \geq 0} \operatorname{Rep}(\mathcal{S}_n^\Lambda).$$

Thus, $\mathfrak{F}^\Lambda$ is a free $\mathbb{Z}[q, q^{-1}]$-module of infinite rank. Let $\mathscr{P}^\Lambda = \bigcup_{n \geq 0} \mathscr{P}_n^\Lambda$. The Fock space $\mathfrak{F}^\Lambda$ is equipped with the following distinguished bases:

- The irreducible modules: $\{\, [L^{\boldsymbol{\mu}}] \mid \boldsymbol{\mu} \in \mathscr{P}^\Lambda \,\}$.
- The standard modules $\{\, [\Delta^{\boldsymbol{\mu}}] \mid \boldsymbol{\mu} \in \mathscr{P}^\Lambda \,\}$.
- The projective indecomposable modules $\{\, [P^{\boldsymbol{\mu}}] \mid \boldsymbol{\mu} \in \mathscr{P}^\Lambda \,\}$.
- The twisted tilting modules $\{\, [T_{\boldsymbol{\mu}}] \mid \boldsymbol{\mu} \in \mathscr{P}^\Lambda \,\}$.

These all gives bases for $\mathfrak{F}^\Lambda$ as a $\mathbb{Z}[q, q^{-1}]$-module because the graded decomposition matrix of $\mathcal{S}_n^\Lambda$ is invertible over $\mathbb{Z}[q, q^{-1}]$ by Corollary 7.24.

The aim of this section is to clarify the relationships between these bases and to give an algorithm for computing the graded decomposition numbers of $\mathcal{S}_n^\Lambda$.

There is a natural duality on $\operatorname{Rep}(\mathcal{S}_n^\Lambda)$ which induces an involution on $\mathfrak{F}^\Lambda$. Let M be an $\mathcal{S}_n^\Lambda$-module. Recall that $M^\circledast = \operatorname{Hom}_{\mathbb{C}}(M, \mathbb{C})$ is the contragredient dual of M. Similarly, define $M^\# = \operatorname{Hom}_{\mathcal{S}_n^\Lambda}(M, \mathcal{S}_n^\Lambda)$, where $\mathcal{S}_n^\Lambda$ acts on $M^\#$ by $(f \cdot s)(x) = s^* f(x)$, for $f \in M^\#$ and $x \in M, s \in \mathcal{S}_n^\Lambda$. Then $\#$ restricts to a duality on $\operatorname{Proj}(\mathcal{S}_n^\Lambda)$.

7.26. **Lemma.** *Suppose that M is an $\mathcal{S}_n^\Lambda$-module. Then*

$$([P^\#], [M]) = \overline{([P], [M^\circledast])},$$

for all $[P] \in \operatorname{Proj}(\mathcal{S}_n^\Lambda)$ and $[M] \in \operatorname{Rep}(\mathcal{S}_n^\Lambda)$. Moreover, if $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$ then $(L^{\boldsymbol{\mu}})^\circledast \cong L^{\boldsymbol{\mu}}$, $(T_{\boldsymbol{\mu}})^\circledast \cong T_{\boldsymbol{\mu}}$ and $(P^{\boldsymbol{\mu}})^\# \cong P^{\boldsymbol{\mu}}$.

Proof. The first statement is well-known; see, for example, [11, Lemma 2.5]. This implies that $(P^{\boldsymbol{\mu}})^\# \cong P^{\boldsymbol{\mu}}$ since $(L^{\boldsymbol{\mu}})^\circledast \cong L^{\boldsymbol{\mu}}$ by Theorem 2.5. Finally, $(T_{\boldsymbol{\mu}})^\circledast \cong T_{\boldsymbol{\mu}}$ by Theorem 5.22. □

A map $f: M \longrightarrow N$ of $\mathbb{Z}[q, q^{-1}]$-modules is **semilinear** if it is $\mathbb{Z}$-linear and $f(q^k m) = q^{-k} f(m)$, for all $m \in M$ and $k \in \mathbb{Z}$.

7.27. **Lemma.** *The maps $\circledast$ and $\#$ induce semilinear involutions on $\mathfrak{F}^\Lambda$ such that*

$$(M\langle d\rangle)^\circledast \cong M^\circledast\langle -d\rangle \qquad \text{and} \qquad (N\langle d\rangle)^\# \cong N^\#\langle -d\rangle,$$

for all $[M] \in \operatorname{Rep}(\mathcal{S}_n^\Lambda)$, $[N] \in \operatorname{Proj}(\mathcal{S}_n^\Lambda)$ and $d \in \mathbb{Z}$.

