Valuation of Real Estate Leases

Peter Buchen and Kevin Ho-Shon

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Abstract

The arbitrage free prices of real estate leases satisfy the Black-Scholes partial differential equation with an upper reflecting boundary condition. This upper boundary corresponds to rent conditions of new entry into the market. We show, in this paper, how to solve such problems using the Method of Images. It transpires that for reflecting boundary value problems, we need a continuous distribution of images, as opposed to the simple point images obtained for absorbing boundary conditions. We derive the corresponding Equivalent European Payoff and use it to price a simple lease contract of fixed maturity. The solution can be expressed as a portfolio of simple power binaries.

1 Introduction

The valuation of a real estate lease in a real options setting has been considered in detail by Grenadier [9]. The emphasis of that paper was the extension to real estate markets, of the theory of dynamic equilibrium in a competitive industry, as outlined in Dixit and Pindyck [1]. Earlier related ideas were attributed to Smith [2]. Grenadier then applied this theory to value a fixed term lease on a property and also to a number of more complex lease contracts.

It transpires that lease agreements in this framework satisfy the classical Black-Scholes partial differential equation (BS-pde) with terminal payoff function depending on the specifics of the contract and with an upper reflecting boundary condition. In the theory of pde's this last condition is known as a Neumann boundary condition and is relatively rare in Black-Scholes derivative pricing – absorbing BC's being much more common. Barrier options, for example are well known derivatives which satisfy the BS-pde with an absorbing (*aka* a Dirichlet) boundary condition at the barrier. While it is fairly straightforward to price derivatives with absorbing BC's, this is not the case for reflecting BC's. Grenadier prices the lease contracts utilising the discounted expectation of the payoff under the risk neutral (*i.e.* arbitrage free) measure. This expectation is taken under the assumption the instantaneous rent process follows geometrical Brownian motion with an upper reflecting boundary. The resulting analysis involves 'grueling integration' [8], so that Grenadier simply states the result without providing the details.

The contribution of this paper is to present a complete analysis of such reflecting boundary value problems for the BS-pde. We employ the method of images for pde's to solve the general problem and derive the price of a fixed term lease as an example of the method. This method has considerable advantages over traditional methods in that it requires no complex integrations and the final outcome has a simple economic interpretation in terms of portfolios of so called, *power binaries* and their images. It is gratifying that our solution can be shown to be equivalent to Grenadier's, apart from a couple of typographical errors which we report later in the text.

Buchen [6] shows how to price barrier options by finding the equivalent European payoff at the expiration time. This payoff involves certain binary options (*i.e.* options which pay out only if the underlying is above or below a given exercise price) and their images. Image options and the Method of Images is described in section 5. The equivalent European option is an otherwise identical option, but without the presence of the barrier. This is an extremely useful concept, since it allows the solution of complex boundary value problems in terms of simpler terminal value problems.

We derive in this paper the Method of Images and corresponding equivalent payoff for reflecting boundary value problems. These turns out to be more complex than the related absorbing boundary value problem. For one thing, absorbing BV problems for the Black-Scholes equation have simple point images. Reflecting BV problems on the other hand, require a continuous distribution of images. In this regard, the equivalent payoffs for reflecting BV problems is closer to those for lookback options than for knock-out barrier options (*e.g.* see Buchen and Konstandatos [7]). We show that the arbitrage free price of a lease on a given property can be expressed as an equivalent portfolio of power binaries and their images. We derive explicit formulae for these power binaries and hence find a closed form expression for the corresponding lease value.

This paper briefly describes the lease model in sections 2 and 3 and how it impacts on the Sydney CBD commercial office space market. In section 4 we derive the pde for the fixed term lease value, while in section 5 we describe the method of images needed to solve it. Proofs are relegated to an Appendix. The conclusion of the paper reports on some computations and their relation to Grenadier's results.

2 The Lease Model

When a tenant leases a property, possession of the property is taken for a fixed term equal to the duration of the lease. At the end of the term, the property is returned to the owner. In exchange for the use of the property, the tenant pays an agreed premium equal to the value of the lease. The premium is usually paid in advance in installments called rent. To simplify the analysis, rent is abstracted to a continuous stochastic process to indicate the value of real estate, similar to the stock price process as an indicator of company value. Although the instantaneous rent is not observable, it is nevertheless a useful concept in the lease model presented. If one further assumes a homogeneous market in which all firms make identical decisions and the real estate supply is constant, the instantaneous rent can be used as a proxy for real estate demand.

