AN INFRASOLVMANIFOLD WHICH DOES NOT BOUND

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Abstract. Orientable 4-dimensional infrasolvmanifolds bound orientably. We show that every non-orientable 4-dimensional infrasolvmanifold M with $\beta = \beta_1(M;\mathbb{Q}) > 0$ or with geometry $\mathbb{N}il^4$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$ bounds. However there are $\mathbb{S}ol_1^4$-manifolds which are not boundaries. The question remains open for $\mathbb{N}il^3 \times \mathbb{E}^1$-manifolds. Any possible counter-examples have severely constrained fundamental groups. We also find simple cobounding 5-manifolds for all but five of the 74 flat 4-manifolds, and investigate which flat 4-manifolds embed in $\mathbb{R}^n$, for $n = 5, 6$ or 7.

1. introduction

Flat n-manifolds are boundaries [8]. This result has been extended to restricted classes of infranilmanifolds [7, 12]. We shall show that it does not extend to all infrasolvmanifolds. Since every 3-manifold bounds, and every orientable 3-manifold bounds orientably, dimension 4 is the first case of interest. Here there is a geometric simplification. Every 4-dimensional infrasolvmanifold is either a mapping torus or the union of two twisted I-bundles. Simple algebraic arguments show that every such mapping torus bounds, while a geometric construction applies to many of the others. We find severe constraints on possible counter-examples, which lead to a $\mathbb{S}ol_1^4$-manifold which is not a boundary. In the latter part of the paper we seek explicit constructions of 5-manifolds with boundary a given flat 4-manifold, and we consider also the related question of which flat 4-manifolds embed in low codimensions.

Every infrasolvmanifold is finitely covered by a quotient $\Gamma\backslash S$, where Γ is a discrete cocompact subgroup of a 1-connected solvable Lie group S [1]. Such quotients are parallelizable, and so the rational Pontrjagin classes of infrasolvmanifolds are 0. In particular, orientable 4-dimensional infrasolvmanifolds have signature $\sigma = 0$. Therefore they

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bound orientably, and those with $w_2 = 0$ bound as *Spin*-manifolds, since Ω_4 and Ω_4^{Spin} are detected by σ.

Non-orientable bordism is detected by Stiefel-Whitney numbers. In our context, the only Stiefel-Whitney class of interest is w_1^4. It follows easily that every 4-dimensional infrasolvmanifold M with $\beta = \beta_1(M;\mathbb{Q}) > 0$ bounds non-orientably. (This class includes all $\mathbb{S}ol^4_{m,n}$-manifolds with $m \neq n$ and all $\mathbb{S}ol^4_0$-manifolds.) If $\beta = 0$ then $\pi = \pi_1(M) \cong A *_C B$, where A, B and C are fundamental groups of 3-dimensional infranilmanifolds and $[A : C] = [B : C] = 2$. In §4–§9 we use a construction based on mapping cylinders of double covers to show that many such manifolds bound. In particular, all $\mathbb{N}il^4$- and $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifolds bound. We do not yet have a complete result for the remaining two geometries.

In §10 we show that if $\beta \geq 2$ (and in many cases with $\beta = 1$) then M is also the total space of an S^1-bundle over a closed 3-manifold, and so bounds the associated disc bundle. If the S^1-bundle space M is orientable then so is the disc bundle space. In §11 we show that the mapping cylinder construction applies to most of the 24 flat 4-manifolds which are not S^1-bundle spaces. Closed hypersurfaces in euclidean spaces bound. In §12 we show that, with one possible exception, all flat 4-manifolds embed in $\mathbb{R}^7$, while between 24 and 56 embed in $\mathbb{R}^6$ and between 11 and 13 embed in $\mathbb{R}^5$.

2. SOLVABLE LIE GEOMETRIES OF DIMENSION 4

If G is a group let $G', \zeta G$ and $\sqrt{G}$ denote its commutator subgroup, centre and Hirsch-Plotkin radical, respectively. Let $G^{ab} = G/G'$ be the abelianization, and let $I(G) = \{g \in G \mid \exists n > 0,\ g^n \in G'\}$ be the isolator subgroup. This is clearly a characteristic subgroup, since $G/I(G)$ is the maximal torsion-free abelian quotient of G. If S is a subset of G then $\langle S\rangle$ shall denote the subgroup of G generated by S, and $\langle\langle S\rangle\rangle$ shall denote the normal closure of $\langle S\rangle$. We use the notation of Chapter 8 of [9] for automorphisms of flat 3-manifold groups.

Every 4-dimensional infrasolvmanifold is geometric. There are six relevant families of geometries: $\mathbb{E}^4$, $\mathbb{N}il^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol^4_0$, $\mathbb{S}ol^4_1$ and $\mathbb{S}ol^4_{m,n}$. (The final family includes the product geometry $\mathbb{S}ol^3 \times \mathbb{E}^1 = \mathbb{S}ol^4_{m,m}$, for all $m > 0$, as a somewhat exceptional case.)

Let G be a 1-connected solvable Lie group of dimension 4 with a left invariant metric, corresponding to a geometry $\mathbb{G}$ of solvable Lie type. Let $Isom(\mathbb{G})$ be the group of isometries, and let $K_G < Isom(\mathbb{G})$ be the stabilizer of the identity in G. Let $\pi < Isom(\mathbb{G})$ be a discrete subgroup which acts freely and cocompactly on G, and let $M = \pi\backslash G$. If $\beta =$

$\beta_1(M;\mathbb{Q}) \geq 1$ then M is the mapping torus of a self-diffeomorphism of a $\mathbb{E}^3$-, $\mathbb{N}il^3$- or $\mathbb{S}ol^3$-manifold. If $\beta = 1$ the mapping torus structure is essentially unique. If $\beta \geq 2$ then M also fibres over the torus T, with fibre T or the Klein bottle Kb.

All orientable $\mathbb{S}ol_0^4$-manifolds are coset spaces $\pi\backslash\widetilde{G}$ with π a discrete subgroup of a 1-connected solvable Lie group $\widetilde{G}$, which in general depends on π. (See page 138 of [9].) In all other cases, the translation subgroup $G \cap \pi$ is a lattice in G, and is a characteristic subgroup of π [4]. If G is nilpotent then $G \cap \pi = \sqrt{\pi}$; in general, $\sqrt{\pi} \leq G \cap \pi$, and the holonomy $\pi/G \cap \pi$ is finite.

If $g : X \to X$ is a self-homeomorphism let $M(g) = X \times [0,1]/(z,0) \sim (g(z),1)$ be the mapping torus of g, and let $[x,t]$ be the image of (x,t) in $M(g)$. If $f : Y \to Z$ let $MCyl(f)$ be the mapping cylinder of f.

3. Stiefel-Whitney classes and the cases with $\beta \geq 1$

We give first some simple observations on the Stiefel-Whitney classes of 4-manifolds, which we shall use to show that 4-dimensional infrasolvmanifolds with $\beta \geq 1$ are boundaries.

Lemma 3.1. *Let M be a closed 4-manifold and $w_i = w_i(M)$. Then $w_4 = w_2^2 + w_1^4$ and $w_1 w_3 = 0$.*

Proof. The Wu formulae give $v_1 = w_1$, $v_2 = w_2 + w^2$, $w_3 = Sq^1 w_2$ and $w_4 = w_2^2 + w^4$, since $v_3 = v_4 = 0$. Hence $Sq^1 z = w_1 z$, for $z \in H^3(M;\mathbb{F}_2)$. If $x \in H^1(M;\mathbb{F}_2)$ then $Sq^1(xw_2) = x^2 w_2 + xSq^1 w_2$. Therefore

$$xw_3 = (w_1 x + x^2)w_2 = (w_1 x + x^2)^2 + (w_1 x + x^2)w_1^2 = x^4 + w_1 x^3.$$

In particular, $w_1 w_3 = w^4 + w^4 = 0$. □

If M is a 4-dimensional infrasolvmanifold then $w_4(M) = 0$, since $w_4(M) \cap [M]$ is the reduction of $\chi(M) = 0$ *mod* (2). Therefore $w_1^4 = w_1^2 w_2 = w_2^2$ is the only Stiefel-Whitney class of interest.