Proof. It follows easily from the definitions that $\circledast$ is a duality on $\operatorname{Rep}(\mathcal{S}_n^\Lambda)$ and that $\#$ is a duality on $\operatorname{Proj}(\mathcal{S}_n^\Lambda)$. This immediately implies that $\circledast$ induces an involution on $\mathfrak{F}^\Lambda$ with the required properties. Moreover, $\#$ extends to an automorphism of $\mathfrak{F}^\Lambda$ because $\{\, [P^{\boldsymbol{\mu}}] \mid \boldsymbol{\mu} \in \mathscr{P}^\Lambda \,\}$ is a $\mathbb{Z}[q, q^{-1}]$-basis of $\mathfrak{F}^\Lambda$. The map induced by $\#$ is an involution because $(P^{\boldsymbol{\mu}})^\# \cong P^{\boldsymbol{\mu}}$ by Lemma 7.26, for $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$. $\square$

We emphasize that both of these maps are semilinear – that is, $\mathbb{Z}$-linear but not $\mathbb{Z}[q, q^{-1}]$-linear. This is implicit in the displayed equation of Lemma 7.27 because, for example, $(q[M])^\circledast = [M\langle 1\rangle]^\circledast = [M^\circledast\langle -1\rangle] = q^{-1}[M^\circledast]$.

Recall from section 2.1 that the **bar involution** on $\mathbb{Z}[q, q^{-1}]$ is the $\mathbb{Z}$-linear automorphism of $\mathbb{Z}[q, q^{-1}]$ determined by $\overline{q} = q^{-1}$. A Laurent polynomial $f(q)$ in $\mathbb{Z}[q, q^{-1}]$ is **bar invariant** if $f(q) = \overline{f(q)}$.

7.28. **Lemma.** *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$. Then*

$$[\Delta^{\boldsymbol{\lambda}}]^\circledast = [\Delta^{\boldsymbol{\lambda}}] + \sum_{\substack{\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda \\ \boldsymbol{\lambda} \rhd \boldsymbol{\mu}}} f_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)[\Delta^{\boldsymbol{\mu}}] \quad \text{and} \quad [\Delta^{\boldsymbol{\lambda}}]^\# = [\Delta^{\boldsymbol{\lambda}}] + \sum_{\substack{\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda \\ \boldsymbol{\lambda} \lhd \boldsymbol{\mu}}} g_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)[\Delta^{\boldsymbol{\mu}}],$$

for some Laurent polynomials $f_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q), g_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in \mathbb{Z}[q, q^{-1}]$.

Proof. Recall that $\big(d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\big)_{\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda}$ is the graded decomposition matrix of $\mathcal{S}_n^\Lambda$. Let $\big(e_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\big)_{\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda}$ be the inverse graded decomposition matrix. Using Lemma 7.28 we compute:

$$[\Delta^{\boldsymbol{\lambda}}]^\circledast = \Big(\sum_{\substack{\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda \\ \boldsymbol{\lambda} \unrhd \boldsymbol{\mu}}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)[L^{\boldsymbol{\mu}}]\Big)^\circledast = \sum_{\substack{\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda \\ \boldsymbol{\lambda} \unrhd \boldsymbol{\mu}}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q^{-1})[L^{\boldsymbol{\mu}}]$$

since $(L^{\boldsymbol{\mu}})^\circledast \cong L^{\boldsymbol{\mu}}$ by Theorem 4.25. Therefore,

$$\begin{aligned}
[\Delta^{\boldsymbol{\lambda}}]^\circledast &= \sum_{\substack{\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda \\ \boldsymbol{\lambda} \unrhd \boldsymbol{\mu}}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q^{-1}) \sum_{\substack{\boldsymbol{\nu} \in \mathscr{P}_n^\Lambda \\ \boldsymbol{\mu} \unrhd \boldsymbol{\nu}}} e_{\boldsymbol{\nu}\boldsymbol{\mu}}(q)[\Delta^{\boldsymbol{\nu}}] \\
&= [\Delta^{\boldsymbol{\lambda}}] + \sum_{\substack{\boldsymbol{\nu} \in \mathscr{P}_n^\Lambda \\ \boldsymbol{\lambda} \rhd \boldsymbol{\nu}}} \Big(\sum_{\substack{\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda \\ \boldsymbol{\lambda} \unrhd \boldsymbol{\mu} \unrhd \boldsymbol{\nu}}} e_{\boldsymbol{\nu}\boldsymbol{\mu}}(q) d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q^{-1})\Big)[\Delta^{\boldsymbol{\nu}}],
\end{aligned}$$

