PARALLELIZABILITY OF 4-DIMENSIONAL INFRASOLVMANIFOLDS

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Abstract. We show that if M is an orientable 4-dimensional infrasolvmanifold and either $\beta = \beta_1(M;\mathbb{Q}) \geq 2$ or M is a $\mathbb{S}ol_0^4$- or a $\mathbb{S}ol_{m,n}^4$-manifold (with $m \neq n$) then M is parallelizable. There are non-parallelizable examples with $\beta = 1$ for each of the other solvable Lie geometries $\mathbb{E}^4$, $\mathbb{N}il^4$, $\mathbb{S}ol_1^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$ and $\mathbb{S}ol^3 \times \mathbb{E}^1$. We also determine which non-orientable flat 4-manifolds have a Pin^+- or Pin^--structure, and consider briefly this question for the other cases.

1. Introduction

A closed smooth 4-manifold M is parallelizable if and only if it is orientable ($w_1(M) = 0$), Spin ($w_2(M) = 0$) and $\chi(M) = \sigma(M) = 0$. Putrycz and Szczepański have shown that just three orientable flat 4-manifolds are not Spin and thus are not parallelizable [8]. The three exceptional cases are mapping tori of isometries of the Hantsche-Wendt flat 3-manifold. Thus a mapping torus of an orientation-preserving self-diffeomorphism of a parallelizable manifold need not be parallelizable.

Orientable nilmanifolds (in any dimension) and orientable solvmanifolds of dimension at most 4 are parallelizable, but there is a closed orientable solvmanifold of dimension 5 which is not Spin [1]. Thus the result of [8] is best possible, in terms of dimension and simplicity of structure.

In this note we shall consider the other 4-dimensional infrasolvmanifolds. These are all geometric [5], and have $\chi = \sigma = 0$. Our starting point is the observation is that the structure group of the tangent bundle of such a manifold M is contained in the image in $O(4)$ of the isometries which fix the identity. We show that if $\beta = \beta_1(M;\mathbb{Q}) \geq 2$ then M is parallelizable, but there are non-parallelizable examples with $\beta = 1$ in most cases. In §10 we give simple explicit models of the $Z/2Z$-extensions Pin^+ and Pin^- of $O(4)$. In §11 these are used to determine

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which non-orientable flat 4-manifolds admit Pin^+- or Pin^--structures. We also show that (with two possible exceptions) every non-orientable flat 4-manifold has a Pin^c-structure. Finally, we consider briefly similar issues for the other geometric 4-manifolds with $\chi = 0$.

In particular, we show that if an orientable 4-manifold is the total space of a torus bundle over the torus then it is Spin. (See Corollary 9.2.) I would like to thank Ron Stern for raising the more general question "is there an orientable aspherical surface bundle over the torus which is not Spin?" (still unsettled) that prompted this work.

2. The tangent bundle

If π is a group let π', $\zeta\pi$ and $\sqrt{\pi}$ denote the commutator subgroup, the centre and the Hirsch-Plotkin radical of π, respectively. Let $G^{ab} = G/G'$ be the abelianization of G. If S is a subset of π then $\langle S\rangle$ shall denote the subgroup of π generated by S.

Let G be a connected Lie group and let $\mathfrak{g}$ be the Lie algebra of G. Let $\pi < G \rtimes Aut(G)$ be a discrete subgroup which acts freely and cocompactly, and let $p : G \to M = \pi\backslash G$ be the canonical submersion. Let $\gamma = \pi/G \cap \pi$. Then γ is isomorphic to a subgroup of $Aut(G)$, and so acts on $\mathfrak{g}$.

Theorem 2.1. *The classifying map for the tangent bundle TM factors through $K(\gamma, 1)$.*

Proof. Let L_g be the diffeomorphism of G given by left translation $L_g(h) = gh$, for all $g, h \in G$, and let $L_{g*} = D(L_g)$ be the induced automorphism of $T(G)$. Let $\tau : G \times \mathfrak{g} \cong TG$ be the trivialization of the tangent bundle given by $\tau(g, v) = L_{g*}(v)$, for all $g \in G$ and $v \in \mathfrak{g} = T_1(G)$.

The group $G \rtimes Aut(G)$ acts on G via $(h, \alpha)(g) = h\alpha(g)$, for all $g, h \in G$ and $\alpha \in Aut(G)$. Since $\alpha(gh) = \alpha(g)\alpha(h)$, the chain rule gives $D(\alpha)_g L_{g*} = L_{\alpha(g)*}D(\alpha)_1$. Therefore $D(h, \alpha)_g(\tau(g, v)) =$

$$L_{h*}D(\alpha)_g(L_{g*}(v)) = L_{h*}L_{\alpha(g)*}D(\alpha)_1(v) = \tau((h, \alpha)(g), D(\alpha)_1(v)).$$

Hence $\tau^{-1}(D(h, \alpha)_g(\tau(g, v))) = (h\alpha(g), D(\alpha)_1(v))$, for all $g, h \in G$, $\alpha \in Aut(G)$ and $v \in T_1(G)$. In particular, τ is G-equivariant, and induces a parallelization of the covering space $\widehat{M} = (G \cap \pi)\backslash G$. Moreover, the structure group for TM is contained in the image of γ in $Aut(\mathfrak{g})$, and so the classifying map for TM factors through $K(\gamma, 1)$. □

Corollary 2.2. *If the epimorphism $\pi \to \gamma$ factors through $\mathbb{Z}$ then $w_1(M)^2 = w_2(M) = 0$.*

Proof. This is clear, since $H^2(\mathbb{Z}; \mathbb{F}_2) = 0$. □

In the cases of interest to us γ shall be a finite group. Hence the rational Pontrjagin classes of M are 0. In our investigations of the Stiefel-Whitney classes we may assume furthermore that γ is a finite 2-group. For the inclusion of the preimage of the 2-Sylow subgroup $S \leq \gamma$ into π induces isomorphisms on cohomology with coefficients $\mathbb{F}_2$, since $[\gamma : S]$ is odd.

Let T and Kb denote the torus and Klein bottle, respectively.

3. *Spin*(4)

Let S^3 be the group of quaternions of norm 1. Then $Spin(4) \cong S^3 \times S^3$, and the covering map $\rho : Spin(4) \to SO(4)$ is given by $\rho(U,V)(q) = VqU^{-1}$, for all $U, V \in S^3$ and $q \in \mathbb{H}$. Let Δ_Q be the image of the quaternionic group $Q(8) = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ under the diagonal embedding in $Spin(4)$. Then $\rho(\Delta_Q)$ is the subgroup of $SO(4)$ generated by the diagonal matrices $diag[1,-1,-1,1]$ and $diag[1,1,-1,-1]$.

Lemma 3.1. *An element σ of order 2 in $SO(4)$ is the image of an element of order 2 in $Spin(4)$ if and only if $\sigma = \pm I$.*

Proof. It is easily seen that $\rho(U,V)^2 = I$ if and only if $U^2 = V^2 = \pm 1$. If $U^2 = V^2 = 1$ then $U, V = \pm 1$ and $\rho(U,V) = \pm I$. □

Thus if σ has order 2 and trace 0 it does not lift to an element of order 2.

Theorem 3.2. *Suppose that $\gamma = Z/2Z$. If the epimorphism $\pi \to \gamma$ factors through $Z/4Z$ then $w_2(M) = 0$. Otherwise, $w_2(M) = 0$ if and only if the image of γ in $SO(4)$ is contained in $\pm I$.*

Proof. If $\gamma = Z/2Z$ then $H^*(\gamma; \mathbb{F}_2) = \mathbb{F}_2[x]$ is generated in degree 1. However if $u \in H^1(Z/4Z; \mathbb{F}_2)$ then $u^2 = 0$.

If the epimorphism $\pi \to \gamma$ does not factor through $Z/4Z$ then $w_2(M) = 0$ if and only if there is an element of order 2 in $Spin(4)$ whose image in $SO(4)$ generates the image of γ. This is possible if and only if the image of γ in $SO(4)$ is $\pm I$, by Lemma 3.1. □

The following simple lemma is a special case of a more general result about nilpotent groups.

Lemma 3.3. *Let G be a finite 2-group with a normal subgroup $K = Z/2Z$ such that the quotient epimorphism of G onto G/K induces isomorphisms $G^{ab}/2G^{ab} \cong (G/K)^{ab}/2(G/K)^{ab}$. If $f : H \to G$ is a homomorphism such that $pf(H) = G/K$ then $f(H) = G$.*

Proof. The index $[G : f(H)]$ is at most 2, and so $f(H)$ is normal in G. Since $G/f(H)$ has order at most 2 and trivial abelianization, it is trivial, and so $f(H) = G$. $\square$

4. Solvable lie geometries of dimension 4

Suppose now that G a 1-connected solvable Lie group of dimension 4, corresponding to a geometry $\mathbb{G}$ of solvable Lie type. There are six relevant families of geometries: $\mathbb{E}^4$, $\mathbb{N}il^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol_0^4$, $\mathbb{S}ol_1^4$ and $\mathbb{S}ol_{m,n}^4$. (The final family includes the product geometry $\mathbb{S}ol^3 \times \mathbb{E}^1 = \mathbb{S}ol_{m,m}^4$, for all $m > 0$, as a somewhat exceptional case.) We shall use the parametrizations (diffeomorphisms from $\mathbb{R}^4$ to the model space) given in [10], unless otherwise indicated. (See also Chapter 7 of [4].)

