CYCLOTOMIC SOLOMON ALGEBRAS

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ABSTRACT. This paper introduces an analogue of the Solomon descent algebra for the complex reflection groups of type $G(r,1,n)$. As with the Solomon descent algebra, our algebra has a basis given by sums of 'distinguished' coset representatives for certain 'reflection subgroups'. We explicitly describe the structure constants with respect to this basis and show that they are polynomials in r. This allows us to define a deformation, or q-analogue, of these algebras which depends on a parameter q. We determine the irreducible representations of all of these algebras and give a basis for their radicals. Finally, we show that the direct sum of cyclotomic Solomon algebras is canonically isomorphic to a concatenation Hopf algebra.

1. INTRODUCTION

In a seminal paper [27], Solomon showed that the group algebra of any finite Coxeter group has a remarkable subalgebra, the **Solomon descent algebra**. In this paper we construct a similar subalgebra of the complex reflection group of type $G(r,1,n)$ and show that this algebra shares many of properties of the Solomon descent algebras.

Solomon showed that each descent algebra has a distinguished basis for which he gave an explicit description of the structure constants. This distinguished basis is given by the sums of the distinguished coset representatives of the parabolic subgroups. Solomon gave a basis for the radical of the descent algebra and he constructed a natural homomorphism from the descent algebra into the parabolic Burnside ring of the associated Coxeter group. As a consequence, it follows that the irreducible representations of the Solomon descent algebras are all one dimensional and that, in characteristic zero, they are naturally indexed by the conjugacy classes of the parabolic subgroups.

There has been an explosion of research into the descent algebras of Coxeter groups since Solomon discovered them; see, for example, [2, 5, 6, 7, 8, 10, 25]. The study of the Solomon descent algebras of the symmetric groups has been even more intense because of connections between these algebras and free Lie algebras, 0-Hecke algebras, noncommutative and quasi-symmetric functions [1, 13, 15, 22], the representation theory of the symmetric group, and card shuffling and associated random walks [4, 17].

The algebra that we construct in this paper is in many ways a natural generalization of the Solomon algebra of the symmetric groups. The **cyclotomic Solomon algebra** $\mathsf{Sol}(G_{r,n})$ is a subalgebra of the group algebra of the complex reflection group $G_{r,n}$ of type $G(r,1,n)$. Like Solomon, we define our algebra to be the subalgebra of the group algebra of $G_{r,n}$ with basis the 'distinguished' coset representatives of a natural class of subgroups of $G_{r,n}$. It turns out that many natural choices of subgroups, and coset representatives for these subgroups, do not yield a subalgebra of the group algebra (see Remark 8.10). We show, however, that with respect to the 'right' length function, the sums of the minimal length coset representatives of the *standard reflection subgroups* of $G_{r,n}$ give rise to a subalgebra of $\mathbb{Z}G_{r,n}$ which is free of rank $2\cdot 3^{n-1}$. We give an explicit formula for the structure constants for this basis which is similar to Solomon's formula for the structure constants of the descent algebra of the symmetric group $\mathfrak{S}_n$.

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One surprising feature of the cyclotomic Solomon algebras $\mathsf{Sol}(G_{r,n})$ is that the structure constants of these algebras for $n \geq 0$ are polynomials in r which are independent of n. As a consequence, these algebras admit a simultaneous deformation $\mathsf{Sol}_q(n)$ which depends on a parameter q. For fixed $n \geq 0$, we show that the algebras $\mathsf{Sol}_q(n)$ are free of rank $2 \cdot 3^{n-1}$. We construct and classify the irreducible representations of these algebras over an arbitrary field, and hence give a basis for the radical of $\mathsf{Sol}_q(n)$.

A remarkable result of Gessel [16] shows that there is a natural duality between the Hopf algebra of quasi-symmetric functions and the descent Hopf algebra. This led Malvenuto and Reutenauer [22] to show that the direct sum of these algebras under the shuffle (or convolution) product is a Hopf subalgebra of the Hopf algebra of permutations. This Hopf algebra is dual to the Hopf algebra of quasi-symmetric functions and it is isomorphic to the Hopf algebra of non-commutative symmetric functions [15]. These results are important because they relate the coproduct of the quasi-symmetric functions with the product in the descent algebras.

Baumann and Hohlweg[3] showed that there is a similar Hopf algebra structure under the shuffle product on the space $\mathscr{G}(r) = \bigoplus_{n\geq 0} \mathbb{Z}G_{r,n}$ of coloured permutations. We prove that the direct sum of the cyclotomic Solomon algebras $\mathsf{Sol}(r) = \bigoplus_{n\geq 0} \mathbb{Z}\,\mathsf{Sol}(G_{r,n})$ is a Hopf subalgebra of $\mathscr{G}(r)$. We show that $\mathsf{Sol}(r)$ is a concatenation Hopf algebra and that $\mathsf{Sol}(r)$ has a second bialgebra structure which has the same coproduct as $\mathscr{G}(r)$ but where the product map is induced by group multiplication. We expect that the Hopf algebra $\mathsf{Sol}(r)$ is dual to the Hopf algebra of quasisymmetric functions of type B considered by Hsiao and Petersen [18].

Different generalizations of the Solomon algebras have been considered by other authors, the most striking of which are the Mantaci-Reutenauer algebras [23]. It is natural to ask whether the cyclotomic Solomon algebras and the Mantaci-Reutenauer algebras are isomorphic, at least for type B_n, since they are both free of rank $2 \cdot 3^{n-1}$. We show in Remark 8.10 that, in general, these two algebras are not isomorphic. Example 8.9 shows that, in stark contrast to the Solomon descent algebra, there is no map from $\mathsf{Sol}(G_{r,n})$ into the character ring of $G_{r,n}$.

This paper is organized as follows. In the second section we introduce the complex reflection groups $G_{r,n}$ and set our notation. In section 3 we define and classify the standard reflection subgroups of $G_{r,n}$ and section 4 shows that every coset of a reflection subgroup has a unique element of minimal length. Sections 4 and 5 give combinatorial descriptions of the coset and double representatives of the reflection subgroups. This combinatorics turns out to be closely related to the structure constants of the cyclotomic Solomon algebras, which are finally introduced in section 6. The first main result of the paper, Theorem 6.8, determines the structure constants of the cyclotomic Solomon algebras, hence showing that they are in fact subalgebras of $G_{r,n}$. In section 7 we investigate the 'generic' cyclotomic Solomon algebras and in section 8 we construct and classify the irreducible representations of the cyclotomic algebras and their deformations. In section 9 we show that the direct sum of the cyclotomic algebras gives rise to a concatenation Hopf algebra which is a Hopf subalgebra of the Hopf algebras of coloured permutations. Finally, in section 10 we give a second combinatorial interpretation of the structure constants of the cyclotomic Solomon algebras. We use this to show that the direct sum of the cyclotomic Hopf algebras comes equipped with a second bialgebra structure which has the same coproduct but where the product map is induced by group multiplication.

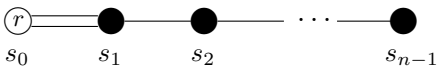
2. Complex reflection groups of type $G(r, 1, n)$

This paper is concerned with certain subalgebras of the group algebra of the complex reflection groups of type $G(r, 1, n)$, in the Shephard–Todd classification of the finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ which are generated by (pseudo) reflections. In this section we introduce these groups and study a length function on them.

Fix positive integers r and n. The complex reflection group of type $G(r,1,n)$ is the group $G_{r,n}$ which is generated by elements $s_0, s_1, \ldots, s_{n-1}$ subject to the relations

$$s_0^r = 1 = s_i^2 \qquad\qquad s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$$
$$s_i s_j = s_j s_i, \qquad\qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

where $1 \le i \le j-1 \le n-1$. This presentation is very similar to the presentation of a Coxeter group; indeed, if $r \le 2$ then $G_{r,n}$ is a Coxeter group. Accordingly, we encode this presentation in the following "cyclotomic Dynkin diagram":



The node labeled by r indicates that the generator s_0 has order r; otherwise, this graph gives the presentation of $G_{r,n}$ in exactly the same way as a Dynkin diagram gives the presentation of the corresponding Coxeter group. If $r = 1$ then $G_{1,n}$ is isomorphic to the symmetric group of degree n.

From the presentation of $G_{r,n}$ it is evident that there is a homomorphism from the symmetric group $\mathfrak{S}_n$ into $G_{r,n}$ which is determined by mapping each transposition $(i, i+1)$ to s_i, for $i = 1, \ldots, n-1$. In fact, this map is injective so we can – and do – identify $\mathfrak{S}_n$ with the subgroup $\langle s_1, \ldots, s_{n-1}\rangle$ via this homomorphism.

The symmetric group $\mathfrak{S}_n$ acts on $\{1, 2, \ldots, n\}$ from the right. We write this action exponentially. Thus, $w \in \mathfrak{S}_n$ sends the integer i to i^w, for $1 \le i \le n$.

Define $t_1 = s_0$ and $t_{i+1} = s_i t_i s_i$, for $1 \le i < n$. Using the relations it is easy to see that $t_i t_j = t_j t_i$, for all i, j. It follows that the subgroup $T = \langle t_1, \ldots, t_n\rangle$ is abelian and, further, one can show that $T \cong (\mathbb{Z}/r\mathbb{Z})^n$. It is easy to see that

$$t_i\, w = w\, t_{i^w}, \qquad \text{for all } w \in \mathfrak{S}_n \text{ and } 1 \le i \le n, \tag{2.1}$$

Hence, T is a normal subgroup of $G_{r,n}$. With a little more work we obtain the following description of $G_{r,n}$ as an (internal) semidirect product, or wreath product:

$$G_{r,n} = T \rtimes \mathfrak{S}_n = \langle s_0\rangle \wr \langle s_1, \ldots, s_{n-1}\rangle \cong (\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n. \tag{2.2}$$

Let $\mathbb{Z}_r^n = \{\, \alpha = (\alpha_1, \ldots, \alpha_n) : 0 \le \alpha_i < r \,\}$. For $\alpha \in \mathbb{Z}_r^n$ let $t^\alpha = t_1^{\alpha_1} \ldots t_n^{\alpha_n}$. Then, as a set, $G_{r,n} = \{\, t^\alpha w : \alpha \in \mathbb{Z}_r^n \text{ and } w \in \mathfrak{S}_n \,\}$ and $|G_{r,n}| = r^n n!$.

Let $\Pi = \Pi_{r,n} = \{t_1, \ldots, t_n, s_1, \ldots, s_{n-1}\}$. Then Π generates $G_{r,n}$ because $\{s_0 = t_1, s_1, \ldots, s_{n-1}\}$ generates $G_{r,n}$.

2.3. **Definition.** *The* Π*–**length function** on* $G_{r,n}$ *is the function* $\ell = \ell_\Pi : G_{r,n} \longrightarrow \mathbb{N}$ *given by* $\ell(g) = \min\{\, k \ge 0 : g = r_i \ldots r_k,$ *for some* $r_i \in \Pi \,\}$.

2.4. *Remark.* Let $S_0 = \{s_0, s_1, \ldots, s_{n-1}\}$. Bremke and Malle [11] have studied the length function $\ell_0 : G_{r,n} \longrightarrow \mathbb{N}$ which is defined by

$$\ell_0(g) = \min\{\, k \ge 0 : g = r_i \ldots r_k, \text{ for some } r_i \in S_0 \,\}.$$

By definition, $\ell(g) \le \ell_0(g)$, for all $g \in G_{r,n}$. Furthermore, it is not hard to see that $\ell(g) \equiv \ell_0(g) \pmod 2$. Moreover, if $w \in \mathfrak{S}_n$ then

$$\ell(w) = \ell_0(w) = \#\{\, (i,j) : 1 \le i < j \le n \text{ and } i^w > j^w \,\}.$$

(The last equality is well–known; see, for example, [24, Prop. 1.3].) Hence, Proposition 2.5 below gives an effective way of computing the Π–length function on $G_{r,n}$.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_r^n$ we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

2.5. **Proposition.** *Suppose that* $\alpha \in \mathbb{Z}_r^n$ *and* $w \in \mathfrak{S}_n$. *Then* $\ell(t^\alpha w) = |\alpha| + \ell(w)$.

Proof. By definition $\ell(t^\alpha w) \leq |\alpha| + \ell(w)$. Conversely, suppose that $t^\alpha w = r_1 \dots r_k$, for some $r_i \in \Pi$. Using (2.1) we can move each $t_i \in \{r_1, \dots, r_k\}$ to give a new word in which all of the elements of T appear on the left. As every element of $G_{r,n}$ can be written uniquely in the form $t^\beta v$, for $\beta \in \mathbb{Z}_r^n$ and $v \in \mathfrak{S}_n$, this new word must be $t^\alpha w$. By (2.1), this rewriting process does not increase the Π–length of the word, however, it may decrease the Π–length if some cancellation occurs. Hence, $k \geq |\alpha| + \ell(w)$, completing the proof. □

2.6. **Corollary.** *Suppose that $\alpha \in \mathbb{Z}_r^n$ and that $w \in \mathfrak{S}_n$. Then*

$$\ell(t_j \cdot t^\alpha w) = \begin{cases} \ell(t^\alpha w) + 1, & \text{if } \alpha_j < r-1, \\ \ell(t^\alpha w) - r + 1, & \text{if } \alpha_j = r-1, \end{cases}$$

for $1 \leq j \leq n$ and $\ell(s_i \cdot t^\alpha w) = |\alpha| + \ell(s_i w)$, for $1 \leq i < n-1$.

Note that $t^\alpha w \cdot t_j = t_{j^{w^{-1}}} \cdot t^\alpha w$ by (2.1) and $\ell(t^\alpha w \cdot s_i) = |\alpha| + \ell(ws_i)$, for $1 \leq i < n-1$ and $1 \leq j \leq n$. Hence, Corollary 2.6 can be used to compute $\ell(g \cdot t^\alpha w)$ and $\ell(t^\alpha w \cdot g)$, for any $g \in G_{r,n}$.

It is sometimes convenient to describe $G_{r,n}$ combinatorially as a set of 'words'. Fix a primitive r^{th} root of unity $\zeta = \exp(2\pi i/r) \in \mathbb{C}$ and set

$$\mathbf{n} = \{1, 2, \dots, n\} \qquad \text{and} \qquad \mathbf{n}_\zeta = \{\, m\zeta^i : m \in \mathbf{n} \text{ and } 0 \leq i < r \,\}.$$

Recall that if $z \in \mathbb{C}$ then $|z|$ is the complex modulus of z. In particular, if $m\zeta^i \in \mathbf{n}_\zeta$ then $|m\zeta^i| = m$. Define a **word** in $\mathbf{n}_\zeta$ to be an element of the set

$$\mathcal{G}_{r,n} = \{\, \underline{\omega} = (\omega_1, \dots, \omega_n) : \omega_i \in \mathbf{n}_\zeta \text{ and } \{|\omega_1|, \dots, |\omega_n|\} = \mathbf{n} \,\}.$$

If $\omega = (\omega_1, \dots, \omega_n)$ is a word then we abuse notation and write $\omega = \omega_1 \dots \omega_n$.

There is a faithful right action of $G_{r,n}$ on $\mathcal{G}_{r,n}$ given by

$$\omega_1 \dots \omega_n \cdot t^\alpha w = \zeta^{\alpha_1} \omega_{1^w} \dots \zeta^{\alpha_n} \omega_{n^w},$$

for $\alpha \in \mathbb{Z}_r^n$ and $w \in \mathfrak{S}_n$. Consequently, there is a natural bijection $G_{r,n} \to \mathcal{G}_{r,n}$ given by $t^\alpha w \mapsto 1 \dots n \cdot t^\alpha w$, so that $|\mathcal{G}_{r,n}| = r^n n! = |G_{r,n}|$. Thus, we have described the regular representation of $G_{r,n}$ as the permutation representation on the set of words $\mathcal{G}_{r,n}$. Equivalently, $G_{r,n}$ is the group of permutations of $\mathbf{n}_\zeta$ such that $(m\zeta^i)^g = m^g \zeta^i$, for all $m \in \mathbf{n}$, $0 \leq i < r$ and $g \in G_{r,n}$.

3. Reflection subgroups

Recall that $\Pi = \{t_1, \dots, t_n, s_1, \dots, s_{n-1}\}$. In this section we define the reflection subgroups of $G_{r,n}$ and show that every coset of a reflection subgroup contains a unique element of minimal Π–length.

3.1. **Definition.** *A (**standard**) **reflection subgroup** of $G_{r,n}$ is a subgroup which is generated by a subset of Π.*

Geometrically, a reflection subgroup of $G_{r,n}$ should be any subgroup which is generated by elements which act by (pseudo) reflections in the reflection representation of $G_{r,n}$. All of the elements of Π act as reflections in the reflection representation of $G_{r,n}$, so every standard reflection subgroup is a reflection subgroup in this geometric sense. If $r > 2$ then it is not difficult to see that there are 'geometric reflection subgroups' of $G_{r,n}$ which are not standard reflection subgroups.

If $J \subseteq \Pi$ let $G_J = \langle J \rangle$ be the corresponding (standard) reflection subgroup of $G_{r,n}$. This notation is inherently ambiguous because it can happen that $G_J = G_K$ even though $J \neq K$, for $J, K \subseteq \Pi$. For example, $G_\Pi = G_{r,n} = G_{S_0}$ (recall that $S_0 = \{s_0, s_1, \dots, s_{n-1}\}$), and yet $\Pi \neq S_0$ if $n > 1$. We start our study of the reflection subgroups by resolving this ambiguity.

A **composition** of n is a sequence $\mu = (\mu_1, \dots, \mu_k)$ of positive integers which sum to n. A **signed composition** of n is a sequence of non–zero integers $\mu = (\mu_1, \dots, \mu_k)$ such that $|\mu| = |\mu_1| + \cdots + |\mu_k| = n$. Let $\Lambda_n^{\pm}$ be the set of signed compositions of n and let Λ_n be the set of compositions of n. Then $\Lambda_n \subseteq \Lambda_n^{\pm}$.

If $\mu = (\mu_1, \dots, \mu_k) \in \Lambda_n^{\pm}$ let $\mu^+ = (|\mu_1|, \dots, |\mu_k|)$ and $-\mu = (-\mu_1, \dots, -\mu_k)$. Then $\mu^+ \in \Lambda_n$ is a composition of n and $-\mu \in \Lambda_n^{\pm}$. We set $|\mu|^+ = \frac{1}{2}\sum_{i=1}^{k}(\mu_i^+ + \mu_i)$, so that $|\mu|^+$ is the sum of the positive parts of μ. Similarly, let $|\mu|^- = \frac{1}{2}\sum_{i=1}^{k}(\mu_i^+ - \mu_i)$ be the absolute value of the sum of the negative parts of μ. Then $|\mu| = |\mu|^- + |\mu|^+ = n$. Finally, set $\overline{\mu}_0 = 0$ and $\overline{\mu}_i = |\mu_1| + \cdots + |\mu_i|$, for $i \geq 1$.

3.2. **Definition.** *Suppose that $\mu = (\mu_1, \dots, \mu_k) \in \Lambda_n^{\pm}$ is a signed composition. Define*

$$\Pi_\mu = \bigcup_{1 \leq i \leq k} \{s_{\overline{\mu}_{i-1}+1}, \dots, s_{\overline{\mu}_i - 1}\} \cup \bigcup_{\substack{1 \leq i \leq k \\ \mu_i > 0}} \{t_{\overline{\mu}_{i-1}+1}, \dots, t_{\overline{\mu}_i}\}.$$

Then $\Pi_\mu \subseteq \Pi$ so we set $G_\mu = G_{\Pi_\mu}$.

Let $S = \{s_1, \dots, s_{n-1}\} \subseteq \Pi$. Suppose that $\mu \in \Lambda_n^{\pm}$. Then $\Pi_\mu \subseteq S$ if and only if $-\mu \in \Lambda_n$. In general, $\Pi_\mu \subseteq \Pi$ and the reflection subgroup G_μ is conjugate to the reflection subgroup

$$G_\mu \cong \prod_{\mu_i > 0} G_{r,\mu_i} \times \prod_{\mu_j < 0} \mathfrak{S}_{-\mu_j}$$

of $G_{r,n}$. Moreover, $\{\, G_\mu : \mu \in \Lambda_n^{\pm} \,\}$ is the complete set of reflection subgroups of $G_{r,n}$.