where the last line follows because both the graded decomposition matrix and its inverse are triangular with respect to dominance by Corollary 4.34. The formula for $[\Delta^{\boldsymbol{\lambda}}]^\#$ is proved in exactly the same way by first writing $[\Delta^{\boldsymbol{\lambda}}] = \sum_{\boldsymbol{\lambda} \unlhd \boldsymbol{\mu}} d_{\boldsymbol{\mu}\boldsymbol{\lambda}}(q)[P^{\boldsymbol{\mu}}]$. $\square$

By a well-known result of Lusztig [34, Lemma 24.2.1], Lemma 7.28 implies that $\mathfrak{F}^\Lambda$ has several uniquely determined 'canonical bases' which are invariant under $\circledast$ and $\#$. Using Corollary 7.24 we can describe these bases explicitly. Let $\mathfrak{F}_0^\Lambda$ be the $\mathbb{Z}[q]$-sublattice of $\mathfrak{F}^\Lambda$ with basis the images of the standard modules $\{\, [\Delta^{\boldsymbol{\mu}}] \mid \boldsymbol{\mu} \in \mathscr{P}^\Lambda \,\}$ in $\mathfrak{F}^\Lambda$. Similarly, let $\mathfrak{F}_\infty^\Lambda$ be the $\mathbb{Z}[q^{-1}]$-sublattice of $\mathfrak{F}^\Lambda$ spanned by these elements.

7.29. **Theorem.** *Suppose that $e=0$ and $\mathcal{Z}=\mathbb{C}$. Then the three bases*

$$\{\,[P^{\boldsymbol{\mu}}]\mid \boldsymbol{\mu}\in\mathscr{P}^\Lambda\,\},\quad \{\,[L^{\boldsymbol{\mu}}]\mid \boldsymbol{\mu}\in\mathscr{P}^\Lambda\,\}\quad \text{and}\quad \{\,[T_{\boldsymbol{\mu}}]\mid \boldsymbol{\mu}\in\mathscr{P}^\Lambda\,\}$$

are "canonical bases" of $\mathfrak{F}^\Lambda$ which, for $\boldsymbol{\mu}\in\mathscr{P}^\Lambda$, are uniquely determined by:

a) $[P^{\boldsymbol{\mu}}]^\# = [P^{\boldsymbol{\mu}}]$ *and* $[P^{\boldsymbol{\mu}}]\equiv[\Delta^{\boldsymbol{\mu}}]$ (*mod* $q\mathfrak{F}^\Lambda_0$).
b) $[L^{\boldsymbol{\mu}}]^\circledast = [L^{\boldsymbol{\mu}}]$ *and* $[L^{\boldsymbol{\mu}}]\equiv[\Delta^{\boldsymbol{\mu}}]$ (*mod* $q\mathfrak{F}^\Lambda_0$).
c) $[T_{\boldsymbol{\mu}}]^\circledast = [T_{\boldsymbol{\mu}}]$ *and* $[T_{\boldsymbol{\mu}}]\equiv[\Delta^{\boldsymbol{\mu}}]$ (*mod* $q^{-1}\mathfrak{F}^\Lambda_\infty$).