While there are many factors that determine real estate value in practice, the model assumes that property value depends only on the rental income it can generate. Commercial office space comes closer to this ideal than other forms of real estate such as residential property, where subjective issues (*e.g.* water views) strongly influence value. The Property Council of Australia classifies commercial office space and produces a valuation index for the different classes. Within each class, the basic unit of office space can be considered to be approximately homogeneous (see `www.propertyoz.com.au` for details). The index involves both temporal and regional averaging, the specifics of which go beyond the scope of this paper, but is broadly consistent with the above model.

Under the above simplifying assumptions, Dixit and Pindyck [1] derive a long term equilibrium value $H(x)$ for the fair value of a property as a function of the rent x. This derivation, which is based on Black-Scholes option pricing concepts, is briefly reviewed in the next section. Grenadier [9] extended this model to pricing a fixed term lease on a property. In Grenadier's approach, which has its foundations in the work of Smith [2], the lease is shown to be equivalent to a portfolio which is long the property and short a European call option on the value of the property $H(x)$, with zero strike price and expiry date coinciding with the end of the lease. The call option will, naturally, always be exercised at expiry and corresponds to the fact the property will be returned to the owner when the lease expires. If $C(x, t; T)$ denotes the value of this call option at time $t < T$, where T is the termination date of the lease, then the corresponding lease value is given by

$$L(x, t : T) = H(x) - C(x, t; T) \tag{1}$$

The model permits valuation of both $H(x)$ and $C(x, t; T)$ as solutions of Black-Scholes differential equations under suitable boundary conditions. Dixit and Pindyck argue that there should exist an upper threshold b on the rent x, which if reached, will trigger new entry into the market. As any one new firm enters the market, the rent will decrease along the demand curve that applies for that instant. Hence if the rent ever climbs to this threshold, it is immediately brought back down to a slightly lower level. In technical terms, the threshold b becomes an upper reflecting barrier for the rent process. Furthermore, under the homogeneous market assumption, industry equilibrium requires a firm's threshold at b to be equal to its rational (risk-neutral) expectation of all other firms' thresholds.

Mathematically, a reflecting boundary condition at $x = b$ for a derivative contract of value $V(x, t)$ is given by $\partial V(b, t)/\partial x = 0$. This can be contrasted with an absorbing BC (*e.g.* for a knock-out barrier option) where $V(b,t) = 0$.

3 Equilibrium Property Value

Assume the instantaneous rent process X_t follows geometrical Brownian motion described by the stochastic differential equation (sde)

$$dX_t = X_t[\mu dt + \sigma\, dB_t] \tag{2}$$

where μ is the drift rate and σ is the volatility, both assumed constant. B_t is a standard Brownian motion (under the real world measure) and for any fixed $t > 0$, is Gaussian with zero mean and variance equal to t, that is, $B_t \sim N(0,t)$.

Dixit and Pindyck [1] show that, under this rent process the long term equilibrium value $H(x,t)$ of property satisfies the time independent Black-Scholes equation (it is actually an ode)

$$\tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 H}{\partial x^2} + (r-q)x\frac{\partial H}{\partial x} - rH + x = 0 \tag{3}$$

where q is the yield on the property (see also Patel and Sing [5]). This pde is almost identical to the standard time independent Black-Scholes pde for a derivative on a dividend paying asset. The only difference is the final added term x which arises from the rent paid in the associated instantaneous hedge portfolio used to derive it.

The BC's for (3) are determined as follows. First $H(x) = 0$ when $x = 0$, since a property that attracts no rent will also have no value. As argued above we also have the reflecting BC at $x = b$, where the derivative with respect to x

vanishes. The natural domain of the ode is therefore $0 < x < b$.

Is such a reflecting barrier observable in the market? It is a moot point whether such a boundary can ever really be observed. Figure 1 shows the median rent for the Sydney CBD[1] total office space corrected for inflation for the years 1986 to 2005. There is a clear fall in rent during the early 1990's and a hint of a resistance level, if not an actual threshold, after 1997.