Lemma 3.2. *Let M be a closed n-manifold and $x \in H^1(M;\mathbb{F}_2)$. If $n > 2$ and $x^{n-1} \neq 0$ then $x^n \neq 0$.*

Proof. This follows easily from the non-degeneracy of Poincaré duality, since $x^2 \neq 0$ and $H^1(M;\mathbb{F}_2)$ is generated by x and $\mathrm{Ker}(x \cup -)$. □

Lemma 3.3. *If N is a non-orientable 3-manifold then $\beta_1(N;\mathbb{Q}) > 0$.*

Proof. This is clear, since $\chi(N) = 0$ and $H_3(N;\mathbb{Q}) = 0$. □

Similarly, if M is an orientable 4-manifold with $\chi(M) = 0$ then $\beta_1(M;\mathbb{Q}) > 0$.

Lemma 3.4. *If a manifold M fibres over an r-manifold, with orientable fibre, then $w_1(M)^{r+1} = 0$.*

Proof. This is clear, since $w_1(M)$ is induced from a class on the base of the fibration. □

Theorem 3.5. *Let M be a 4-dimensional infrasolvmanifold with $\beta = \beta_1(M;\mathbb{Q}) > 0$. Then $M = \partial W$ for some 5-manifold W.*

Proof. The manifold M is the mapping torus of a (based) self diffeomorphism f of a closed 3-manifold N. Let $\pi = \pi_1(M)$ and $\nu = \pi_1(N)$. Then π and ν are virtually polycyclic, and $\pi \cong \nu \rtimes_\theta \mathbb{Z}$, where $\theta = \pi_1(f)$. If N is not orientable then $I(\nu) < \nu$, by Lemma 3.3, and so $I(\nu) \cong \mathbb{Z}$, $\mathbb{Z}^2$ or $\pi_1(Kb) = \mathbb{Z} \rtimes_{-1} \mathbb{Z}$. In the latter case $I(I(\nu)) \cong \mathbb{Z}$. In all cases, M fibres over a lower-dimensional manifold with orientable fibre, and so $w_1^4 = 0$, by Lemma 3.4. Therefore all the Stiefel-Whitney numbers of M are 0, and so $M = \partial W$ for some 5-manifold W. □

If M is a non-orientable $\mathbb{S}ol_1^4$-manifold then $\beta = 0$. There are non-orientable manifolds with $\beta > 0$ for each of the other geometries.

For all but three flat 4-manifolds, either $w_1^2 = 0$ or $w_2 = 0$ or $w_1^2 = w_2$ [10]. Hence $w_1^4 = 0$, so all Stiefel-Whitney numbers are 0, and the manifold bounds. Two more are total spaces of S^1-bundles, and so bound the associated disc bundles. Thus only the example with group $G_6 *_\phi B_4$ requires further argument. (See the next section.)

All $\mathbb{S}ol_{m,n}^4$-manifolds (with $m \neq n$) and all $\mathbb{S}ol_0^4$-manifolds are mapping tori of self-diffeomorphisms of $\mathbb{R}^3/\mathbb{Z}^3$. (See Corollary 8.4.1 of [9].) Thus they all bound.

We may assume henceforth that $\beta = 0$ (so the manifolds considered are not orientable) and the geometry is $\mathbb{N}il^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol_1^4$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$. (However we shall also consider $\mathbb{E}^4$ in some detail.)

We shall need the following more specialized lemmas later.

Lemma 3.6. *Let $w : \pi \to \mathbb{F}_2 = Z/2Z$ be a homomorphism. Then $p : \pi \to G = \pi/\langle k^2 \mid w(k) = 0\rangle$ induces an isomorphism $H^1(G;\mathbb{F}_2) \cong H^1(\pi;\mathbb{F}_2)$. If $p^*(uw) = 0$ in $H^2(\pi;\mathbb{F}_2)$ then $uw = 0$ in $H^2(G;\mathbb{F}_2)$.*

Proof. If $p^*(uw) = 0$ in $H^2(\pi;\mathbb{F}_2)$ there is a function $f : \pi \to \mathbb{F}_2$ such that $u(g)w(g') = f(g) + f(g') - f(gg')$, for all $g, g' \in \pi$. Let $K = \mathrm{Ker}(w)$ and $H = \langle k^2 \mid w(k) = 0\rangle$. Then $f|_K$ is a homomorphism, and so $f(h) = 0$, for all $h \in H$. Hence $f(g) = f(gh)$, for all $g \in \pi$ and $h \in H$. Thus f factors through a function $\bar{f} : G \to \mathbb{F}_2$, and so $uw = 0$ in $H^2(G;\mathbb{F}_2)$. □

The next lemma uses the non-degeneracy of Poincaré duality.

Lemma 3.7. *Let M be a non-orientable closed 4-manifold with $\chi(M) = 0$, and let $w = w_1(M)$. Suppose that $H^1(M;\mathbb{F}_2) = \langle u, w\rangle$, where $u^2 = 0$. Then*

(1) *if $w^2 \neq 0$ and $uw \neq 0$, then $w^3 = 0$.*
(2) *if $w^2 \neq 0$ and $uw = 0$ then $w^4 \neq 0 \Leftrightarrow w_2(M) \neq 0$ or w^2.*

Proof. (1). Since $u.uw^2 = u^2w^2 = 0$ and $w.uw^2 = Sq^1(uw^2) = u^2w^2 = 0$, we have $uw^2 = 0$, by Poincaré duality. Now $\beta_2(M,\mathbb{F}_2) = 2\beta_1(M,\mathbb{F}_2) - 2 = 2$. Since $uw.w^2 = uw.uw = 0$ but $uw \neq 0$ and $w^2 \neq 0$ we must have $uw = w^2$, by Poincaré duality. Hence $w^3 = uw^2 = 0$.

(2). Let $v = w_2(M) + w^2 = v_2(M)$. If $w_2(M) \neq 0$ or w^2 then $H^2(M;\mathbb{F}_2) = \langle w^2, v\rangle$. Since $\chi(M) = 0$ we have $v^2 = w_4 = 0$. Therefore $w^4 = (w^2)^2 = w^2v \neq 0$, by Poincaré duality. The converse is clear, since $v_2^2 = w_4 = 0$. □

The second condition may be generalized as follows. Let $H^i = H^i(M;\mathbb{F}_2)$ for $i = 1$ and 2. If $w_1^2 \neq 0$, $w_1 \cup - : H^1 \to H^2$ has rank 1, w_2 is not in the image of $H^1 \odot H^1$ and $H^2 = \langle H^1 \odot H^1, w_2\rangle$, then $w_1^4 \neq 0$. However these conditions are harder to check if $\beta_1(\pi;\mathbb{F}_2) > 2$.

There are two (flat) 4-manifolds which fibre over T with fibre Kb, and thus bound, but for which none of the conditions $w_1^2 = 0$, $w_2 = 0$ or $w_2 = w_1^2$ hold. Thus these conditions are not necessary for a 4-manifold to bound. Nevertheless, manifolds which are not mapping tori and whose orientable double covers are not Spin 4-manifolds may provide non-bounding examples.

4. 4-manifolds with $\chi = \beta = 0$

If M is a closed 4-manifold with $\chi(M) = 0$ and $\beta = 0$ then M is non-orientable, and there is an epimorphism $f : \pi \to D_\infty$, where $D_\infty = Z/2Z * Z/2Z$ is the infinite dihedral group, by Lemma 3.14 of [9]. Hence $\pi \cong A *_C B$, where $C = \mathrm{Ker}(f)$ and $[A : C] = [B : C] = 2$. Since $D_\infty \cong \mathbb{Z} \rtimes Z/2Z$, the group π has a subgroup of index 2 which is a semidirect product $C \rtimes \mathbb{Z}$. Since $\beta = 0$ the Mayer-Vietoris sequence for the homology of π gives an epimorphism from $H_1(C;\mathbb{Q})$ to $H_1(A;\mathbb{Q}) \oplus H_1(B;\mathbb{Q})$, and so $\beta_1(A;\mathbb{Q}) + \beta_1(B;\mathbb{Q}) \leq \beta_1(C;\mathbb{Q})$.

If, moreover, M is an infrasolvmanifold then A, B and C are the fundamental groups of 3-dimensional infrasolvmanifolds X, Y and Z, say, and $M = MCyl(c) \cup_Z MCyl(d)$, where $c : Z \to X$ and $d : Z \to Y$ are double covers. The next two lemmas are clear.