Proof. The existence and uniqueness of bases of $\mathfrak{F}^\Lambda$ with these properties follows from what is by now a standard argument (see [34, Lemma 24.2.1]), using the triangularity of the involutions $\circledast$ and $\#$ from Lemma 7.28. If $\boldsymbol{\mu}\in\mathscr{P}^\Lambda_n$ then $[P^{\boldsymbol{\mu}}]^\# = [P^{\boldsymbol{\mu}}]$, $(L^{\boldsymbol{\mu}})^\circledast\cong L^{\boldsymbol{\mu}}$ and $(T_{\boldsymbol{\mu}})^\circledast\cong T_{\boldsymbol{\mu}}$ by Lemma 7.26. Furthermore, $[P^{\boldsymbol{\mu}}]=\sum_{\boldsymbol{\lambda}} d_{\boldsymbol{\lambda\mu}}(q)[\Delta^{\boldsymbol{\lambda}}]$ and $[L^{\boldsymbol{\mu}}]=\sum_{\boldsymbol{\lambda}} e_{\boldsymbol{\lambda\mu}}(q)[\Delta^{\boldsymbol{\lambda}}]$, where $d_{\boldsymbol{\lambda\mu}}(q)$ and $e_{\boldsymbol{\lambda\mu}}(q)$ are polynomials in $\mathbb{Z}[q]$ with constant term $d_{\boldsymbol{\lambda\mu}}(0)=\delta_{\boldsymbol{\lambda\mu}}=e_{\boldsymbol{\lambda\mu}}(q)$ by Corollary 7.24. Therefore, if $\boldsymbol{\mu}\in\mathscr{P}^\Lambda$ then $[P^{\boldsymbol{\mu}}]$ and $[L^{\boldsymbol{\mu}}]$ belong to $\mathfrak{F}^\Lambda_0$ and, moreover,

$$[P^{\boldsymbol{\mu}}]\equiv[\Delta^{\boldsymbol{\mu}}] \pmod{q\mathfrak{F}^\Lambda_0}\quad\text{and}\quad [\Delta^{\boldsymbol{\mu}}]\equiv[L^{\boldsymbol{\mu}}] \pmod{q\mathfrak{F}^\Lambda_0}$$

Hence, parts (a) and (b) follow. Finally, the twisted tilting module $[T_{\boldsymbol{\mu}}]\in\mathfrak{F}^\Lambda_\infty$ and $[T_{\boldsymbol{\mu}}]\equiv[\Delta^{\boldsymbol{\mu}}] \pmod{q^{-1}\mathfrak{F}^\Lambda_\infty}$ Theorem 5.22. This completes the proof. □

We call $\{\,[P^{\boldsymbol{\mu}}]\mid\boldsymbol{\mu}\in\mathscr{P}^\Lambda\,\}$ the **canonical basis** of $\mathfrak{F}^\Lambda$ and $\{\,[L^{\boldsymbol{\lambda}}]\mid\boldsymbol{\mu}\in\mathscr{P}^\Lambda\,\}$ the **dual canonical basis** because these two bases are dual under the Cartan pairing on $\mathfrak{F}^\Lambda$. By Theorem 5.19, Ringel duality induces a automorphism of $\mathfrak{F}^\Lambda$ which interchanges, setwise, the canonical basis $\{[P^{\boldsymbol{\mu}}]\}$ and the basis $\{[T_{\boldsymbol{\lambda}}]|\boldsymbol{\lambda}\}$ of twisted tilting modules. We remark that Theorem 7.29 should lift to a categorification of the canonical bases of $\mathfrak{F}^\Lambda$ as a $U_q(\mathfrak{gl}_\infty)$-module.

7.30. *Remark.* Abusing notation slightly, let $\#$ be the involution on $\operatorname{Rep}(\mathcal{R}^\Lambda_n)$ defined by $M^\# = \operatorname{HOM}_{\mathcal{R}^\Lambda_n}(M,\mathcal{R}^\Lambda_n)$. Then, as noted in [11, Remark 4.7], it follows from Theorem 3.20 and [40, Theorem 3.1] that there is an isomorphism of functors $\#\cong\langle 2\operatorname{def}\beta\rangle\circ\circledast$. Therefore,

$$\{\,q^{\operatorname{def}\beta}[D^{\boldsymbol{\mu}}]\mid\boldsymbol{\mu}\in\mathcal{K}^\Lambda_\beta\text{ for }\beta\in Q^+\,\}$$

is a $\#$-invariant basis of $\bigoplus_{n\ge0}\operatorname{Rep}(\mathcal{R}^\Lambda_n)$ which has similar uniqueness properties to the twisted tilting module basis of $\mathfrak{F}^\Lambda$. Similarly, $\{\,q^{-\operatorname{def}\beta}[Y^{\boldsymbol{\mu}}]\mid\boldsymbol{\mu}\in\mathcal{K}^\Lambda_\beta\,\}$ is a 'canonical' $\circledast$-invariant basis of $\bigoplus_{n\ge0}\operatorname{Rep}(\mathcal{R}^\Lambda_n)$.