Numerous other factors influence the Sydney commercial real estate market. The introduction of capital gains tax in 1985 and compulsory superannuation for all employees in 1992 are two important examples. Newell [4] identifies superannuation in Australia as being a "key driver" behind the growth in the commercial property market with "95% of superannuation funds having a specific allocation to property". Figure 2 shows the total supply and total occupancy of Sydney CBD office space from 1990 to 2005. In the first half of the 1990 decade, supply increased without an increase in occupancy. Subsequently, total supply behaved like the supremum of occupancy. This is consistent with Grenadier's model assumption that demand indirectly drives supply. There is also empirical evidence for this assumption, as noted by Higgins [3]: "the movement of office capital values leads to new orders and consequently new office space."

Figure 3. shows that the yield on commercial real estate in the Sydney CBD has remained pretty well constant at around 7% since 1997.

[1] Central Business District

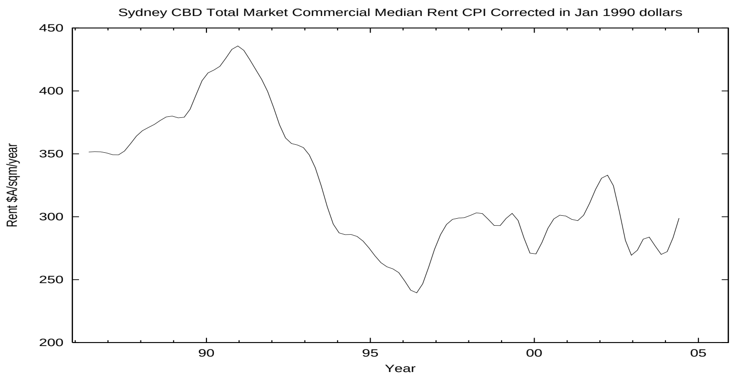

Figure 1: Commercial real estate rent from the Property Council of Australia

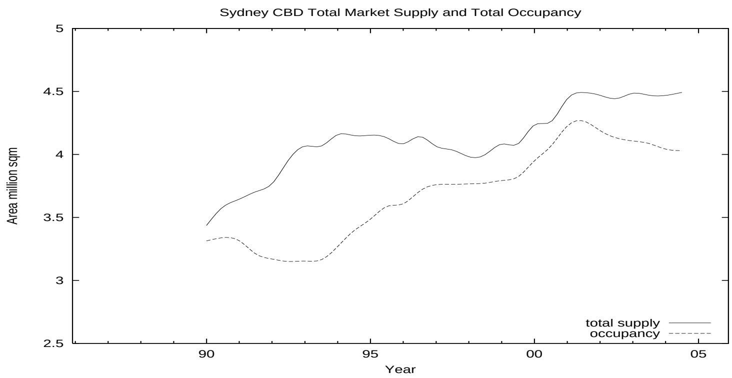

Figure 2: Commercial office total supply and occupancy from the Property Council of Australia

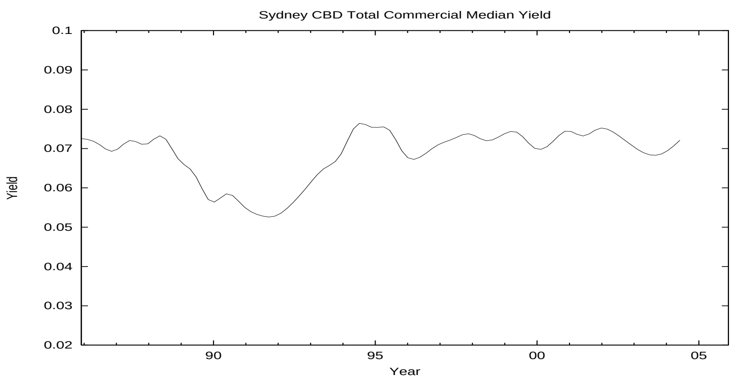

Figure 3: Commercial real estate yield from the Property Council of Australia

The unique solution of (3) is

$$H(x) = \frac{b}{q}\left[\frac{x}{b} - \frac{1}{\beta}\left(\frac{x}{b}\right)^{\beta}\right] \tag{4}$$

where β is the positive root of the quadratic

$$\tfrac{1}{2}\sigma^2\beta^2 + (r - q - \tfrac{1}{2}\sigma^2)\beta - r = 0 \tag{5}$$

The first term in (4), *i.e.* x/q, is the value of the property generated from continuous rent to perpetuity. The second term is a correction for the presence of the upper threshold at $x = b$. Note that $H \to x/q$ as $b \to \infty$, that is as the threshold disappears.