Let $Isom^+(\mathbb{G})$ be the group of orientation-preserving isometries, and let $K_G < Isom^+(\mathbb{G})$ be the stabilizer of the identity in G. Let $\pi < Isom^+(\mathbb{G})$ be a discrete subgroup which acts freely and cocompactly on G. If $M = \pi\backslash G$ then $\beta = \beta_1(M;\mathbb{Q}) \geq 1$ and M is the mapping torus of a self-diffeomorphism of a $\mathbb{E}^3$-, $\mathbb{N}il^3$- or $\mathbb{S}ol^3$-manifold. If $\beta = 1$ the mapping torus structure is essentially unique. If $\beta \geq 2$ then M is also the total space of a torus bundle over the torus.

All orientable $\mathbb{S}ol_0^4$-manifolds are coset spaces $\pi\backslash\widetilde{G}$ with π a discrete subgroup of a 1-connected solvable Lie group $\widetilde{G}$, which in general depends on π. (See page 138 of [4]). Thus all are parallelizable, by Theorem 2.1. In all other cases, the translation subgroup $G \cap \pi$ is a lattice in G. If G is nilpotent then $G\cap\pi = \sqrt{\pi}$; in general, $\sqrt{\pi} \leq G \cap \pi$, and γ is finite.

The groups of $\mathbb{S}ol_{m,n}^4$-manifolds (with $m \neq n$) have the form $\mathbb{Z}^3 \rtimes_\theta \mathbb{Z}$, and so the epimorphism $\pi \to \gamma$ factors through $\mathbb{Z}$. (This holds also for $\mathbb{S}ol_0^4$-manifolds. See Corollary 8.4.1 of [4].) Similarly, if M is an orientable $\mathbb{E}^4$-manifold and $\beta \geq 2$ the epimorphism factors through $\mathbb{Z}$. Thus all such manifolds are parallelizable.

If M is a $\mathbb{N}il^4$-manifold and $\beta = 2$ or is a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold and $\beta = 3$ then π is nilpotent, so $\gamma = 1$ and M is parallelizable, by Theorem 2.1.

We shall consider the other possibilities below.

5. $\mathbb{E}^4$-manifolds

There are 13 orientable flat 4-manifolds with $\beta = 1$ and holonomy a 2-group, nine with holonomy $(Z/2Z)^2$ and four with holonomy D_8. Seven of these are T-bundles over Kb, and the holonomy representation factors through $Z \rtimes_{-1} Z$. In these cases it is not hard to lift the holonomy representation to $Spin(4)$. The remaining six are mapping tori of diffeomorphisms of the Hantzsche-Wendt flat 3-manifold HW. Let $G_6 = \pi_1(HW)$.

In [8] it is shown directly that just three orientable flat 4-manifolds are not Spin manifolds. We shall identify the groups of the three exceptional cases as semidirect products $G_6 \rtimes \mathbb{Z}$. In our presentations for these groups we shall represent the standard basis of the translation subgroup $\mathbb{Z}^4$ by w, x, y, z and the other generators by m, n, p, q, r (rather than by t_1, t_2, t_3, t_4 and $\gamma_1, \gamma_2, F_1, F_2$, respectively, as in [8]). We shall use the notation of §8.2 of [4] for automorphisms of G_6. (Note however that Γ_7 and Γ_9 are here used as in [8], and are not the nilpotent 3×3 matrix groups of [4].)

The first example is the group Γ_7 with presentation

$$\langle \mathbb{Z}^4, m, n \mid m^2 = w,\ n^2 = y^{-1}z,\ (mn)^2 = xy^{-1}z,\ mxm^{-1} = x^{-1},$$

$$mym^{-1} = z^{-1},\ nwn^{-1} = w^{-1},\ nxn^{-1} = x^{-1},\ ny = yn,\ nz = zn\rangle$$

and holonomy $(Z/2Z)^2$. The infinite cyclic subgroups generated by each of w, x, yz and $y^{-1}z$ are all normal. Let $\tilde{n} = ny$. The subgroup generated by $\{m, \tilde{n}\}$ is normal, contains w, $x = (m\tilde{n})^2$ and $yz = \tilde{n}^2$, and the quotient is infinite cyclic, generated by the image of y. This subgroup has the presentation $\langle m, \tilde{n} \mid\ \tilde{n}m^2\tilde{n}^{-1} = m^{-2},\ m\tilde{n}^2m^{-1} = \tilde{n}^{-2}\rangle$, and so is isomorphic to G_6. Since $ymy^{-1} = \tilde{n}^2m$ and $y\tilde{n} = \tilde{n}y$ we see that $\Gamma_7 \cong G_6 \rtimes_e \mathbb{Z}$.

The corresponding flat 4-manifold has a number of distinct Seifert fibrations. In particular, it is Seifert fibred over the silvered annulus $\mathbb{A} = S^1 \times \mathbb{I}$, since $\Gamma_7/\langle w, yz\rangle$ has the presentation

$$\langle y, m\tilde{n} \mid\ ym = my,\ y\tilde{n} = \tilde{n}y,\ m^2 = \tilde{n}^2 = 1\rangle,$$

and so $\Gamma_7/\langle w, yz\rangle \cong \mathbb{Z} \times D_\infty$, where $D_\infty = Z/2Z * Z/2Z$ is the infinite dihedral group.

The second example is the group Γ_9 with presentation

$$\langle \mathbb{Z}^4, p, q \mid p^2 = x^{-1}y,\ q^2 = x^{-1}z,\ (pq)^2 = wx^{-1}z,$$

$$pwp^{-1} = w^{-1},\ pxp^{-1} = xy^{-1}z^{-1},\ pyp^{-1} = z^{-1},$$

$$qwq^{-1} = w^{-1},\ qxq^{-1} = y,\ qzq^{-1} = x^{-1}yz\rangle$$

and holonomy $(Z/2Z)^2$. The infinite cyclic subgroups generated by each of w and yz are normal. Let $\tilde{q} = xq$. The subgroup generated by $\{p, \tilde{q}\}$ is normal, contains $w = (p\tilde{q})^2$ and $yz = \tilde{q}^2$, and the quotient is infinite cyclic, generated by the image of x. This subgroup has the presentation $\langle p, \tilde{q} \mid\ p\tilde{q}^2p^{-1} = \tilde{q}^{-2},\ \tilde{q}p^2\tilde{q}^{-1} = p^{-2}\rangle$, and so is isomorphic to G_6. Since $xpx^{-1} = \tilde{q}^2p$ and $x\tilde{q}x^{-1} = p^{-2}\tilde{q}$ we see that $\Gamma_7 \cong G_6 \rtimes_{bce} \mathbb{Z}$.

The corresponding flat 4-manifold is also Seifert fibred over $\mathbb{A}$, since $\Gamma_7/\langle w, yz\rangle$ has the presentation

$$\langle x, p, \tilde{q} \mid \ xp = px,\ xp\tilde{q} = \tilde{q}xp,\ \tilde{q}^2 = (p\tilde{q})^2 = 1\rangle,$$

and so $\Gamma_7/\langle w, yz\rangle \cong \mathbb{Z} \times D_\infty$.

The third example is the group Δ_4 with presentation

$$\langle \mathbb{Z}^4, q, r \mid q^2 = x^{-1}z,\ r^2 = y,\ (qr)^4 = w^{-1}x^{-1}z,$$
$$qwq^{-1} = w^{-1},\ qxq^{-1} = y,\ qzq^{-1} = x^{-1}yz,$$
$$rwr^{-1} = w^{-1},\ rxr^{-1} = z^{-1},\ ry = yr\rangle$$

and holonomy D_8. The infinite cyclic subgroups generated by w and $x^{-1}z$ are normal, as is the subgroup generated by $\{xy^{-1}, yz\}$. Let $s = qy$, $t = rsr^{-1}$, $u = xy^{-1}$ and $v = yz$. Then $s^2 = v$, $t^2 = u^{-1}$ and $(st)^2 = w^{-1}$. The subgroup generated by $\{s, t\}$ is normal, and has the presentation $\langle s, t \mid \ st^2s^{-1} = t^{-2},\ ts^2t^{-1} = s^{-2}\rangle$, and so is isomorphic to G_6. The quotient is infinite cyclic, generated by the image of r. Since $rsr^{-1} = t$ and $rtr^{-1} = st^2$ we see that $\Delta_4 \cong G_6 \rtimes_{ei} \mathbb{Z}$.

The quotient $\Delta_4/\langle w, x^{-1}z\rangle$ has the presentation

$$\langle x, q, r \mid \ rxr^{-1} = x^{-1},\ q^2 = (qr)^4 = 1,\ qxq^{-1} = r^2\rangle.$$

Hence the corresponding flat manifold is Seifert fibred over $\mathbb{D}(\overline{2}, \overline{4})$.

The quotient $\Delta_4/\langle x^{-1}y, yz\rangle$ has the presentation

$$\langle y, r, s, t \mid \ yr = ry,\ ys = sy, ,\ s^2 = 1,\ t = rsr^{-1}\rangle,$$

and so is a semidirect product $D_\infty \rtimes \mathbb{Z}$. Thus the manifold is also Seifert fibred over the silvered Möbius band $\mathbb{M}b$.