7.5. **An LLT algorithm for $\mathcal{S}^\Lambda_n$.** If $\Lambda=\Lambda_0$ then $\mathcal{H}^\Lambda_n\cong\mathcal{R}^\Lambda_n$ is isomorphic to the Iwahori-Hecke algebra of the symmetric group. In this case, Lascoux, Leclerc and Thibon [32] have given an efficient algorithm for computing the canonical bases of the irreducible $U_q(\widehat{\mathfrak{sl}})$-module $L(\Lambda_0)$. By Ariki's Theorem [1,11], the LLT algorithm computes the (graded) decomposition matrices of the Iwahori-Hecke algebra of the symmetric group.

In this section we give an LLT-like algorithm for computing the canonical basis of $\mathfrak{F}^\Lambda$. By Theorem 7.29 this gives an algorithm for computing the graded decomposition numbers of $\mathcal{R}^\Lambda_n$ and $\mathcal{S}^\Lambda_n$. To this end, if $f(q)=\sum_{d\in\mathbb{Z}}f_dq^d$ is a non-zero Laurent polynomial in $\mathbb{Z}[q,q^{-1}]$ let $\operatorname{mindeg} f(q)=\min\{\,d\in\mathbb{Z}\mid f_d\ne0\,\}$.

Suppose that $\boldsymbol{\mu}\in\mathscr{P}^\Lambda_\beta$, for $\beta\in Q^+$. Recall from (5.1) that $Z^{\boldsymbol{\mu}}=\Psi^{\boldsymbol{\mu}}\mathcal{S}^\Lambda_\beta$.

7.31. **Lemma.** *Suppose that $\boldsymbol{\mu}\in\mathscr{P}^\Lambda$. Then $(Z^{\boldsymbol{\mu}})^\#\cong Z^{\boldsymbol{\mu}}$ and*

$$Z^{\boldsymbol{\mu}} = P^{\boldsymbol{\mu}}\oplus\bigoplus_{\boldsymbol{\lambda}\rhd\boldsymbol{\mu}} p_{\boldsymbol{\mu\lambda}}(q)P^{\boldsymbol{\lambda}},$$

for some bar invariant polynomials $p_{\boldsymbol{\lambda\mu}}(q)\in\mathbb{N}[q,q^{-1}]$.

Proof. By definition, $Z^{\boldsymbol{\mu}}$ is a direct summand of $\mathcal{S}^\Lambda_\beta$, so $(Z^{\boldsymbol{\mu}})^\# \cong Z^{\boldsymbol{\mu}}$. We already noted in (5.4) that $Z^{\boldsymbol{\mu}} = P^{\boldsymbol{\mu}} \oplus \bigoplus_{\boldsymbol{\lambda}} p_{\boldsymbol{\lambda\mu}}(q)P^{\boldsymbol{\lambda}}$, for some Laurent polynomials $p_{\boldsymbol{\lambda\mu}}(q) \in \mathbb{N}[q]$, because $Z^{\boldsymbol{\mu}}$ is projective. In view of Lemma 7.26 these polynomials are bar invariant. □

Next observe that (5.3) implies that in $\mathfrak{F}^\Lambda$

$$[Z^{\boldsymbol{\mu}}] = [\Delta^{\boldsymbol{\mu}}] + \sum_{\substack{\boldsymbol{\nu} \rhd \boldsymbol{\mu} \\ \mathfrak{s} \in \operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\nu})}} q^{\deg \mathfrak{s} - \deg \mathfrak{t}^{\boldsymbol{\mu}}} [\Delta^{\boldsymbol{\nu}}]. \tag{7.32}$$

We now show how to use Lemma 7.31 and (7.32) to inductively compute $[P^{\boldsymbol{\mu}}]$, for $\boldsymbol{\mu} \in \mathscr{P}^\Lambda_n$, as a linear combination of standard modules in $\mathfrak{F}^\Lambda$. Since $[P^{\boldsymbol{\mu}}] = \sum_{\boldsymbol{\lambda}} d_{\boldsymbol{\lambda\mu}}(q)[\Delta^{\boldsymbol{\lambda}}]$ this will give an algorithm for computing the graded decomposition numbers of $\mathcal{S}^\Lambda_\beta$.