4 The Lease Framework

Consider a self-financing portfolio that is long a derivative $V(x,t)$ and short h units of rent $x = X_t$. The dynamics of the process X_t is assumed to follow the sde (2). This portfolio has current value

$$P(t) = V(X_t, t) - hX_t \tag{6}$$

Holding this portfolio to time $t + dt$, the value will then be

$$P(t+dt) = V(X_t + dX_t, t + dt) - h(X_t + dX_t) - hqX_t\,dt \tag{7}$$

The last term is the yield generated by the rent process over the time interval $[t, t+dt)$. Subtracting these two expressions and employing Itô's Lemma gives, conditional on $x = X_t$,

$$\begin{aligned} dP &= dV(x,t) - hdX_t - hqx\,dt \\ &= (V_t + \tfrac{1}{2}\sigma^2 x^2 V_{xx} - hqx)dt + (V_x - h)dX_t \end{aligned} \tag{8}$$

The standard Black-Scholes hedging argument now proceeds by choosing $h = V_x$. The portfolio becomes riskless, since the only stochastic term is dX_t, and in order to avoid arbitrage the portfolio must earn the risk free rate r. That is, from (6) and (8)

$$dP = rPdt = r(V - xV_x)dt = V_t + \tfrac{1}{2}\sigma^2 x^2 V_{xx} - qxV_x$$

This leads to the usual Black-Scholes pde for the derivative, *viz*

$$\mathcal{L}V(x,t) = \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + (r-q)x\frac{\partial V}{\partial x} - rV = 0 \tag{9}$$

Hence all derivatives that depend only on the rent process satisfy the Black-Scholes pde. Different derivatives will be distinguished only by their payoffs at maturity T and their boundary conditions. For the lease problem of interest here, the zero strike call option on the value of the property $C(x,t;T)$, will satisfy the Black-Scholes pde $\mathcal{L}C(x,t;T) = 0$ in the domain $0 < x < b,\ t < T$ subject to the BC's

$$C(0,t;T) = 0; \qquad C(x,T;T) = H(x); \qquad C_x(b,t;T) = 0 \tag{10}$$

where $H(x)$ given by (4), is the long term property value and $C_x(b,t;T) = 0$ specifies the reflecting BC at the barrier $x = b$.

The next section describes the mathematical tools needed to solve this pde for $C(x,t;T)$ in its domain of definition and also derives this solution as a portfolio of power binaries.

5 Solving for the Lease Value

5.1 Method of Images

Solutions of boundary value problems for pde's are often facilitated using so called *image solutions*. Buchen [6] has shown that if $V(x,t)$ is any solution of the Black-Scholes pde (9), then

$$\overset{*}{V}(x,t) = \mathcal{I}[V(x,t)] = \left(\frac{x}{b}\right)^{\alpha} V\left(\frac{b^2}{x}, t\right); \qquad \alpha = \frac{2(r-q)}{\sigma^2} - 1 \tag{11}$$

is the image solution relative to $x = b$. The image solution has a number of important properties. These include

1. $\overset{**}{V}(x,t) = \mathcal{I}[\overset{*}{V}(x,t)] = \mathcal{I}^2 V(x,t) = V(x,t)$
2. $V = \overset{*}{V}$ when $x = b$
3. If $V(x,t)$ solves the BS-pde with payoff $V(x,T) = F(x)$, then $\overset{*}{V}(x,t)$ solves the BS-pde with payoff $\overset{*}{V}(x,T) = \overset{*}{F}(x)$.

The Method of Images for the absorbing BV problem: $\mathcal{L}V = 0$ in $x < b,\ t < T$ with $V(x,T) = F(x)$ and $V(b,t) = 0$, has solution given by $V(x,t) = V_b(x,t) - \overset{*}{V}_b(x,t)$, where $V_b(x,t)$ solves the terminal value problem $\mathcal{L}V_b = 0$ in $x > 0,\ t < T$ with $V_b(x,T) = F(x)\mathbb{I}(x < b)$. The payoff $F(x)\mathbb{I}(x<b)$ for $V_b(x,T)$ is termed a down-type binary on $F(x)$; the holder receives $F(x)$ at the expiry date T, but only if the underlying asset (here, the rent) is below the barrier level b. Otherwise the holder gets nothing.