Lemma 4.1. *If $c : Z \to X$ is a double cover of an n-manifold X then $MCyl(c)$ is an $(n+1)$-manifold with boundary Z. If Z is connected* *the mapping cylinder is orientable if and only if X is non-orientable and c is the orientable double cover.* □

In particular, if f is an orientation-preserving self-diffeomorphism of a 3-manifold N then $M(f^2)$ bounds a non-orientable 5-manifold.

Lemma 4.2. *Let X and Y be connected $(n-1)$-manifolds which have double covers $c : Z \to X$ and $d : Z \to Y$ with the same domain, and let $M = MCyl(c) \cup_Z MCyl(d)$. Suppose that X, Y and Z each bound n-manifolds $\widehat{X}$, $\widehat{Y}$ and $\widehat{Z}$, and that c and d extend to double covers $\widehat{c} : \widehat{Z} \to \widehat{X}$ and $\widehat{d} : \widehat{Z} \to \widehat{Y}$. Let $W = MCyl(\widehat{c}) \cup_{\widehat{Z}} MCyl(\widehat{d})$. Then $\partial W = M$. If c and d are the orientable covers of non-orientable manifolds then W and M are orientable.* □

We shall show that this construction applies to many 4-dimensional infrasolvmanifolds.

Theorems 8.4–8.9 of [9] limit the possibilities for A, B and C. In particular, if C is virtually $\mathbb{Z}^3$ but π is not virtually abelian then C has holonomy of order ≤ 2. There are four such, two orientable: $\mathbb{Z}^3$ and $G_2 = \mathbb{Z}^2 \rtimes_{-I} \mathbb{Z}$, and two non-orientable: $B_1 = \mathbb{Z} \times \pi_1(Kb)$ and B_2. Similarly, if C is a $\mathbb{N}il^3$-group but π is not virtually nilpotent then $[C : \sqrt{C}] \leq 2$. We shall not need to consider the possibility that C be a $\mathbb{S}ol^3$-group.

We note also the following simple result.

Lemma 4.3. *If $\pi \cong A *_C B$ where $[A : C] = [B : C] = 2$ and A, B and C are the groups of 3-dimensional infranilmanifolds then the holonomy of A maps injectively to the holonomy of π.* □

5. AMALGAMATION OVER FLAT 3-MANIFOLD GROUPS

If $C = \mathbb{Z}^3$ then A and B have holonomy of order ≤ 2. Since $\beta_1(A;\mathbb{Q})$ and $\beta_1(B;\mathbb{Q}) \geq 1$ and $\beta_1(A;\mathbb{Q}) + \beta_1(B;\mathbb{Q}) \leq 3$, we may assume that $A \cong G_2$ and B is not $\mathbb{Z}^3$. Let f, g and h be the involutions of $S^1 \times D^2$ given by $f(u,v) = (\bar{u},\bar{v})$, $g(u,v) = (u,\bar{v})$ and $h(u,v) = (\bar{u},uv)$, for all $(u,v) \in S^1 \times D^2$. The boundaries of the mapping tori $M(f)$, $M(g)$ and $M(h)$ are the flat 3-manifolds with groups G_2, B_1 and B_2, respectively, and in each case the mapping torus is doubly covered by $S^1 \times D^2 \times S^1$, with boundary the 3-torus $\mathbb{R}^3/\mathbb{Z}^3$. Therefore the mapping cylinder construction shows that M is a boundary.

If $C = G_2$ then $\beta_1(C;\mathbb{Q}) = 1$. We may assume that $A = G_6$ and B is one of G_2, G_4, G_6, B_3 or B_4. If $B = G_2 \cong C$ then the inclusion of C into B induces an isomorphism $C/I(C) \cong B/I(B)$, and is induced by a double cover from $M(f)$ to itself. Non-orientable 3-manifolds bound non-orientable 4-manifolds, and their orientable double covers bound the orientable double covers of such manifolds. If f is the involution of $S^1 \times D^2$ defined above then $M(f)$ has an orientation-preserving free involution given by $[u, v, t] \mapsto [-u, \bar{v}, -t]$. The quotient manifold has boundary HW, the Hantzsche-Wendt flat 3-manifold with group G_6. Thus the mapping cylinder construction applies, provided $B \ncong G_4$.

If $C = B_1$ or B_2 then A and B must be B_3 or B_4, and $I(I(A)) = I(I(B)) = I(C) \cong \mathbb{Z}$. Hence $\pi/I(C) \cong A/I(C) *_{\mathbb{Z}^2} B/I(C)$ and so is a 3-manifold group. The manifold M is then the total space of an S^1-bundle. (The mapping cylinder construction can also be used here.)

There remains the possibility that $A = G_6$, $B = G_4$ and $C = G_2$. In this case the holonomy group $Z/4Z$ of G_4 does not act diagonally, and there is no obvious construction of a 4-manifold with boundary the flat 3-manifold with group G_4. Instead we may use algebraic arguments. The group π then has a presentation

$$\langle t, x, y, z \mid xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = x^{-2},\ z = xy,\ tx^2t^{-1} = x^{2m}y^{2p},$$

$$ty^2t^{-1} = x^{2n}y^{-2m},\ tzt^{-1} = x^{-2r}y^{2s}z,\ t^2 = x^{2a}y^{2b}z\rangle,$$

where $a, b, m, n, p, \in \mathbb{Z}$, $r = (m-1)a + pb$, $s = -na + (m+1)b$ and $m^2 + np = -1$. (We may assume also that $0 \leq a, b \leq 1$.) Here $C = \langle x^2, y^2, z\rangle$, and $\pi/C \cong D_\infty$ is generated by the images of t and x. The automorphism of $\sqrt{C} = \langle x^2, y^2, z^2\rangle$ determined by conjugation by tx has eigenvalues $m \pm \sqrt{m^2+1}$. If $m = 0$ then π is virtually abelian, and the corresponding manifold M is flat. In this case π is also isomorphic to $G_2 *_{\mathbb{Z}^3} B_2$, and so M bounds. Otherwise, π is not virtually nilpotent, and M is a $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold.

The generators t, x and y in this presentation represent orientation-reversing elements of π. If m is even, or if m is odd and n, p are both even, then $\pi/\pi' \cong (Z/4Z)^2$, and so $w_1^2 = 0$. Thus we may asume that m, n are odd (and hence p is even). In this case $\pi/\pi' \cong Z/8Z \oplus Z/2Z$, where the summands are generated by the images of tx^{-1} and x, respectively. Thus $w = w_1$ is projection onto the second summand. Let $u : \pi \to Z/2Z$ be the homomorphism determined by $u(t) = 1$ and $u(x) = 0$. Let $H = \langle k^2 \mid w(k) = 0\rangle$, as in Lemma 3.6. Then $G = \pi/H \cong Z/4Z \oplus Z/2Z$, and so $u^2 = 0$ and $uw \neq 0$ in $H^2(G; \mathbb{F}_2)$. Hence $uw \neq 0$ in $H^2(\pi; \mathbb{F}_2)$, by Lemma 3.6, and so $w^3 = 0$, by part (1) of Lemma 3.7. Thus all such manifolds bound.

These results apply immediately to the flat 4-manifolds with $\beta = 0$. In the next section we shall use them to confirm that all $\mathbb{N}il^4$- and $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifolds are boundaries.

6. $\mathbb{N}il^4$- and $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifolds

Let M be a $\mathbb{N}il^4$-manifold and let C be the centralizer of $I(\sqrt{\pi}) \cong \mathbb{Z}^2$ in $\sqrt{\pi}$. Then $C \cong \mathbb{Z}^3$, and $1 < \zeta\sqrt{\pi} < I(\sqrt{\pi}) < C < \sqrt{\pi}$ is a characteristic series with all successive quotients $\mathbb{Z}$. (See Theorem 1.5 of [9].) In particular, C is normal in π and π/C has two ends. The preimage in π of any finite normal subgroup of π/C is a flat 3-manifold group which is normal in π. This must be $\mathbb{Z}^3$, by Theorem 8.4 of [9], and so π/C has no non-trivial finite normal subgroup. Hence $\pi/C \cong \mathbb{Z}$ or D_∞, and $[\pi : \sqrt{\pi}]$ divides 4. In particular, if $\beta = 0$ the mapping cylinder construction of §4 applies, and so all $\mathbb{N}il^4$-manifolds bound. (Note that since $\zeta\sqrt{\pi} \cong \mathbb{Z}$ the result of [7] applies here if and only if either $\pi = \sqrt{\pi}$ or $\pi/\sqrt{\pi} = Z/2Z$ and acts by inversion on $\zeta\sqrt{\pi}$.)