If $\boldsymbol{\mu}$ is maximal in $\mathscr{P}^\Lambda_\beta$, with respect to dominance, then $Z^{\boldsymbol{\mu}} = P^{\boldsymbol{\mu}} = \Delta^{\boldsymbol{\mu}}$ by Lemma 7.31. So $[P^{\boldsymbol{\mu}}] = [\Delta^{\boldsymbol{\mu}}]$ in this case and there is nothing to do.

Now suppose that $\boldsymbol{\mu}$ is not maximal in $\mathscr{P}^\Lambda_n$ and that $[P^{\boldsymbol{\lambda}}]$ is known whenever $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$ and $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$. By (7.32) we can write

$$[Z^{\boldsymbol{\mu}}] = [\Delta^{\boldsymbol{\mu}}] + \sum_{\boldsymbol{\nu} \rhd \boldsymbol{\mu}} z_{\boldsymbol{\nu\mu}}(q)[\Delta^{\boldsymbol{\nu}}]$$

for some Laurent polynomials $z_{\boldsymbol{\nu\mu}}(q) \in \mathbb{N}[q, q^{-1}]$ which are not all zero since $\boldsymbol{\mu}$ is not maximal in $\mathscr{P}^\Lambda_\beta$. Let $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$ be any multipartition such that $z_{\boldsymbol{\lambda\mu}}(q) \neq 0$ and

$$\operatorname{mindeg} z_{\boldsymbol{\lambda\mu}}(q) \leq \operatorname{mindeg} z_{\boldsymbol{\nu\mu}}(q),$$

for all $\boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta$. Let $d = \operatorname{mindeg} z_{\boldsymbol{\lambda\mu}}(q)$.

If $d > 0$ then $[Z^{\boldsymbol{\mu}}] \equiv [\Delta^{\boldsymbol{\mu}}] \pmod{\mathfrak{F}^\Lambda_0}$ by (7.32). Now $[Z^{\boldsymbol{\mu}}]^\# = [Z^{\boldsymbol{\mu}}]$, by Lemma 7.31, so this forces $[Z^{\boldsymbol{\mu}}] = [P^{\boldsymbol{\mu}}]$ because in this case $[Z^{\boldsymbol{\mu}}]$ satisfies the two properties which uniquely determine $[P^{\boldsymbol{\mu}}]$ in Theorem 7.29(a).

Now suppose that $d \leq 0$. Let $z^{(d)}_{\boldsymbol{\lambda\mu}}$ be the coefficient of q^d in $z_{\boldsymbol{\lambda\mu}}(q)$ and set

$$p^{(d)}_{\boldsymbol{\lambda\mu}} = \begin{cases} z^{(d)}_{\boldsymbol{\lambda\mu}}(q^d + q^{-d}), & \text{if } d < 0, \\ z^{(d)}_{\boldsymbol{\lambda\mu}}, & \text{if } d = 0. \end{cases}$$

Since $[P^{\boldsymbol{\nu}}] \equiv [\Delta^{\boldsymbol{\nu}}] \pmod{\mathfrak{F}^\Lambda_0}$ for all $\boldsymbol{\nu} \in \mathscr{P}^\Lambda_\beta$, the minimally of d together with Lemma 7.31 implies that $p^{(d)}_{\boldsymbol{\lambda\mu}} P^{\boldsymbol{\lambda}}$ is a direct summand of $Z^{\boldsymbol{\mu}}$. Since $[P^{\boldsymbol{\lambda}}]$ is known by induction we can now replace $[Z^{\boldsymbol{\mu}}]$ with $[Z^{\boldsymbol{\mu}}] - p^{(d)}_{\boldsymbol{\lambda\mu}}[P^{\boldsymbol{\lambda}}]$, which is still #-invariant. By repeating this process of stripping off the bar invariant minimal degree terms we can rewrite $[Z^{\boldsymbol{\mu}}]$ as a linear combination of canonical bases elements as in Lemma 7.31. This recursively computes $[P^{\boldsymbol{\mu}}]$ and so determines the graded decomposition numbers $d_{\boldsymbol{\lambda\mu}}(q)$.

Note that the Laurent polynomials $p_{\boldsymbol{\lambda\mu}}(q)$ in Lemma 7.31 are given by $p_{\boldsymbol{\lambda\mu}}(q) = \sum_{d \leq 0} p^{(d)}_{\boldsymbol{\lambda\mu}}$. Hence, this algorithm also decomposes $Z^{\boldsymbol{\mu}}$ into a direct sum of projective modules.