This solution is unique and solves the problem of pricing an up-and-out barrier option with arbitrary payoff $F(x)$. The Method of Images implies the following important result for absorbing BV problems in the Black-Scholes framework.

Theorem 1. *Equivalent payoff for absorbing BV problems*
The equivalent European payoff for an absorbing BV problem for the Black-Scholes pde in $x < b$, with expiry T payoff $F(x)$ is given by

$$V_{\mathrm{eq}}(x,T) = F(x)\mathbb{I}(x<b) - \overset{*}{F}(x)\mathbb{I}(x>b) \tag{12}$$

The last term above follows from the observation $\mathcal{I}[F(x)\mathbb{I}(x<b)] = \overset{*}{F}(x)\mathbb{I}(x>b)$. This theorem allows us to solve absorbing BV problems by solving a related terminal value problem, *i.e.* a European option without the presence of the barrier at $x = b$.

The lease problem we are concerned with however is not an absorbing BV problem but rather a reflecting BV problem. The corresponding result for such problems is a central result of this paper and is given by:

Theorem 2. *Equivalent payoff for reflecting BV problems*
The equivalent European payoff for a reflecting BV problem for the Black-Scholes pde in $x < b$, with expiry T payoff $F(x)$ is given by

$$V_{\mathrm{eq}}(x,T) = F(x)\mathbb{I}(x<b) + \overset{*}{F}(x)\mathbb{I}(x>b) + \alpha\mathbb{I}(x>b)\int_b^x \overset{*}{F}(y)\frac{dy}{y} \tag{13}$$

Proof: See Appendix. □

The equivalent payoff for reflecting BV problems is seen to be somewhat more complex than that for absorbing BV problems. The extra term involving an integral of $\overset{*}{F}(y)$ shows that the payoff now involves a continuum of images, as well as the single point image $\overset{*}{F}(x)$.

For the lease problem, the payoff is $F(x) = H(x)$ given by (4). Define $F_a(x) = (x/b)^a$. Then $H(x) = b/q[F_1(x) - \beta^{-1}F_\beta(x)]$. By linearity, if $C_a(x,t)$ solves the reflecting BV problem with payoff $F_a(x)$, then

$$C(x,t;T) = b/q[C_1(x,t) - \beta^{-1}C_\beta(x,t)] \tag{14}$$

solves the Black-Scholes pde with BC's (10).

We now evaluate the equivalent payoff for the reflecting BV problem with payoff $F_a(x)$, using (13) and $\mathcal{I}[(x/b)^a] = (b/x)^{a+\alpha}$ to obtain

$$C_a^{\rm eq}(x,T) = (x/b)^a \mathbb{I}(x<b) + (b/x)^{a+\alpha}\mathbb{I}(x>b) + \alpha K \mathbb{I}(x>b)$$

where

$$K = \int_b^x \left(\frac{b}{y}\right)^{a+\alpha} \frac{dy}{y} = \tfrac{\alpha}{a+\alpha}\left[1-(b/x)^{a+\alpha}\right] \tag{15}$$

Hence

$$\begin{aligned} C_a^{\rm eq}(x,T) &= (x/b)^a \mathbb{I}(x<b) + \tfrac{a}{a+\alpha}(b/x)^{a+\alpha}\mathbb{I}(x>b)] + \tfrac{\alpha}{a+\alpha}\mathbb{I}(x>b) \\ &= (x/b)^a \mathbb{I}(x<b) + \tfrac{a}{a+\alpha}\mathcal{I}[(x/b)^a\, \mathbb{I}(x<b)] + \tfrac{\alpha}{a+\alpha}\mathbb{I}(x>b) \end{aligned} \tag{16}$$

This is the equivalent payoff we are seeking and it only remains to determine the present value of a European derivative with this payoff. To this end, observe that each term above is a special case of a *power binary* defined by the terminal payoff

$$P_b^s(x,T;c) = (x/b)^c\, \mathbb{I}(sx>sb) \tag{17}$$

The holder of such a contract receives $(x/b)^c$ at time T, but only if the asset price x is above b (up-type if $s=+1$) or below b (down-type if $s=-1$). For these up and down binaries the barrier price b plays the role of an exercise price. The parameter $s=\pm 1$ defines the binary type.