If M is a $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold then $\sqrt{\pi} \cong \mathbb{Z}^3$ and the quotient $\pi/\sqrt{\pi}$ has two ends. Therefore $\pi \cong A *_C B$, where $\sqrt{\pi} \le C$, $[C : \sqrt{\pi}]$ is finite and $[A : C] = [B : C] = 2$, since we are assuming that $\beta = 0$. Since π is not virtually nilpotent, $[C : \sqrt{\pi}] \le 2$, by Theorem 8.4 of [9]. In all cases M is a boundary, by the results of §4.

7. amalgamation over $\mathbb{N}il^3$-manifold groups

The other cases that we shall need to consider are when A, B and C are fundamental groups of $\mathbb{N}il^3$-manifolds. These have canonical Seifert fibrations, with base a flat 2-orbifold with no reflector curves. (There are seven such orbifolds: T, Kb, $S(2,2,2,2)$, $P(2,2)$, $S(2,4,4)$, $S(2,3,6)$ and $S(3,3,3)$.) The quotients $\overline{A} = A/\zeta\sqrt{A}$, $\overline{B} = B/\zeta\sqrt{B}$ and $\overline{C} = C/\zeta\sqrt{C}$ are the orbifold fundamental groups of the bases. If the image of $g \in A$ generates a maximal finite cyclic subgroup of $\overline{A}$ then $\zeta\sqrt{A} \le \langle g \rangle$, since $\langle g, \zeta\sqrt{A} \rangle$ is torsion-free and virtually $\mathbb{Z}$.

Lemma 7.1. *Suppose that $\pi \cong A *_C B$, where C is a $\mathbb{N}il^3$-group and $A = \langle C, t \rangle$ and $B = \langle C, u \rangle$, with $t^2, u^2 \in C$. Then*

(1) *if $[\sqrt{A} : \sqrt{C}] = 2$ or if $C = \sqrt{C}$ and $A/\zeta\sqrt{A} \cong \mathbb{Z}^2 \rtimes_{-I} Z/2Z$ then the automorphism of $\sqrt{C}/\zeta\sqrt{C}$ induced by conjugation by tu has finite order;*

(2) *if π is not virtually nilpotent then $\sqrt{A} = \sqrt{B} = \sqrt{C}$;*

(3) *if the inclusion of C into each of A and B induces isomorphisms $C/\zeta\sqrt{C} \cong A/\zeta\sqrt{A}$ and $C/\zeta\sqrt{C} \cong B/\zeta\sqrt{B}$ then M bounds.*

Proof. If $[\sqrt{A} : \sqrt{C}] = 2$ then $t \in \sqrt{A}$, and so t centralizes $\sqrt{C}/\zeta\sqrt{C}$. If C is nilpotent and $A/\zeta\sqrt{A} \cong \mathbb{Z}^2 \rtimes_{-I} Z/2Z$ then t acts via $-I$ on $\sqrt{C}/\zeta\sqrt{C}$. Since $u^2 \in C$ and $[C : \sqrt{C}]$ is finite, in each case some power of tu acts trivially on $\sqrt{C}/\zeta\sqrt{C}$. Hence π is virtually nilpotent.

Part (2) is an immediate consequence of part (1).

The hypotheses of part (3) imply that $\pi/\zeta\sqrt{C} \cong C/\zeta\sqrt{C} \times D_\infty$. (Hence π is virtually a product $\sqrt{C} \times \mathbb{Z}$.) Let $N = K(C,1)$ and let ι be the free involution of $N \times D^2$ which is the antipodal map on the S^1 fibres of N and reflection across a diameter of D^2. Then the quotient $N \times D^2/\langle\iota\rangle$ is a 5-manifold with boundary $M = K(\pi,1)$. □

As in the flat case, $\beta_1(A;\mathbb{Q}) + \beta_1(B;\mathbb{Q}) \leq \beta_1(C;\mathbb{Q}) \leq 2$. If $C = \sqrt{C}$ we may assume that either $A = \sqrt{A}$ and $K(B,1)$ has base $S(2,2,2,2)$, or the bases for $K(A,1)$ and $K(B,1)$ are Kb or $S(2,2,2,2)$.

If $[C : \sqrt{C}] = 2$ then $K(C,1)$ has base $S(2,2,2,2)$ or Kb. In the first case $K(A,1)$ and $K(B,1)$ have base $S(2,2,2,2)$, $P(2,2)$ or $S(2,4,4)$. In the second case we may assume that $K(A,1)$ has base $P(2,2)$ and $K(B,1)$ has base Kb or $P(2,2)$.

Lemma 7.2. *Suppose that $\pi \cong A *_C B$, where C is a $\mathbb{N}il^3$-group and $A = \langle C,t\rangle$ and $B = \langle C,u\rangle$, with $t^2, u^2 \in C$. Then $w_1^2 = 0$ if either*

1. *$q = [\zeta\sqrt{C} : \zeta\sqrt{C} \cap \sqrt{C}']$ is even, and either $C = \sqrt{C}$ or $t^n, u^n \in \zeta\sqrt{C}$ for some $n \geq 2$; or*
2. *$C = \sqrt{C}$ and $K(A,1)$ and $K(B,1)$ fibre over Kb; or*
3. *$K(C,1)$ has base $S(2,2,2,2)$ and $K(A,1)$ and $K(B,1)$ both have base $S(2,4,4)$; or*
4. *$K(C,1)$ has base $S(2,2,2,2)$ and $K(A,1)$ and $K(B,1)$ both have base $P(2,2)$.*

Proof. Since $\mathbb{N}il^3$-manifolds are orientable the orientation reversing elements of π are of the form xc, where $x \in (A \cup B) \setminus C$ and $c \in C$. In each case, such elements have images in π/π' of order divisible by 4. □

This does not always hold if $K(A,1)$ has base $P(2,2)$ and $K(B,1)$ has base $S(2,4,4)$. When $\zeta\sqrt{A} = \zeta\sqrt{B} = \zeta\sqrt{C}$ and $K(C,1)$ and $K(A,1)$ have bases $S(2,2,2,2)$ and $P(2,2)$, respectively, the automorphism of $\sqrt{C}/\zeta\sqrt{C}$ induced by tu has matrix

$$\xi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} m & p \\ n & -m \end{smallmatrix}\right) = \left(\begin{smallmatrix} m & p \\ -n & m \end{smallmatrix}\right),$$

where $m^2 + np = 1$ if $K(B,1)$ hase base $P(2,2)$ and $m^2 + np = -1$ if $K(B,1)$ has base $S(2,4,4)$. If $m = 0$ this has finite order, and so M is a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold. If $m = \pm 1$ and $np = 0$ then $K(B,1)$ must also have base $P(2,2)$, and M is a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold if $n = p = 0$, and is a $\mathbb{N}il^4$-manifold if one of n or p is not 0. In all these cases $w_1^2 = 0$, and so M bounds. Otherwise (if $m^2 = 1$ and $np = -2$, or if $|m| > 1$) the eigenvalues of ξ are not roots of unity, and so M is a $\mathbb{S}ol_1^4$-manifold.

If $[C:\sqrt{C}]>2$ then M must be a $\mathbb{N}il^3\times\mathbb{E}^1$-manifold. These cases are considered in the next section. (In most such cases part (3) of Lemma 7.1 applies.)