7.33. *Remark.* Note that $(E^{\boldsymbol{\mu}})^\circledast \cong E^{\boldsymbol{\mu}}$ by Theorem 5.14. An equivalent version of this algorithm computes $[T^{\boldsymbol{\mu}}]$ by applying the same "straightening algorithm" to the element $[E^{\boldsymbol{\mu}}] = [E^{\boldsymbol{\mu}}]^\circledast$, where we use Corollary 5.12 in place of (7.32) and Corollary 5.12 in place of Lemma 7.31.

7.34. **Example** Suppose that $e = 0$, $\Lambda = 3\Lambda_0$ and that $\beta = \alpha_{-1} + 3\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$. Then $\mathcal{S}^\Lambda_\beta$ is a block of defect 4. The maximal multipartition in $\mathscr{P}^\Lambda_\beta$ is $(4, 2|1|0)$ so

$P^{(4,2|1|0)} = \Delta^{(4,2|1|0)}$. Taking $\boldsymbol{\mu} = (4,1|1|1)$ the tableaux in $\mathrm{Std}^{\boldsymbol{\mu}}(\mathscr{P}^{\Lambda}_{\beta})$ are

$$\left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5\\\cline{1-1}\end{array}\;\middle|\;\boxed{6}\;\middle|\;\boxed{7}\right)\quad\left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\;\middle|\;\boxed{\text{-}}\;\middle|\;\boxed{7}\right)$$

$$\left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&7\\\cline{1-2}\end{array}\;\middle|\;\boxed{6}\;\middle|\;\boxed{\text{-}}\right)\quad\left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\;\middle|\;\boxed{7}\;\middle|\;\boxed{\text{-}}\right)$$

Therefore, $[Z^{\boldsymbol{\mu}}] = [\Delta^{(4,1|1|1)}] + q[\Delta^{(4,2|0|1)}] + (q^2+1)[\Delta^{(4,2|1|0)}]$. Applying our algorithm, $[Z^{\boldsymbol{\mu}}] = [P^{\boldsymbol{\mu}}] + [P^{(4,2|1|0)}]$. Using our LLT algorithm, the full graded decomposition matrix of $\mathcal{S}^{\Lambda}_{\beta}$ in characteristic zero is:

$$\begin{array}{r|ccccccccccccccc}
(0|\ 1\ |4,2) & 1 \\
(0|4,2|1) & q & 1 \\
(1|\ 0\ |4,2) & q & . & 1 \\
(1|\ 1\ |4,1) & q^2 & . & q & 1 \\
(1|\ 1^2\ |4) & . & . & . & q & 1 \\
(1|\ 4\ |1^2) & . & . & . & q & . & 1 \\
(1|4,1|1) & q^2 & q & q & q^2 & q & q & 1 \\
(1|4,2|0) & q^3 & q^2 & q^2 & . & . & . & q & 1 \\
(1^2|\ 1\ |4) & . & . & q & q^2 & q & . & . & . & 1 \\
(1^2|\ 4\ |1) & . & . & q^2 & q^3 & q^2 & q^2 & q & . & q & 1 \\
(4|\ 1\ |1^2) & . & . & q & q^2 & . & q & . & . & . & . & 1 \\
(4|\ 1^2\ |1) & . & . & q^2 & q^3 & q^2 & q^2 & q & . & . & . & q & 1 \\
(4,1|\ 1\ |1) & q^2 & q & q^3+q & q^4 & q^3 & q^3 & q^2 & . & q^2 & q & q^2 & q & 1 \\
(4,2|\ 0\ |1) & q^3 & q^2 & q^2 & . & . & . & . & . & . & . & . & . & q & 1 \\
(4,2|\ 1\ |0) & q^4 & q^3 & q^3 & . & . & . & q^2 & q & . & q & . & q & q^2 & q & 1
\end{array}$$

The Kleshchev multipartitions in this block are $(0|1|4,2)$ and $(1|1|4,1)$. If $\ell = 2$ then the graded decomposition numbers of $\mathcal{S}^{\Lambda}_{\beta}$ are always monomials in q by [14, (5.14)]. This is one of the smallest examples of a block $\mathcal{R}^{\Lambda}_{\beta}$ that has a graded decomposition number which is not a monomial. ◊

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