3.3. **Proposition.** *Suppose that $n \geq 1$, $r \geq 2$ and that $J \subseteq \Pi$. Then $G_J = G_\mu$, for a unique signed composition $\mu \in \Lambda_n^{\pm}$. Consequently, $G_{r,n}$ has $2 \cdot 3^{n-1}$ distinct reflection subgroups.*

Proof. We prove both statements in the Proposition by induction on n. If $n = 1$ then $G_\emptyset = G_{(1)}$ and $G_\Pi = G_{(-1)}$ are the only reflection subgroups of $G_{r,1}$ so the Proposition holds. In particular, $G_{r,1}$ has $|\Lambda_1^{\pm}| = 2$ reflection subgroups.

Suppose then that $n > 1$ and observe that $\Pi_{r,n} = \Pi_{r,n-1} \cup \{s_{n-1}, t_n\}$. Let $G' = G_{r,n-1}$. Then $G_\mu \cap G'_{r,n}$ is a reflection subgroup of G' and, by induction on n, every reflection subgroup is of the form G'_μ, for some $\mu = (\mu_1, \dots, \mu_k) \in \Lambda_{n-1}^{\pm}$. Now, $t_{n-1} \in G'_\mu$ if and only if $\mu_k > 0$, so one can check that

$$\langle G'_{(\mu_1,\dots,\mu_k)}, s_{n-1}, t_n \rangle = \begin{cases} \langle G'_{(\mu_1,\dots,\mu_{k-1},\mu_k)}, s_{n-1} \rangle, & \text{if } \mu_k > 0, \\ \langle G'_{(\mu_1,\dots,\mu_{k-1},-\mu_k)}, s_{n-1} \rangle, & \text{if } \mu_k < 0. \end{cases}$$

Consequently, the reflection subgroups of $G_{r,n}$ are precisely the groups G'_μ, $\langle G'_\mu, t_n \rangle$, and $\langle G'_\mu, s_{n-1} \rangle$, where $\mu = (\mu_1, \dots, \mu_k) \in \Lambda_{n-1}^{\pm}$. Moreover, by (2.2) the groups arising this way for different $\mu \in \Lambda_{n-1}^{\pm}$ are all distinct. Applying the definitions,

$$\begin{aligned} G_{(\mu_1,\dots,\mu_k,-1)} &= G'_{(\mu_1,\dots,\mu_k)}, \\ G_{(\mu_1,\dots,\mu_k,1)} &= \langle G'_{(\mu_1,\dots,\mu_k)}, t_n \rangle, \\ G_{(\mu_1,\dots,\mu_k+\varepsilon_k)} &= \langle G'_{(\mu_1,\dots,\mu_k)}, s_{n-1} \rangle, \end{aligned}$$

where $\varepsilon_k = 1$ if $\mu_k > 0$ and $\varepsilon_k = -1$ if $\mu_k < 0$. Hence, the reflection subgroups of $G_{r,n}$ are naturally indexed by the signed compositions of n. Consequently, by induction, $G_{r,n}$ has $3|\Lambda_{n-1}^{\pm}| = 2 \cdot 3^{n-1}$ reflection subgroups. □

It follows from the definitions and Proposition 3.3 that Π_μ is the unique maximal subset of Π (under inclusion) which generates the reflection subgroup G_μ. In contrast, if $\mu \in \Lambda_n^{\pm}$ then the reader can check that there are $\prod_{i:\mu_i>0} \mu_i$ distinct minimal subsets of Π which generate G_μ. Thus, the (minimal) subsets of Π which generate the reflection subgroups are, in general, not unique.

4. DISTINGUISHED COSET REPRESENTATIVES.

In this section we describe, both algebraically and combinatorially, a set of 'distinguished' coset representatives for the reflection subgroups of $G_{r,n}$.

Fix a composition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n. Then $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k}$ is a parabolic, or Young subgroup of $\mathfrak{S}_n$. According to our conventions $\mathfrak{S}_\lambda = G_{-\lambda}$, so $\mathfrak{S}_\lambda$ is a reflection subgroup of $G_{r,n}$. Let

$$\mathscr{D}_\lambda = \{\, d \in \mathfrak{S}_n : \ell(d) \le \ell(w) \text{ for all } w \in \mathfrak{S}_\lambda d \,\}.$$

Then, as is well–known, $\mathscr{D}_\lambda$ is a complete set of right coset representatives for $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$. Moreover, if $d \in \mathscr{D}_\lambda$ then d is the unique element of minimal length in the coset $\mathfrak{S}_\lambda d$; see, for example, [14, Prop. 2.1.1]. It is not hard to see that $T\mathscr{D}_\lambda$ is a complete set of minimal length coset representatives for $G_{-\lambda} = \mathfrak{S}_\lambda$ in $G_{r,n}$. We want to generalize this observation to all reflection subgroups.

Recall that if $\mu = (\mu_1, \dots, \mu_k) \in \Lambda_n^\pm$ then $\mu^+ = (|\mu_1|, \dots, |\mu_k|)$ is a composition of n. Consulting the definitions, $G_\mu \cap \mathfrak{S}_n = \mathfrak{S}_{\mu^+}$. Similarly, define

$$\begin{aligned} T_\mu = G_\mu \cap T &= \langle t_i \mid t_i \in G_\mu \rangle \\ &= \langle t_i \mid \overline{\mu}_{j-1} < i \le \overline{\mu}_j \text{ for some } j \text{ with } \mu_j > 0 \rangle. \end{aligned}$$

Then, $T_\mu \cong (\mathbb{Z}/r\mathbb{Z})^{|\mu|^+}$.

With this notation, (2.2) gives the following description of G_μ as a semidirect product of T_μ and $\mathfrak{S}_{\mu^+}$.

4.1. **Lemma.** *Suppose that $\mu \in \Lambda_n^\pm$. Then $G_\mu = T_\mu \rtimes \mathfrak{S}_{\mu^+}$.*

Since $T \cong (\mathbb{Z}/r\mathbb{Z})^n$ is an abelian group, every subgroup of T is a normal subgroup of T. In particular, if G_μ is a reflection subgroup of $G_{r,n}$ then T_μ is normal in T and $T/T_\mu \cong (\mathbb{Z}/r\mathbb{Z})^{|\mu|^-} \cong T_{-\mu}$. Further, $T_\mu T_{-\mu} = T = T_{-\mu}T_\mu$, for all $\mu \in \Lambda_n^\pm$.

Mimicking the definition of $\mathscr{D}_{\mu^+}$ we have:

4.2. **Definition.** *Suppose that $\mu \in \Lambda_n^\pm$. Set*

$$\mathscr{E}_\mu = \{\, e \in G_{r,n} : \ell(e) \le \ell(g) \text{ for all } g \in G_\mu e \,\}.$$

We can now prove the main result of this section which shows that $\mathscr{E}_\mu$ is a (distinguished) set of coset representatives for G_μ in $G_{r,n}$.

4.3. **Theorem.** *Suppose that $\mu \in \Lambda_n^\pm$. Then $\mathscr{E}_\mu = T_{-\mu} \times \mathscr{D}_{\mu^+}$ and $\mathscr{E}_\mu$ is a complete set of right coset representatives for G_μ in $G_{r,n}$.*

Proof. We first show that $T_{-\mu} \times \mathscr{D}_{\mu^+}$ is a complete set of coset representatives for G_μ in $G_{r,n}$. Suppose that $t^\alpha w \in G_{r,n}$, where $\alpha \in \mathbb{Z}_r^n$ and $w \in \mathfrak{S}_n$. Define $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_r^n$ by

$$\beta_i = \begin{cases} \alpha_i, & \text{if } t_i \notin G_\mu \iff t_i \in T_{-\mu}, \\ 0, & \text{if } t_i \in G_\mu \iff t_i \notin T_{-\mu}. \end{cases}$$

Then, by definition, $t^\beta \in T_{-\mu}$. Moreover, $G_\mu t^\alpha w = G_\mu t^\beta w$ and $\ell(t^\alpha w) \ge \ell(t^\beta w)$, with equality if and only if $\alpha = \beta$.

Write $w = vd$, where $v \in \mathfrak{S}_{\mu^+}$ and $d \in \mathscr{D}_{\mu^+}$. Let $\gamma = \beta v = (\beta_{1^v}, \dots, \beta_{n^v})$. Then $t^\beta v = v t^\gamma$, by (2.1), so that $t^\gamma = v^{-1} t^\beta v \in T_{-\mu}$ since $\mathfrak{S}_\mu$ centralizes $T_{-\mu}$. Consequently, $G_\mu t^\alpha w = G_\mu t^\gamma d$, where $t^\gamma \in T_{-\mu}$ and $d \in \mathscr{D}_{\mu^+}$. However, by Lemma 4.1,

$$[G_{r,n} : G_\mu] = [T : T_{-\mu}] \cdot [\mathfrak{S}_n : \mathfrak{S}_{\mu^+}] = \#(T_{-\mu} \times \mathscr{D}_{\mu^+}).$$

Therefore, $T_{-\mu} \times \mathscr{D}_{\mu^+}$ is a complete set of right coset representatives for G_μ in $G_{r,n}$.

It remains to prove that $\mathscr{E}_\mu = T_{-\mu} \times \mathscr{D}_{\mu^+}$. Suppose that, as above, we have $G_\mu t^\alpha w = G_\mu t^\gamma d$, for $\alpha \in \mathbb{Z}_r^n$, $w \in \mathfrak{S}_n$, $t^\gamma \in T_{-\mu}$ and $d \in \mathscr{D}_{\mu^+}$. The argument of the first paragraph shows that $\ell(t^\gamma d) \le \ell(t^\alpha w)$ with equality if and only if $t^\alpha \in T_{-\mu}$ and $w \in \mathscr{D}_{\mu^+}$. That is, if and only if $\alpha = \gamma$ and $w = d$. Hence, $\mathscr{E}_\mu = T_{-\mu} \times \mathscr{D}_{\mu^+}$ as claimed. □

Theorem 4.3 shows that every coset of a reflection subgroup contains a unique element of minimal Π–length. We call $\mathscr{E}_\mu$ the set of **distinguished coset representatives** for G_μ in $G_{r,n}$.

4.4. **Example** Suppose that $r \geq 2$ and consider $G_{r,2} = (\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_2$. Then $\Pi = \{t_1, t_2, s_1\}$ and $G_{r,2}$ has six reflection subgroups. The following table describes these groups and the corresponding sets of distinguished right coset representatives.

μ	G_μ	Π_μ	$\mathscr{E}_\mu$
$(-1,-1)$	1	$\emptyset$	$T \times \mathfrak{S}_2$
$(1,-1)$	$\{\, t_1^k : 0 \leq k < r \,\}$	$\{t_1\}$	$\langle t_2 \rangle \times \mathfrak{S}_2$
$(-1,1)$	$\{\, t_2^k : 0 \leq k < r \,\}$	$\{t_2\}$	$\langle t_1 \rangle \times \mathfrak{S}_2$
$(1,1)$	T	$\{t_1, t_2\}$	$\mathfrak{S}_2$
(-2)	$\mathfrak{S}_2$	$\{s_1\}$	T
(2)	$T \rtimes \mathfrak{S}_2$	$\{t_1, t_2, s_1\}$	1

For each reflection subgroup we have given the factorization of $\mathscr{E}_\mu$ from Theorem 4.3. Observe that the reflection subgroups do not depend in a crucial way on r and that $|\mathscr{E}_\mu| = |G_{r,n}|/|G_\mu|$ is a polynomial in r, for $\mu \in \Lambda_n^\pm$ (and $r \geq 2$). $\Diamond$

We now give combinatorial interpretations of the set of distinguished coset representatives $\mathscr{E}_\mu$, for $\mu \in \Lambda_n^\pm$, which is similar to the description of $\mathscr{D}_{\mu^+}$ in terms of row standard tableaux (see [24, Prop. 3.3]).

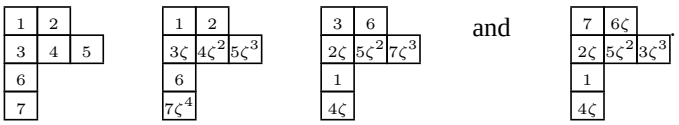
Fix a composition $\lambda \in \Lambda_n$. The **diagram** of λ is the set

$$[\lambda] = \{\, (i,j) \in \mathbb{N}^2 : 1 \leq j \leq \lambda_i \text{ and } 1 \leq i \leq \ell(\lambda) \,\}.$$

Here $\ell(\lambda)$ is the number of non–zero parts of λ. We think of $[\lambda]$ as being an array of boxes in the plane.

Now suppose that $\mu \in \Lambda_n^\pm$. A μ–**tableau** is a map $\mathfrak{t} : [\mu^+] \longrightarrow \mathbf{n}_\zeta$. We identify a μ–tableau with a diagram for μ^+ which is labeled by elements of $\mathbf{n}_\zeta$. If $\mathfrak{t}$ is a μ–tableau let $|\mathfrak{t}|$ be the tableau obtained by taking the complex modulus of the entries in $\mathfrak{t}$; that is, $|\mathfrak{t}|(x) = |\mathfrak{t}(x)|$, for all $x \in [\mu^+]$.

4.5. **Example** Let $\mu = (2,-3,1,-1)$. Then four μ–tableaux are:



$\Diamond$

As remarked at the end of section 2 we can think of $G_{r,n}$ as the group of permutations of $\mathbf{n}_\zeta$ such that $(m\zeta^i)^g = m^g\zeta^i$, for all $m\zeta^i \in \mathbf{n}_\zeta$ and all $g \in G_{r,n}$. Consequently, $G_{r,n}$ acts on the set of μ–tableaux by composition of maps. Thus, if $\mathfrak{t}$ is a μ–tableau and $g \in G_{r,n}$ then $\mathfrak{t}^g$ is the tableau with $\mathfrak{t}^g(x) = \mathfrak{t}(x)^g$, for $x \in [\mu^+]$.

Let $\mathfrak{t}^\mu$ be the μ–tableau which has the numbers $1, \dots, n$ entered in order, from left to right and then top to bottom, along the rows of $[\mu^+]$. The first μ–tableau in Example 4.5 is $\mathfrak{t}^\mu$ when $\mu = (-2,3,-1,1)$.

So far none of the combinatorial definitions above distinguish between compositions and signed compositions. We now single out a set of μ–tableaux that are in bijection with $\mathscr{E}_\mu$. First, define a total order $\preceq$ on $\mathbf{n}_\zeta$ by declaring that $a\zeta^i \preceq b\zeta^j$ if $a < b$, or $a = b$ and $i > j$. Then

$$\zeta^{m-1} \preceq \zeta^{m-2} \preceq \cdots \preceq \zeta \preceq 1 \preceq 2\zeta^{m-1} \preceq \cdots \preceq 2 \preceq \cdots \preceq n\zeta^{m-1} \preceq \cdots \preceq n.$$

4.6. **Definition.** *Suppose that $\mu \in \Lambda_n^\pm$. A μ–tableau $\mathfrak{t}$ is **row standard** if it satisfies the following three conditions:*

a) *The set of entries in the tableau $|\mathfrak{t}|$ is $\{1, \dots, n\}$.*
b) *The entries in row i of $\mathfrak{t}$ belong to $\{1, \dots, n\}$ whenever $\mu_i > 0$.*

c) *In each row the entries of* $\mathfrak{t}$ *appear, from left to right, in increasing order with respect to* $\preceq$.

For example, the first three of the $(2,-3,1,-1)$–tableaux in Example 4.5 are row standard. The last tableau in this example is not row standard because it fails conditions (b) and (c).

The action of $G_{r,n}$ on the set of μ–tableau which satisfy condition (a) of Definition 4.6 gives a realization of the regular representation of $G_{r,n}$. Consequently, the map $g \mapsto \mathfrak{t}^\mu g$, for $g \in G_{r,n}$, is a bijection from $G_{r,n}$ to the set of these μ–tableaux. If $\mathfrak{t}$ is such a μ–tableau let $d_\mathfrak{t}$ be the unique element of $G_{r,n}$ such that $\mathfrak{t} = \mathfrak{t}^\mu d_\mathfrak{t}$.

4.7. **Proposition.** *Suppose that* $\mu \in \Lambda_n^\pm$. *Then*

$$\mathscr{E}_\mu = \{\, d_\mathfrak{t} : \mathfrak{t} \text{ is a row standard } \mu\text{–tableau} \,\}.$$

Proof. By definition, the orbit $\mathfrak{t}^\mu G_\mu = \{\, \mathfrak{t}^\mu g : g \in G_\mu \,\}$ of $\mathfrak{t}^\mu$ under G_μ consists of all those tableaux which can be obtained by permuting the entries of each row of $\mathfrak{t}^\mu$ and multiplying the entries in row i by a power of ζ when $\mu_i > 0$. Consequently, $\mathfrak{t}^\mu$ is the unique row standard μ–tableaux in $\mathfrak{t}^\mu G_\mu$, so that each right coset of G_μ in $G_{r,n}$ contains a unique element e such that $\mathfrak{t}^\mu e$ is row standard. Now, $\mathscr{E}_\mu = T_{-\mu}\mathscr{D}_{\mu^+}$ by Theorem 4.3 and $T_{-\mu}$ acts on $\mathfrak{t}^\mu$ by multiplying the entries in row i by different powers of ζ when $\mu_i < 0$. If $d \in \mathscr{D}_{\mu^+}$ then it is well–known that the entries in $\mathfrak{t}^\mu d$ increase from left to right along each row; see, for example, [24, Prop. 3.3]. Hence, $\mathfrak{t}^\mu e$ is row standard whenever $e \in \mathscr{E}_\mu$. This completes the proof. □

In the case of the symmetric groups the set of distinguished coset representatives can be described combinatorially in terms of 'descents'. Explicitly, if $w \in \mathfrak{S}_n$ then its **descent set** is

$$\mathsf{Des}(w) = \{\, s \in S : \ell(sw) < \ell(w) \,\} = \{\, s_i : 1 \le i < n \text{ and } i^w > (i+1)^w \,\}.$$

If μ is a composition of n then the connection between distinguished coset representatives and descents is that

$$\mathscr{D}_\mu = \{\, d \in \mathfrak{S}_n : \mathsf{Des}(d) \subseteq S - \Pi_{-\mu} \,\}. \tag{4.8}$$

There is an analogous description of $\mathscr{E}_\mu$, for $\mu \in \Lambda_n^\pm$. If $\alpha \in \mathbb{Z}_r^n$ define the **colour** of t^α to be the set $\mathsf{Col}(t^\alpha) = \{\, t_i \in T : \alpha_i > 0 \,\}$. Then using Theorem 4.3 it is easy to see that if $\mu \in \Lambda_n^\pm$ then

$$\mathscr{E}_\mu = \{\, t^\alpha w \in G_{r,n} : \mathsf{Col}(\alpha) \cup \mathsf{Des}(w) \subseteq \Pi - \Pi_{-\mu} \,\}.$$

We remark that it is easy to rephrase this last statement combinatorially in terms of words in $\mathcal{G}_{r,n}$.

4.9. *Remark.* It is easy to check that $\mathscr{E}_\mu^{-1} = \mathscr{D}_\mu^{-1} \times T_{-\mu} = \mathscr{D}_\mu^{-1} \times T_{-\mu}$ is a complete set of left coset representatives for G_μ in $G_{r,n}$. Moreover, $e \in \mathscr{E}_\mu^{-1}$ if and only if $\ell(e) \le \ell(g)$ for all $g \in eG_\mu$, so every left coset of G_μ in $G_{r,n}$ contains a unique element of minimal Π–length.

4.10. *Remark.* Mak [20] has shown that every coset of a reflection subgroup contains a unique element of minimal length with respect to the length function ℓ_0 defined in Remark 2.4. Mak's set of coset representatives is different from $\mathscr{E}_\mu$. Nonetheless, it does admit a factorization which is similar to the factorization of $\mathscr{E}_\mu$ given in Theorem 4.3. To describe this if $\mu = (\mu_1, \dots, \mu_k) \in \Lambda_n^\pm$ then set

$$\mathscr{E}'_\mu = \prod_{\substack{k \ge j \ge 1 \\ \mu_j < 0}} \prod_{\overline{\mu}_j \ge i > \overline{\mu}_{j-1}} \Big[\{1\} \cup \{\, s_{\overline{\mu}_{j-1}+1} s_{\overline{\mu}_{j-1}+2} \dots s_{i-1} t_i^k : 1 \le k < r \,\} \Big] \times \mathscr{D}_\mu,$$

where the product is taken in order from left to right in terms of decreasing values of i. One can show that $\mathscr{E}'_\mu$ is Mak's set of coset representatives for G_μ in $G_{r,n}$. As we will never need this result we leave the proof to the reader.