The mapping cylinder construction appears to have limited applicability here. Let Θ_m and Ψ_n be the self-diffeomorphisms of $S^1\times D^2$ given by $\Theta_m(u,d)=(u,u^m d)$ and $\Psi_n(u,d)=(\bar{u},u^n\bar{d})$, for all $(u,d)\in S^1\times D^2$, respectively, and let $\theta_m=\Theta_m|_T$ and $\psi_n=\Psi_n|_T$ be the restrictions to $T=\partial(S^1\times D^2)$. The mapping tori $M(\Theta_m)$ and $M(\Psi_n)$ are D^2-bundles over T and Kb, respectively. The double covers of $M(\Theta_m)$ are all diffeomorphic to $M(\Theta_{2m})$, while the double covers of $M(\Psi_n)$ are diffeomorphic to $M(\Theta_{2n})$ or $M(\Psi_{2n})$. In particular, if $C=\sqrt{A}=\sqrt{B}$ and $K(A,1)$ and $K(B,1)$ each fibre over Kb then M bounds.

8. $\mathbb{N}il^3\times\mathbb{E}^1$-MANIFOLDS

If M is an infranilmanifold with holonomy a finite 2-group which acts effectively on $\zeta\sqrt{\pi}$ then M bounds, by Proposition 1.3 of [7]. (The hypotheses of the later result of [12] imply that M must be an orientable $\mathbb{N}il^3\times\mathbb{E}^1$-manifold, and so this is of limited interest for our problem.)

Let M be a $\mathbb{N}il^3\times\mathbb{E}^1$-manifold. Then $\sqrt{\pi}\cong\Gamma_q\times\mathbb{Z}$, for some $q\geq 1$, and so $\zeta\sqrt{\pi}\cong\mathbb{Z}^2$ and $\sqrt{\pi}/\zeta\sqrt{\pi}\cong\mathbb{Z}^2$. Moreover, $I(\sqrt{\pi})\cong\mathbb{Z}$ and $I(\sqrt{\pi})<\zeta\sqrt{\pi}$. Let $\theta:\pi\to Aut(\zeta\sqrt{\pi})$, $\bar{\theta}:\pi\to Aut(\zeta\sqrt{\pi}/I(\sqrt{\pi}))$ and $\psi:\pi\to Aut(\sqrt{\pi}/\zeta\sqrt{\pi})$ be the homomorphisms induced by conjugation in π. Since $I(\sqrt{\pi})$ is a characteristic subgroup of π, the image of θ lies in the diagonal group $(Z/2Z)^2$ of $GL(2,\mathbb{Z})$. The manifold M is non-orientable if and only if $\bar{\theta}$ is nontrivial. (In that case the holonomy $\gamma=\pi/\sqrt{\pi}$ acts by inversion on the Euclidean factor of $Nil^3\times\mathbb{R}$.)

Let $K=\mathrm{Ker}(\theta)$. Then $\sqrt{K}=\sqrt{\pi}$, since $\sqrt{\pi}\leq K\trianglelefteq\pi$. Moreover, $\zeta\sqrt{\pi}\leq\zeta K\leq\sqrt{K}$, and so $\zeta K=\zeta\sqrt{\pi}$. The quotient $K/\zeta K$ is a flat 2-orbifold group with holonomy $K/\sqrt{K}$. Since K acts trivially on ζK this orbifold is orientable, and so $K/\sqrt{K}$ is cyclic, of order 1, 2, 3, 4 or 6. The preimage in π of any finite normal subgroup of $\pi/I(\sqrt{\pi})$ is an infinite cyclic normal subgroup, and therefore is $I(\sqrt{\pi})$. Therefore the induced action of γ on $\sqrt{\pi}/I(\sqrt{\pi})$ is effective, and so $(\psi,\bar{\theta}):\gamma\to GL(2,\mathbb{Z})\times\mathbb{Z}^\times$ is injective. Hence γ is isomorphic to a subgroup of $D_{2n}\times Z/2Z$, for $n=4$ or 6. All the possibilities are realized, except for the products $D_{2n}\times Z/2Z$, with $n=3$, 4 or 6 [5].

Although some $\mathbb{N}il^3\times\mathbb{E}^1$-groups with $\beta=0$ are amalgamated free products $\pi\cong A*_C B$ with A,B and C virtually $\mathbb{Z}^3$, the cases with $A=G_6$, $B=G_4$ and $C=G_2$ do not arise here, and so the corresponding manifolds bound. Thus we may assume that $\pi\cong A*_C B$, where A, B and C are fundamental groups of $\mathbb{N}il^3$-manifolds. If $K(C,1)$ has base

$P(2,2)$, $S(2,4,4)$ or $S(2,3,6)$ then $\overline{A} = \overline{B} = \overline{C}$, and so M bounds, by part (3) of Lemma 7.1. However, if $K(C,1)$ has base $S(3,3,3)$ then $K(A,1)$ or $K(B,1)$ could have base $S(2,3,6)$. In this case there are non-normal subgroups of index 3, with similar structures $\widetilde{A} *_{\sqrt{C}} \widetilde{B}$, where $K(\widetilde{A},1)$ and $K(\widetilde{B},1)$ have base T or $S(2,2,2,2)$. Since coverings of odd degree induce isomorphisms on cohomology with coefficients $\mathbb{F}_2$, we may further assume that $[C : \sqrt{C}] \leq 2$, and that $\gamma = \pi/\sqrt{\pi}$ is a 2-group, of order dividing 8.

If $\gamma = Z/2Z$ then γ must act trivially on $I(\sqrt{\pi})$ and via $-I_3$ on $\sqrt{\pi}/I(\sqrt{\pi}) \cong \mathbb{Z}^3$ (since $\beta = 0$). Thus γ acts effectively on $\zeta\sqrt{\pi}$, and so M bounds, by Proposition 1.3 of [7]. Thus we may assume that either $\gamma = (Z/2Z)^2$ and $\zeta\pi = I(\sqrt{\pi})$ (i.e., γ does not act effectively on $\zeta\sqrt{\pi}$) or $\gamma = Z/4Z$, $Z/4Z \oplus Z/2Z$, $(Z/2Z)^3$ or D_8.

If $C = \sqrt{C}$ then the orientable double cover of M is a Spin 4-manifold. If, moreover, either $K(A,1)$ and $K(B,1)$ both fibre over Kb or $q = [\zeta\sqrt{C} : \zeta\sqrt{C} \cap \sqrt{C}']$ is even then $w_1^2 = 0$ and so M bounds, by part (1) of Lemma 7.2. If $K(C,1)$ has base $S(2,2,2,2)$ and $\sqrt{A} = \sqrt{B} = \sqrt{C}$ (and π is virtually nilpotent) then $w_1^2 = 0$. There are mapping tori of self-diffeomorphisms of such $K(C,1)$ which are not Spin [10]. Thus the cases when $K(A,1)$ and $K(C,1)$ have base $S(2,2,2,2)$ may give examples of $\mathbb{N}il^3 \times \mathbb{E}^1$-manifolds which are not boundaries.

9. $\mathbb{S}ol_1^4$-MANIFOLDS

If M is a $\mathbb{S}ol_1^4$-manifold then $\sqrt{\pi} \cong \Gamma_q$ for some $q \geq 1$, and $\pi/\sqrt{\pi}$ has two ends. Therefore $\pi \cong A *_C B$, where $[A : C] = [B : C] = 2$, $\sqrt{\pi} = \sqrt{C}$ and $[C : \sqrt{\pi}]$ is finite. Thus A, B and C are fundamental groups of $\mathbb{N}il^3$-manifolds. Since π is not virtually nilpotent, $[C : \sqrt{\pi}] \leq 2$, by Theorem 8.4 of [9], and so $[A : \sqrt{\pi}]$ and $[B : \sqrt{\pi}]$ are each ≤ 4. Moreover $\sqrt{A} = \sqrt{B} = \sqrt{C}$, by part (2) of Lemma 7.1. The possibilities are limited further by the fact that π cannot have $\mathbb{Z}^2$ as a normal subgroup, since $\mathbb{S}ol_1^4$-manifolds are not Seifert fibred. In particular, $K(C,1)$ cannot be fibred over Kb, for otherwise the characteristic subgroup $I(C) \cong \mathbb{Z}^2$ would be normal in π.

If $C = \sqrt{\pi}$ then $K(A,1)$ and $K(B,1)$ are S^1-bundles over Kb, by part (1) of Lemma 7.1. The mapping cylinder construction then applies to show that M bounds. If $[C : \sqrt{\pi}] = 2$ then $K(C,1)$ has base $S(2,2,2,2)$, and so $K(A,1)$ and $K(B,1)$ have bases $P(2,2)$ or $S(2,4,4)$. If the bases are the same then $w_1^2 = 0$, by parts (3) and (4) of Lemma 7.2, and so M bounds. There remains the possibility that $K(A,1)$ has base $S(2,4,4)$ and $K(B,1)$ has base $P(2,2)$.