Theorem 3. *Present value of a power binary*
The arbitrage free price of the power binary, defined by (17), is given, in the Black-Scholes framework, by

$$P_b^s(x,t;c) = (x/b)^c\, e^{\mu(c)\tau}\, \mathcal{N}[sd(x,\tau;c)] \tag{18}$$

for all $t < T$, where $\tau = T - t$, $\mathcal{N}(d)$ is the cumulative normal distribution function and

$$\mu(c) = \tfrac{1}{2}\sigma^2 c^2 + (r - q - \tfrac{1}{2}\sigma^2)c - r \tag{19}$$

$$d(x,\tau;c) = \frac{\log(x/b) + [r - q + (c - \frac{1}{2})\sigma^2]\tau}{\sigma\sqrt{\tau}} \tag{20}$$

Proof: See Appendix. □

Observe that the three values of c implicit in the call option value are $c = (1, \beta, 0)$, with corresponding μ values $\mu(c) = (-q, 0, -r)$. ¿From (17), the present value of $C_a(x,t)$ is given by

$$C_a(x,t) = P_b^-(x,t;a) + \frac{a}{a+\alpha}\overset{*}{P}{}_b^-(x,t;a) + \frac{\alpha}{a+\alpha}P_b^+(x,t;0) \tag{21}$$

Substituting into (14), we then finally obtain the expression

$$\begin{aligned} C(x,t;T) = \frac{b}{q}\Big[&P_b^-(x,t;1) + \tfrac{1}{\alpha+1}\overset{*}{P}{}_b^-(x,t;1) + \tfrac{\alpha}{\alpha+1}P_b^+(x,t;0) \\ &-\tfrac{1}{\beta}P_b^-(x,t;\beta) - \tfrac{1}{\alpha+\beta}\overset{*}{P}{}_b^-(x,t;\beta) - \tfrac{\alpha}{\beta(\alpha+\beta)}P_b^+(x,t;0)\Big] \end{aligned} \tag{22}$$

The call option is seen to be a portfolio of power binaries (and their images), all with the same exercise price b, but with different power indices $c = (1, \beta, 0)$. Note further that although the two expressions in $P_b^+(x,t;0)$ can be combined, we leave them as above, in order to facilitate comparison with Grenadier's result. One final point to observe is that the images in (22) can be evaluated using (11) and leads to the identity

$$\overset{*}{P}{}_b^s(x,t;c) = P_b^{-s}(x,t;-c-\alpha) \tag{23}$$

Thus the image of a power binary is itself a power binary but of opposite type.

The lease value can be graphed against the rent and against the time remaining in the lease. These plots are shown in figure (4) and figure (5).

These graphs are scaled to the value of the reflecting barrier for clarity. The values of the default parameters used were $q = 0.07$, $r = 0.05$, $\sigma = 0.15$ and $T = 5$years.