5. Double Coset representatives

Our next aim is to describe the double cosets of reflection subgroups. In order to do this we first recall some well–known facts about the symmetric group $\mathfrak{S}_n$. Suppose that μ and ν are compositions of n. Then $\mathfrak{S}_\mu$ and $\mathfrak{S}_\nu$ are Young, or parabolic, subgroups of $\mathfrak{S}_n$. Set $\mathscr{D}_{\mu\nu} = \mathscr{D}_\mu \cap \mathscr{D}_\nu^{-1}$. Then $\mathscr{D}_{\mu\nu}$ is a complete set of $(\mathfrak{S}_\mu, \mathfrak{S}_\nu)$–double coset representatives in $\mathfrak{S}_n$; see, for example, [24, Prop. 4.4]. Moreover, if $d \in \mathscr{D}_{\mu\nu}$ then $d^{-1}\mathfrak{S}_\mu d \cap \mathfrak{S}_\nu$ is a Young subgroup of $\mathfrak{S}_n$; see, for example, [24, Lemma 4.3]. Define $\mu d \cap \nu$ to be the unique composition of n such that $\mathfrak{S}_{\mu d\cap\nu} = d^{-1}\mathfrak{S}_\mu d \cap \mathfrak{S}_\nu$. We remark that the composition $\mu d \cap \nu$ can be determined by comparing the row stabilizers of the tableaux $\mathfrak{t}^\mu d$ and $\mathfrak{t}^\nu$.

5.1. **Lemma.** *Suppose that $\mu, \nu \in \Lambda_n^\pm$ and $d \in \mathscr{D}_{\mu^+\nu^+}$. Then $d^{-1}G_\mu d \cap G_\nu$ is a reflection subgroup of $G_{r,n}$.*

Proof. The group G_ν consists of those elements of $G_{r,n}$ which act on $\mathfrak{t}^\nu$ by first multiplying each entry of row l by possibly different powers of ζ, if $\nu_l > 0$, and then permuting the entries in each row of the resulting tableaux. Similarly, the group $d^{-1}G_\mu d$ consists of those elements of $G_{r,n}$ which act on the row standard tableau $\mathfrak{t}^\mu d$ by multiplying each entry of row k by different powers of ζ, if $\mu_k > 0$, and then permuting the entries in each row. Consequently, the subgroup $d^{-1}G_\mu d \cap G_\nu$ is generated by the elements $\{s_i, t_j\}$, where i runs over those integers for which i and $i+1$ are in the same row of $\mathfrak{t}^\nu$ and in the same row of $\mathfrak{t}^\mu d$, and $j \in \mathbf{n}$ is in row l of $\mathfrak{t}^\nu$ with $\nu_l > 0$ and j is in row k of $\mathfrak{t}^\mu d$ with $\mu_k > 0$ (cf. the proof of [24, Lemma 4.3]). Therefore, $d^{-1}G_\mu d \cap G_\nu = G_\sigma$, where σ is the unique signed composition such that $\sigma^+ = \mu^+ d \cap \nu^+$ and $\sigma_i > 0$ if and only if $\nu_j > 0$ and $\mu_k > 0$, where $\overline{\sigma}_i$ appears in row j of $\mathfrak{t}^\nu$ and row k of $\mathfrak{t}^\mu d$. □

Suppose that $d \in \mathscr{D}_{\mu^+\nu^+}$, for $\mu, \nu \in \Lambda_n^\pm$. Then $d^{-1} \in \mathscr{D}_{\nu^+\mu^+}$, since $\mathscr{D}_{\nu^+\mu^+} = \mathscr{D}_{\mu^+\nu^+}^{-1}$. Therefore, $G_\mu \cap dG_\nu d^{-1}$ is also a reflection subgroup of $G_{r,n}$.

5.2. **Definition.** *Suppose that $\mu, \nu \in \Lambda_n^\pm$ and $d \in \mathscr{D}_{\mu^+\nu^+}$. Then $\mu d \cap \nu$ is the signed composition of n such that $G_{\mu d\cap\nu} = d^{-1}G_\mu d \cap G_\nu$ and $\mu \cap d\nu$ is the signed composition such that $G_{\mu\cap d\nu} = G_\mu \cap dG_\nu d^{-1}$.*

Note that the proof of Lemma 5.1 gives a recipe for computing $\mu d \cap \nu$. Note also that $\mu d \cap \nu = d^{-1}\mu \cap \nu$, for $d \in \mathscr{D}_{\mu^+\nu^+}$ and $\mu, \nu \in \Lambda_n^\pm$.

We now describe a set of (G_μ, G_ν)–double coset representatives. We do this by generalizing the description of the double cosets of the Young subgroups of the symmetric group in terms of row semistandard tableaux.

5.3. **Definition.** *Suppose that $\mu \in \Lambda_n^\pm$. A μ–tableau $\mathsf{T} : [\mu^+] \longrightarrow \mathbf{n}_\zeta$ is **row semistandard** if*

a) *The entries in row i of T belong to $\{1, \dots, n\}$ whenever $\mu_i > 0$.*
b) *The entries in each row of T appear in weakly increasing order, from left to right, with respect to $\preceq$.*

There is a map from the set of row semistandard tableau to the set of row standard tableaux. To define this first observe that a row semistandard μ–tableau T determines a unique total order $<_\mathsf{T}$ on $[\mu^+]$ where $x <_\mathsf{T} x'$, for $x, x' \in [\mu^+]$, if

a) $|\mathsf{T}(x)| < |\mathsf{T}(x')|$, or
b) $|\mathsf{T}(x)| = |\mathsf{T}(x')|$ and x is in an earlier row of $[\mu^+]$ than x', or
c) $|\mathsf{T}(x)| = |\mathsf{T}(x')|$ and x and x' are in the same row and x is to the left of x'.

Let $x_1 <_{\mathsf{T}} \cdots <_{\mathsf{T}} x_n$ be the nodes in $[\mu^+]$. Then the μ–tableau T^* is defined by the requirements that $|\mathsf{T}^*(x_i)| = i$ and $\arg \mathsf{T}^*(x_i) = \arg \mathsf{T}(x_i)$, for $1 \le i \le n$. By construction, T^* is a row standard μ–tableaux. Moreover, it is easy to see that the map $\mathsf{T} \mapsto \mathsf{T}^*$ is injective.

5.4. **Definition.** *Suppose that $\mu, \nu \in \Lambda_n^{\pm}$ and let T be a μ–tableau. Then T has **type** ν if*

a) $|\nu_j| = \#\{\, x \in [\mu^+] : |\mathsf{T}(x)| = j \,\}$, *for $j \ge 1$.*
b) *If $\nu_j > 0$ then $\nu_j = \#\{\, x \in [\mu^+] : \mathsf{T}(x) = j \,\}$.*

*Let $\mathcal{T}_\zeta(\mu,\nu) = \{\, \mathsf{T} : [\mu^+] \longrightarrow \mathbf{n}_\zeta : \mathsf{T}$ is row semistandard μ–tableau of type $\nu \,\}$. If μ and ν are **compositions** let $\mathcal{T}(\mu,\nu) = \{\, \mathsf{T} : [\mu^+] \longrightarrow \mathbf{n} : \mathsf{T}$ is row semistandard μ–tableau of type $\nu \,\}$.*

See Example 5.9 below for these definitions in action.

5.5 ([24, Prop. 4.4])**.** *Suppose that $\mu, \nu \in \Lambda_n$. Then*

$$\mathscr{D}_{\mu\nu} = \{\, d_{\mathsf{T}^*} : \mathsf{T} \in \mathcal{T}(\mu,\nu) \,\}$$

is a complete set of $(\mathfrak{S}_\mu, \mathfrak{S}_\nu)$ double coset representatives in $\mathfrak{S}_n$. Moreover, if $d \in \mathscr{D}_{\mu\nu}$ then $\ell(d) \le \ell(w)$, for all $w \in \mathfrak{S}_\mu d \mathfrak{S}_\nu$, with equality if and only if $w = d$.

If $\mathfrak{t}$ is a row standard tableau let $\nu(\mathfrak{t})'$ be the tableau obtained by replacing each entry $m\zeta^a$ in $\mathfrak{t}$ with $k\zeta^{a'}$ if m appears in row k of $\mathfrak{t}^\nu$, where $a' = 0$ if $\nu_k > 0$ and $a' = a$ otherwise. Now define $\nu(\mathfrak{t})$ to be the row semistandard tableau obtained by reordering the entries in each row of $\nu(\mathfrak{t})'$ so that they are in increasing order. Then $\nu(\mathfrak{t})$ is a row semistandard tableau of type ν.

For example, let $\nu = (2,-2,1)$ and $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 3 & 4\zeta^a & 5\zeta^b \\ \hline 1 & 2 \\ \cline{1-2}\end{array}$, where $0 \le a, b < r$. Then, by definition, $\nu(\mathfrak{t})' = \begin{array}{|c|c|c|}\hline 2 & 2\zeta^a & 3 \\ \hline 1 & 1 \\ \cline{1-2}\end{array}$ and $\nu(\mathfrak{t}) = \begin{array}{|c|c|c|}\hline 2\zeta^a & 2 & 3 \\ \hline 1 & 1 \\ \cline{1-2}\end{array}$.

5.6. **Proposition.** *Suppose that μ and ν are signed compositions of n and let*

$$\mathscr{E}_{\mu\nu} = \{\, d_{\mathsf{T}^*} : \mathsf{T} \in \mathcal{T}_\zeta(\mu,\nu) \,\}.$$

Then $\mathscr{E}_{\mu\nu}$ is a complete set of (G_μ, G_ν) double coset representatives in $G_{r,n}$. Moreover, if $e \in \mathscr{E}_{\mu\nu}$ then $\ell(e) \le \ell(g)$, for all $g \in G_\mu e G_\nu$.

Proof. By Proposition 4.7 the right cosets of G_μ in $G_{r,n}$ are naturally indexed by the row standard μ–tableaux. Hence, the (G_μ, G_ν)–double cosets are indexed by the G_ν–orbits of the row standard μ–tableaux. Using the definitions it is easy to see that two μ–tableaux $\mathfrak{s}$ and $\mathfrak{t}$ belong to the same G_ν–orbit if and only if $\nu(\mathfrak{s}) = \nu(\mathfrak{t})$. Moreover, if $\mathfrak{t}$ is row standard then $\nu(\mathfrak{t})$ is row semistandard. Finally, if T is a row semistandard μ–tableau of type ν then T^* is a row standard μ–tableau such that $\mathsf{T} = \nu(\mathsf{T}^*)$. Hence, $\mathscr{E}_{\mu\nu}$ is a complete set of (G_μ, G_ν)–double coset representatives in $G_{r,n}$.

To complete the proof we need to show that if $\mathsf{T} \in \mathcal{T}_\zeta(\mu,\nu)$ then d_{T^*} is an element of minimal length in the double coset $G_\mu d_{\mathsf{T}^*} G_\nu$. For convenience, let $d = d_{|\mathsf{T}^*|}$. Then, $d \in \mathscr{D}_{\mu^+\nu^+}$ by (5.5). Now, by the last paragraph $G_\mu d_{\mathsf{T}^*} G_\nu = \bigcup_{\mathfrak{t}} G_\mu d_{\mathfrak{t}}$, where $\mathfrak{t}$ runs over the row standard μ–tableau $\mathfrak{t}$ such that $\nu(\mathfrak{t}) = \mathsf{T}$. By definition, $d_{\mathsf{T}^*} = t_1^{\alpha_1} \dots t_n^{\alpha_n} d$, where if $x \in [\mu^+]$ then $\mathsf{T}^*(x) = \zeta^{\alpha_i} i^d$ if and only if $\mathsf{T}(x) = \zeta^{\alpha_i} k$ and i^d is in row k of $\mathfrak{t}^\nu$. Now suppose that $\mathfrak{t}$ is any row standard μ–tableaux such that $\nu(\mathfrak{t}) = \mathsf{T}$. Then, using (5.5) again, $d_{\mathfrak{t}} = t_1^{\beta_1} \dots t_n^{\beta_n} d_{|\mathfrak{t}|} = t_1^{\beta_1} \dots t_n^{\beta_n} du$, for some $u \in \mathfrak{S}_\nu$ and where $\beta_i = \alpha_{i^w}$, for some $w \in \mathfrak{S}_n$ (since $\nu(\mathfrak{t}) = \mathsf{T}$). Therefore,

$$\begin{aligned}\ell(d_{\mathfrak{t}}) &= \beta_1 + \dots + \beta_n + \ell(du) = \alpha_1 + \dots + \alpha_n + \ell(du)\\ &\ge \alpha_1 + \dots + \alpha_n + \ell(d) = \ell(d_{\mathsf{T}^*}),\end{aligned}$$

with equality if and only if $u = 1$. By Theorem 4.3, $d_{\mathfrak{t}}$ is the unique element of minimal length in the coset $G_\mu d_{\mathfrak{t}}$, for each such $\mathfrak{t}$. Therefore, $\ell(d_{\mathsf{T}^*}) \le \ell(g)$ for all $g \in G_\mu d_{\mathsf{T}^*} G_\nu$ as claimed. □

Note that we are not claiming that each double coset of two reflection subgroups of $G_{r,n}$ contains a unique element of minimal length. Indeed, the proof of Proposition 5.6 shows that if T is a row semistandard μ–tableau of type ν then the double coset $G_\mu d_{\mathsf{T}^*} G_\nu$ contains more than one element of minimal length if and only if there exist integers b, c, not both zero, such that $m\zeta^b$ and $m\zeta^c$ appear in the same row of T, for some $m \in \mathbf{n}$. For future comparison we make this statement explicit.

If $d \in \mathscr{D}_{\mu^+\nu^+}$ let $\mathsf{T}_d \in \mathcal{T}(\mu^+, \nu^+)$ be the unique row semistandard tableau such that $d = d_{\mathsf{T}_d^*}$ as in (5.5). If $X \subseteq G_{r,n}$ let $X^{-1} = \{\, g : g^{-1} \in X \,\}$.

5.7. **Lemma.** *Suppose that $\mu, \nu \in \Lambda_n^\pm$. Then*

$$\mathscr{E}_{\mu\nu} = \coprod_{d \in \mathscr{D}_{\mu^+\nu^+}} \left\{ t_1^{\alpha_1} \dots t_n^{\alpha_n} \in T_{-\mu \cap d(-\nu)} \;\middle|\; \begin{array}{l} \alpha_i \le \alpha_j \text{ whenever } i^d \text{ and } j^d \text{ are in the} \\ \text{same row of } \mathsf{T}_d^* \text{ and the same row of } \mathfrak{t}^\nu \end{array} \right\} d$$

Moreover,

$$\mathscr{E}_\mu \cap \mathscr{E}_\nu^{-1} = \coprod_{d \in \mathscr{D}_{\mu^+\nu^+}} T_{-\mu\cap d(-\nu)} d = \{\, e \in G_{r,n} : \ell(e) \le \ell(g) \text{ for all } g \in G_\mu e G_\nu \,\}$$

is the set of elements in $G_{r,n}$ which are of minimal length in their (G_μ, G_ν)–double coset.

Proof. Observe that $\mathscr{D}_{\mu^+\nu^+} = \mathscr{D}_{(-\mu)^+(-\nu)^+}$. Therefore, if $d \in \mathscr{D}_{\mu^+\nu^+}$ then the signed composition $-\mu \cap d(-\nu)$ in the statement of the Corollary makes sense by Definition 5.2. (Note, however, that the two signed compositions $-\mu\cap d(-\nu)$ and $-(\mu\cap d\nu)$ are *not* equal in general.)

By Proposition 5.6, we have $\mathscr{E}_{\mu\nu} = \{\, d_{\mathsf{T}^*} : \mathsf{T} \in \mathcal{T}_\zeta(\mu, \nu) \,\}$. Fix a row semistandard μ–tableau T of type ν. Then, as in the proof of Proposition 5.6, $d_{\mathsf{T}^*} = t_1^{\alpha_1} \dots t_n^{\alpha_n} d$, where $d = d_{|\mathsf{T}^*|} \in \mathscr{D}_{\mu^+\nu^+}$ and, for all $x \in [\mu^+]$ if $\mathsf{T}^*(x) = \zeta^{\alpha_i} i^d$ then $\mathsf{T}(x) = \zeta^{\alpha_i} k$ where i^d is in row k of $\mathfrak{t}^\nu$. In particular, $\alpha_i = 0$ if $t_i \in T_\mu$ or if $t_{i^d} \in T_\nu$. Therefore, $\alpha_i > 0$ only if $t_i \in T_{-\mu} \cap dT_{-\nu}d^{-1} = T_{-\mu\cap d(-\nu)}$. If $t_i \in T_{-\mu\cap d(-\nu)}$ then the integer α_i can take any value in $\{0, \dots, r-1\}$ provided that this is compatible with T being row semistandard. That is, we require that $\alpha_i \le \alpha_j$ whenever i^d and j^d are in the same row of T^* and in the same row of $\mathfrak{t}^\nu$. This gives the decomposition of $\mathscr{E}_{\mu\nu}$ in the statement of Lemma.

For the final claim, suppose that $d \in \mathscr{D}_{\mu^+\nu^+}$ and let $\mathsf{T} = \nu(\mathfrak{t}^\mu d)$. By the last paragraph, if $t \in T$ then $\nu(\mathfrak{t}^\mu t d_{\mathsf{T}^*}) = \mathsf{T}$ if and only if $t \in T_{-\mu\cap d(-\nu)}$. By the last paragraph again, if $t \in T_{-\mu\cap d(-\nu)}d$ then td is an element of minimal length in the double coset $G_\mu t d G_\nu$. That $\mathscr{E}_\mu \cap \mathscr{E}_\nu^{-1} = \coprod_d T_{-\mu\cap d(-\nu)}d$ is now follows from the definition of row semistandard tableaux. □

5.8. **Corollary.** *Suppose that $\mu, \nu \in \Lambda_n^\pm$ and $d \in \mathscr{D}_{\mu^+\nu^+}$. Then $G_{r,n}$ contains $|T_{-\mu\cap d(-\nu)}|$ elements of the form $t^\alpha d$ which are of minimal length in their (G_μ, G_ν)–double coset, for some $\alpha \in \mathbb{Z}_r^n$. Moreover, if $\mathsf{T} = \nu(\mathfrak{t}^\mu d)$ then $|T_{-\mu\cap d(-\nu)}| = r^{\mathsf{wt}(\mathsf{T})}$, where $\mathsf{wt}(\mathsf{T})$ is the number of pairs (i, j) such that j appears in row i of T and $\mu_i < 0$ and $\nu_j < 0$.*

Proof. That $|T_{-\mu\cap d(-nu)}|$ counts number of elements of the form $t^\alpha d$ which are of minimal length in their (G_μ, G_ν)–double coset is immediate from Lemma 5.7. The second claim follows from the observation that the tableaux $\{\, \mathfrak{t}^\mu t d : t \in T_{-\mu\cap d(-\nu)} \,\}$ differ only in that any of the numbers appearing in row i of $\mathfrak{t}^\mu d$ and row j of $\mathfrak{t}^\nu$ can be multiplied by arbitrary powers of ζ whenever $\mu_i < 0$ and $\nu_j < 0$. □

5.9. **Example** Suppose that $r \ge 2$ and $n = 5$ and let $\mu = (3, -2)$ and $\nu = (-2, -2, 1)$. Then the set of row semistandard μ–tableaux T of type ν, together with the corresponding row standard tableau T^* and the coset representatives $d_{\mathsf{T}^*} \in \mathscr{E}_{\mu\nu} = \mathscr{E}_{(3,-2)(-2,-2,1)}$, is as

follows (we set $d = d_{|\mathsf{T}^*|}$).