Theorem 9.1. *Let M be a $\mathbb{S}ol_1^4$-manifold with $\pi = \pi_1(M) \cong A *_C B$, where $K(A,1)$ is Seifert fibred over $S(2,4,4)$ and $K(B,1)$ is Seifert fibred over $P(2,2)$. If $q = [\zeta\sqrt{C} : \zeta\sqrt{C} \cap \sqrt{C}']$ is odd then M bounds if and only if $w_1^2 = 0$.*

Proof. Since $K(C,1)$ is a double cover of each of $K(A,1)$ and $K(B,1)$, it is Seifert fibred over $S(2,2,2,2)$, and $\sqrt{A} = \sqrt{B} = \sqrt{C}$. The orbifold fundamental groups of the bases $\overline{A} = \pi^{orb}(S(2,4,4))$ and $\overline{B} = \pi^{orb}(P(2,2))$ have presentations $\langle a, x \mid a^4 = (a^2x)^2,\ [x, axa^{-1}] = 1\rangle$ and $\langle j, u \mid j^2 = (ju^2)^2 = 1\rangle$, and their maximal abelian normal subgroups are $\langle x, axa^{-1}\rangle$ and $\langle u^2, (ju)^2\rangle$, respectively.

After suitable normalizations we may assume that A has a presentation

$$\langle a, x, y \mid y = axa^{-1},\ [x, y] = a^{4q},\ a^2xa^{-2} = x^{-1}\ \rangle,$$

and that $C = \langle a^2, x, y\rangle$. We may then assume that B has a presentation

$$\langle j, k, x, y \mid [x, y] = j^{2q},\ jxj^{-1} = x^{-1}, jyj^{-1} = y^{-1},\ kxk^{-1} = x^m y^n j^{2e},$$
$$kyk^{-1} = x^p y^{-m} j^{2f},\ k^2 = x^r y^s j^{2g},\ (jk)^2 = x^t y^u j^{2h}\ \rangle,$$

where m is odd and p and n are even (since $\left(\begin{smallmatrix} m & p \\ n & -m \end{smallmatrix}\right)$ must be conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$), and $ru - ts = \pm 1$. Here C is the subgroup $\langle j, x, y\rangle$, and we may identify j with a^2. Hence π has a presentation

$$\langle a, k, x, y \mid axa^{-1} = y,\ a^2xa^{-2} = x^{-1},\ kxk^{-1} = x^m y^n a^{4e},$$
$$kyk^{-1} = x^p y^{-m} a^{4f},\ k^2 = x^r y^s a^{4g},\ (a^2k)^2 = x^t y^u a^{4h},\ [x, y] = a^{4q}\ \rangle.$$

Abelianizing this presentation gives $[x] = [y]$, $4q[a] = 0$, $2[x] = 0$, $(m+n+1)[x] = 4e[a]$, $(m+p+1)[x] = 4f[a]$, $2[k] = (r+s)[x] + 4g[a]$ and $2[k] = (t+u)[x] + 4(h-1)[a]$. Since $m+n+1$ and $m+p+1$ are even two of these simplify to $4e[a] = 4f[a] = 0$. Moreover $2q[k] = q[x]$.

Since $r+s$ and $t+u$ cannot both be even, we can solve for $[x]$ in terms of $[a]$ and $[k]$. If they are both odd then $\pi/\pi' \cong Z/4\tilde{q}Z \oplus Z/4Z$, where $\tilde{q} = h.c.f.\{q, e, f, g-h+1\}$, and then $w_1^2 = 0$. Otherwise $\pi/\pi' \cong Z/4\tilde{q}Z \oplus Z/2Z$, where $\tilde{q}$ divides $h.c.f.\{q, e, f\}$, and $w_1^2 \neq 0$. If (say) $r+s$ is even then $2([k] - 2g[a]) = 0$ and so ka^{-2g} is an orientation reversing element with image in π/π' of order 2.

The projection to the quotient $\pi/\langle\langle a^4, (ak)^2, x\rangle\rangle \cong D_8$ induces an isomorphism $H^1(D_8; \mathbb{F}_2) \cong H^1(\pi; \mathbb{F}_2) = \langle u, w\rangle$. Since $uw = 0$ in $H^2(D_8; \mathbb{F}_2)$ it follows that $uw = 0$ in $H^2(\pi; \mathbb{F}_2)$ also.

The orientable double cover of M is the mapping torus of the self-diffeomorphism of $K(C,1)$ corresponding to $t = ak$, and is not a Spin manifold, since q is odd. (See §7 of [10].) Therefore $w_2(M) \neq 0$ or w^2. It now follows from part (2) of Lemma 3.7 that $w^4 \neq 0$, and so M does not bound. $\square$

In particular, the $\mathbb{S}ol_1^4$-manifold M whose group has presentation

$$\langle a, k, x, y \mid axa^{-1} = y,\ a^2xa^{-2} = x^{-1},\ kxk^{-1} = x^3y^{-4},\ kyk^{-1} = x^2y^{-3},$$

$$k^2 = xy^{-1},\ (a^2k)^2 = xy^{-2},\ [x, y] = a^4\ \rangle.$$

is not a boundary.

10. S^1-bundle spaces

In many cases a 4-dimensional infrasolvmanifold M is the boundary of the total space of a D^2-bundle over a 3-manifold.

In all, 50 of the 74 flat 4-manifolds are total spaces of S^1-bundles. The exceptions have $\beta \leq 1$, and are three with group $G_2 \rtimes \mathbb{Z}$ (all non-orientable), three with group $G_3 \rtimes \mathbb{Z}$ (all orientable), two with group $G_4 \rtimes \mathbb{Z}$ (both orientable), one with group $G_5 \rtimes \mathbb{Z}$ (orientable), twelve with group $G_6 \rtimes \mathbb{Z}$ (seven orientable) and three with $\beta = 0$ and groups $G_2 *_\phi B_2$, $G_6 *_\phi B_3$ and $G_6 *_\phi B_4$ (all non-orientable). In §11 we shall show that the mapping cylinder construction applies to most of these.

Coset spaces of $Nil^3 \times \mathbb{R}$ or $Sol^3 \times \mathbb{R}$ are products $N \times S^1$, with N a Nil^3- or Sol^3-coset space, respectively, and so bound $N \times D^2$. Coset spaces of Nil^4 or Sol_1^4 are also S^1-bundle spaces, since the action of the centre $\mathbb{R}$ induces a free S^1-action on the coset space. A $\mathbb{N}il^4$-manifold is such a coset space if and only if $\beta = 2$, while a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold is such a coset space if and only if $\beta = 3$. These coset spaces are orientable, and so bound orientably.

If M is a $\mathbb{N}il^4$-manifold or a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold, but is not a coset space, then $\beta \leq 1$ or $\beta \leq 2$, respectively. If M is non-orientable and $\beta > 0$, or if M is an orientable $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold and $\beta = 2$, then $\pi \cong \nu \rtimes_\theta \mathbb{Z}$, where $\nu = \mathbb{Z}^3, G_2, B_1$ or B_2. (See Theorems 8.4 and 8.9 of [9].) The manifold M is the mapping torus of a self-diffeomorphism of the corresponding flat 3-manifold N. (If M is orientable then $\nu = \mathbb{Z}^3$ or G_2, and if M is a non-orientable $\mathbb{N}il^4$-manifold then $\nu = \mathbb{Z}^3$.) If $\nu = \mathbb{Z}^3$ or G_2 then $\theta|_{I(\nu)}$ has an eigenvalue ± 1, since π is virtually nilpotent. (If $\beta = 1$ and $\nu = \mathbb{Z}^3$ the eigenvalue must be -1.) The quotient of π by the corresponding infinite cyclic normal subgroup is torsion-free, and so M is also the total space of an S^1-bundle over a closed 3-manifold. A similar result holds if $\nu = B_1$ or B_2, for in these cases $I(\nu) \cong \mathbb{Z}$.