6. $\mathbb{N}il^4$-manifolds

The group Nil^4 is the semidirect product $\mathbb{R}^3 \rtimes_\theta \mathbb{R}$, where $\theta(t) = exp(tJ) = I + tJ + \frac{t^2}{q}J^2$ is the 1-parameter group determined the nilpotent matrix

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $A = diag[1, -1, 1]$. Then $AJ = -JA$, so $A\theta(t)A^{-1} = \theta(-t)$. The stabilizer of the identity in $Isom(\mathbb{N}il^4)$ is generated by the commuting involutions α and ω defined by $\alpha(v, t) = (Av, -t)$ and $\omega(v, t) = (-v, t)$, for $(v, t) \in \mathbb{R}^3 \rtimes_\theta \mathbb{R}$. These act in the obvious diagonal fashion on $\mathfrak{g}$, and the orientation-reversing involution ω acts non-trivially on the +1-eigenspace of α. (See [10].) In particular, the group of left translations of Nil^4 has index 2 in $Isom^+(\mathbb{N}il^4)$, and the image in $SO(4)$ of the generator α of the quotient $Isom^+(\mathbb{N}il^4)/Nil^4 = Z/2Z$ has trace 0.

If M is an orientable $\mathbb{N}il^4$-manifold with $\beta = 1$ then M is the mapping torus of a self-diffeomorphism of a $\mathbb{N}il^3$-manifold. The fibre N is Seifert fibred over T or the flat 2-orbifold $S(2,2,2,2)$, by Theorem 8.5 of [4]. If N is an S^1-bundle over T then $\nu = \pi_1(N)$ is nilpotent and the mapping torus M is parallelizable, by Corollary 2.2. If N is Seifert fibred over $S(2,2,2,2)$ then ν has a presentation

$$\langle w, x, y \mid wxw^{-1} = x^{-1}w^{2a},\ wyw^{-1} = y^{-1}w^{2b},\ w^2x = xw^2,\ w^2y = yw^2,$$
$$xyx^{-1}y^{-1} = w^{2q}\rangle.$$

We may assume that $0 \leq a, b \leq 1$. The exponent q is nonzero, for otherwise N would be flat. (In fact $\frac{q}{2}$ is the Euler invariant of the Seifert fibration.) Since ν is torsion-free, $(xyw)^2 = w^{2(q+a+b+1)}$ and w^2 is central, $q+a+b$ must be even. Let $t \in \pi$ represent a generator of $\pi/\nu \cong \mathbb{Z}$. The automorphism A of the canonical subquotient $\sqrt{\nu}/\zeta\sqrt{\nu} \cong \mathbb{Z}^2$ induced by t must have infinite order and both eigenvalues equal. (See Theorem 8.5 of [4].) On replacing t by tw, if necessary, we may assume that the eigenvalues are 1 and $(A - I)^2 = 0$. Then $\sqrt{\pi} = \langle\sqrt{\nu}, t\rangle$ and $\gamma = \pi/\sqrt{\pi} = Z/2Z$ is generated by the image of w. Since the image of w in $SO(4)$ has trace 0 it follows from Theorem 3.2 that $w_2(M) = 0$ if and only if the image of w in π/π' has order divisible by 4.

In particular, if M is parallelizable and $\beta = 1$ then q must be even. Therefore the group with presentation

$$\langle t, w, x, y \mid\ tw = wt,\ tx = xt,\ tyt^{-1} = xy,\ wxw^{-1} = x^{-1}w^{-2},$$
$$wyw^{-1} = y^{-1},\ xyx^{-1}y^{-1} = w^2\rangle$$

is the group of an orientable $\mathbb{N}il^4$-manifold which is not parallelizable.

7. $\mathbb{S}ol^4_1$-MANIFOLDS

The group of left translations of Sol^4_1 has index 4 in $Isom^+(\mathbb{S}ol^4_1)$, and the image of the quotient $Isom^+(\mathbb{S}ol^4_1)/Sol^4_1$ in $SO(4)$ is the subgroup of diagonal matrices $\rho(\Delta_Q) \cong (Z/2Z)^2$.

If M is an orientable $\mathbb{S}ol^4_1$-manifold then $\beta = 1$ and $M = M(\theta)$ is the mapping torus of a self-diffeomorphism of a $\mathbb{N}il^3$-manifold. The fibre N is again an S^1-bundle over T or Seifert fibred over $S(2,2,2,2)$, by Theorem 8.5 of [4], and we shall use the notation $a, b, q, A, \dots$ of §5 above here. If N is an S^1-bundle over T then $\nu = \pi_1(N)$ is nilpotent and M is parallelizable, by Corollary 2.2. If N is Seifert fibred over $S(2,2,2,2)$ and the automorphism A has determinant $+1$ then $\gamma = Z/2Z$ and Theorem 3.2 applies, and we again see that $w_2(M) = 0$ if and only if q is even.

Otherwise, N is Seifert fibred over $S(2,2,2,2)$, the automorphism A has determinant -1 and infinite order, and the image of γ in $SO(4)$ is $\rho(\Delta_Q)$. An obvious necessary condition for M to be parallelizable is that $w_2(M_2)=0$, where $M_2=M(\theta^2)$. For this double cover $\gamma=Z/2Z$, and so q must be even. Then $a=b$, and we may assume that π has the presentation

$$\langle t,w,x,y \mid twt^{-1}=x^m y^n w^p,\ txt^{-1}=x^e y^f w^{2r},\ \ tyt^{-1}=x^g y^h w^{2s},$$
$$wxw^{-1}=x^{-1}w^{2a},\ \ wyw^{-1}=y^{-1}w^{2a},\ \ w^2x=xw^2,\ \ w^2y=yw^2,$$
$$xyx^{-1}y^{-1}=w^{2q}.\rangle$$

Since w^2 generates the centre of ν and $eh-fg=-1$ we must have $(x^m y^n w^p)^2=w^{-2}$, and so

$$x^m y^n x^{-m} y^{-n} w^{2(p+am+an)}=w^{-2}.$$

Using $x^i y^j = w^{2qij} y^j x^i$ gives

$$p=-(qmn+a(m+n)+1),$$

which must be odd, and so $a(m+n)$ is even. Conjugating the defining relations for ν by t leads to further numerical constraints on the exponents m,n,r and s. These are that q divides $2r+a(e+f+1)$ and $2s+a(g+h+1)$, and

$$n=fh(e-g)+\frac{1}{q}((2r+a)h-(2s+a)f-a)$$

$$m=eg(f-h)+\frac{1}{q}((2r+a)g-(2s+a)e+a).$$

(Since $GL(2,\mathbb{F}_2)\cong Sym(3)$ and passing to a cover of odd degree induces isomorphisms on $H^*(-;\mathbb{F}_2)$, we may also assume that the image of A in $GL(2,\mathbb{F}_2)$ has order 1 or 2, i.e., that $e+h$ is even.)

Since no proper subgroup of $Q(8)$ maps onto $(Z/2Z)^2$, the manifold M is parallelizable if and only if there is an epimorphism from π to $Q(8)$ which maps $\{t,w\}$ to a generating set. Any such map factors through the quotient π/C, where C is the subgroup normally generated by t^2w^{-2}, $twt^{-1}w$ and all fourth powers.

It may be easily verified that if q is a multiple of 4, $a=r=s=0$, $m=eg(f-h)$ and $n=fh(e-g)$ then the above constraints are satisfied. On setting $x=y=1$ and $t^2=w^2$ in the resulting presentation for π we obtain the presentation

$$\langle t,w \mid twt^{-1}=w^{-1},\ t^2=w^2\rangle$$

for $Q(8)$, and so M is parallelizable.

On the other hand, if $q=2$, $e+h\equiv 0\ mod\ (4)$, f is odd, $r=0$ and s is odd then the quotient π/C has a presentation

$$\langle t,w,x,y \mid twt^{-1}=w^{-1},\ t^2=w^2,\ txt^{-1}=x^ey^f,\ tyt^{-1}=x^gy^hw^2,$$
$$x^4=y^4=1,\ wxw^{-1}=x^{-1},\ wyw^{-1}=y^{-1},\ xy=yx,\ x^my^n=w^2\rangle.$$

(Here $m=eg(f-h)-se$ and $n=fh(e-g)-sf$.) The centrality of t^2 implies that

$$x=t^2xt^{-2}=(x^ey^f)^e(x^gy^hw^2)^f=x^{e^2+fg}y^{f(e+h)}w^{2f}=xw^2,$$

(since $e^2+fg=e(e+h)+1$), and so $w^2=1$. Hence no such manifold is parallelizable.

8. $\mathbb{N}il^3\times\mathbb{E}^1$-manifolds

Since all isometries of $\mathbb{N}il^3$ act orientably, $Isom^+(\mathbb{N}il^3\times\mathbb{E}^1)\cong Isom(\mathbb{N}il^3)\times\mathbb{R}$, and the quotient of $Isom(\mathbb{N}il^3)$ by its subgroup of left translations is isomorphic to $O(2)$. The induced action on the abelianization $\mathbb{R}^2$ is the standard action by rotations and reflections, and the action on the centre $\mathbb{R}$ is multiplication by the determinant. We shall use the parametrization of $Nil^3\times\mathbb{R}\cong\mathbb{R}\times Nil^3$ for which the first coordinate represents the Euclidean factor, the middle coordinates represent the image of Nil^3 in its abelianization and the final coordinate corresponds to the centre. Then reflections across the axes of $\mathbb{R}^2$ have image in $\rho(\Delta_Q)$.