T	T^*	d_{T^*}	$\lvert T_{-\mu\cap d(-\nu)}\rvert$	$\mu \cap d\nu$
$\begin{array}{\|c\|c\|c\|}\hline 1&1&2\\\hline 2\zeta^a&3\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4\zeta^a&5\\\cline{1-2}\end{array}$	t_4^a	r	$(-2,-1^3)$
$\begin{array}{\|c\|c\|c\|}\hline 1&1&3\\\hline 2\zeta^b&2\zeta^c\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&5\\\hline 3\zeta^b&4\zeta^c\\\cline{1-2}\end{array}$	$t_4^b t_5^c s_3 s_4$	r^2	$(-2,1,-2)$
$\begin{array}{\|c\|c\|c\|}\hline 1&2&2\\\hline 1\zeta^a&3\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2\zeta^a&5\\\cline{1-2}\end{array}$	$t_4^a s_3 s_2$	r	$(-1,-2,-1^2)$
$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 1\zeta^a&2\zeta^b\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&5\\\hline 2\zeta^a&4\zeta^b\\\cline{1-2}\end{array}$	$t_4^a t_5^b s_3 s_2 s_4$	r^2	$(-1^2,1,-1^2)$
$\begin{array}{\|c\|c\|c\|}\hline 2&2&3\\\hline 1\zeta^b&1\zeta^c\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 3&4&5\\\hline 1\zeta^b&2\zeta^c\\\cline{1-2}\end{array}$	$t_4^b t_5^c s_3 s_2 s_4 s_3 s_1 s_2$	r^2	$(-2,1,-2)$

where $0 \le a,b,c < r$ and $b \le c$. We use exponentials in the signed compositions to indicate consecutive repeated parts. Therefore, there are $2r^2+3r$ (G_μ, G_ν)–double cosets in $G_{r,n}$. When checking the entries in this table observe that the signed composition $\mu \cap d\nu = \mu \cap \nu d^{-1}$ can be computed by intersecting G_μ with the 'row stabilizer' of $\mathfrak{t}^\nu d^{-1}$ as in the proof of Lemma 5.1. Note that $|T_{-\mu\cap d(-\nu)}|$ can be computed without finding $-\mu \cap d(-\nu)$ by using Corollary 5.8. ◊

5.10. *Remark.* If μ and ν are compositions of n then $\mathscr{D}_{\mu\nu} = \mathscr{D}_\mu \cap \mathscr{D}_\nu^{-1}$ is a complete set of minimal length $(\mathfrak{S}_\mu, \mathfrak{S}_\nu)$–double coset representatives in $\mathfrak{S}_n$ by (5.5). In contrast, it is not hard to show that $\mathscr{E}_{\mu\nu} \subseteq \mathscr{E}_\mu \cap \mathscr{E}_\nu^{-1}$ with $\mathscr{E}_\mu \cap \mathscr{E}_\nu^{-1}$ being strictly bigger than $\mathscr{E}_{\mu\nu}$ in general. For example, if we take $\mu = (3,-2)$ and $\nu = (-2^2,1)$ then $|\mathscr{E}_\mu \cap \mathscr{E}_\nu^{-1}| = 3r^2+2r$, whereas $|\mathscr{E}_{\mu\nu}| = 2r^2+3r$ by Example 5.9. So $\mathscr{E}_{\mu\nu} \subsetneq \mathscr{E}_\mu \cap \mathscr{E}_\nu^{-1}$ since $r > 1$.

6. The cyclotomic Solomon algebra

Suppose that R is a commutative ring (with one) and let $RG_{r,n}$ be the group ring of $G_{r,n}$ over R. In this section we use the distinguished coset representatives of the reflection subgroups of $G_{r,n}$ to define an analogue of Solomon's descent algebra for the complex reflection group $G_{r,n}$.

Recall that for each reflection subgroup G_μ of $G_{r,n}$ we have a distinguished set $\mathscr{E}_\mu$ of right coset representatives, for $\mu \in \Lambda_n^\pm$. Define

$$E_\mu = \sum_{e \in \mathscr{E}_\mu} e \in RG_{r,n}.$$

The main aim of this paper is to understand the subalgebra of $RG_{r,n}$ which is generated by these elements.

6.1. **Definition.** *Suppose that $r > 1$. The **cyclotomic Solomon algebra***

$$\mathsf{Sol}(G_{r,n}) = \mathsf{Sol}_R(G_{r,n})$$

is the subalgebra of $RG_{r,n}$ generated by $\{ E_\mu : \mu \in \Lambda_n^\pm \}$.

It is not yet clear that $\mathsf{Sol}(G_{r,n})$ is a subalgebra of $RG_{r,n}$; this will be established in Theorem 6.8 below. We begin by taking advantage of the factorization of $\mathscr{E}_\mu$ given by Theorem 4.3. To do this, for $i = 1,\dots,n$ and $\lambda \in \Lambda_n$ define

$$F_i = \sum_{k=0}^{r-1} t_i^k \qquad \text{and} \qquad D_\lambda = \sum_{d \in \mathscr{D}_\lambda} d,$$

Then F_i and D_λ are both elements of $RG_{r,n}$.

6.2. **Lemma.** *Suppose that $1 \le i,j \le n$ and that $w \in \mathfrak{S}_n$. Then*

a) *$F_iF_j = F_jF_i$ and $F_i^2 = rF_i$.*

b) $F_i w = w F_{i^w}$.

Proof. As T is an abelian group part (a) is true and part (b) is immediate from the definitions and (2.1). □

Hence, if $1 \leq i \leq n$ then F_i is a multiple of an idempotent if the characteristic of R does not divide r and, otherwise, it is a nilpotent element of $RG_{r,n}$.

Suppose that $\mu \in \Lambda_n^{\pm}$. In order to factorize E_μ set

$$F_{-\mu} = \prod_{t_i \in T_{-\mu}} F_i = \prod_{i:\mu_i<0} F_{\overline{\mu}_{i-1}+1} \cdots F_{\overline{\mu}_i}.$$

Then, by Lemma 6.2(a), $\big(F_{-\mu}\big)^2 = r^{|\mu|^-} F_{-\mu}$.

By Lemma 6.2, $\mathfrak{S}_n$ acts on $\{F_1, \dots, F_n\}$ by conjugation. If $w \in \mathfrak{S}_n$ and $i \in \mathbf{n}$ then we set $F_i^w = w^{-1} F_i w = F_{i^w}$. Similarly, if $\mu \in \Lambda_n^{\pm}$ let

$$F_{-\mu}^w = \prod_{t_i \in T_{-\mu}} F_i^w.$$

Then $F_{-\mu} w = w F_{-\mu}^w$, for all $w \in \mathfrak{S}_n$, by Lemma 6.2(b).

6.3. **Lemma.** *Suppose that $\mu \in \Lambda_n^{\pm}$ is a signed composition of n. Then:*

a) $E_\mu = F_{-\mu} D_{\mu^+}$.

b) *If $w \in \mathfrak{S}_{\mu^+}$ then $F_{-\mu}^w = F_{-\mu}$, so that $F_{-\mu} w = w F_{-\mu}$.*

Proof. Part (a) is an immediate consequence of the factorization $\mathscr{E}_\mu = T_{-\mu} \times \mathscr{D}_{\mu^+}$ of $\mathscr{E}_\mu$ given by Theorem 4.3. For part (b), use Lemma 6.2(b) and the fact that the elements of the two subgroups $\mathfrak{S}_\mu$ and $T_{-\mu}$ commute. □

Definition 6.1 is motivated by Solomon's [27] definition of the descent algebra of a finite Coxeter group. As an important special case, the **Solomon descent algebra** $\mathsf{Sol}(\mathfrak{S}_n)$ of $\mathfrak{S}_n$ is the subalgebra of $R\mathfrak{S}_n$ generated by $\{ D_\lambda : \lambda \in \Lambda_n \}$. The next result, due to Solomon, shows that $\{ D_\lambda : \lambda \in \Lambda_n \}$ is basis of $\mathsf{Sol}(\mathfrak{S}_n)$.

6.4 (Solomon [27, Theorem 1])**.**

a) *The set $\{ D_\mu : \mu \in \Lambda_n \}$ is linearly independent in $\mathsf{Sol}(\mathfrak{S}_n)$.*

b) *Suppose that μ and ν are composition of n. Then*

$$D_\mu D_\nu = \sum_{d \in \mathscr{D}_{\mu\nu}} D_{\mu \cap d\nu}.$$

By the remarks before Lemma 5.1, part (b) is equivalent to the following formula:

$$D_\mu D_\nu = \sum_{\sigma \in \Lambda_n} d_{\mu\nu\sigma} D_\sigma,$$

where $d_{\mu\nu\sigma} = \#\{ d \in \mathscr{D}_{\mu\nu} : \mathfrak{S}_\sigma = \mathfrak{S}_\mu \cap d^{-1}\mathfrak{S}_\nu d \}$. In fact, Solomon proved an analogous result for an arbitrary finite Coxeter group W, where the Young subgroups $\mathfrak{S}_\mu$ are replaced with the parabolic subgroups of W and D_μ by the sum of the 'distinguished' (left) coset representatives which are of minimal length in their coset.

As we now recall, part (a) of Solomon's theorem is easy to prove. Recall that $S = \{s_1, \dots, s_{n-1}\}$. For each composition $\mu \in \Lambda_n$ let $S_\nu = \Pi_{-\nu}$, so that $S_\nu \subseteq S$. Now define $Y_\mu \in R\mathfrak{S}_n \subset RG_{r,n}$ by

$$Y_\mu = \sum_{\substack{w \in \mathfrak{S}_n \\ \mathsf{Des}(w) = S_\mu}} w.$$

By (6.4) the descent sets partition $\mathfrak{S}_n$, so the set $\{ Y_\mu : \mu \in \Lambda_n \}$ is linearly independent in $R\mathfrak{S}_n$. By (6.4) again, we can write

$$D_\mu = \sum_{\substack{\nu \in \Lambda_n \\ S_\nu \subseteq S - S_\mu}} Y_\nu.$$

Hence, $\{ D_\mu : \mu \in \Lambda_n \}$ is a linearly independent subset of $R\mathfrak{S}_n$, as claimed.

We build upon this idea to prove that the E_μ's are linearly independent.

6.5. **Proposition.** *The set $\{ E_\mu : \mu \in \Lambda_n^\pm \}$ is linearly independent in $\mathsf{Sol}(G_{r,n})$.*

Proof. Suppose that there exist scalars $a_\mu \in R$ such that

$$\sum_{\mu\in\Lambda_n^\pm} a_\mu E_\mu = 0.$$

By Lemma 6.3, $E_\mu = F_{-\mu}D_{\mu^+}$. Therefore, the last displayed equation becomes

$$\begin{aligned} 0 &= \sum_{\mu\in\Lambda_n^\pm} a_\mu F_{-\mu}D_{\mu^+} = \sum_{\mu\in\Lambda_n^\pm} a_\mu F_{-\mu} \sum_{\substack{\nu\in\Lambda_n\\ S_\nu\subseteq S-S_{\mu^+}}} Y_\nu \\ &= \sum_{\nu\in\Lambda_n} \Big(\sum_{\substack{\mu\in\Lambda_n^\pm\\ S_{\mu^+}\subseteq S-S_\nu}} a_\mu F_{-\mu} \Big) Y_\nu \end{aligned}$$

Now, $RG_{r,n} = \bigoplus_{t\in T} tR\mathfrak{S}_n$, as an R–module, and $\{ Y_\nu : \nu \in \Lambda_n \}$ is a linearly independent subset of $R\mathfrak{S}_n$. Therefore, for any composition $\nu \in \Lambda_n$ we must have

$$0 = \sum_{\substack{\mu\in\Lambda_n^\pm\\ S_{\mu^+}\subseteq S-S_\nu}} a_\mu F_{-\mu}. \tag{6.6}$$

We use this equation to argue by induction on ν to show that $a_\mu = 0$ for all $\mu \in \Lambda_n^\pm$.

First suppose that $\nu = (n)$. Then $S_\nu = S$ and the summation in (6.6) becomes a sum over those signed compositions μ with $S_{\mu^+} = \emptyset$. Hence, $\mu^+ = (1^n)$ and (6.6) becomes

$$0 = \sum_{\substack{\mu\in\Lambda_n^\pm\\ \mu^+=(1^n)}} a_\mu F_{-\mu} = \sum_{\substack{\mu\in\Lambda_n^\pm\\ \Pi_\mu\subseteq\{t_1,\dots,t_n\}}} a_\mu F_{-\mu}.$$

Each monomial $t_{i_1}\dots t_{i_k}$, where $1 \le i_1 < \dots < i_k \le n$, occurs in a unique $F_{-\mu}$ when $\mu^+ = (1^n)$. Hence, $a_\mu = 0$ for all $\mu \in \Lambda_n^\pm$ with $\mu^+ = (1^n)$, as claimed.

Now suppose that $\nu \ne (n)$. By induction we may assume that $a_\mu = 0$ whenever $S_{\mu^+} \subsetneq S - S_\nu$. Therefore, by (6.6) we have

$$0 = \sum_{\substack{\mu\in\Lambda_n^\pm\\ S_{\mu^+}=S-S_\nu}} a_\mu F_{-\mu} = \sum_{\substack{\mu\in\Lambda_n^\pm\\ \Pi_\mu-(S-S_\nu)\subseteq\{t_1,\dots,t_n\}}} a_\mu F_{-\mu}.$$

So, by exactly the same argument as before, $a_\mu = 0$ whenever $\mu^+ = \nu$. Hence, $a_\mu = 0$, for all $\mu \in \Lambda_n^\pm$, and $\{ E_\mu : \mu \in \Lambda_n^\pm \}$ is linearly independent as required. □

The next result that we need amounts to a proof of part (b) of Solomon's theorem (6.4). Once again, we state the result only for the symmetric group even though it is valid for an arbitrary finite Coxeter group. All of the results quoted in (6.7) follow easily from the fact that $\mathfrak{S}_{\mu d\cap\nu} = d^{-1}\mathfrak{S}_\mu d \cap \mathfrak{S}_\nu$, for $d \in \mathscr{D}_{\mu\nu}$.

If $\mu, \nu \in \Lambda_n$ and $\mathfrak{S}_\nu \subseteq \mathfrak{S}_\mu$ then we write $\nu \subseteq \mu$ and set $\mathscr{D}_\nu^\mu = \mathscr{D}_\nu \cap \mathfrak{S}_\mu$. It is easy to check that $\mathscr{D}_\nu^\mu$ is a complete set of coset representatives for $\mathfrak{S}_\nu$ in $\mathfrak{S}_\mu$.

6.7 (Bergeron, Bergeron, Howlett and Taylor [5, Lemmas 2.2 and 2.4])**.** *Suppose that μ and ν are compositions of n. Then*

a) *If $\sigma \subseteq \nu$ then $\mathscr{D}_\sigma = \mathscr{D}_\sigma^\nu \mathscr{D}_\nu$.*
b) *$\mathscr{D}_\mu = \coprod_{d\in\mathscr{D}_{\mu\nu}} d\mathscr{D}_{\mu d\cap\nu}^\nu$.*
c) *If $d \in \mathscr{D}_\mu$ and μd is a composition of n (that is, $d^{-1}\mathfrak{S}_\mu d = \mathfrak{S}_\sigma$ for some $\sigma \in \Lambda_n$), then $\mathscr{D}_\mu = d\mathscr{D}_{\mu d}$.*

We can now establish one of the main results of this paper.

6.8. **Theorem.** *Suppose that $r > 1$ and that μ and ν are signed compositions of n. Then*

$$E_\mu E_\nu = \sum_{d \in \mathscr{D}_{\mu^+\nu^+}} |T_{-\mu \cap d(-\nu)}|\, E_{\mu \cap d\nu}.$$

Proof. We use most of the results in this section to compute $E_\mu E_\nu$:

$$\begin{aligned}
E_\mu E_\nu &= F_{-\mu} D_{\mu^+} F_{-\nu} D_{\nu^+}, && \text{by Lemma 6.3(a)},\\
&= \sum_{d \in \mathscr{D}_{\mu^+\nu^+}} F_{-\mu} d D^{\nu^+}_{\mu^+ d \cap \nu^+} F_{-\nu} D_{\nu^+}, && \text{by Lemma 6.7(b)},\\
&= \sum_{d \in \mathscr{D}_{\mu^+\nu^+}} F_{-\mu} d F_{-\nu} D^{\nu^+}_{\mu^+ d \cap \nu^+} D_{\nu^+}, && \text{by Lemma 6.3(b)},\\
&= \sum_{d \in \mathscr{D}_{\mu^+\nu^+}} F_{-\mu} F_{-\nu}^{d^{-1}} d D_{\mu^+ d \cap \nu^+}, && \text{by Lemma 6.7(b)},\\
&= \sum_{d \in \mathscr{D}_{\mu^+\nu^+}} F_{-\mu} F_{-\nu}^{d^{-1}} D_{\mu^+ \cap d\nu^+}, && \text{by Lemma 6.7(c)}.
\end{aligned}$$

Fix $d \in \mathscr{D}_{\mu^+\nu^+}$ and consider $F_{-\mu} F_{-\nu}^{d^{-1}}$. Now $F_i^2 = rF_i = |T_i| F_i$, for $1 \le i \le n$. So,

$$F_{-\mu} F_{-\nu}^{d^{-1}} = |T_{-\mu} \cap d T_{-\nu} d^{-1}| \prod_{t_i \in T_{-\mu} \cap d T_{-\nu} d^{-1}} F_i.$$

First, $T_{-\mu} \cap d T_{-\nu} d^{-1} = T_{-\mu \cap d(-\nu)}$ since $d \in \mathscr{D}_{\mu^+\nu^+} = \mathscr{D}_{(-\mu)^+(-\nu)^+}$. Next, the subgroup of T generated by $T_{-\mu}$ and $d T_{-\nu} d^{-1}$ is $T_{-(\mu \cap d\nu)}$ since $t_i \in T_{-(\mu\cap d\nu)} = T/T_{\mu \cap d\nu}$ if and only if $t_i \notin T_\mu$ and $t_i \notin d T_\nu d^{-1}$. Therefore, $F_{-\mu} F_{-\nu}^{d^{-1}} = |T_{-\mu \cap d(-\nu)}| F_{-(\mu \cap d\nu)}$. Hence, using Lemma 6.3 once more,

$$E_\mu E_\nu = \sum_{d \in \mathscr{D}_{\mu^+\nu^+}} |T_{-\mu \cap d(-\nu)}|\, F_{-(\mu\cap d\nu)} D_{(\mu \cap d\nu)^+} = \sum_{d \in \mathscr{D}_{\mu^+\nu^+}} |T_{-\mu \cap d(-\nu)}|\, E_{\mu \cap d\nu},$$

as required. □

Corollary 5.8 gives a combinatorial interpretation of the structure constants $|T_{-\mu \cap d(-\nu)}|$ of $\mathsf{Sol}(G_{r,n})$. This shows that Theorem 6.8 is a direct generalization of (6.4)(b). A second combinatorial interpretation of the structure constants $|T_{-\mu \cap d(-\nu)}|$ is given in Proposition 10.3 below.

Combining Theorem 6.8 and Proposition 6.5 we obtain the following.

6.9. **Corollary.** *Suppose that $r > 1$. The cyclotomic Solomon algebra $\mathsf{Sol}(G_{r,n})$ is a subalgebra of $RG_{r,n}$ which is free as an R–module of rank $2 \cdot 3^{n-1}$.*

6.10. **Example** Suppose that $r > 1$. Then, by Example 5.9, we have

$$E_{(3,-2)} E_{(-2^2,1)} = 2r^2 E_{(-2,1,-2)} + r E_{(-2,-1^3)} + r E_{(-1,-2,-1^2)} + r^2 E_{(-1^2,1,-1^2)}.$$

See Example 10.4 for a second way of computing this product using Proposition 10.3.

Notice that by (6.4) and Theorem 6.8 we can recover the multiplication in $\mathsf{Sol}(\mathfrak{S}_n)$ by setting $r = 1$ and identifying μ and μ^+, for all $\mu \in \Lambda_n^\pm$, so that

$$D_{(3,2)} D_{(2^2,1)} = 2D_{(2,1,2)} + D_{(2,1^3)} + D_{(1,2,1^2)} + D_{(1^5)}.$$

◊

7. The generic cyclotomic Solomon algebra

By Theorem 6.8, if $r > 1$ then the structure constants of the algebra $\mathsf{Sol}(G_{r,n})$ are polynomials in r. Consequently, these algebras admit a simultaneous deformation.

Recall that if $r \geq 2$ and $\mu, \nu \in \Lambda_n^{\pm}$ then $|T_{-\mu \cap d(-\nu)}| = r^{\mathsf{wt}(\mathsf{T}^d)}$ by Corollary 5.8.