Orientable $\mathbb{N}il^3 \times \mathbb{E}^1$- and $\mathbb{N}il^4$-manifolds with $\beta = 1$, and all orientable $\mathbb{S}ol_1^4$-manifolds (which have $\beta = 1$) are mapping tori of diffeomorphisms of $\mathbb{N}il^3$-manifolds. If the fibre is a Nil^3-coset space, with group $\nu = \sqrt{\nu}$, then $\pi/I(\nu)$ is torsion-free, and so the 4-manifold is the total space of an S^1-bundle over a $\mathbb{N}il^3$-manifold. However if $\nu \neq \sqrt{\nu}$

then π has no infinite cyclic normal subgroup with torsion-free quotient, and the manifold is not an S^1-bundle space.

If M is a $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold then $\beta \leq 2$, and if $\beta = 2$ then $\pi \cong \mathbb{Z}^3 \rtimes_\theta \mathbb{Z}$. In this case θ has an eigenvalue 1, and so M is an S^1-bundle space. This is also the case if $\beta = 1$ and $\pi \cong \mathbb{Z}^3 \rtimes_\theta \mathbb{Z}$, as one eigenvalue of θ must be ± 1. Otherwise either $\beta = 1$ and $\pi \cong \sigma \rtimes \mathbb{Z}$, where σ is the group of a $\mathbb{S}ol^3$-manifold, or $\beta = 0$.

11. MAPPING CYLINDER CONSTRUCTIONS

The mapping cylinder construction of Lemma 4.1 and 4.2 apply to many of the flat 4-manifolds which are not realizable by S^1-bundle spaces. We note here the following variation: if $c : Z \to X$ is a double cover and f is a self-diffeomorphism X such that $f_* c_* \pi_1(Z) = c_* \pi_1(Z)$ then f extends to a self-diffeomorphism F of $MCyl(c)$, and so $M(f) = \partial M(F)$.

All the mapping tori of self-diffeomorphisms of orientable flat 3-manifolds with cyclic holonomy and $\beta = 1$ also fibre over Kb, and so their groups map onto D_∞. The groups $G_6 \rtimes_\theta \mathbb{Z}$ corresponding to the outer automorphism classes $\theta = a, ab, i$ and ei also map onto D_∞. The groups corresponding to $cej, abcej$ and j have abelianization $\mathbb{Z}$, and so Lemma 4.2 does not apply to these. The classes $ace = (ci)^2$, $bce = (ei)^2$ and and $abcej = j^4$ are squares in $Out(G_6)$ (as are $1 = 1^2$ and $ab = (cei)^2$). These bound, since $M(f^2)$ bounds the mapping cylinder of the canonical double cover of $M(f)$. (Since cei and ci are orientation-reversing, two of these mapping cylinders are orientable.) The classes a, ce, cei, ci and j are not squares, since they are orientation-reversing. The classes i and ei are not squares, as they have order 4 and $Out(G_6)$ has no elements of order 8. The class cej is not a square, as it has order 6 and $Out(G_6)$ has no elements of order 12.

The mapping cylinder construction applies to show that each of the four flat 4-manifolds with $\beta = 0$ is a boundary. There remain five flat 4-manifolds (corresponding to ce, cei, cej, ci and j) for which we do not yet have simple cobounding 5-manifolds, and a further two orientable flat 4-manifolds (corresponding to $abcej$ and bce) for which we do not have simple orientable cobounding 5-manifolds.

12. EMBEDDING FLAT 4-MANIFOLDS IN $\mathbb{R}^n$

If a closed smooth n-manifold embeds in $\mathbb{R}^k$ then the kth normal Stiefel-Whitney classes $\overline{w}_k(M)$ is 0, since this is the *mod*-(2) normal Euler class. (See Theorem 10.2 of [11].) This necessary condition is also sufficient when $n = 4$ and $k = 3$: a closed smooth 4-manifold

M embeds in $\mathbb{R}^7$ if and only if $\overline{w}_3(M) = 0$ [6]. (Note that $\overline{w}_3(M) = w_3(M)+w_1(M)^3 = Sq^1 w_2(M)+w_1(M)^3$, by the Whitney sum theorem and the Wu formulae.) In particular, every orientable closed smooth 4-manifold embeds in $\mathbb{R}^7$. An orientable closed smooth 4-manifold M embeds in $\mathbb{R}^6$ if and only if $w_2(M) = 0$ and $\sigma(M) = 0$ [2]. However, there is as yet no general criterion for non-orientable 4-manifolds to embed in $\mathbb{R}^6$.

It follows from these results (and Lemma 3.1) that if a 4-dimensional infrasolvmanifold M is a boundary and $w_3(M) = 0$ then M embeds in $\mathbb{R}^7$, since $w_1^4 = 0$ implies $w_1^3 = 0$, by Lemma 3.2, and then $\overline{w}_3(M) = 0$. If M is orientable then it embeds in $\mathbb{R}^6$ if and only if $w_2(M) = 0$.

In [10] it is shown that w_2 is integral (and hence $w_3 = 0$) for all but at most two flat 4-manifolds. The exceptions have groups $\pi = G_6 \rtimes_{ci} \mathbb{Z}$ or $G_6 *_\phi B_4$. When $\pi = G_6 \rtimes_{ci} \mathbb{Z}$, the Wang sequence for π as an extension of $\mathbb{Z}$ and the Universal Coefficient Theorem imply that $H^2(\pi; Z/4Z) \cong (Z/4Z)^2$ maps onto $H^2(\pi; \mathbb{F}_2)$. Therefore $w_3 = Sq^1 w_2 = 0$. Thus, with one possible exception, every 4 flat 4-manifold embeds smoothly in $\mathbb{R}^7$.

Three orientable flat 4-manifolds have $w_2 \neq 0$; they are mapping tori of self-diffeomorphisms of HW, corresponding to $\theta = e, bce$ or ei in $Out(G_6)$. The other 24 embed in $\mathbb{R}^6$. Since $\overline{w_2}(M) = w_2(M)+w_1(M)^2$, non-orientable flat 4-manifolds which embed in $\mathbb{R}^6$ must have Pin^--structures. This condition excludes 15 of the 47 non-orientable flat 4-manifolds, but we do not know whether all the others embed in $\mathbb{R}^6$.

If M embeds in $\mathbb{R}^5$ then it bounds a compact region and is s-parallelizable. Thus M is parallelizable if also $\chi(M) = 0$. Moreover, if X and Y are the closures of the components of $S^5 \setminus M$ then X and Y are connected and $H^1(X) \oplus H^1(Y) \cong H^1(M)$. In particular, if $\beta = 1$ then M has an essentially unique infinite cyclic covering M', and this bounds a covering of X, say. Let t generate the covering group, and let T be the maximal finite submodule of $H_1(M; \Lambda)$. Then Poincaré duality with coefficients in the group ring $\Lambda = \mathbb{Z}[t, t^{-1}]$ and the Universal coefficient spectral sequence together give an isomorphism $T \cong \overline{Ext^2_\Lambda(T, \Lambda)}$. This is equivalent to a non-degenerate pairing $\ell_p : T \times T \to \mathbb{Q}/\mathbb{Z}$, with an isometric action of the covering group. When M' is homotopy equivalent to a 3-manifold this pairing is the standard torsion linking pairing on M', with the action of the covering group $\langle t \rangle$. (In knot theory this pairing is known as the Farber-Levine pairing.) If $M = \partial W$ and p extends to a homomorphism from $\pi_1(W)$ to $\mathbb{Z}$ then $K = \mathrm{Ker}(: T \to H_1(W; \Lambda)$ is a submodule which is its own annihilator with respect to ℓ_p. Hence ℓ_p is metabolic.

Every closed 3-manifold N embeds in $\mathbb{R}^5$ [13]. The normal bundle of an embedding $j : N \to \mathbb{R}^5$ is classified by an Euler class $e(j) \in H^2(N;\mathbb{Z}^w) \cong H_1(N;\mathbb{Z})$. If M is the boundary of a regular neighbourhood of j then M is the total space of an S^1-bundle over N, and $e(j)$ is also the class of the corresponding extension of $\pi_1(N)$ by $\mathbb{Z}$. If N is orientable the normal bundle is trivial, and so $M = N \times S^1$.