If M is an orientable $\mathbb{N}il^3\times\mathbb{E}^1$-manifold but is not a nilmanifold then $\beta=1$ or 2, $\gamma=\pi/\sqrt{\pi}\neq 1$ and γ is finite cyclic or finite dihedral. Moreover M is the mapping torus of a self-diffeomorphism of a $\mathbb{E}^3$- or $\mathbb{N}il^3$-manifold N. If N is flat then it is the 3-torus or the half-turn flat 3-manifold, by Theorem 8.5 of [4]. In this case $\beta\geq 2$, as may be seen by considering the Jordan normal form of $\theta|_{\sqrt{\nu}}$ (and using the facts that π is orientable and not virtually abelian).

Theorem 8.1. *If M is an orientable $\mathbb{N}il^3\times\mathbb{E}^1$-manifold and $\beta=\beta_1(M;\mathbb{Q})=2$ then M is parallelizable.*

Proof. Let $\overline{\pi}$ be the image of π under projection to the the first factor. The induced action of $\pi/\sqrt{\pi}$ on $\sqrt{\pi}$ is effective, since M is orientable. Therefore the holonomy $\gamma=\pi/\sqrt{\pi}$ is isomorphic to a finite subgroup of $GL(2,\mathbb{Z})$. If it is not cyclic then $\overline{\pi}$ has finite abelianization. Therefore if $\beta=2$ then γ is cyclic, and M is parallelizable, by Theorem 3.2. □

If $\nu=\pi_1(N)$ is nilpotent then γ is cyclic and M is again parallelizable. Thus we may assume for the remainder of this section that $\beta=1$ and the fibre N is a $\mathbb{N}il^3$-manifold, with $\nu\neq\sqrt{\nu}$.

We may also assume that $[\nu:\sqrt{\nu}]$ is a 2-group. The elements of finite order in $\nu/\zeta\sqrt{\nu}$ act orientably on $\sqrt{\nu}/\zeta\sqrt{\nu}\cong\mathbb{Z}^2$, by Theorem 8.5(3) of [4], and so $\nu/\sqrt{\nu}\cong Z/2Z$, $Z/4Z$ or $(Z/2Z)^2$. The image of $\nu/\sqrt{\nu}$ in $GL(2,\mathbb{Z})$ is normalized by the matrix A corresponding to the action of a generator of $\pi/\nu\cong\mathbb{Z}$ on $\sqrt{\nu}/\zeta\sqrt{\nu}$.

If the image of $\nu/\sqrt{\nu}$ in $O(2)$ lies in $SO(2)$ then $\nu/\sqrt{\nu}$ is cyclic. If it is generated by $-I$ then we may assume that $A=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. Otherwise it has order 4, and is generated by the rotation $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, and we may assume that $A=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

If $\nu/\sqrt{\nu}\cong Z/2Z$ and its image is not contained in $SO(2)$ then $\nu/\zeta\sqrt{\nu}\cong\pi_1(Kb)=\mathbb{Z}\rtimes_{-1}\mathbb{Z}$, and $\nu/\sqrt{\nu}$ acts on $\sqrt{\nu}/\zeta\sqrt{\nu}$ via a diagonal matrix $D=\pm\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. We may assume that $A=-D$.

If $\nu/\sqrt{\nu}\cong(Z/2Z)^2$ then its image in $GL(2,\mathbb{Z})$ is the diagonal subgroup, and we may assume that $A=I$ or $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

In each case, if A is diagonal then $\gamma=\rho(\Delta_Q)$, and we may hope to apply Theorem 3.2. In particular, the group with presentation

$$\langle t,w,x,y\mid twt^{-1}=y^{-1}w^{-1},\, txt^{-1}=xw^2,\ tyt^{-1}=y^{-1},$$
$$wxw^{-1}=x^{-1},\ wyw^{-1}=y^{-1},\ xyx^{-1}y^{-1}=w^2\rangle$$

is the group of an orientable $\mathbb{N}il^3\times\mathbb{E}^1$-manifold which is not parallelizable.

In the remaining cases $A=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and the image of γ in $SO(4)$ is not in the standard diagonal subgroup. Let $\mathbf{r}=\frac{1}{\sqrt{2}}(1+\mathbf{k})$ and $\mathbf{R}=(\mathbf{r},\mathbf{r})$ in $Spin(4)$. Then $\rho(\mathbf{R})$ induces rotation through $\frac{\pi}{2}$ in the middle two coordinates. If $\gamma\cong(Z/2Z)^2$ then conjugation by $\rho(\mathbf{R})$ diagonalizes the image of γ, and we may argue as before.

Otherwise, $\gamma\cong D_8$. The subgroup of S^3 generated by $Q(8)$ and $\mathbf{r}$ is isomorphic to the generalized quaternionic group $Q(16)$, with presentation

$$\langle\xi,\eta\mid\xi^2=\eta^4,\ \xi\eta\xi^{-1}=\eta^{-1}\rangle.$$

It now follows from Lemma 3.3 that $w_2(M)=0$ if and only if the homomorphism from π to $\gamma=\pi/\sqrt{\pi}$ factors through $Q(16)$.

9. $\mathbb{S}ol^3\times\mathbb{E}^1$-manifolds

Let G be the subgroup of $GL(4,\mathbb{R})$ of bordered matrices $\left(\begin{smallmatrix}D&\xi\\0&1\end{smallmatrix}\right)$, where $D=diag[\pm e^{at},\pm1,\pm e^{-at}]$ and $\xi\in R^3$. The model space for the geometry $\mathbb{S}ol^3\times\mathbb{E}^1$ is the subgroup $Sol^3\times\mathbb{R}$ of G with positive diagonal entries, and $Isom(\mathbb{S}ol^3\times\mathbb{E}^1)$ is generated by G and the involution which sends (x,y,z,t) to $(z,y,x,-t)$. The quotient of $Isom^+(\mathbb{S}ol^3\times\mathbb{E}^1)$ by its subgroup of left translations is isomorphic to D_8.

If M is an orientable $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold then $\beta = 1$ or 2, and M is the mapping torus of a self-diffeomorphism of a $\mathbb{E}^3$- or $\mathbb{S}ol^3$-manifold N.

Theorem 9.1. *If M is an orientable $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold and $\beta = \beta_1(M;\mathbb{Q}) = 2$ then M is parallelizable.*

Proof. If $\beta = 2$ then the elements of π are isometries (σ, τ), where σ is an orientation-preserving isometry of Sol^3 and τ is a translation of $\mathbb{R}$. The eigenvalues of σ acting on the commutator subgroup $Sol^{3\prime} = \mathbb{R}^2$ are either both positive or both negative. It follows that the holonomy group γ is $Z/2Z$, and so M is parallelizable, by Theorem 3.2. $\square$

Corollary 9.2. *If an orientable 4-manifold M is the total space of a T-bundle over T then it is parallelizable.*

Proof. This follows from [8] if M is flat, and from Theorem 2.1 if M is a nilmanifold. Otherwise, $\beta = 2$ and M is a $\mathbb{N}il^3 \times \mathbb{E}^1$- or $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold, and Theorem 8.1 or Theorem 9.1 applies. $\square$

If $\beta = 1$ and $\pi \cong \mathbb{Z}^3 \rtimes \mathbb{Z}$ then γ is cyclic and M is parallelizable, by Theorem 3.2.

Otherwise the fibre N is either the half-turn flat manifold or is a $\mathbb{S}ol^3$-manifold, by Theorem 8.5 of [4], and so $\beta_1(N;\mathbb{Q}) \leq 1$. Suppose first that N is a T-bundle over S^1, so that $\nu \cong \mathbb{Z}^2 \rtimes_W \mathbb{Z}$, where $W = -I$ if N is the half-turn flat manifold and $\det(W) = +1$ and $|\mathrm{tr}(W)| > 2$ otherwise. Then π has a presentation

$$\langle t, w, x, y \mid twt^{-1} = x^m y^n w^{-1},\ txt^{-1} = x^e y^f,\ tyt^{-1} = x^g y^h,$$

$$wxw^{-1} = x^p y^q,\ wyw^{-1} = x^r y^s,\ xy = yx\rangle$$

where $eh - fg = -1$ (since $\beta = 1$ and M is orientable). In this case γ has image $\rho(\Delta_Q)$ in $SO(4)$. Then π maps onto $\pi_1(Kb) = \mathbb{Z} \rtimes_{-1} \mathbb{Z}$, and M is parallelizable if and only if the image of w in π/π' has order divisible by 4. This holds if and only if the class of (m, n) *mod* (2) is not in the image of $\left(\begin{smallmatrix} e-1 & f \\ g & h-1 \end{smallmatrix}\right)$. In particular, if $e + h$ is odd then M is not parallelizable.

The remaining possibility is that N is the union of two copies of the mapping cylinder of the orientable double cover of the Klein bottle, and ν is a torsion-free extension of D_∞ by $\mathbb{Z}^2$. In this case the image of γ in $SO(4)$ is generated by $\rho(\Delta_Q)$ and the involution $\rho(\mathbf{s}, -\mathbf{s})$, where $\mathbf{s} = \mathbf{ir} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$. The preimage of this group in $Spin(4)$ is again isomorphic to $Q(16)$, and we again find that $w_2(M) = 0$ if and only if the homomorphism from π to γ factors through $Q(16)$.