Let x be an indeterminate over $\mathbb{Z}$ and suppose that $\mu, \nu, \sigma \in \Lambda_n^{\pm}$. Define polynomials $d_{\mu\nu\sigma}(x) \in \mathbb{N}[x]$ by

$$d_{\mu\nu\sigma}(x) = \sum_{\substack{d \in \mathscr{D}_{\mu^+\nu^+} \\ \sigma = \mu \cap d\nu}} x^{\mathsf{wt}(\mathsf{T}^d)}.$$

We abuse notation and consider $d_{\mu\nu\sigma}(x)$ to be a polynomial over R. For $q \in R$ we let $d_{\mu\nu\sigma}(q)$ be the evaluation of this polynomial at q. Then, by Theorem 6.8,

$$E_\mu E_\nu = \sum_{\sigma \in \Lambda_n^{\pm}} d_{\mu\nu\sigma}(r) E_\sigma.$$

7.1. **Definition.** *Suppose that $n \geq 1$ and that R is a commutative ring. The **cyclotomic Solomon algebra with parameter** $q \in R$ is the R–algebra $\mathsf{Sol}_q(n) = \mathsf{Sol}_{R,q}(n)$ with generating set $\{ E_\mu : \mu \in \Lambda_n^{\pm} \}$ and relations*

$$E_\mu E_\nu = \sum_{\sigma \in \Lambda_n^{\pm}} d_{\mu\nu\sigma}(q) E_\sigma,$$

*for $\mu, \nu \in \Lambda_n^{\pm}$. The **generic cyclotomic Solomon algebra** is the $\mathbb{Z}[x]$–algebra $\mathsf{Sol}_x(n)$, where x is an indeterminate over $\mathbb{Z}$.*

We are abusing notation slightly in Definition 7.1 because from here onwards E_μ is a generator of $\mathsf{Sol}_q(n)$ and not necessarily the element defined in the previous section. This abuse is justified by the following result.

7.2. **Corollary.** *Suppose that $q = r \cdot 1_R$, where $r > 1$. Then $Sol_q(n)$ and $\mathsf{Sol}(G_{r,n})$ are canonically isomorphic R–algebras where the isomorphism $\mathsf{Sol}_q(n) \to \mathsf{Sol}(G_{r,n})$ is given by $E_\mu \mapsto E_\mu$, for $\mu \in \Lambda_n^{\pm}$.*

Proof. By Theorem 6.8 there is a natural surjection $\mathsf{Sol}_q(n) \longrightarrow \mathsf{Sol}(G_{r,n})$. By Corollary 6.9 this map is an isomorphism. □

The explicit description of the algebra $\mathsf{Sol}(G_{r,n})$ as a subalgebra of the group algebra $RG_{r,n}$ makes the algebra $\mathsf{Sol}(G_{r,n})$ slightly easier to work with than the more general algebras $\mathsf{Sol}_q(n)$. For example, we know that $E_\mu = F_{-\mu} D_{\mu^+}$ in $RG_{r,n}$ but we have no such factorization in general. As we will soon see, however, almost all of the properties of the algebras $\mathsf{Sol}(G_{r,n})$ hold for the algebras $\mathsf{Sol}_q(n)$.

7.3. **Proposition.** *Suppose that $n \geq 1$ and that $q \in R$. Then*

a) *$\mathsf{Sol}_q(n)$ is free as an R–module with basis $\{ E_\mu : \mu \in \Lambda_n^{\pm} \}$. In particular, $\mathsf{Sol}_q(n)$ has rank $2 \cdot 3^{n-1}$.*
b) *$\mathsf{Sol}_q(n) \cong \mathsf{Sol}_x(n) \otimes_{\mathbb{Z}[x]} R$, where R is considered as a $\mathbb{Z}[x]$–module by letting x act on R as multiplication by q (and $1 \in \mathbb{Z}$ acts as multiplication by 1_R).*
c) *$\mathsf{Sol}_q(n)$ is a unital associative R–algebra with multiplicative identity $E_{(n)}$.*

Proof. First consider the generic Solomon algebra over $\mathbb{Z}[x]$. Suppose that

$$\sum_{\mu \in \Lambda_n^{\pm}} f_\mu(x) E_\mu = 0,$$

for some $f_\mu(x) \in R[x]$. Then $f_\mu(r) = 0$, for $r = 2, 3, 4 \ldots$ and all $\mu \in \Lambda_n^{\pm}$, by Corollary 7.2 and Proposition 6.5. As non–zero polynomials have only finitely many roots, we conclude that $f_\mu(x) = 0$, for all $\mu \in \Lambda_n^{\pm}$. Consequently, $\mathsf{Sol}_x(n)$ is free as a $\mathbb{Z}[x]$–module with basis $\{ E_\mu : \mu \in \Lambda_n^{\pm} \}$.

Now fix $q \in R$ and consider R as a $\mathbb{Z}[x]$–module by letting x act on R as multiplication by q (and $1 \in \mathbb{Z}$ act as multiplication by 1_R). Then the R–algebra $\mathsf{Sol}_x(n) \otimes_{\mathbb{Z}[x]} R$ is free as an R–module with basis $\{ E_\mu \otimes 1 : \mu \in \Lambda_n^\pm \}$ and it satisfies the relations of $\mathsf{Sol}_q(n)$. As $\mathsf{Sol}_q(n)$ is spanned by the elements $\{ E_\mu : \mu \in \Lambda_n^\pm \} \subseteq \mathsf{Sol}_q(n)$ it follows that $\mathsf{Sol}_q(n) \cong \mathsf{Sol}_x(n) \otimes_{\mathbb{Z}[x]} R$. This proves (a) and (b).

To prove (c) it is now enough to prove the corresponding statements for the generic Solomon algebra $\mathsf{Sol}_x(n)$. We first show that $E_{(n)}$ is the identity element of $\mathsf{Sol}_x(n)$. This is equivalent to the polynomial identities

$$d_{\mu(-n)\alpha}(x) = \delta_{\mu\alpha} = d_{\alpha(-n)\mu}(x),$$

for all $\mu, \alpha \in \Lambda_n^\pm$. All of these identities follow directly from the definitions because $T_{(-n)\cap d(-\nu)} = 1 = T_{-\mu\cap d'(-n)}$, for all $\mu, \nu \in \Lambda_n^\pm$, $d \in \mathscr{D}_{(n)\mu^+}$ and $d' \in \mathscr{D}_{\nu^+(n)}$. Similarly, the associativity of $\mathsf{Sol}_x(n)$ is equivalent to the polynomial identities

$$\sum_{\alpha,\beta\in\Lambda_n^\pm} d_{\mu\nu\alpha}(x) d_{\alpha\sigma\beta}(x) = \sum_{\alpha,\beta\in\Lambda_n^\pm} d_{\mu\alpha\beta}(x) d_{\nu\sigma\alpha}(x),$$

for all $\mu, \nu, \sigma \in \Lambda_n^\pm$. As in the first paragraph of the proof, by Corollary 7.2 these identities hold when $x = 2, 3, \ldots$ since the algebras $\mathsf{Sol}(G_{r,n})$ are associative for $r \geq 2$. As these identities hold for infinitely many values of x, they lift to the required polynomial identities. □

Part (b) of the Proposition justifies our calling the $\mathbb{Z}[x]$–algebra $\mathsf{Sol}_x(n)$ the *generic* cyclotomic Solomon algebra.

As we next describe, the algebras $\mathsf{Sol}_q(n)$ have many interesting subalgebras.

7.4. **Lemma.** *Suppose that $d_{\mu\nu\sigma}(q) \neq 0$, for $\mu, \nu, \sigma \in \Lambda_n^\pm$. Then $\Pi_\sigma = \Pi_\mu \cap d\Pi_\nu d^{-1}$, for some $d \in \mathscr{D}_{\mu^+\nu^+}$*

Proof. By definition, the polynomial $d_{\mu\nu\sigma}(x)$ is non–zero only if $G_\sigma = G_\mu \cap dG_\nu d^{-1}$ for some $d \in \mathscr{D}_{\mu^+\nu^+}$. Consequently, if $d_{\mu\nu\sigma}(q) \neq 0$ then $\Pi_\sigma = \Pi_\mu \cap d\Pi_\nu d^{-1}$, for some $d \in \mathscr{D}_{\mu^+\nu^+}$. □

Notice, in particular, that this implies that the poset structure on $\Lambda_n^\pm$ given by defining $\mu \preceq \nu$ whenever $\Pi_\nu \subseteq \Pi_\mu$ is compatible with the ideal structure of $\mathsf{Sol}_q(n)$.

7.5. **Proposition.** *Suppose that $n \geq 1$ and that $q \in R$. Then $\mathsf{Sol}_q(n)$ has a filtration by two–sided ideals*

$$\mathsf{Sol}_q(n) = \mathscr{S}_0 \supset \cdots \supset \mathscr{S}_n \supset 0$$

where $\mathscr{S}_i$ is the R–submodule of $\mathsf{Sol}_q(n)$ with basis $\{ E_\mu : \mu \in \Lambda_n^\pm$ such that $|\mu|^+ \geq i \}$, for $i = 0, \ldots, n$.

Proof. By Lemma 7.4, $d_{\mu\nu\sigma}(q) \neq 0$ only if $\Pi_\sigma = \Pi_\mu \cap d\Pi_\nu d^{-1}$, for some $d \in \mathscr{D}_{\mu^+\nu^+}$. Consulting the definitions, $|\mu|^+ = |\Pi_\mu \cap T|$. Therefore, $d_{\mu\nu\sigma}(q) \neq 0$ only if $|\sigma|^+ \leq \min\{|\mu|^+, |\nu|^+\}$. Hence, $\mathscr{S}_i$ is a two–sided ideal of $\mathsf{Sol}(G_{r,n})$, for $0 \leq i \leq n$, and the Proposition follows. □

7.6. **Proposition.** *Suppose that $n \geq 1$ and that $q \in R$. Let*

$$\mathsf{Sol}_q^+(n) = \sum_{\mu\in\Lambda_n} RE_\mu$$

$$\mathsf{Sol}_q^\pm(n) = \sum_{\pm\mu\in\Lambda_n} RE_\mu$$

$$\mathsf{Sol}_q^1(n) = \sum_{\substack{\mu\in\Lambda_n^\pm \\ \mu_i>0 \text{ for } i>1}} RE_\mu.$$

Then $\mathsf{Sol}_q^+(n)$, $\mathsf{Sol}_q^\pm(n)$ *and* $\mathsf{Sol}_q^1(n)$ *are all subalgebras of* $\mathsf{Sol}_q(n)$. *Moreover,* $\mathsf{Sol}_q^+(n)$ *is naturally isomorphic to* $\mathsf{Sol}(\mathfrak{S}_n)$ *via the* R*–linear map* $E_\mu \mapsto D_\mu$, *for* $\mu \in \Lambda_n$.

Proof. All of these results can be proved directly using the definition of the polynomials $d_{\mu\nu\sigma}(x)$, for $\mu, \nu, \sigma \in \Lambda_n^\pm$. Note that $\mathsf{Sol}_q^+(n) = \mathscr{S}_n$ in the notation of Proposition 7.5, so in this case the result is already known. The isomorphism $\mathsf{Sol}_q^+(n) \cong \mathsf{Sol}(\mathfrak{S}_n)$ is trivial because if $\mu \in \Lambda_n$ then $T_{-\mu} = 1$, so that $E_\mu = D_\mu$ by Lemma 6.3. □

8. The representation theory of $\mathsf{Sol}_q(n)$

In this section we construct all of the irreducible representations of the algebras $\mathsf{Sol}_q(n)$ over an arbitrary field. Even though $\mathsf{Sol}_q(n)$ is, in general, not commutative, it turns out that every irreducible $\mathsf{Sol}_q(n)$–module is one dimensional — so that $\mathsf{Sol}_q(n)$ is a basic algebra for all n and q. As an application of these results we give a basis for the radical of $\mathsf{Sol}_q(n)$ when R is an arbitrary field.

Let $\sim$ be the equivalence relation on the set of signed compositions where two signed compositions are $\sim$–equivalent if one can be obtained by reordering the parts of the other. More explicitly, if $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_l)$ then $\lambda \sim \mu$ if and only if $k = l$ and $\lambda_i = \mu_{i^v}$, for some $v \in \mathfrak{S}_k$.

8.1. **Lemma.** *Suppose that* $\lambda, \mu \in \Lambda_n^\pm$. *Then the following are equivalent:*

a) $\lambda \sim \mu$;
b) $G_\lambda = w^{-1} G_\mu w$, *for some* $w \in \mathfrak{S}_n$;
c) $G_\lambda = g^{-1} G_\mu g$, *for some* $g \in G_{r,n}$.

Proof. We leave the proof for the reader. □

8.2. **Lemma.** *Suppose that* $\lambda, \mu \in \Lambda_n^\pm$. *Then*

a) *If* $\mu \not\sim \lambda$ *then* $d_{\mu\lambda\mu}(q) \neq 0$ *only if* $|\Pi_\lambda| > |\Pi_\mu|$.
b) *If* $\mu \sim \lambda$ *then* $d_{\mu\alpha\mu}(q) = d_{\lambda\alpha\lambda}(q)$, *for all* $\alpha \in \Lambda_n^\pm$.

Proof. By Lemma 7.4 $d_{\mu\lambda\mu}(q) \neq 0$ only if $\Pi_\mu = \Pi_\mu \cap d\Pi_\lambda d^{-1}$, for some $d \in \mathscr{D}_{\mu^+\lambda^+}$. Hence, part (a) follows since $\lambda \not\sim \mu$.

Consulting the definition of the polynomials $d_{\mu\nu\sigma}(x)$, to prove (b) it is enough to show that if $r \geq 2$ then in the group $G_{r,n}$ we have

$$(\dagger) \qquad \sum_{\substack{d \in \mathscr{D}_{\lambda^+\alpha^+} \\ \lambda = \lambda \cap d\alpha}} |T_{-\lambda \cap d(-\alpha)}| = \sum_{\substack{d \in \mathscr{D}_{\mu^+\alpha^+} \\ \mu = \mu \cap d\alpha}} |T_{-\mu \cap d(-\alpha)}|.$$

We prove this by showing that the 'obvious' bijection $\mathscr{D}_{\lambda^+\alpha^+} \longrightarrow \mathscr{D}_{\mu^+\alpha^+}$ preserves each of the summands in this equation.

First note that by Lemma 8.1 we can find an element $w \in \mathfrak{S}_n$ such that $G_\lambda = w^{-1} G_\mu w$ since $\lambda \sim \mu$. That is, $T_\lambda \mathfrak{S}_{\lambda^+} = w^{-1} T_\mu w \cdot w^{-1} \mathfrak{S}_{\mu^+} w$, so that $T_\lambda = w^{-1} T_\mu w$ and $\mathfrak{S}_{\lambda^+} = w^{-1} \mathfrak{S}_{\mu^+} w$. Consequently, the map $\mathfrak{S}_{\lambda^+} \backslash \mathfrak{S}_n / \mathfrak{S}_{\alpha^+} \to \mathfrak{S}_{\mu^+} \backslash \mathfrak{S}_n / \mathfrak{S}_{\alpha^+}$ given by $C \mapsto wC$ defines a bijection since if $d \in \mathscr{D}_{\lambda^+\alpha^+}$ then $\mathfrak{S}_{\lambda^+} d \mathfrak{S}_{\alpha^+} = w^{-1} \mathfrak{S}_{\mu^+} w d \mathfrak{S}_{\alpha^+}$. Let $d \mapsto d'$ be map from $\mathscr{D}_{\lambda^+\alpha^+}$ to $\mathscr{D}_{\mu^+\alpha^+}$ determined by $\mathfrak{S}_{\mu^+} w d \mathfrak{S}_{\alpha^+} = \mathfrak{S}_{\mu^+} d' \mathfrak{S}_{\alpha^+}$.

Now fix $d \in \mathscr{D}_{\lambda^+\alpha^+}$ such that $\lambda = \lambda \cap d\alpha$. Then

$$\begin{aligned} T_{-\lambda \cap d(-\alpha)} &= T_{-\lambda} \cap d T_{-\alpha} d^{-1} \\ &= w^{-1} T_{-\mu} w \cap d T_{-\alpha} d^{-1} \\ &= w^{-1} \Big(T_{-\mu} \cap w d T_{-\alpha} (wd)^{-1} \Big) w. \end{aligned}$$

Write $wd = ud'v$, for $u \in \mathfrak{S}_{\mu^+}$ and $v \in \mathfrak{S}_{\alpha^+}$. Then we have

$$\begin{aligned} T_{-\lambda\cap d(-\alpha)} &= w^{-1}\Big(T_{-\mu} \cap (ud'v)T_{-\alpha}v^{-1}(d')^{-1}u^{-1}\Big)w, \\ &= w^{-1}u\Big(T_{-\mu} \cap d'T_{-\alpha}(d')^{-1}\Big)u^{-1}w, \\ &= w^{-1}u\Big(T_{-\mu} \cap T_{d'(-\alpha)}\Big)u^{-1}w, \end{aligned}$$

where the second equality follows because $\mathfrak{S}_\mu$ normalizes $T_{-\mu}$ and the last equality follows because $\mathfrak{S}_\alpha$ normalizes $T_{-\alpha}$. Hence, we have shown that $|T_{-\lambda\cap d(-\alpha)}| = |T_{-\mu\cap d'(-\alpha)}|$, for all $d \in \mathscr{D}_{\lambda^+\alpha^+}$. This establishes (†), so the Lemma is proved. □

8.3. **Theorem.** *Suppose that R is a field, $q \in R$ and $n \geq 0$.*

a) *If $\lambda \in \Lambda_n^\pm$ then $\mathsf{Sol}_q(n)$ has a unique one dimensional representation $I(\lambda)$ upon which E_α acts as multiplication by $d_{\lambda\alpha\lambda}(q)$, for $\alpha \in \Lambda_n^\pm$.*
b) *Every irreducible representation of $\mathsf{Sol}_q(n)$ is isomorphic to $I(\lambda)$, for some $\lambda \in \Lambda_n^\pm$.*
c) *If $\lambda \sim \mu$ then $I(\lambda) \cong I(\mu)$.*

Proof. Choose a total order $\geq$ on $\Lambda_n^\pm$ such that $|\Pi_\lambda| \geq |\Pi_\mu|$ whenever $\lambda > \mu$, for $\lambda, \mu \in \Lambda_n^\pm$. Let $\mathscr{S}_\lambda$ be the R–submodule of $\mathsf{Sol}_q(n)$ with basis $\{ E_\mu : \lambda \geq \mu \in \Lambda_n^\pm \}$ and let $\mathscr{S}'_\lambda$ be the R–submodule with basis $\{ E_\mu : \lambda > \mu \in \Lambda_n^\pm \}$. Then $\mathscr{S}_\lambda$ and $\mathscr{S}'_\lambda$ are both right $\mathsf{Sol}_q(n)$–modules by Lemma 7.4. Hence the quotient module $I(\lambda) = \mathscr{S}_\lambda/\mathscr{S}'_\lambda = R(E_\lambda + \mathscr{S}'_\lambda)$ is one dimensional $\mathsf{Sol}_q(n)$–module. By definition, if $\alpha \in \Lambda_n^\pm$ then E_α acts on $I(\lambda)$ as multiplication by $d_{\lambda\alpha\lambda}(q)$. Hence, $I(\lambda)$ is the one dimensional $\mathsf{Sol}_q(n)$–module described in part (a).

Now suppose that $\Lambda_n^\pm = \{\lambda_1 > \lambda_1 > \cdots > \lambda_N\}$, where $N = 2 \cdot 3^{n-1} = \dim \mathsf{Sol}_q(n)$. Then

$$\mathsf{Sol}_q(n) = \mathscr{S}_{\lambda_1} \supset \mathscr{S}_{\lambda_2} \supset \cdots \supset \mathscr{S}_{\lambda_N} \supset 0$$

is a filtration of $\mathsf{Sol}_q(n)$ by two–sided ideals with quotients $\mathscr{S}_{\lambda_i}/\mathscr{S}_{\lambda_{i+1}} \cong I(\lambda_i)$, since $\mathscr{S}_{\lambda_{i+1}} = \mathscr{S}'_{\lambda_i}$. As every irreducible $\mathsf{Sol}_q(n)$–module arises as a composition factor of $\mathsf{Sol}_q(n)$ part (b) now follows.