The six orientable flat 4-manifolds which are products $N \times S^1$ (with groups $G_i \times \mathbb{Z}$, for $1 \leq i \leq 6$) all embed in $\mathbb{R}^5$. Since $G_3^{ab} \cong \mathbb{Z} \oplus Z/3Z$ and $G_4^{ab} \cong \mathbb{Z} \oplus Z/2Z$, the flat 4-manifolds with groups $G_i \rtimes_\theta \mathbb{Z}$ (for $i = 3$ or 4) and $\beta = 1$ do not embed in $\mathbb{R}^5$. The group $G_6^{ab} \cong (Z/4Z)^2$ does not have a subgroup which is its own annihilator with respect to the torsion linking pairing of HW, and so no flat 4-manifold with group $G_6 \rtimes \mathbb{Z}$ and $\beta = 1$ can embed in $\mathbb{R}^5$. However, such considerations do not apply to the flat 4-manifold with group $G_5 \rtimes_\theta \mathbb{Z}$ and $\beta = 1$, since $G_5^{ab} \cong \mathbb{Z}$ is torsion-free. In this case $H_1(\pi) \cong \mathbb{Z} \oplus Z/2Z$ is the sum of two cyclic groups. Since the corresponding flat 4-manifold M has $w_2(M) = 0$ and $\sigma(M) = 0$, it embeds in $\mathbb{R}^5$, by Theorem 6.2 of [3].

If $\pi \cong \mathbb{Z}^3 \rtimes_T \mathbb{Z}$ has cyclic holonomy and $\beta = 2$, then any basis for $\pi/I(\pi) \cong \mathbb{Z}^2$ will contain at least one element whose image generates the holonomy. Therefore if M embeds in S^5 with closed complementary regions X and Y there will be an infinite cyclic cover M' with fundamental group an orientable flat 3-manifold group with the same holonomy, which bounds an infinite cyclic cover of X, say. This is again impossible if the holonomy has order 3 or 4.

The remaining six orientable flat 4-manifolds are mapping tori of self-diffeomorphisms of the half-turn flat 3-manifold, with groups $G_2 \rtimes_\theta \mathbb{Z}$, and five of these have $\beta = 1$. These also fibre over non-orientable flat 3-manifolds. In three of these cases the group is a semidirect product $\mathbb{Z} \rtimes_w B_i$, where $w = w_1(B_2)$ and $2 \leq i \leq 4$. These correspond to S^1-bundles with a section, i.e., to bundles with Euler class 0. We shall show that they each embed in $\mathbb{R}^5$.

If a flat 4-manifold M is the boundary of a regular neighbourhood of an embedding j of a non-orientable flat 3-manifold N in $\mathbb{R}^5$, then $\pi = \pi_1(M)$ is a non-trivial extension of $\pi_1(N)$ by $\mathbb{Z}$, $\beta = \beta_1(N)$ and $e(j)$ must have finite order. In particular, if $\pi_1(N) = B_1$ or B_2 then $\pi \cong G_2 \times \mathbb{Z}$ or $\mathbb{Z} \rtimes_w B_2$. The semidirect product is the only orientable, virtually abelian extension of B_2 by $\mathbb{Z}$, since $H_1(B_2;\mathbb{Z})$ is torsion-free. If $\pi_1(N) = B_3$ or B_4 then $\beta = 1$, $\pi \cong G_2 \rtimes_\theta \mathbb{Z}$ and the holonomy is $(Z/2Z)^2$.

Since Kb embeds in G_2, $Kb \times S^1$ embeds in $\mathbb{R}^5$ with normal Euler class 0, and so the flat 4-manifold with group $\mathbb{Z} \rtimes_w B_1$ embeds. (This is of course $G_2 \times S^1$.) Let R be the orientation preserving involution of

$D^2 \times D^2$ which swaps the factors. Then R restricts to an orientation-reversing involution of $T = S^1 \times S^1$, and $M(R_T) \cong K(B_2, 1)$ embeds in $M(R) \cong S^1 \times D^4 \subset \mathbb{R}^5$. Since this embedding can be isotoped off itself, the flat 3-manifold $K(B_2, 1)$ embeds in $\mathbb{R}^5$, with normal Euler class 0.

Two of the non-orientable flat 3-manifolds fibre over the torus, while the other two fibre over the Klein bottle. Let $p_i : E_i \to F$ be the projection of the associated $\mathbb{R}^2$-bundle, let $s : F \to E_i$ be the 0-section, and let $j_i : K(B_i, 1) \to E_i$ be the natural inclusion of the unit circle bundle. Note that j_i may be isotoped to a disjoint nearby embedding. Let η_i be the line bundle over F with $w_1(\eta_i) = s^*w_1(E_i)$. Then the Whitney sum $p_i \oplus \eta_i$ is an $\mathbb{R}^3$-bundle over F, with orientable total space $\widehat{E_i} = E(p_i \oplus \eta_i)$.

If $i = 2$ or 4 the fibres of the projections $p_i j_i$ have image 0 in $H_1(B_i; \mathbb{F}_2)$, and so $p_i j_i$ induces isomorphisms $H^q(F; \mathbb{F}_2) \cong H^q(B_i; \mathbb{F}_2)$, for $q \leq 2$. Since $w_2 = w_1^2$ for any 3-manifold, by the Wu relations, the Whitney sum formula gives $w_2(\widehat{E_i}) = 0$. Regular neighbourhoods of any embedding of T or Kb in $\mathbb{R}^5$ are D^3-bundles with parallelizable total space. Therefore if $i = 2$ or 4 then $\widehat{E_i}$ embeds in $\mathbb{R}^5$. Hence the flat 3-manifold $K(B_i, 1)$ also embeds in $\mathbb{R}^5$, with normal Euler class 0. The boundary of a regular neighbourhood is an orientable flat 4-manifold with group $\mathbb{Z} \rtimes_w B_i$.

When $i = 1$ or 3 it is not so clear that $w_2(\widehat{E_i}) = 0$. Instead we use more explicit constructions. We have already done this for $i = 1$. We may embed Kb in $S^1 \times D^3$ as the subset $\{(u^2, x, yu) \mid u \in S^1,\ x, y \in \mathbb{R},\ x^2 + y^2 = 1\}$. Let h be the orientation-preserving diffeomorphism of $S^1 \times D^3$ given by $h(u, x, y, z) = (\bar{u}, x, y, -z)$. Then h reverses the S^1 factor, $h(Kb) = Kb$ and h fixes pointwise the fibre of Kb over $u = 1$. The mapping torus $M(h)$ is an orientable D^3-bundle over Kb, and $M(h|_{Kb}) = B_3$. Since $h|_\partial$ has 1-dimensional fixed point set, the boundary of $M(h)$ is the orientable S^2-bundle over Kb with $w_2 = 0$, and so $w_2(M(h)) = 0$. Therefore $M(h)$ embeds in $\mathbb{R}^5$ as a regular neighbourhood of an embedding of Kb. Hence $K(B_3, 1)$ also embeds in $\mathbb{R}^5$, with normal Euler class 0. The boundary of a regular neighbourhood is an orientable flat 4-manifold with group $\mathbb{Z} \rtimes_w B_3$.

One of the three remaining groups $G_2 \rtimes \mathbb{Z}$ has abelianization $\mathbb{Z} \oplus Z/4Z$. The corresponding flat 4-manifold embeds in $\mathbb{R}^5$, by Theorem 6.2 of [3]. The group is a non-split extension of B_4 by $\mathbb{Z}$, and so the normal Euler class is a non-zero torsion class.

The two undecided cases have groups with presentations

$$\langle t, x, y, z \mid txt^{-1} = x^{-1}yz,\ ty = yt,\ tzt^{-1} = z^{-1},$$

$$xyx^{-1} = y^{-1},\ xzx^{-1} = z^{-1}, yz = zy\rangle$$

and

$$\langle t, x, y, z \mid txt^{-1} = x^{-1},\ tyt^{-1} = z,\ tzt^{-1} = y,$$
$$xyx^{-1} = y^{-1},\ xzx^{-1} = z^{-1}, yz = zy\rangle,$$

respectively. These manifolds are *Spin*, and so embed in $\mathbb{R}^6$. In each case the Farber-Levine pairing is metabolic, and so provides no obstruction to an embedding in $\mathbb{R}^5$. On the other hand, the abelianizations each need at least three generators, and so the result of [3] does not apply.

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