10. Pin^+ and Pin^-

There are two central extensions of $O(n)$ by $Z/2Z$ which are of topological interest. A closed n-manifold M admits a (tangential) $Pin(n)$-structure if and only if $w_2(M)=0$, while it admits a $Pin(n)^-$-structure if and only if $w_2(M)+w_1(M)^2=0$ [7]. In each case the orientable cover must be a $Spin$-manifold. If $w_1(M)^2=0$ then M either admits both structures or neither. If $w_1(M)^2\neq 0$ then M admits at most one of these structures.

The orthogonal groups $O(n)$ are generated by reflections. Let R_i be reflection in the ith coordinate of $\mathbb{R}^n$. Then every reflection is a conjugate of R_1 in $O(n)$. The groups $Pin(n)^+$ and $Pin(n)^-$ are each generated by $Spin(n)$ and an element e_+ or e_- (respectively) such that $e_\pm^2\in Spin(n)$. The covering epimorphism from $Spin(n)$ to $SO(n)$ extends to epimorphisms $\rho_\pm : Pin(n)^\pm\to O(n)$, and we may assume that $\rho_\pm(e_\pm)=R_1$. Hence $\rho_\pm(e_\pm)^2=I$, and so $e_\pm^2\in\zeta Spin(n)$.

When $n=4$ we must have $e_\pm^2=(\mathbf{1},\mathbf{1})=1$ or $e_\pm^2=(-\mathbf{1},-\mathbf{1})$. In this dimension R_1 is reflection across the hyperplane of pure quaternions, i.e., $R_1(q)=-\overline{q}$, for all $q\in\mathbb{H}$. Therefore $R_1(UqV^{-1})=VR_1(q)U^{-1}$, for all $\mathbf{q}\in\mathbb{H}$ and $U,V\in S^3$. Therefore we must have $e_\pm(U,V)e_\pm^{-1}=\varepsilon(V,U)$, where $\varepsilon=1$ or -1. The coefficient ε is independent of (V,U), since $Spin(4)$ is connected, and must in fact be 1, since conjugation by $e_\pm$ fixes $e_\pm^2$.

This leads to a simple explicit description of the groups $Pin^\pm=Pin(4)^\pm$. We define Pin^+ to be the semidirect product $Spin(4)\rtimes Z/2Z$ with respect to the natural involution of $Spin(4)$ which sends (U,V) to (V,U), for all $U,V\in S^3$. Thus Pin^+ is generated by $Spin(4)$ and an element c_+ such that $c_+^2=1$. We let Pin^- be the group generated by $Spin(4)$ and an element c_- inducing the same involution, but such that $c_-^2=(-\mathbf{1},-\mathbf{1})$. (Thus c_- has order 4.) The double cover $\rho : Spin(4)\to SO(4)$ extends to epimorphisms $\rho_\pm : Pin^\pm\to O(4)$ by setting $\rho_\pm(c_\pm)(\mathbf{q})=-\overline{\mathbf{q}}$, for all $\mathbf{q}\in\mathbb{H}$. These are (central) extensions of $O(4)$ by $Z/2Z$.

The groups Pin^+ and Pin^- are in fact isomorphic. (This may be a peculiarity of low dimensions.) The map from Pin^+ to Pin^- which is the identity on $Spin(4)$ and sends c_+ to $\widetilde{c}=c_-(\mathbf{i},\mathbf{i})$ defines an isomorphism. However the extensions $\rho_\pm$ are not equivalent. In particular, $\rho_+^{-1}(\langle R_1\rangle)\cong(Z/2Z)^2$, while $\rho_-^{-1}(\langle R_1\rangle)\cong Z/4Z$.

In each case, the preimage in $Pin^\pm$ of a finite subgroup $F<O(4)$ is a (necessarily central) extension of F by $Z/2Z$. In particular, if a 4-dimensional infrasolvmanifold $M=\pi\backslash G$ admits a $Pin^\pm$-structure the image of $G\cap\pi$ in $Pin^\pm$ must be central, of order at most 2. Thus

we may simplify our search for such structures by imposing suitable additional relations on π.

With the exception of six cases, the holonomy groups of non-orientable flat 4-manifolds are either cyclic or have 2-Sylow subgroup $(Z/2Z)^k$ for some $k \leq 3$. The following lemma applies to these groups.

Lemma 10.1. *If $A \cong (Z/2Z)^k < O(4)$ then A is diagonalizable.*

(1) *If $A = \langle R_1, R_2R_3\rangle$ then $\rho_\pm^{-1}(A) \cong Z/4Z \oplus Z/2Z$;*
(2) *if $A = \langle R_1, R_1R_2\rangle$ then $\rho_+^{-1}(A) \cong D_8$ and $\rho_-^{-1}(A) \cong Q(8)$;*
(3) *if $A = \langle -R_1, R_1R_2\rangle$ then $\rho_+^{-1}(A) \cong Q(8)$ and $\rho_-^{-1}(A) \cong D_8$;*
(4) *if $A = \langle R_1, R_2, R_3\rangle$ the preimages have presentations*

$$\langle \tau, \xi, \eta \mid \tau\xi\tau^{-1} = \xi^{-1},\ \tau\eta = \eta\tau,\ \xi^2 = (\xi\eta)^2 = \eta^2,\ \tau^2 = \xi^k\rangle,$$

where $k = 0$ for $\rho_+^{-1}(A)$ and $k = 2$ for $\rho_-^{-1}(A)$;
(5) *if $A = \langle R_1, R_2R_3, R_2R_4\rangle$ then $\rho_+^{-1}(A) \cong Z/2Z \times Q(8)$, while $\rho_-^{-1}(A)$ has presentation*

$$\langle \tau, \xi, \eta \mid \tau\xi = \xi\tau,\ \tau\eta = \eta\tau,\ \tau^2 = \xi^2 = (\xi\eta)^2 = \eta^2\rangle.$$

In cases (4) *and* (5), $\det(\rho_\pm(\xi)) = \det(\rho_\pm(\eta)) = 1$ *and* $\det(\rho_\pm(\tau)) = -1$, *while the elements of order* 2 *in the preimage in Pin^- are in the subgroup $Spin(4)$, in all cases.*

Proof. If α is a generator of A then $\mathbb{R}^4$ is the orthogonal direct sum of the $+1$- and -1-eigenspaces of α. The first assertion follows by induction on k, since the generators commute, and thus preserve such eigenspaces.

Since $R_1 = \rho_\pm(c_\pm)$ and $R_2R_3 = \rho(\mathbf{k}, \mathbf{k})$, and $c_\pm$ fixes the diagonal, the preimages $\rho_\pm^{-1}(\langle R_1, R_2R_3\rangle)$ are abelian.

On the other hand, $R_1R_2 = \rho(\mathbf{i}, -\mathbf{i})$. In this case $c_\pm$ inverts $(\mathbf{i}, -\mathbf{i})$, and so the preimages are nonabelian. The rest of parts (1) and (2) follow on considering the order of $c_\pm$. Part (3) is similar. (Note that $c_+(\mathbf{1}, -\mathbf{1})$ has order 4 in $Pin^+$and $c_-(\mathbf{1}, -\mathbf{1})$ has order 2 in Pin^-, and $\rho_\pm(c_\pm(\mathbf{1}, -\mathbf{1})) = -R_1$.)

Since $A = \langle R_1, R_1R_2, R_2R_3\rangle$, part (4) is a consequence of parts (1) and (2).

Part (5) is also a consequence of parts (1) and (2). The final assertion follows on considering the order of $c_\pm$. □

In each case ρ induces isomorphisms $\rho^{-1}(A)^{ab}/2\rho^{-1}(A)^{ab} \cong A/2A$. Thus we may use this result in conjunction with Lemma 3.3. We shall also use repeatedly the simple observation that $w_1^2 = 0 \Leftrightarrow w_1$ factors through $Z/4Z \Rightarrow$ no orientation-reversing element of π has image of order 2 in π/π'.

11. $Pin^{\pm}$ STRUCTURES ON FLAT 4-MANIFOLDS

In the following analysis of which flat 4-manifolds admit such structures we shall use the presentations for their fundamental groups and notation for automorphisms of flat 3-manifold groups given in Chapter 8 of [4]. (See also [2] for explicit realizations of these groups as subgroups of $E(4) = \mathbb{R}^4 \rtimes O(4)$.)

There are eight non-orientable groups of the form $\mathbb{Z}^3 \rtimes_A \mathbb{Z}$, two with $\beta = 3$ and six with $\beta = 1$. All have $w_1^2 = w_2 = 0$, by Corollary 2.2. Thus flat 4-manifolds with such groups have both structures.

There are eleven non-orientable flat 4-manifolds with $\beta = 2$. In each case the holonomy is $(Z/2Z)^2$, and is conjugate to $\langle R_1, R_1R_2\rangle$ in $O(4)$. Hence the preimages in $Pin^{\pm}$ are non-abelian. Five of these manifolds are T-bundles over T, with $w_1^2 = 0$. Two of the groups are semidirect products $\mathbb{Z}^2 \rtimes \mathbb{Z}^2$, and the image of a section generates the holonomy. In these cases the holonomy does not factor through a non-abelian nilpotent group, by Lemma 3.3. Therefore the corresponding manifolds admit neither structure. For the other three such T-bundles the holonomy factors through the group with presentation

$$\langle t, u \mid t[t,u] = [t,u]t,\ u[t,u] = [t,u]u,\ [t,u]^2 = 1\rangle,$$

which maps onto each of $Q(8)$ and D_8. The corresponding manifolds have both structures.