Finally, if $\lambda \sim \mu$ then $I(\lambda) \cong I(\mu)$ by Lemma 8.2(b). Hence, part (c) holds. □

8.4. **Corollary.** *Every field is a splitting field for $\mathsf{Sol}_q(n)$.*

Proof. Suppose that D is an irreducible $\mathsf{Sol}_q(n)$–module. Then D is one dimensional by the Proposition, and hence absolutely irreducible. □

If A is an algebra over a field then let $\operatorname{Rad} A$ be its **radical**. Thus, $\operatorname{Rad} A$ is the unique maximal nilpotent ideal of A and A is semisimple if and only if $\operatorname{Rad} A = 0$. Recall that $a \in A$ is **nilpotent** if $a^k = 0$, for some $k > 0$, whereas an ideal I of A is **nilpotent** if $I^k = 0$ for some $k > 0$.

8.5. **Corollary.** *Suppose that R is a field. Then $\operatorname{Rad} \mathsf{Sol}_q(n)$ is the set of nilpotent elements in $\mathsf{Sol}_q(n)$.*

Proof. By definition every element of $\operatorname{Rad} \mathsf{Sol}_q(n)$ is nilpotent. To prove the converse let M be the number of irreducible $\mathsf{Sol}_q(n)$–modules. By Theorem 8.3 every irreducible $\mathsf{Sol}_q(n)$–module is one dimensional. Therefore, $\mathsf{Sol}_q(n)/\operatorname{Rad} \mathsf{Sol}_q(n) \cong R^M$ by the Wedderburn Theorem. In particular, $\mathsf{Sol}_q(n)/\operatorname{Rad} \mathsf{Sol}_q(n)$ contains no nilpotent elements, so the result follows. □

8.6. **Corollary.** *Suppose that R is a field and $q \in R$. Then $\mathsf{Sol}_q(n)$ is semisimple if and only if $n = 1$ and $q \neq 0$.*

Proof. If $n \geq 2$ then $\mathsf{Sol}_q(n)$ is not semisimple because there exist distinct signed compositions $\lambda, \nu \in \Lambda_n^{\pm}$ such that $\lambda \sim \mu$. Therefore, $E_\lambda - E_\mu \in \operatorname{Rad} \mathsf{Sol}_q(n)$, so that $\operatorname{Rad} \mathsf{Sol}_q(n) \neq 0$. If $n = 1$ then a quick calculation verifies that $I(1) \cong I(-1)$ if and only if $q = 0$ which implies the result. □

Each $\sim$–equivalence class of $\Lambda_n^{\pm}$ contains a unique signed composition $\mu = (\mu_1, \dots, \mu_k)$ such that $\mu_1 \geq \cdots \geq \mu_k$. If $\mu \in \Lambda_n^{\pm}$ and $\mu_1 \geq \cdots \geq \mu_k$ then we call μ a **signed partition** of n. Let $\Lambda_n^{\oplus}$ be the set of all signed partitions of n. By the remarks above, the $\mathfrak{S}_n$–conjugacy classes of reflection subgroups of $G_{r,n}$ are indexed by the signed partitions of n. We note that $\Lambda_n^{\oplus}$ is naturally in bijection with the set of bipartitions of n, however, for us the signed partitions are more natural because we have already defined a reflection subgroup G_λ for each $\lambda \in \Lambda_n^{\oplus}$.

8.7. **Theorem.** *Suppose that R is a field of characteristic zero and that $q \neq 0$. Then*

$$\{\, I(\lambda) : \lambda \in \Lambda_n^{\oplus} \,\}$$

is a complete set of pairwise non–isomorphic irreducible $\mathsf{Sol}_q(n)$–modules.

Proof. As the $\sim$–equivalence classes of $\Lambda_n^{\pm}$ are indexed by the signed partitions of n, $\{\, I(\lambda) : \lambda \in \Lambda_n^{\oplus} \,\}$ is a complete set of irreducible $\mathsf{Sol}_q(n)$–modules by parts (b) and (c) of Theorem 8.3. It remains then to show that if $\lambda, \mu \in \Lambda_n^{\oplus}$ then $I(\lambda) \ncong I(\mu)$ if $\lambda \neq \mu$. Now, R is a field of characteristic zero and $q \neq 0$, so $d_{\lambda\nu\lambda}(q) \neq 0$ if and only if $d_{\lambda\nu\lambda}(x) \neq 0$, for $\lambda, \nu \in \Lambda_n^{\pm}$. However, $d_{\lambda\lambda\lambda}(x) \in 1 + x\mathbb{N}[x]$ since $1 \in \mathscr{D}_{\lambda^+\lambda^+}$ and $\Pi_\lambda = \Pi_\lambda \cap 1 \cdot \Pi_\lambda \cdot 1^{-1}$. Therefore, $d_{\lambda\lambda\lambda}(q) \neq 0$ and so, using Lemma 8.2(a) again, if $\lambda \neq \mu$ then $I(\lambda) \ncong I(\mu)$. □

8.8. **Corollary.** *Suppose that R is a field of characteristic zero and $q \neq 0$. Then*

$$\{\, E_\lambda - E_\mu : \lambda \in \Lambda_n^{\oplus}, \mu \in \Lambda_n^{\pm}, \lambda \sim \mu \text{ and } \lambda \neq \mu \,\}$$

is a basis of $\operatorname{Rad} \mathsf{Sol}_q(n)$. *Consequently,* $\dim \mathsf{Sol}_q(n)/\operatorname{Rad} \mathsf{Sol}_q(n) = |\Lambda_n^{\oplus}|$.

Proof. Suppose that $\lambda \sim \mu$ where $\lambda \in \Lambda_n^{\oplus}$, $\mu \in \Lambda_n^{\pm}$ and $\lambda \neq \mu$. Then, by Theorem 8.7 and Lemma 8.2, $E_\lambda - E_\mu$ acts as multiplication by zero on every irreducible $\mathsf{Sol}_q(n)$–module. Therefore, $E_\lambda - E_\mu$ belongs to $\operatorname{Rad} \mathsf{Sol}_q(n)$ whenever $\lambda \cong \mu$. Consequently, $\dim \mathsf{Sol}_q(n)/\operatorname{Rad} \mathsf{Sol}_q(n) \leq |\Lambda_n^{\oplus}|$. However, $\dim \mathsf{Sol}_q(n)/\operatorname{Rad} \mathsf{Sol}_q(n) = |\Lambda_n^{\oplus}|$ by Theorem 8.7, so the result follows. □

Suppose that R is a field of characteristic zero and that $q \neq 0$. Define the **character table** of $\mathsf{Sol}_q(n)$ to be the matrix

$$\mathbf{C}_q(n) = \big(d_{\lambda\mu\lambda}(q)\big)_{\lambda,\mu \in \Lambda_n^{\oplus}}.$$

Then $\mathbf{C}_q(n)$ is the character table of $\mathsf{Sol}_q(n)/\operatorname{Rad} \mathsf{Sol}_q(n)$, by Theorem 8.7, so it completely determines the maximal semisimple quotient of $\mathsf{Sol}_q(n)$. The character table $\mathbf{C}_q(n)$ is explicitly known for all $q \neq 0$ and all $n \geq 1$ since the polynomials $d_{\lambda\mu\sigma}(x)$ are explicitly known for all $\lambda, \mu, \sigma \in \Lambda_n^{\pm}$ by Corollary 5.8.

8.9. **Example** Suppose that R is a field of characteristic zero and that $q = 2 = n$. Then $\mathsf{Sol}_2(2) \cong \mathsf{Sol}(G_{2,2})$ and the character table $\mathbf{C}_2(2)$ of $\mathsf{Sol}_2(2)$ is the following matrix.

	(2)	(1^2)	$(1,-1)$	(-2)	(-1^2)
(2)	1				
(1^2)	1	2			
$(1,-1)$	1	2	2		
(-2)	1	.	.	4	
(-1^2)	1	2	4	4	8

As all of the diagonal entries of $\mathbf{C}_q(2)$ are powers of 2 it follows that if R is any field of characteristic *different from* 2 then $\{ I(\lambda) : \lambda \in \Lambda_n^{\oplus} \}$ is a complete set of pairwise non–isomorphic irreducible $\mathsf{Sol}_q(2)$–modules. If R is a field of characteristic 2 then $I(2)$ is the only irreducible $\mathsf{Sol}_q(2)$–module. This is in agreement with Theorem 8.11 below.

By comparing the character table of $\mathsf{Sol}(G_{2,2})$ with the character table of the group $G_{2,2}$ (the Coxeter group of type B_2) it is easy to see that there cannot be a ring homomorphism from $\mathsf{Sol}(G_{2,2})$ into the character ring of $G_{2,2}$. This is in marked contrast with the Solomon algebras of Coxeter groups for which such a homomorphism always exists. ◊

8.10. *Remark.* As discussed in Remark 4.10, Mak has shown that the cosets of the reflection subgroups of $G_{r,n}$ have a unique element of minimal length with respect to the Bremke–Malle length function ℓ_0 (see Remark 2.4). For each $\mu \in \Lambda_n^{\pm}$ let $\mathscr{E}'_\mu$ be Mak's set of distinguished coset representatives for G_μ and let $E'_\mu = \sum_{e \in \mathscr{E}'_\mu} e \in RG_{r,n}$. Define

$$\Sigma'(G_{r,n}) = \sum_{\mu \in \Lambda_n^{\pm}} RE'_\mu.$$

If $r > 2$ then $\Sigma'(G_{r,n})$ is not, in general, a subalgebra of $RG_{r,n}$. The smallest counter example occurs when $r = n = 3$.

Now suppose that $r = 2$. Then $G_{2,n}$ is a Coxeter group of type B_n and Bonnafé and Hohlweg [9] have shown that $\Sigma'(G_{2,n})$ is a subalgebra of $RG_{2,n}$ and, moreover, that $\Sigma'(G_{2,n})$ is isomorphic to the Mantaci-Reutenauer algebra [23]. Now, the algebras $\mathsf{Sol}(G_{2,n})$ and $\Sigma'(G_{2,n})$ are both free of rank $2 \cdot 3^{n-1}$, so it is natural to ask whether these algebras are isomorphic. In fact, $\mathsf{Sol}(G_{2,n}) \ncong \Sigma'(G_{2,n})$ if $n > 1$. This can be proved by induction on n starting from the following observation. Bonnafé and Hohlweg have shown in [9, Table V] that the following matrix is the character table of the semisimple quotient of $\Sigma'(G_{2,2})$.

	(2)	(1^2)	$(1,-1)$	(-2)	(-1^2)
(2)	1				
(1^2)	1	2			
$(1,-1)$	1	2	2		
(-2)	1	.	.	2	
(-1^2)	1	2	4	4	8

Observe that the $\big((-2),(-2)\big)$–entry in this character table is different to the corresponding entry in the character table of $\mathsf{Sol}(G_{2,2})$ given in Example 8.9. Therefore, $\mathsf{Sol}(G_{2,2})$ and $\Sigma'(G_{2,2})$ are not isomorphic algebras because they have non–isomorphic maximal semisimple quotients.

We close this section by classifying the irreducible $\mathsf{Sol}_q(n)$–modules over an arbitrary field. This classification is a direct generalization of the corresponding results for the descent algebra of the symmetric groups [2] – although our proofs are necessarily different because there is no homomorphism from $\mathsf{Sol}_q(n)$ into the character ring of $G_{r,n}$.

For $\lambda \in \Lambda_n$ let $N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda) = \{ w \in \mathfrak{S}_n : \mathfrak{S}_\lambda = w^{-1}\mathfrak{S}_\lambda w \}$ be the normalizer of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$.

8.11. **Theorem.** *Suppose that R is field, $q \in R$ and $\lambda \in \Lambda_n^{\oplus}$. Then the following are equivalent:*

a) $d_{\lambda\lambda\lambda}(q) = 0$*;*
b) $q^{|\lambda|^-}[N_{\mathfrak{S}_n}(\mathfrak{S}_{\lambda^+}) : \mathfrak{S}_{\lambda^+}] = 0$ *in R;*
c) $E_\lambda \in \operatorname{Rad} \mathsf{Sol}_q(n)$*;*
d) E_λ *is nilpotent; and,*
e) $I(\lambda) \cong I(\mu)$*, for some $\mu \in \Lambda_n^{\oplus}$ with $|\Pi_\mu| > |\Pi_\lambda|$.*

Proof. By definition,

$$d_{\lambda\lambda\lambda}(q) = \sum_{\substack{d\in\mathscr{D}_{\lambda^+\lambda^+}\\ \lambda=\lambda\cap d\lambda}} |T_{-\lambda\cap d(-\lambda)}| = \sum_{\substack{d\in\mathscr{D}_{\lambda^+\lambda^+}\\ \lambda=\lambda\cap d\lambda}} q^{|\lambda|^-} = q^{|\lambda|^-}[N_{\mathfrak{S}_n}(\mathfrak{S}_{\lambda^+}) : \mathfrak{S}_{\lambda^+}],$$

since $|T_{-\lambda}| = q^{|\lambda|^-}$ and $T_{-\lambda\cap d(-\lambda)} = T_{-\lambda}$ if $\lambda = \lambda \cap d\lambda$. Hence, (a) and (b) are equivalent. Further, (c) and (d) are equivalent by Corollary 8.5.

To complete the proof it is enough to show that (a) $\implies$ (c) $\implies$ (e) $\implies$ (a). In order to do this let $\mathsf{Sol}_q(n) = \mathscr{S}_{\lambda_1} \supset \mathscr{S}_{\lambda_2} \supset \cdots \supset \mathscr{S}_{\lambda_N} \supset 0$ be the filtration of $\mathsf{Sol}_q(n)$ by two sided ideals which was constructed in the proof of Theorem 8.3 using a total order $>$ on $\Lambda_n^\pm$. Recall that $|\Pi_\mu| \geq |\Pi_\nu|$ whenever $\mu > \nu$, for $\mu, \nu \in \Lambda_n^\pm$. Then $\mathscr{S}_{\lambda_i}$ is a subalgebra of $\mathsf{Sol}_q(n)$ which is also a quotient of $\mathsf{Sol}_q(n)$ since $\mathscr{S}_{\lambda_i} \cong \mathsf{Sol}_q(n)/\mathscr{S}_{\lambda_{i+1}}$, for $1 \leq i \leq N$. Therefore, by Theorem 8.3, every irreducible $\mathscr{S}_{\lambda_i}$–module is isomorphic to $I(\mu)$ for some $\mu \in \Lambda_n^\oplus$ with $\mu \geq \lambda_i$, for $1 \leq i \leq N$. In particular, every irreducible $\mathscr{S}_\lambda$–module is isomorphic to $I(\mu)$ for some $\mu \geq \lambda$.

We can now return to the proof of the Theorem.

First, suppose (a) holds, so that $d_{\lambda\lambda\lambda}(q) = 0$. By definition, if $\mu \in \Lambda_n^\oplus$ then E_λ acts on $I(\mu)$ as multiplication by $d_{\mu\lambda\mu}(q)$. By Lemma 8.2(a), if $\mu > \lambda$ then E_λ acts on $I(\mu)$ as multiplication by 0, whereas E_λ acts on $I(\lambda)$ as multiplication by 0 since $d_{\lambda\lambda\lambda}(q) = 0$. Therefore, $E_\lambda \in \operatorname{Rad}\mathscr{S}_\lambda$ and (c) holds because $\operatorname{Rad}\mathscr{S}_\lambda \subseteq \operatorname{Rad}\mathsf{Sol}_q(n)$.

Next, suppose that (c) holds. Then E_λ belongs to the radical of $\mathscr{S}_\lambda$. Now, $\mathscr{S}_\lambda \subset \mathscr{S}_{\lambda_{l-1}}$ so, as vector spaces, $\operatorname{Rad}\mathscr{S}_\lambda = RE_\lambda + \operatorname{Rad}\mathscr{S}_{\lambda_{l-1}}$. On the other hand, $\dim\mathscr{S}_\lambda = \dim\mathscr{S}_{\lambda_{l-1}} + 1$, so it follows that the algebras $\mathscr{S}_\lambda$ and $\mathscr{S}_{\lambda_{l-1}}$ have the same number of irreducible modules. Hence, $I(\lambda) \cong I(\mu)$ for some signed partition $\mu > \lambda$. That is, (e) holds.

Finally, assume that (e) holds. Then $I(\lambda) \cong I(\mu)$, for some signed partition $\mu > \lambda$. Therefore, E_λ acts on these modules as multiplication by $d_{\lambda\lambda\lambda}(q) = d_{\mu\lambda\mu}(q)$. Consequently, $d_{\lambda\lambda\lambda}(q) = 0$ by Lemma 8.2, so (a) holds.

This completes the proof of the Theorem. □

In the following Corollaries note that the integer $d_{\lambda\lambda\lambda}(q) = q^{|\lambda|^-}[N_{\mathfrak{S}_n}(\mathfrak{S}_{\lambda^+}) : \mathfrak{S}_{\lambda^+}]$ is explicitly known by Theorem 8.11 (and Corollary 5.8).

8.12. **Corollary.** *Suppose that R is a field and $q \in R$. Then*

$$\{\, I(\lambda) : \lambda \in \Lambda_n^\oplus \text{ and } d_{\lambda\lambda\lambda}(q) \neq 0 \,\}$$

is a complete set of pairwise non–isomorphic irreducible $\mathsf{Sol}_q(n)$–modules.

Proof. This follows from Theorem 8.11 and Theorem 8.3. □

Similarly, combining the Theorem 8.11 with Corollary 8.5 and Corollary 8.8, we obtain the general description of the radical of $\mathsf{Sol}_q(n)$ when R is a field.

8.13. **Corollary.** *Suppose that R is a field and $q \in R$. Then*

$$\{\, E_\lambda - E_\mu : \lambda \in \Lambda_n^\oplus, \mu \in \Lambda_n^\pm, \lambda \sim \mu \text{ and } \lambda \neq \mu \,\} \bigcup \{\, E_\lambda : \lambda \in \Lambda_n^\oplus \text{ and } d_{\lambda\lambda\lambda}(q) = 0 \,\}$$

is a basis of $\operatorname{Rad}\mathsf{Sol}_q(x)$.

Finally, we can use Theorem 8.11 to describe the radical and irreducible modules for each of the subalgebras of $\mathsf{Sol}_q(n)$ described in Proposition 7.6. For brevity we state only the following result.

8.14. **Corollary.** *Suppose that R is a field, $n \geq 1$ and $q \in R$. Let A be one of the subalgebras $\mathsf{Sol}_q^+(n)$, $\mathsf{Sol}_q^\pm(n)$, $\mathsf{Sol}_q^1(n)$ of $\mathsf{Sol}_q(n)$. Then* $\operatorname{Rad} A = A \cap \operatorname{Rad}\mathsf{Sol}_q(n)$.

9. The Hopf algebra of cyclotomic Solomon algebras

In this section we fix $r > 1$ and show that the direct sum of cyclotomic Solomon algebras $\bigoplus_{n\geq 0}\mathsf{Sol}(G_{r,n})$ is a concatenation Hopf algebra, where $G_{r,0} = \{1_{G_{r,0}}\}$ is the trivial group. Further, this Hopf algebra is a Hopf subalgebra of the Hopf algebra of colored permutations introduced by Baumann and Hohlweg [3].

Most of the results in this section hold over an arbitrary integral domain, however, the main results of this section (Theorem 9.7 and Corollary 9.8), hold only in characteristic zero. Consequently, for this section we fix a field $\Bbbk$ of characteristic zero and we work only over this field. Thus, all tensor products are over $\Bbbk$, all modules are $\Bbbk$-vector spaces and all algebras are $\Bbbk$–algebras. In particular, the cyclotomic Solomon algebras $\mathsf{Sol}(G_{r,n}) = \mathsf{Sol}_{\Bbbk}(G_{r,n})$ are $\Bbbk$–algebras.

We first recall some general facts about bialgebras and Hopf algebras.

A **$\Bbbk$–coalgebra** is a triple (A, δ, ε) consisting of a $\Bbbk$–vector space A together with two linear maps $\delta : A \longrightarrow A \otimes A$ (comultiplication) and $\varepsilon : A \longrightarrow \Bbbk$ (the counit) such that

$$(\delta \otimes \mathsf{id}_A) \circ \delta = (\mathsf{id}_A \otimes \delta) \circ \delta \qquad \text{and} \qquad (\varepsilon \otimes \mathsf{id}_A) \circ \delta = (\mathsf{id}_A \otimes \varepsilon) \circ \delta,$$

where id_A is the identity map on A.