The other six are Kb-bundles over T, and their groups have presentations

$$\langle t, x, y, z \mid txt^{-1} = xz^a,\ tyt^{-1} = yz^b,\ tzt^{-1} = z^w,$$

$$xyx^{-1} = y^{-1}z^c,\ xz = zx,\ yzy^{-1} = z^{-1}\rangle,$$

where $w = 1$ or -1 and $2a = (1-w)c$. Two are products, $\mathbb{Z} \times B_3$ and $\mathbb{Z} \times B_4$. The manifold with group $\mathbb{Z} \times B_4$ has both structures, since $w_2 = w_1^2 = 0$. In all other cases, one of the exponents b or c is 0, so the orientation-reversing generator y has order 2 in the abelianization π/π'. Hence $w_1^2 \neq 0$. For the group $\mathbb{Z} \times B_3$ and the third example with exponent $w = 1$, the subgroup generated by $\{x, y\}$ is $\pi_1(Kb) = \mathbb{Z} \rtimes_{-1} \mathbb{Z}$, with presentation $\langle x, y \mid xyx^{-1} = y^2\rangle$. Adjoining the relation $x^2 = y^2$ gives $Q(8)$. Thus the corresponding manifolds have Pin^--structures, but no Pin^+-structure.

If $w = -1$ then $\sqrt{\pi}$ is generated by $\{ty, x^2, y^2, z\}$. For the two with exponent $a = c = 0$ the group π has no non-abelian quotient in which the image of ty is central. Thus the corresponding manifolds admit neither structure. Otherwise $a = 1$ and $b = 0$, and the holonomy factors through $\pi/\langle\langle ty, x^2z, y^2, z^2\rangle\rangle \cong D_8$. Hence the corresponding manifold has a Pin^+-structure.

There are 27 nonorientable flat 4-manifolds with $\beta = 1$. These are semidirect products $\pi = I(\pi) \rtimes_\theta \mathbb{Z}$, where θ is an automorphism of a flat 3-manifold group $I(\pi)$. If $I(\pi)$ is orientable then $w_1^2 = 0$ and $I(\pi) \cong G_1 = \mathbb{Z}^3$, $G_2 = \mathbb{Z}^2 \rtimes_{-I} \mathbb{Z}$ or G_6. Otherwise, $I(\pi) \cong B_1 = \mathbb{Z} \times (\mathbb{Z} \rtimes_{-1} \mathbb{Z})$, B_2, B_3 or B_4. The six examples with $I(\pi) = \mathbb{Z}^3$ have already been considered.

Five have $I(\pi) = G_2$. Let A be the image of the automorphism $\theta|_{G_2'}$ in $PSL(2,\mathbb{Z})$. Then A has order 1, 2 or 3. If $A = I$ the holonomy is conjugate to $\langle R_1, R_2R_3\rangle \cong (Z/2Z)^2$. Therefore the preimages in $Pin^\pm$ are abelian. If $txt^{-1} = x^{-1}$ the image of the orientation preserving subgroup in any abelian quotient π of exponent 4 has order 2. The corresponding manifold admits neither structure. If $txt^{-1} = xy$ then the image of x in π/π' has order 4 and so the holonomy factors through $(Z/4Z)^2$. The corresponding manifold has both structures. If A has order 2 the holonomy is $Z/4Z$, and factors through $\mathbb{Z}$, so $w_2 = w_1^2 = 0$, by Corollary 2.2. The two corresponding manifolds have both structures. If A has order 3 then $txt^{-1} = x^{-1}$. In this case we may pass to the normal subgroup $G_2 \rtimes_{\theta^3} \mathbb{Z}$ of index 3. In this subgroup $A = I$ and $txt^{-1} = x^{-1}$, and so the corresponding manifold admits neither structure.

Five have $I(\pi) = G_6$. The outer automorphism class of θ is a, ce, cei, ci or j. Since we may pass to subgroups of odd finite index, and $j^3 = abce = icei$, it shall suffice to consider the first four. The first pair have holonomy conjugate to $\langle R_1, R_2, R_3\rangle \cong (Z/2Z)^3$. If $\theta = a$ the holonomy factors through the group with presentation

$$\langle t, x, y \mid txt^{-1} = x^{-1},\ ty = yt,\ x^2 = (xy)^2 = y^2\rangle,$$

which clearly maps onto each of the groups $\rho_\pm^{-1}(\langle R_1, R_2, R_3\rangle)$ (as presented in part (4) of Lemma 10.1) via $t \mapsto \tau$, $x \mapsto \xi$ and $y \mapsto \eta$. If $\theta = ce$ the holonomy factors through the group with presentation

$$\langle t, x, y \mid txt^{-1} = x^{-1},\ tyt^{-1} = y^{-1},\ x^2 = (xy)^2 = y^2\rangle,$$

which maps onto each of the groups $\rho_\pm^{-1}(\langle R_1, R_2, R_3\rangle)$ via $t \mapsto \tau\xi$, $x \mapsto \xi$ and $y \mapsto \eta$. Thus each of the manifolds corresponding to $\theta = a, ce$ or j has both structures.

The other pair have holonomy D_8. Since $(ci)^2 = ace = j^{-1}ej$ in $Out(G_6)$, the group $G_6 \rtimes_{ci} \mathbb{Z}$ has a subgroup of index 2 which is isomorphic to $\Gamma_7 = G_6 \rtimes_e \mathbb{Z}$, the first of the non-$Spin$ examples of [8]. Thus the manifold corresponding to $\theta = ci$ admits neither structure.

The group $G_6 \rtimes_{cei} \mathbb{Z}$ has presentation

$$\langle t, x, y \mid txt^{-1} = (xy)^2 y,\ tyt^{-1} = y^2 x,\ xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = x^{-2}\rangle.$$

The holonomy is conjugate to the subgroup of $O(4)$ generated by R_3R_4, R_2R_4 and the reflection that swaps the second and third coordinates. (These are the images of x, y and t, respectively.) Thus the preimages of the holonomy in $Pin^{\pm}$ are quotients of $Q(8) \rtimes_\alpha \mathbb{Z}$, where α is the involution that swaps a pair of generators. Now $G_6/\langle x^2, y^2, (xy)^2\rangle \cong Q(8)$ and so $\pi/\langle x^2, y^2, (xy)^2\rangle \cong Q(8) \rtimes_\alpha \mathbb{Z}$ also. It follows easily that the manifold corrsponding to cei has both structures.

The two groups with $I(\pi) = B_1$ and holonomy $(Z/2Z)^2$ have presentations

$$\langle t, x, y, z \mid txt^{-1} = x^{-1},\ tyt^{-1} = y^{-1}z^\delta,\ tz = zt,$$
$$xy = yx,\ xz = zx,\ yzy^{-1} = z^{-1}\rangle,$$

where $\delta = 0$ or 1. The generators t, x and z are orientation-preserving, but the orientation-reversing element y has image of order $2+2\delta$ in π/π'. Hence $w_1^2 \neq 0$ if $\delta = 0$ and $w_1^2 = 0$ if $\delta = 1$. The holonomy is conjugate to $\langle R_1, R_2R_3\rangle$, and factors through $\pi/\langle x, z\rangle \cong \mathbb{Z} \oplus (Z/(2+2\delta)Z)$. The corresponding manifold has a Pin^+-structure if $\delta = 0$ and has both structures if $\delta = 1$.

The two groups with $I(\pi) = B_1$ and holonomy $Z/4Z \oplus Z/2Z$ have presentations

$$\langle t, x, y, z \mid txt^{-1} = x^{-1}y^{-2},\ tyt^{-1} = xyz^\delta,\ tz = zt,$$
$$xy = yx,\ xz = zx,\ yzy^{-1} = z^{-1}\rangle,$$

where $\delta = 0$ or 1. The generators t, x and z are orientation-preserving, while y is orientation-reversing and has image of order 2 in π/π'. Hence $w_1^2 \neq 0$. The holonomy is conjugate to the subgroup of $O(4)$ generated by R_1 (the image of y) and the block-diagonal matrix $\rho(\mathbf{t}, \mathbf{t})$, where $\mathbf{t} = \frac{1}{\sqrt{2}}(\mathbf{1} - \mathbf{k})$ (the image of t). Now $\mathbf{t}^2 = -\mathbf{k}$, so $\mathrm{Ker}(\rho)$ is generated by $(\mathbf{t}, \mathbf{t})^4$. The subgroup of Pin^+ generated by $(\mathbf{t}, \mathbf{t})$ and c_+ is isomorphic to $Z/8Z \oplus Z/2Z$, which is a quotient of π/π' in each case. (The subgroup of Pin^- generated by $(\mathbf{t}, \mathbf{t})$ and c_- is also isomorphic to $Z/8Z \oplus Z/2Z$, but not compatibly with w_1.) The corresponding manifolds have Pin^+-structures.

The group with $I(\pi) = B_2$ and holonomy $(Z/2Z)^2$ has presentation

$$\langle t, x, y, z \mid txt^{-1} = x^{-1},\ tyt^{-1} = y^{-1},\ tz = zt,$$
$$xyx^{-1} = yz,\ xz = zx,\ yzy^{-1} = z^{-1}\rangle.$$

The generators t, x and z are orientation-preserving, while y reverses the orientation and has image of order 2 in π/π'. Hence $w_1^2 \neq 0$. The holonomy is conjugate to $\langle R_1, R_2R_3\rangle$, and factors through $\pi/\langle x, z\rangle \cong \mathbb{Z} \oplus (Z/2Z)$. The corresponding manifold has a Pin^+-structure.