A **$\Bbbk$–bialgebra** is a coalgebra (A, δ, ε) such that A is a $\Bbbk$–algebra and the structure maps $\delta : A \longrightarrow A \otimes A$ and $\varepsilon : A \longrightarrow \Bbbk$ are algebra homomorphisms. A **Hopf algebra** is a quadruple $(A, \delta, \varepsilon, S)$ where (A, δ, ε) is a bialgebra and $S : A \longrightarrow A$ (the antipode) is a linear map such that $\mu(S \otimes \mathsf{id}_A)\delta = \eta\varepsilon = \mu(1 \otimes S)\delta$. Here $\mu : A \otimes A \longrightarrow A : (a, b) \mapsto ab$ is the multiplication map and $\eta : \Bbbk \longrightarrow A; 1 \mapsto 1_A$ is the unit map for the algebra A.

Finally, a **graded bialgebra** is a triple (A, δ, ε) where $A = \bigoplus_{n\in\mathbb{N}} A_n$ is $\mathbb{N}$–graded bialgebra and the maps δ and ε are graded (degree zero) vector space homomorphisms. A **graded Hopf** algebra is a graded bialgebra which is equipped with an antipode which is a graded vector space homomorphism of degree zero. A graded bialgebra, or a graded Hopf algebra, $A = \bigoplus_{n\geq 0} A_n$ is **connected** if $A_0 = \Bbbk$.

Following Baumann and Hohlweg [3], we next define the (graded connected) Hopf algebra of coloured permutations. This will require some preparation. As a graded vector space this Hopf algebra is the direct sum of the group algebras of groups $G_{r,n}$:

$$\mathscr{G}(r) := \bigoplus_{n\geq 0} \Bbbk G_{r,n}.$$

We need some more notation before we can describe the Hopf algebra structure on $\mathscr{G}(r)$.

First, suppose that m and n are non–negative integers. Then $G_{r,m} \times G_{r,n}$ is naturally isomorphic to the reflection subgroup $G_{(m,n)}$ of $G_{r,m+n}$. By identifying $G_{r,m} \times G_{r,n}$ and $G_{(m,n)}$ we have an embedding $G_{r,m} \times G_{r,n} \hookrightarrow G_{r,m+n}$. Explicitly, this embedding sends the generators $\{s_0, \ldots, s_{m-1}\}$ of $G_{r,m}$ to $\{s_0, \ldots, s_{m-1}\}$ in $G_{r,m+n}$ and the generators $\{s_0, \ldots, s_{n-1}\}$ of $G_{r,n}$ to $\{t_{m+1}, s_{m+1}, \ldots, s_{m+n-1}\}$, respectively.

By Proposition 4.7 there is a natural bijection between the set $\mathscr{E}_{(m,n)} = \mathscr{D}_{(m,n)}$ of right coset representatives of $G_{(m,n)}$ in $G_{r,n}$ and the set of row standard (m, n)–tableau. The product $*$ on the Hopf algebra $\mathscr{G}(r)$ is the bilinear map determined by

$$u * v = \sum_{e\in\mathscr{E}_{(m,n)}} (u \times v)e = (u \times v)E_{(m,n)},$$

for $u \in G_{r,m}$, $v \in G_{r,n}$ and where $u \times v$ is multiplication inside $G_{r,m+n}$ (the *internal product* on $\mathscr{G}(r)$). The product $*$ on $\mathscr{G}(r)$ is called the **shuffle product**, or the *external product*, on $\mathscr{G}(r)$ because, by Proposition 4.7, $\mathscr{E}_{(m,n)}$ is in bijection with the ways of shuffling the two sets $\{1, \ldots, m\}$ and $\{m + 1, \ldots, m + n\}$ together. It is easy to check that $E_{(0)} = 1_{G_{r,0}} \in \mathsf{Sol}(G_{r,0})$ is the unit for the shuffle product.

To define the coproduct on $\mathscr{G}(r)$ observe that for $m = 0, \ldots, n$ any element $g \in G_{r,n}$ can be written uniquely in the form $g = e_m^{-1}(g_{(m)} \times g_{(n)})$, where $g_{(m)} \in G_{r,m}$, $g_{(n)} \in$

$G_{r,n}$ and $e_m \in \mathscr{E}_{(m,n)}$. Using this notation, the **coproduct** Δ on $\mathscr{G}(r)$ is the linear map determined by

$$\Delta(g) = \sum_{m=0}^{n} g_{(m)} \otimes g_{(n)},$$

for $g \in G_{r,n}$.

9.1. **Example** In order to better distinguish between the elements $G_{r,n}$ for different values of n recall from the end of section 2 that there is a natural bijection between $G_{r,n}$ and the set of words $\mathcal{G}_{r,n} = \{\, \underline{\omega} = \omega_1 \dots \omega_n : \omega_i \in \mathbf{n}_\zeta \text{ and } \{|\omega_1|, \dots, |\omega_n|\} = \mathbf{n} \,\}$. To give an example of the shuffle product and the coproduct on $\mathscr{G}(r)$ we identify $G_{r,n}$ and $\mathcal{G}_{r,n}$ using this bijection.

Suppose that $0 \le a, b, c, d < r$. Then, using the identification above,

$$\begin{aligned} 1\zeta^a\, 2\zeta^b * 2\zeta^c\, 1\zeta^d &= 1\zeta^a 2\zeta^b 4\zeta^c 3\zeta^d + 1\zeta^a 3\zeta^b 4\zeta^c 2\zeta^d + 1\zeta^a 4\zeta^b 3\zeta^c 2\zeta^d \\ &\quad + 2\zeta^a 3\zeta^b 4\zeta^c 1\zeta^d + 2\zeta^a 4\zeta^b 3\zeta^c 1\zeta^d + 3\zeta^a 4\zeta^b 2\zeta^c 1\zeta^d \end{aligned}$$

and

$$\begin{aligned} \Delta(2\zeta^a 3\zeta^b 1\zeta^c 4\zeta^d) &= \emptyset \otimes 2\zeta^a 3\zeta^b 1\zeta^c 4\zeta^d + 1\zeta^c \otimes 1\zeta^a 2\zeta^b 3\zeta^d + 2\zeta^a 1\zeta^c \otimes 1\zeta^b 2\zeta^d \\ &\quad + 2\zeta^a 3\zeta^b 1\zeta^c \otimes 1\zeta^d + 2\zeta^a 3\zeta^b 1\zeta^c 4\zeta^d \otimes \emptyset, \end{aligned}$$

where $\emptyset$ is the empty word in $G_{r,0}$. ◊

As remarked above, $E_{(0)} = 1_{G_{r,0}}$ is the multiplicative unit for the shuffle product. The counit of $\mathscr{G}(r)$ is the linear map $\varepsilon : \mathscr{G}(r) \longrightarrow \Bbbk$ defined by

$$\varepsilon(w) = \begin{cases} 1 & \text{if } w = E_{(0)} \in G_{r,0} \\ 0 & \text{otherwise.} \end{cases}$$

9.2. **Theorem** (Baumann and Hohlweg [3, Theorem 1])**.** *The triple $(\mathscr{G}(r), \Delta, \varepsilon)$ is a graded connected bialgebra.*

In fact, $(\mathscr{G}(r), \Delta, \varepsilon)$ is a Hopf algebra at least when $\Bbbk$ is a field because every connected $\mathbb{N}$-graded $\Bbbk$-bialgebra is a Hopf algebra; see [28, Ex. 1, page 238].

We remind the reader that $r > 1$ is fixed throughout this section.

9.3. **Definition.** *The **cyclotomic Hopf algebra** is the graded vector space*

$$\mathsf{Sol}(r) = \bigoplus_{n \ge 0} \mathsf{Sol}(G_{r,n}).$$

The cyclotomic Hopf algebra is naturally graded with $\mathsf{Sol}(r)_n = \mathsf{Sol}(G_{r,n})$ and, as a vector space, $\mathsf{Sol}(r)_n$ is finite dimensional with basis $\{\, E_\mu : \mu \in \Lambda_n^{\pm} \,\}$. For convenience, we set $E_n = E_{(n)}$, for $n \in \mathbb{Z}$.

Our next aim is to show that $\mathsf{Sol}(r)$ is a Hopf subalgebra of $\mathscr{G}(r)$. We begin with a Lemma which generalizes 6.7(a).

9.4. **Lemma.** *Suppose that $\alpha, \beta \in \Lambda_n^{\pm}$ with $G_\alpha \subseteq G_\beta$. Then $\mathscr{E}_\alpha^\beta = \mathscr{E}_\alpha \cap G_\beta$ is a complete set of minimal length right coset representatives for G_α in G_β and $\mathscr{E}_\alpha = \mathscr{E}_\alpha^\beta \mathscr{E}_\beta$.*

Proof. It is clear that $\mathscr{E}_\alpha^\beta$ is a complete set of right coset representatives for G_α in G_β. Moreover, by definition, if $e \in \mathscr{E}_\alpha^\beta$ then e is the unique element of minimal length in the coset $G_\alpha e$. To prove the second statement observe that

$$G_{r,n} = \coprod_{d \in \mathscr{E}_\beta} G_\beta d = \coprod_{d \in \mathscr{E}_\beta} \Big(\coprod_{e \in \mathscr{E}_\alpha^\beta} G_\alpha e \Big) d.$$

So, $\mathscr{E}_\alpha^\beta \mathscr{E}_\beta$ is a complete set of coset representatives for G_α in $G_{r,n}$. Therefore, $\mathscr{E}_\alpha = \mathscr{E}_\alpha^\beta \mathscr{E}_\beta$ since the elements of both sides are of minimal length in their respective cosets. □

9.5. **Proposition.** *Suppose that $\mu \in \Lambda_m^{\pm}$ and $\nu \in \Lambda_n^{\pm}$. Then*

$$E_\mu * E_\nu = E_{\mu \sqcup \nu} \in \mathsf{Sol}(G_{r,n+m})$$

where $\mu \sqcup \nu = (\mu_1, \dots, \mu_l, \nu_1, \dots, \nu_k)$ is the concatenation of two signed permutations.

Proof. By definition, $E_\mu * E_\nu = (E_\mu \times E_\nu)E_{(m,n)}$ where, as above, we interpret $E_\mu \times E_\nu$ as an element of $\Bbbk G_{(m,n)} \subseteq \Bbbk G_{r,n}$. Therefore, it is enough to prove that $\mathscr{E}_{\mu\sqcup\nu} = \mathscr{E}_{\mu\times\nu}^{(m,n)}\mathscr{E}_{(m,n)}$. However, this follows immediately from the previous Lemma because $G_{\mu\sqcup\nu} = G_\mu \times G_\nu \subseteq G_{(m,n)}$. □

Notice that the Proposition says that $\mathsf{Sol}(r)$ is a subalgebra of $\mathscr{G}(r)$ and that, as an algebra, $\mathsf{Sol}(r)$ is freely generated by the elements $\{ E_n : n \in \mathbb{Z} \}$.

9.6. **Proposition.** *Suppose that n is a positive integer. Then*

a) $\Delta(E_n) = \sum_{m=0}^{n} E_m \otimes E_{n-m}$;

b) $\Delta(E_{-n}) = \sum_{m=0}^{n} E_{-m} \otimes E_{m-n}$.

Proof. Part (a) follows directly from the definitions. This result is well known because $E_n = 1_{G_{r,n}}$ is the identity element of $\Bbbk G_{r,n}$, so we omit the details.

For part (b), observe that $E_{-n} = F_{(n)} = \sum_{t\in T} t$. Therefore,

$$\begin{aligned}
\Delta(E_{-n}) &= \sum_{\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_n^r} \Delta(t_1^{\alpha_1}\dots t_n^{\alpha_n}) \\
&= \sum_{\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_n^r} \sum_{m=0}^{n} t_1^{\alpha_1}\dots t_m^{\alpha_m} \otimes t_1^{\alpha_{m+1}}\dots t_{n-m}^{\alpha_m} \\
&= \sum_{m=0}^{n} \sum_{\substack{\beta\in\mathbb{Z}_m^r \\ \gamma\in\mathbb{Z}_{n-m}^r}} t_1^{\beta_1}\dots t_m^{\beta_m} \otimes t_1^{\gamma_1}\dots t_{n-m}^{\gamma_{n-m}} \\
&= \sum_{m=0}^{n} E_{-m} \otimes E_{m-n},
\end{aligned}$$

as required. □

We henceforth adopt the unusual convention that $\sum_{m=a}^{b} f(m) = \sum_{m=b}^{a} f(m)$ if $b < a$. This allows us to write the Proposition 9.6 more compactly as $\Delta(E_n) = \sum_{m=0}^{\mu_k} E_m \otimes E_{\mu_k - m}$, for $n \in \mathbb{Z}$.

As the coproduct is an algebra homomorphism $\mathscr{G}(r) \to \mathscr{G}(r) \otimes \mathscr{G}(r)$ it follows from the last two Propositions that $\mathsf{Sol}(r)$ is a sub-bialgebra of $\mathscr{G}(r)$.

Let $\mathbf{P}$ be a set of non-commuting indeterminates over $\Bbbk$. The **concatenation Hopf algebra** on $\mathbf{P}$ is the free associative $\Bbbk$-algebra $\Bbbk\langle\mathbf{P}\rangle$ on $\mathbf{P}$ with counit ε, where $\varepsilon(f(\mathbf{P})) = f(0)$ is the constant term of $f(\mathbf{P}) \in \Bbbk\langle\mathbf{P}\rangle$, coproduct $\delta(p) = p \otimes 1 + 1 \otimes p$ for any $p \in \mathbf{P}$, and antipode S determined by $S(p_1 \dots p_k) = (-1)^k p_k \dots p_1$, for $p_1, \dots, p_k \in \mathbf{P}$. Any function $\deg : \mathbf{P} \longrightarrow \mathbb{N}$ extends to a degree function on the monomials in $\Bbbk\langle\mathbf{P}\rangle$ by setting $\deg(p_1 \dots p_k) = \deg(p_1) + \dots + \deg(p_k)$. In this way, $\Bbbk\langle\mathbf{P}\rangle = \bigoplus_{n\geq 0} \Bbbk\langle\mathbf{P}\rangle_n$ becomes a graded connected Hopf algebra, where $\Bbbk\langle\mathbf{P}\rangle_n$ is the space of homogeneous polynomials $p_1 \dots p_k$ in $\mathbf{P}$ with $\deg(p_1 \dots p_k) = n$.

We can now prove the main result of this section. Up until now we have not used the assumption that $\Bbbk$ is a field of characteristic zero. This assumption is necessary, however, for the proof of the following Theorem.

9.7. **Theorem.** *Suppose that $\Bbbk$ is a field of characteristic zero. Then $(\mathsf{Sol}(r), \Delta, \varepsilon)$ is isomorphic to the graded connected concatenation Hopf algebra $\Bbbk\langle\mathbf{P}\rangle$ on a set of non-commuting indeterminates $\mathbf{P} = \{ P_n : n \in \mathbb{Z} \setminus \{0\} \}$ where $\deg P_{\pm n} = n$, for $n > 0$.*

Proof. Our argument is modeled on the proof of [22, Theorem 2.1].

Let x be a formal variable and consider the algebra $\mathsf{Sol}(r)[\![x]\!]$ of formal power series in x over $\mathsf{Sol}(r)$, where x commutes with $\mathsf{Sol}(r)$. For each positive integer n define elements $P_{\pm n} \in \mathsf{Sol}(r)[\![x]\!]$ using the generating series

$$\sum_{n>0} P_n x^n = \log(1 + E_1 x + E_2 x^2 + \cdots)$$

and

$$\sum_{n>0} P_{-n} x^n = \log(1 + E_{-1} x + E_{-2} x^2 + \cdots).$$

A straightforward calculation using Proposition 9.5 and the Taylor series expansion of $\log(1+t)$ shows that

$$P_n = \sum_{\alpha\in\Lambda_n} \frac{(-1)^{\ell(\alpha)-1}}{\ell(\alpha)} E_\alpha \quad \text{and} \quad P_{-n} = \sum_{-\alpha\in\Lambda_n} \frac{(-1)^{\ell(\alpha)-1}}{\ell(\alpha)} E_\alpha.$$

(Recall that $\ell(\alpha)$ is the number of non-zero parts in α.) Therefore, $P_n, P_{-n} \in \mathsf{Sol}(G_{r,n})$ are homogeneous of degree n; in particular, $P_{\pm n} \in \mathsf{Sol}(r)$, for all $n > 0$. Consequently, the elements $\{ P_{\pm n} : n > 0 \}$ generate a subalgebra of $\mathsf{Sol}(r)$.

Similarly, since $\sum_{n\geq 0} E_{\pm n} x^n = \exp(\sum_{n>0} P_{\pm n} x^n)$, another completely formal calculation using the Taylor series expansion of $\exp(x)$ and Proposition 9.5 shows that if $n > 0$ then

$$E_n = \sum_{\alpha\in\Lambda_n} \frac{1}{\ell(\alpha)!} P_\alpha \quad \text{and} \quad E_{-n} = \sum_{-\alpha\in\Lambda_n} \frac{1}{\ell(\alpha)!} P_\alpha,$$

where we set $P_\alpha = P_{\alpha_1} * \cdots * P_{\alpha_k}$, for $\alpha = (\alpha_1, \ldots, \alpha_k) \in \pm\Lambda_n$. Therefore, by the last paragraph, the set $\mathbf{P} = \{ P_n : n \in \mathbb{Z} \setminus \{0\} \}$ freely generates $\mathsf{Sol}(r)$ as an algebra. That is, $\mathsf{Sol}(r) = \langle P_{\pm n} \mid n > 0 \rangle$ as an algebra.

We claim that $\Delta(P_n) = P_n \otimes 1 + 1 \otimes P_n$, for $n \in \mathbb{Z} \setminus \{0\}$. This will complete the proof because it shows that these elements generate a concatenation Hopf algebra $\Bbbk\langle \mathbf{P} \rangle$ inside $\mathsf{Sol}(r)$. Starting from the definition of $P_{\pm n}$ we have that

$$\sum_{n>0} \Delta(P_{\pm n} x^n) = \Delta\big(\sum_{n\geq 0} \log(\sum_{n\geq 0} E_{\pm n} x^n) \big) = \log\big(\sum_{n\geq 0} \Delta(E_{\pm n}) x^n \big),$$

where the last equality follows by the linearity of Taylor expansions since Δ is an algebra homomorphism. Using Proposition 9.6 to expand the right hand side of the last equation, exactly as in the proof of [22, (2.9)], shows that $\Delta(P_{\pm n}) = P_n \otimes 1 + 1 \otimes P_n$. This proves our claim and so completes the proof. □

9.8. **Corollary.** *Suppose that $r > 1$. Then the graded vector space $\mathsf{Sol}(r)$ equipped with the product $*$, coproduct Δ, unit E_0 and counit ε, is a graded connected Hopf subalgebra of $\mathscr{G}(r)$.*

10. A SECOND BIALGEBRA STRUCTURE ON $\mathsf{Sol}(r)$

In this section we show that the cyclotomic Hopf algebra $\mathsf{Sol}(r)$ has a second bialgebra structure with the same coproduct Δ as in section 9, but where the product is inherited from multiplication in the groups $G_{r,n}$, for $r, n \geq 0$. More precisely, the *internal product* is the unique bilinear map $\cdot : \mathscr{G}(r) \longrightarrow \mathscr{G}(r)$ such that if $w \in G_{r,m}$ and $v \in G_{r,n}$ then

$$w \cdot v = \begin{cases} wv, & \text{if } n = m, \\ 0, & \text{otherwise.} \end{cases}$$

We frequently abuse notation and write $xy = x \cdot y$, for $x, y \in \mathscr{G}(r)$. It is straightforward to check that $(\mathsf{Sol}(r), \Delta, \varepsilon, \cdot)$ is a bialgebra.

As each of the group algebras $\Bbbk G_{r,n}$ are associative algebras it follows that $(\mathscr{G}(r), \cdot)$ is an associative algebra. Note, however, that $(\mathscr{G}(r), \cdot)$ does not have a multiplicative unit, so we cannot expect to obtain a second Hopf algebra structure on $\mathsf{Sol}(r)$ in this way. Note also that the internal product $\cdot$ does not respect the grading on $\mathscr{G}(r) = \bigoplus_n \Bbbk G_{r,n}$.

By Theorem 6.8, $\mathsf{Sol}(r)$ is a subalgebra of the algebra $(\mathscr{G}(r), \cdot)$. We will show that $\big(\mathsf{Sol}(r), \cdot, \Delta\big)$ is a sub-bialgebra of $\mathscr{G}(r)$. To prove this we need only show that Δ is an algebra homomorphism with respect to the internal product. The argument that we give generalizes that used by Malvenuto [21, Remark 5.15] to prove the analogous statement for the descent algebra of the symmetric group. We start with some new definitions.