The group with $I(\pi) = B_2$ and holonomy $Z/4Z \oplus Z/2Z$ has presentation

$$\langle t, x, y, z \mid txt^{-1} = x^{-1}y^{-2},\ tyt^{-1} = xy,\ tz = zt,$$
$$xyx^{-1} = yz,\ xz = zx,\ yzy^{-1} = z^{-1}\rangle,$$

The generators t, x and z are orientation-preserving, while y reverses the orientation and has image of order 2 in π/π'. Hence $w_1^2 \neq 0$. The holonomy is conjugate to the subgroup of $O(4)$ generated by R_1 (the image of y) and the block-diagonal matrix $\rho(\mathbf{t}, \mathbf{t})$, where $\mathbf{t} = \frac{1}{\sqrt{2}}(\mathbf{1} - \mathbf{k})$ (the image of t). As for the cases with $I(\pi) = B_1$ and holonomy $Z/4Z \oplus Z/2Z$, the corresponding manifold has a Pin^+-structure.

The four groups with $I(\pi) = B_3$ have holonomy $(Z/2Z)^3$ and presentations

$$\langle t, x, y, z \mid txt^{-1} = x^{-1}y^{2\gamma},\ tyt^{-1} = yz^{\delta},\ tz = zt,$$
$$xyx^{-1} = y^{-1},\ xz = zx,\ yzy^{-1} = z^{-1}\rangle,$$

where γ and δ are 0 or 1. The generators t, x, and y are all orientation-reversing, and y has image of order 2 in π/π'. Hence $w_1^2 \neq 0$. The holonomy is conjugate to $\langle R_1, R_2, R_3\rangle$. In each case there is an epimorphism $f : \pi \to \rho_-^{-1}(\langle R_1, R_2, R_3\rangle)$ which maps t, x, y and z to $\tau, \xi\tau, \tau^{-1}\eta$ and 1, respectively. The corresponding manifolds have Pin^--structures.

The four groups with $I(\pi) = B_4$ have holonomy $(Z/2Z)^3$ and presentations

$$\langle t, x, y, z \mid txt^{-1} = x^{-1}y^{2\delta e},\ tyt^{-1} = y^{e},\ tz = zt,$$
$$xyx^{-1} = y^{-1}z,\ xz = zx,\ yzy^{-1} = z^{-1}\rangle,$$

where δ is 0 or 1 and e is 1 or -1. The generators x and y are orientation-reversing. However, t is orientation-reversing $\Leftrightarrow ty = yt$ (i.e., $e = 1$). There is an orientation reversing element with image of order 2 in π/π' except when $\delta = 0$ and $e = 1$. The holonomy is conjugate to $\langle R_1, R_2, R_3\rangle$. In each case there is an epimorphism $f : \pi \to \rho_-^{-1}(\langle R_1, R_2, R_3\rangle)$, If $e = 1$ we may define f as in the previous paragraph, for $\pi/\langle z\rangle$ is then isomorphic to a similar quotient of a group with $I(\pi) = B_3$. If $e = -1$ and $\delta = 0$ we map t, x, y and z to $\xi, \tau, \tau\eta$ and 1, respectively. If $e = -1$ and $\delta = 1$ we map t, x, y and z to $\eta, \tau, \tau\xi$ and 1, respectively. The corresponding manifolds have Pin^--structures, but only the one corresponding to $e = 1$ and $\delta = 0$ has $w_1^2 = 0$, and thus also has a Pin^+ structure.

The group $G_2 *_\phi B_1$ has holonomy $(Z/2Z)^2$ and presentation

$$\langle s, t, z \mid st^2s^{-1} = t^{-2},\ szs^{-1} = z^{-1},\ ts^2t^{-1} = s^{-2},\ tz = zt\rangle.$$

It has abelianization $(Z/4Z)^2 \oplus Z/2Z$. The generators t and z are orientation-preserving, while s is orientation-reversing. Since z generates the $Z/2Z$ summand of the abelianization, $w_1^2 = 0$. Let $S = s^2$ and $U = (st)^2$. Then $\mathrm{Ker}(w_1) = \langle t, S, U, z\rangle \cong G_2 \times \mathbb{Z}$. The holonomy is conjugate to $\langle -R_1, R_1R_2\rangle$, and factors through G_6, since $\pi/\langle z^2\rangle \cong G_6 \times Z/2Z$. This maps onto each of D_8 and $Q(8)$, compatibly with the orientation conditions of Lemma 10.1, and so the corresponding manifold has both structures.

The group $G_2 *_\phi B_2$ has holonomy D_8 and presentation

$$\langle s, t, z \mid st^2s^{-1} = t^{-2},\ szs^{-1} = z^{-1},\ ts^2t^{-1} = z,\ tzt^{-1} = s^2\rangle.$$

The abelianization is $(Z/4Z)^2$ and so $w_1^2 = 0$. The generators t and z are orientation-preserving, while s is orientation-reversing. Let $S = s^2$ and $U = (st)^2$. Then $\mathrm{Ker}(w_1)$ has presentation

$$\langle t, S, U, z \mid tSt^{-1} = z,\ tzt^{-1} = S,\ tUt^{-1} = U^{-1}S^{-1}z,$$
$$Sz = zS,\ UzU^{-1} = z^{-1},\ USU^{-1} = S^{-1}\rangle.$$

Hence $\mathrm{Ker}(w_1) \cong G_2 \rtimes_\psi \mathbb{Z}$, where $\psi = (\left(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right), -1)$. The translation subgroup $\sqrt{\pi} \cong \mathbb{Z}^4$ is generated by $\{t^2, s^2, (st)^4, z\}$. The images of s and t in the holonomy are the diagonal matrix $R_1\rho(\mathbf{i}, \mathbf{i})$ and the block-diagonal matrix $\rho(\mathbf{t}, \mathbf{t})$, where $\mathbf{t} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$, respectively.

The subgroup of $Pin^\pm$ generated by $(\mathbf{t}, \mathbf{t})$ and $c_\pm(\mathbf{i}, \mathbf{i})$ contains $\mathrm{Ker}(\rho)$, since $\mathbf{t}^2 = -\mathbf{1}$, and is isomorphic to the group with presentation

$$\langle \gamma, \phi \mid \gamma^2 = \phi^{2\pm 2},\ \phi^8 = 1,\ \gamma\phi\gamma^{-1} = \phi^{-1}\rangle,$$

where γ and ϕ correspond to $c_\pm(\mathbf{i}, \mathbf{i})$ and $c_\pm(\mathbf{i}, \mathbf{i})(\mathbf{t}, \mathbf{t})$, respectively. This group is $Q(16)$ for Pin^+ and is D_{16} for Pin^-. The quotient $\pi/\langle s^4, sts^{-1}t, z\rangle \cong \mathbb{Z} \rtimes_{-1} Z/4Z$ maps onto each of these groups compatibly with the orientations, and so the corresponding manifold has both structures.

The groups $G_6 *_\phi B_3$ and $G_6 *_\phi B_4$ have holonomy $(Z/2Z)^3$ and presentations

$$\langle t, x, y \mid xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = x^{-2},\ x^2 = t^2(xy)^{2\delta},\ y^2 = (t^{-1}x)^2,$$
$$t(xy)^2 = (xy)^2t\rangle,$$

where $\delta = 0$ for $G_6 * B_3$ and $\delta = 1$ for $G_6 * B_4$. In each case the abelianization is $Z/4Z \oplus (Z/2Z)^2$. The generators t and x are orientation-preserving, while y is orientation-reversing. Therefore $w_1^2 \neq 0$, since $x^{-1}ty$ has image of order 2 in π/π'.

Let $X = x^2, Z = (xy)^2$ and $U = (ty)^2$. Then $xU = Ux$ and $yUy^{-1} = U^{-1}$. Let $v = t^{-1}x$. In each case $\mathrm{Ker}(w_1)$ has a presentation

$$\langle t, v, U, Z \mid tZ = Zt,\ tUt^{-1} = U^{-1},\ tvt^{-1} = v^{-1}Z^\delta,$$

$$vUv^{-1} = U^{-1},\ vZv^{-1} = Z^{-1}\rangle.$$

Hence $\mathrm{Ker}(w_1) \cong G_2 \rtimes_\theta \mathbb{Z}$, where $\theta = (\left(\begin{smallmatrix} 0 \\ \delta \end{smallmatrix}\right), -I, 1)$.

The holonomy is conjugate to $\langle R_1, R_2R_3, R_2R_4\rangle$, with these generators corresponding to $w = x^{-1}ty$, t and x, respectively. The first, second and last of the five relations are satisfied in any group of exponent 4 in which all squares are central. If we rewrite the other two relations in terms of t, w and x we obtain the relations $x^2 = t^2(xt^{-1}xw)^{2\delta}$ and $wt^{-1}xw = t^{-1}x$. If moreover w is central then it must have order 2. But then the first relation becomes $x^2 = t^{2-2\delta}x^{4\delta}$. Thus if $\delta = 0$ both relations are satisfied in $\rho_+^{-1}(\langle R_1, R_2R_3, R_2R_4\rangle) \cong Z/2Z \times Q(8)$, under the epimorphism sending w, t, x to τ, ξ, η, respectively. Hence $w_2(G_6 * B_3) = 0$. Thus the manifold corresponding to $G_6 *_\phi B_3$ has a Pin^+-structure, but it does not have a Pin^--structure. The manifold corresponding to $G_6 *_\phi B_4$ has neither structure.