A **pseudo signed composition** of n is an element $\mathbf{c} = (c_1, c_2, \dots, c_k) \in \mathbb{Z}^k$, for some $k > 0$, such that $|\mathbf{c}| = |c_1| + |c_2| + \cdots |c_k| = n$. A **pseudo composition** is an element of $\mathbb{N}^k$, for some $k > 0$. The difference between (signed) compositions and pseudo (signed) compositions is that pseudo (signed) compositions can contain zeros. If $\mathbf{c}$ is a pseudo signed composition let $\overline{\mathbf{c}}$ be the signed composition obtained by omitting the zeros from $\mathbf{c}$. For example, if $\mathbf{c} = (-2, 0, 3, 0, 1)$ then $\overline{\mathbf{c}} = (-2, 3, 1)$.

If $\mathbf{c} \in \mathbb{Z}^k$ is a pseudo signed composition then set $E_{\mathbf{c}} = E_{\overline{\mathbf{c}}}$. If $\mathbf{c}, \mathbf{c}' \in \mathbb{Z}^k$ are two pseudo signed composition of the same length then $\mathbf{c} + \mathbf{c}' \in \mathbb{Z}^k$, where addition is defined componentwise. We extend the operation of concatenation to pseudo signed compositions in the obvious way so that if $\mathbf{c} \in \mathbb{Z}^k$ and $\mathbf{c}' \in \mathbb{Z}^l$ then $\mathbf{c} \sqcup \mathbf{c}' \in \mathbb{Z}^{k+l}$.

Two integers c and c' are **sign equivalent**, and we write $c \sim_{\text{sgn}} c'$, if c and c' are both non–negative, or both non-positive. Similarly, two (pseudo) signed compositions $\mathbf{c} = (c_1, \dots, c_k)$ and $\mathbf{c}' = (c'_1, \dots, c'_k)$ are sign equivalent if $c_i \sim_{\text{sgn}} c'_i$, for $i = 1, \dots, k$. Again, we write $\mathbf{c} \sim_{\text{sgn}} \mathbf{c}'$.

10.1. **Proposition.** *Suppose that $\mu \in \Lambda_n^{\pm}$ and that $\ell(\mu) = k$. Then*

$$\Delta(E_\mu) = \sum_{\substack{\mathbf{c}' \sim_{sgn} \mathbf{c}'' \in \mathbb{Z}^k \\ \mu = \mathbf{c}' + \mathbf{c}''}} E_{\mathbf{c}'} \otimes E_{\mathbf{c}''}.$$

Proof. We argue by induction on k. As $\Delta(E_0) = E_0 \otimes E_0$ the case $k = 0$ is clear. So we may assume that $k > 0$. Let $\nu = (\mu_1, \dots, \mu_{k-1})$ so that $\mu = \nu \sqcup (\mu_k)$. Then, by Proposition 9.5 and Proposition 9.6,

$$\begin{aligned}\Delta(E_\mu) &= \Delta(E_\nu * E_{\mu_k}) = \Delta(E_\nu) * \Delta(E_{\mu_k}) \\ &= \Big(\sum_{\substack{\mathbf{c}' \sim_{\text{sgn}} \mathbf{c}'' \in \mathbb{Z}^{k-1} \\ \nu = \mathbf{c}' + \mathbf{c}''}} E_{\mathbf{c}'} \otimes E_{\mathbf{c}''} \Big) * \Big(\sum_{m=0}^{\mu_k} E_m \otimes E_{\mu_k - m} \Big),\end{aligned}$$

by induction on k. (If $\mu_k < 0$ then recall our unusual convention for summations from after Proposition 9.6.) Therefore, using Proposition 9.5 for the second equality,

$$\begin{aligned}\Delta(E_\mu) &= \sum_{\substack{\mathbf{c}' \sim_{\text{sgn}} \mathbf{c}'' \in \mathbb{Z}^{k-1} \\ \nu = \mathbf{c}' + \mathbf{c}''}} \sum_{m=0}^{\mu_k} E_{\mathbf{c}'} * E_m \otimes E_{\mathbf{c}''} * E_{\mu_k - m} \\ &= \sum_{\substack{\mathbf{c}' \sim_{\text{sgn}} \mathbf{c}'' \in \mathbb{Z}^{k-1} \\ \nu = \mathbf{c}' + \mathbf{c}''}} \sum_{m=0}^{\mu_k} E_{\mathbf{c}' \sqcup (m)} \otimes E_{\mathbf{c}'' \sqcup (\mu_k - m)} \\ &= \sum_{\substack{\mathbf{c}' \sim_{\text{sgn}} \mathbf{c}'' \in \mathbb{Z}^k \\ \mu = \mathbf{c}' + \mathbf{c}''}} E_{\mathbf{c}'} \otimes E_{\mathbf{c}''}\end{aligned}$$

as required. □

Let $k, l > 0$ be positive integers and let $\mathsf{Mat}_{kl}(\mathbb{Z})$ be the set of $k \times l$ integer matrices. If $M \in \mathsf{Mat}_{kl}(\mathbb{Z})$ let $\mathsf{row}(M) = (r_1, \dots, r_k)$ be the pseudo composition where r_i is the sum of the absolute entries in row i of M, for $1 \le i \le k$. Similarly, let $\mathsf{col}(M) = (c_1, \dots, c_k)$ be the pseudo composition where c_j is the sum of the absolute values of the entries in column j of M. Finally, if $M \in \mathsf{Mat}_{kl}(\mathbb{Z})$ let $\mathsf{comp}(M)$ be the signed composition obtained by listing the non–zero entries in M in order, from left to right and then top to bottom; thus, if $M = (m_{ij})$ then $\mathsf{comp}(M) = (m_{11}, \dots, m_{1l}, m_{21}, \dots, m_{k1}, \dots, m_{kl})$.

If $\mathbf{c} = (c_1, \dots, c_k)$ is a pseudo signed composition then define $\mathbf{c}^+ = (|c_1|, \dots, |c_k|)$. In the next definition we are most interested in the case when μ and ν are signed compositions. We include pseudo signed compositions in the definition because they are needed in the proof of Theorem 10.5 below.

10.2. **Definition.** *Suppose that $\mu = (\mu_1, \dots, \mu_k)$ and $\nu = (\nu_1, \dots, \nu_l)$ are pseudo signed compositions of n. Let*

$$\mathcal{N}_{\mu\nu} = \left\{ M = (m_{ij}) \in \mathsf{Mat}_{kl}(\mathbb{Z}) \;\middle|\; \begin{array}{c} \mathsf{row}(M) = \mu^+, \text{ and } \mathsf{col}(M) = \nu^+, \\ m_{ij} \le 0 \text{ if } \mu_i < 0 \text{ or if } \nu_j < 0, \\ \text{and } m_{ij} \ge 0 \text{ if } \mu_i > \text{ and } \nu_j > 0 \end{array} \right\}.$$

*Suppose now that $M = (m_{ij}) \in \mathcal{N}_{\mu\nu}$. The **weight** of M is the non-negative integer*

$$\mathsf{wt}(M) = -\sum_{\substack{i:\mu_i<0 \\ j:\nu_j<0}} m_{ij},$$

where in the sum $1 \le i \le k$ and $1 \le j \le l$ (note that $m_{ij} \le 0$ for all such i, j). If μ and ν are signed compositions let T_M be the unique row semistandard tableau in $\mathcal{T}(\mu, \nu)$ such that j appears $|m_{ij}|$ times in row i of T, for $1 \le i \le k$ and $1 \le j \le l$.

Note that if μ and ν are compositions and $M = (m_{ij}) \in \mathcal{N}_{\mu\nu}$ then $\mathsf{wt}(M) = 0$ and $m_{ij} \ge 0$, for $1 \le i \le \ell(\mu)$ and $1 \le j \le \ell(\nu)$.

10.3. **Proposition.** *Suppose that μ and ν are signed compositions of n. Then*

$$E_\mu E_\nu = \sum_{M \in \mathcal{N}_{\mu\nu}} r^{\mathsf{wt}(M)} E_{\mathsf{comp}(M)}$$

Proof. By Theorem 6.8, $E_\mu E_\nu = \sum_{d \in \mathscr{D}_{\mu^+\nu^+}} |T_{-\mu \cap d(-\nu)}|\, E_{\mu \cap d\nu}$. Therefore, to prove the Proposition it is enough to show that there exists a bijection $\mathcal{N}_{\mu\nu} \to \mathscr{D}_{\mu^+\nu^+}; M \mapsto d_M$ such that $\mathsf{comp}(M) = \mu \cap d_M\nu$ and $r^{\mathsf{wt}(M)} = |T_{-\mu \cap d_M(-\nu)}|$.

First, observe that the map $\mathcal{N}_{\mu\nu} \to \mathcal{T}(\mu, \nu); M \mapsto \mathsf{T}_M$ is a bijection because its inverse is the map which sends the tableau $\mathsf{T} \in \mathcal{T}(\mu, \nu)$ to $M_\mathsf{T} = (m_{ij})$, where $|m_{ij}|$ is the number of times that j appears in row i of T, and where the sign of m_{ij} is determined by the constraints on $\mathcal{N}_{\mu\nu}$. Next, by (5.5), the map $\mathcal{T}(\mu, \nu) \to \mathscr{D}_{\mu^+\nu^+}; \mathsf{T} \mapsto d_{\mathsf{T}^*}$ is a bijection. Hence, the map

$$\mathcal{N}_{\mu\nu} \to \mathscr{D}_{\mu^+\nu^+}; M \mapsto d_M = d_{\mathsf{T}^*_M}$$

is a bijection.

Fix $M \in \mathcal{N}_{\mu\nu}$. Then $\mathsf{wt}(M) = \mathsf{wt}(\mathsf{T}_M)$ in the notation of Corollary 5.8, so that $r^{\mathsf{wt}(M)} = |T_{-\mu \cap d_M(-\nu)}|$. Hence, it remains to prove that $\mathsf{comp}(M) = \mu \cap d_M\nu$. The permutation d_M is determined by the row semistandard tableau T which, by the last paragraph, also determines $M = (m_{ij})$. If $m_{ij} \ne 0$ then $|m_{ij}|$ is equal to the number of times that j appears in row i of T. Writing $G_\mu = G_{\mu_1} \times \cdots \times G_{\mu_k}$ and $G_\nu = G_{\nu_1} \times \cdots \times G_{\nu_l}$, and abusing notation slightly, we see that m_{ij} computes the intersection of G_{μ_i} with $d_M G_{\nu_j} d_M^{-1}$; more precisely,

$$G_{\mu_i} \cap d_M G_{\nu_j} d_M^{-1} \cong \begin{cases} G(r, 1, m_{ij}), & \text{if } m_{ij} > 0, \\ \mathfrak{S}_{-m_{ij}}, & \text{if } m_{ij} < 0. \end{cases}$$

Comparing this with the recipe given in the proof of Lemma 5.1 for computing $\mu \cap d_M\nu$ we see that $\mathsf{comp}(M) = \mu \cap d_M\nu$, as required. □

Garsia–Remmel [13, Prop. 1.1] (see also [12, §4]), proved the analogue of this result for the Solomon algebras of the symmetric groups. This is equivalent to special case of Proposition 10.3 when μ and ν are both compositions of n. If $\mu, \nu \in \Lambda_n$ then the bijection $\mathcal{N}_{\mu\nu} \xrightarrow{\simeq} \mathscr{D}_{\mu\nu}$ is well–known; see, for example, [19, Theorem 1.3.10].

10.4. **Example** As in Example 5.9, suppose that $\mu = (3, -2)$ and $\nu = (-2^2, 1)$. The following table lists all of the elements of $\mathcal{N}_{\mu\nu}$, together with the associated signed composition and row semistandard μ–tableau of type ν and the weight of the matrix.

M	$\mathsf{comp}(M)$	T_M	$\mathsf{wt}(M)$
$\left(\begin{smallmatrix}-2&-1&0\\0&-1&-1\end{smallmatrix}\right)$	$(-2, -1^3)$	$\begin{array}{|c|c|c\|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array}$	1
$\left(\begin{smallmatrix}-2&0&1\\0&-2&0\end{smallmatrix}\right)$	$(-2, 1, -2)$	$\begin{array}{|c|c|c\|}\hline 1&1&3\\\hline 2&2\\\cline{1-2}\end{array}$	2
$\left(\begin{smallmatrix}-1&-2&0\\-1&0&-1\end{smallmatrix}\right)$	$(-1, -2, -1^2)$	$\begin{array}{|c|c|c\|}\hline 1&2&2\\\hline 1&3\\\cline{1-2}\end{array}$	1
$\left(\begin{smallmatrix}-1&-1&1\\-1&-1&0\end{smallmatrix}\right)$	$(-1^2, 1, -1^2)$	$\begin{array}{|c|c|c\|}\hline 1&2&3\\\hline 1&2\\\cline{1-2}\end{array}$	2
$\left(\begin{smallmatrix}0&-2&1\\-2&0&0\end{smallmatrix}\right)$	$(-2, 1, -2)$	$\begin{array}{|c|c|c\|}\hline 2&2&3\\\hline 1&1\\\cline{1-2}\end{array}$	2

The reader might like to compare this table with the one given in Example 5.9.

Combining the information above with Proposition 10.3 shows that

$$E_{(3,-2)}E_{(-2^2,1)} = 2r^2E_{(-2,1,-2)} + rE_{(-2,-1^3)} + rE_{(-1,-2,-1^2)} + r^2E_{(-1^2,1,-1^2)}.$$

This calculation agrees with Example 6.10, as it must. ◊

Suppose that $M' = (m'_{ij}), M'' = (m''_{ij}) \in \mathsf{Mat}_{kl}(\mathbb{Z})$, for some $k, l > 0$. Then M' and M'' are **signed equivalent**, and we write $M' \sim_{\mathrm{sgn}} M''$, if $m'_{ij} \sim_{\mathrm{sgn}} m''_{ij}$, for $1 \leq i \leq k$ and $1 \leq j \leq l$.

We can now prove the main result of this section.

10.5. **Theorem.** *Suppose that $r > 1$. Then $\mathsf{Sol}(r)$ equipped with product $\cdot$, coproduct Δ and counit ε, is a bialgebra.*

Proof. As remarked at the beginning of this section, it remains to show that the coproduct $\Delta : \mathsf{Sol}(r) \longrightarrow \mathsf{Sol}(r) \otimes \mathsf{Sol}(r)$ is an algebra homomorphism with respect to the internal product $\cdot$. By linearity it is enough to show that

$$\Delta(E_\mu E_\nu) = \Delta(E_\mu)\Delta(E_\nu),$$

for all signed compositions μ and ν. Further, we may assume that $|\mu| = |\nu|$ since otherwise both sides of this equation are zero. Let $k = \ell(\mu)$ and $l = \ell(\nu)$ and for $M \in \mathcal{N}_{\mu\nu}$ let $\ell(M) = \ell(\mathsf{comp}(M))$. Then, by Proposition 10.3 and Proposition 10.1,

$$\begin{aligned}\Delta(E_\mu E_\nu) &= \sum_{M \in \mathcal{N}_{\mu\nu}} r^{\mathsf{wt}(M)} \Delta(E_{\mathsf{comp}(M)}) \\ &= \sum_{M \in \mathcal{N}_{\mu\nu}} \sum_{\substack{\mathbf{c}' \sim_{\mathrm{sgn}} \mathbf{c}'' \in \mathbb{Z}^{\ell(M)} \\ \mathsf{comp}(M) = \mathbf{c}' + \mathbf{c}''}} r^{\mathsf{wt}(M)} E_{\mathbf{c}'} \otimes E_{\mathbf{c}''},\end{aligned}$$

For the moment, fix a matrix $M \in \mathcal{N}_{\mu\nu}$ and $\mathbf{c}', \mathbf{c}'' \in \mathbb{Z}^{\ell(M)}$ such that $\mathbf{c}' \sim_{\mathrm{sgn}} \mathbf{c}''$ and $\mathsf{comp}(M) = \mathbf{c}' + \mathbf{c}''$. Since $\mathbf{c}' \sim_{\mathrm{sgn}} \mathbf{c}''$ there exist unique matrices $M' = (m'_{ij}), M'' = (m''_{ij}) \in \mathsf{Mat}_{kl}(\mathbb{Z})$ such that $M = M' + M''$, $M \sim_{\mathrm{sgn}} M' \sim_{\mathrm{sgn}} M''$, $\mathsf{comp}(M') = \overline{\mathbf{c}'}$ and

$\mathsf{comp}(M'') = \overline{\mathbf{c}''}$. Note that $\mathsf{wt}(M) = \mathsf{wt}(M') + \mathsf{wt}(M'')$ since $M' \sim_{\mathrm{sgn}} M''$. Therefore, the last equation becomes

$$\Delta(E_\mu E_\nu) = \sum_{M\in\mathcal{N}_{\mu\nu}} \sum_{\substack{M'\sim_{\mathrm{sgn}} M''\\ M'+M''=M}} r^{\mathsf{wt}(M')} E_{\mathsf{comp}(M')} \otimes r^{\mathsf{wt}(M'')} E_{\mathsf{comp}(M'')}$$

For each pair M' and M'' in the second sum let $\mu' = \mathsf{row}(M')$ and $\mu'' = \mathsf{row}(M'')$. Then μ' and μ'' are *pseudo* signed compositions such that $\mu = \mu' + \mu''$ and $\mu' \sim_{\mathrm{sgn}} \mu''$. Similarly, $\nu' = \mathsf{col}(M')$ and $\nu'' = \mathsf{col}(M'')$ are *pseudo* signed compositions such that $\nu = \nu' + \nu''$ and $\nu' \sim_{\mathrm{sgn}} \nu''$. By signed equivalence, $M' \in \mathcal{N}_{\mu'\nu'}$ and $M'' \in \mathcal{N}_{\mu''\nu''}$. Moreover, M' and M'' run through $\mathcal{N}_{\mu'\nu'}$ and $\mathcal{N}_{\mu''\nu''}$, respectively, for all possible μ', μ'', ν' and ν'', as M runs through $\mathcal{N}_{\mu\nu}$. Observe that if $M' \in \mathcal{N}_{\mu'\nu'}$ and $M'' \in \mathcal{N}_{\mu''\nu''}$, for μ', μ'', ν' and ν'' as above, then $M' \sim_{\mathrm{sgn}} M''$ since $\mu' \sim_{\mathrm{sgn}} \mu''$ and $\nu' \sim_{\mathrm{sgn}} \nu''$. Therefore, we can reverse the order of summation in the last displayed equation to obtain

$$\begin{aligned}
\Delta(E_\mu E_\nu) &= \sum_{\substack{\mu'\sim_{\mathrm{sgn}}\mu''\\ \mu=\mu'+\mu''\\ \nu'\sim_{\mathrm{sgn}}\nu''\\ \nu=\nu'+\nu''}} \sum_{\substack{M'\in\mathcal{N}_{\mu'\nu'}\\ M''\in\mathcal{N}_{\mu''\nu''}}} r^{\mathsf{wt}(M')} E_{\mathsf{comp}(M')} \otimes r^{\mathsf{wt}(M'')} E_{\mathsf{comp}(M'')}\\
&= \sum_{\substack{\mu'\sim_{\mathrm{sgn}}\mu''\\ \mu=\mu'+\mu''\\ \nu'\sim_{\mathrm{sgn}}\nu''\\ \nu=\nu'+\nu''}} \Big(\sum_{M'\in\mathcal{N}_{\mu'\nu'}} r^{\mathsf{wt}(M')} E_{\mathsf{comp}(M')}\Big) \otimes \Big(\sum_{M''\in\mathcal{N}_{\mu''\nu''}} r^{\mathsf{wt}(M'')} E_{\mathsf{comp}(M'')}\Big)\\
&= \Big(\sum_{\substack{\mu'\sim_{\mathrm{sgn}}\mu''\\ \mu=\mu'+\mu''}} E_{\mu'} \otimes E_{\mu''}\Big)\Big(\sum_{\substack{\nu'\sim_{\mathrm{sgn}}\nu''\\ \nu=\nu'+\nu''}} E_{\nu'} \otimes E_{\nu''}\Big)\\
&= \Delta(E_\mu)\Delta(E_\nu),
\end{aligned}$$

where the last two equalities follow by Proposition 10.3 and Proposition 10.1 respectively. This completes the proof. □

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