In summary, 23 of the 47 non-orientable flat manifolds have $w_1^2 = w_2 = 0$, seven have $w_1^2 \neq 0$ and $w_2 = 0$, nine have $w_2 = w_1^2 \neq 0$, five have $w_1^2 = 0$ and $w_2 \neq 0$, while for the remaining three w_1^2 and w_2 are distinct and non-zero.

With one exception, the orientable double cover of a non-orientable flat 4-manifold is parallelizable. The only cases that require close inspection are the five with $\beta = 1$ and $\pi \cong G_6 \rtimes_\theta \mathbb{Z}$ and the four with $\beta = 0$. We have observed above that the maximal orientable subgroup of each of the latter four groups is a semidirect product $G_2 \rtimes_\theta \mathbb{Z}$. If $\pi \cong G_6 \rtimes_\theta \mathbb{Z}$ with $\theta = a, ce$ or cei then the orientable double cover is parallelizable, since $a^2 = (ce)^2 = 1$ and $(cei)^2 = ab$. On the other hand, the orientable double cover of the manifold with group $G_6 \rtimes_{ci} \mathbb{Z}$ is not parallelizable, since $(ci)^2 = ace$.

12. Pin^c-STRUCTURE ON FLAT 4-MANIFOLDS

A closed n-manifold M has a Pin^c-structure if $w_2(M)$ is integral. In particular, Pin^+-manifolds also have Pin^c-structures. This also holds for Pin^--manifolds, as a consequence of the following simple lemma.

Lemma 12.1. *If $u \in H^1(G; \mathbb{F}_2)$ then u^2 is the reduction mod (2) of a class in $H^2(G; \mathbb{Z})$.*

Proof. This holds for $G = Z/2Z$, since reduction *mod* (2) induces an isomorphism from $H^2(Z/2Z; \mathbb{Z})$ to $H^2(Z/2Z; \mathbb{F}_2)$, and follows in general by functoriality. □

Orientable 4-manifolds all have $Spin^c$-structures. In this dimension $w_3 = Sq^1 w_2$, by the Wu formulae, and so integrality of w_2 implies that $w_3 = 0$.

There are 8 non-orientable flat 4-manifolds which have $w_2 \neq 0$ or w_1^2. Four are T-bundles over T or Kb. (Two of these are mapping tori of self-diffeomorphisms of the half-turn flat 3-manifold, with group a semidirect product $G_2 \rtimes \mathbb{Z}$.) In these cases the holonomy factors through $\mathbb{Z}^2$ or $\mathbb{Z} \rtimes_{-1} \mathbb{Z}$, and so w_2 is integral, since it is the pullback of an integral class.

The two Kb-bundles over T are also S^1-bundles over $N = S^1 \times Kb$. Let M be either of these manifolds, let ξ be the associated disc bundle, with total space E and projection $p : E \to N$, and let $j : M = \partial E \to E$ be the natural inclusion. Then $w_*(E) = p^*(w_*(\xi) \cup w_*(N))$. Let $v = w_1(\xi)$ and $w = w_1(N)$. Hence $w_2(M) = (pj)^*(vw)$. In a non-orientable 3-manifold such as N, a class $u \in H^2(N; \mathbb{F}_2)$ is integral $\Leftrightarrow Sq^1 u = 0$. Since $Sq^1 u = w_1(N) \cup u$ and $w_1(N)^2 = 0$ it follows that vw is integral, and so $w_2(M)$ is integral, in each case.

There remain the two examples with groups $G_6 \rtimes_{ci} \mathbb{Z}$ and $G_6 *_\phi B_4$. We have not yet been able to decide the issue for these groups.

13. Further considerations

It is clear that the present approach should extend to the other 4-dimensional infrasolvmanifolds. In particular, Corollary 2.2 applies also to all $\mathbb{S}ol_0^4$- and $\mathbb{S}ol_{m,n}^4$-manifolds with $m \neq n$, and so all such manifolds have both Pin^+- and Pin^--structures.

The general pattern for $\mathbb{N}il^4$ is easily outlined. Let M be a non-orientable $\mathbb{N}il^4$-manifold and $\pi = \pi_1(M)$. Then $\pi/\sqrt{\pi} \cong Z/2Z$ or $(Z/2Z)^2$. If $\pi/\sqrt{\pi} \cong Z/2Z$ then M is a Pin^+ manifold, and has both structures if and only if $w_1^2 = 0$. If $\pi/\sqrt{\pi} \cong (Z/2Z)^2$ and M has a $Pin^\pm$ structure then π must map onto $Z/4Z \oplus Z/2Z$. If this condition holds M has one or both of these structures, depending on whether there is an orientation-reversing element with image of order 4.

The other infrasolvmanifold geometries involve more work, since the stabilizer of the identity in the isometry group is larger. (In $Isom(\mathbb{S}ol_1^4)$ it is D_8, in $Isom(\mathbb{S}ol^3 \times \mathbb{E}^1)$ it is $D_8 \times Z/2Z$ and in $Isom(\mathbb{N}il^3 \times \mathbb{E}^1)$ it is $O(2) \times Z/2Z$.) We shall not consider these cases further here.

All orientable 4-manifolds with $\chi = 0$ and $\pi_3 \cong \mathbb{Z}$ are parallelizable, since they have finite covers of odd degree which are homotopy equivalent to mapping tori of involutions of S^3/H, where $H \cong Z/2^m Z$ or $Q(2^n)$ is a subgroup of S^3. (See Chapter 11 of [4].) Thus we may apply Theorem 2.1 and Corollary 2.2 (with $G = S^3 \times \mathbb{R}$).

Manifolds with $\chi = 0$ and $\pi_2 \cong \mathbb{Z}$ are homotopy equivalent to either an S^2- or RP^2-bundle over the torus or Klein bottle, or to an S^2-orbifold bundle over a flat 2-orbifold. With one exception, all are $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds. There are just 23 such homotopy types, and their Stiefel-Whitney classes are determined in [4] and [6].

The other geometries supported by 4-manifolds with $\chi = 0$ are the product geometries $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^3 \times \mathbb{E}^1$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$. These manifolds are finitely covered by products $N \times S^1$, with N a geometric 3-manifold. Do the classifying maps for the tangent bundles factor as in Theorem 2.1? (This is certainly so when N is the coset space for a discrete subgroup of $\widetilde{PSL}(2, \mathbb{R})$, as in [9].)

All flat n-manifolds bound [3]. We note here that when $n = 4$ this is an easy consequence of our calculations, for all but one case. Since χ is even, $w_2^2 = w_4 = 0$, and $w_1 w_3 = w_1 Sq^1 w_2 = 0$, by the Cartan and Wu formulae. For all but five flat 4-manifolds, either $w_1^2 = 0$ or $w_2 = 0$ or $w_1^2 = w_2 = 0$. Hence $w_1^4 = w_1^2 w_2$ is 0, so all Stiefel-Whitney numbers are 0, and the manifold bounds. The total spaces of S^1-bundles bound disc bundles. Thus only the example with group $G_6 *_\phi B_4$ requires further argument.

We close with three related questions:

1. Which 4-dimensional mapping tori are parallelizable? (This seems a natural extension of Stern's question.)
2. Does every 4-dimensional infrasolvmanifold bound? (The only Stiefel-Whitney class of interest is again w_1^4. Orientable 4-dimensional infrasolvmanifolds bound orientably, since $w_1 = 0$ and $\sigma = 0$.)
3. Which geometric 4-manifolds admit Pin^c structures?

References

[1] Auslander, L. and Szczarba, R.H. Characteristic classes of compact solvmanifolds, Ann. Math. 76 (1962), 1–8.

[2] Brown, H., Bülow, R., Neubüser, J., Wondratschek, H. and Zassenhaus, H. *Crystallographic Groups of Four-Dimensional Space*, John Wiley and Sons Inc., New York - London - Sydney - Toronto (1978).

[3] Hamrick, G.C. and Royster, D.C. Flat Riemannian manifolds are boundaries, Invent. Math. 66 (1982), 405–413.

[4] Hillman, J.A. *Four-Manifolds, Geometries and Knots*, GT Monograph vol 5, (2002). (Revision 2007.)

[5] Hillman, J.A. Geometries and infrasolvmanifolds in dimension 4, Geom. Dedicata 129 (2007), 57–72.

[6] Hillman, J.A. S^2-orbifold bundles over 2-orbifolds, (submitted, January 2011).

[7] Kirby, R.C. and Taylor, L.R. *Pin* structures on low-dimensional manifolds, in *Geometry of Low-Dimensional Manifolds:2* (edited by S.K.Donaldson and C.B.Thomas), London Mathematical Society Lecture Notes Series 151, Cambridge University Press, Cambridge – New York – Melbourne (1990), 177–242.

[8] Putrycz, B. and Szczepański, A. Existence of spin structures on flat four-manifolds, Adv. Geom. 10 (2010), 323–332.

[9] Raymond, F. and Vasquez, A.T. 3-manifolds whose universal coverings are Lie groups, Top. Appl. 12 (1981), 161–179.

[10] Wall, C.T.C. Geometric structures on complex analytic surfaces, Topology 25 (1986), 119–153.

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