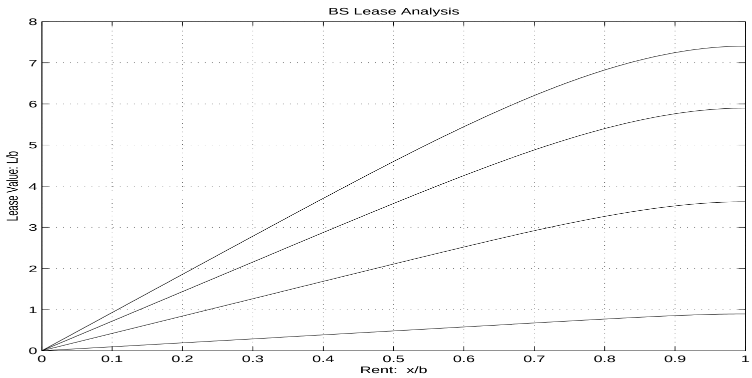

Figure 4: Lease versus Rent

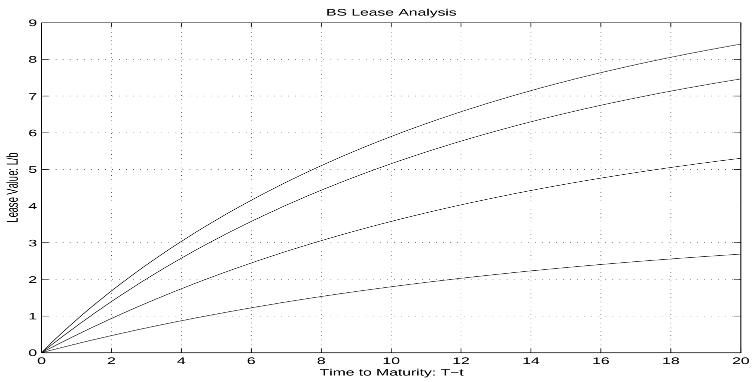

Figure 5: Lease versus Time

6 Comparison with Grenadier

While our graph of Lease versus Time looks quite different to Grenadier [9] figure 1, the two are in fact consistent. In Grenadier's term structure of lease rates, he includes an interest rate factor, $\frac{r}{1-\exp(-r.T)}$, which accounts fully for the difference. Indeed, while our lease curves (fig 5) are all monotone increasing with time, this factor permits humped shaped and monotone decreasing term structures as well .

We can also demonstrate the analytical equivalence of our pricing formula (22) with his equation [9](29) and [9](30), provided we correct a typographical error in his equation [9](30): the factor $[1 - h(w)]$ should read $[1 + h(w)]$.

7 Conclusion

Grenadier's [9] model for pricing real estate leases in the Black-Scholes framework involves the solution of the Black-Scholes partial PDE with an upper reflecting boundary condition. Such boundary conditions are not often seen in applications of financial mathematics.

We have shown in this paper how to solve such problems by extending the Method of Images for absorbing boundary conditions used to price barrier options. The image system turns out to contain both point images and a continuous distribution of images.

We have also derived the corresponding Equivalent European Payoff for this class of problems. This further simplifies the analysis, as demonstrated, for the case of a standard lease with fixed maturity. For this problem the Equivalent European Payoff is found to be that of a portfolio of power binaries and their images. The problem of pricing the lease is therefore reduced to one of pricing a power binary, which is straightforward.

Once the analytical machinery is in place, the only integration needed was an elementary one carried out in equation (15). Our method is therefore considerably simpler than the standard approach adopted by Grenadier and also leads to a simpler interpretation and a more elegant pricing formula.

Acknowledgment

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References

[1] Dixit A.K. and Pindyck R.S. *Investment under Uncertainty.* Princeton University Press, 1994.

[2] Smith C.W. Application of option pricing analysis. *in: J.L. Bicksler , ed., Handbook of Financial Economics(North-Holland, Amsterdam)*, 1979.

[3] Higgins D. Changes in the determinants of commercial property market performance. *Australian Property Journal*, pages 286–290, November 2002.

[4] Newell G. The significance of property in industry based superannuation funds in australia. *Pacific Rim Real Estate Society Conference*, January 2006.

[5] Patel K and Sing T.F. Implied volatility in the u.k. commercial property market: Empirical evidence based on transaction data. *Journal of Real Estate Finance and Economics*, 20(1):5–24, 2000.

[6] Buchen P.W. Image options and the road to barriers. *Risk Magazine*, 14(9):127–130, 2001.

[7] Buchen P.W. and Konstandatos O. A new method of pricing lookback options. *Mathematical Finance*, 15(2):245–259, 2004.

[8] Grenadier S.R. Valuing lease contracts: A real options approach. *The Journal of Financial Economics*, 38:297–331, 1995.

[9] Grenadier S.R. An equilibrium analysis of real estate leases. *The Journal of Business*, 78(4):1173(41 pages), 2005.

A Equivalent Payoff for reflecting BV problems

We prove in this appendix that the equivalent European payoff for a reflecting BV problem for the Black-Scholes pde in $x < b$, with expiry T payoff $F(x)$ is given by

$$V_{\text{eq}}(x,T) = F(x)\mathbb{I}(x<b) + \overset{*}{F}(x)\mathbb{I}(x>b) + \alpha\mathbb{I}(x>b)\int_b^x \overset{*}{F}(y)\frac{dy}{y} \tag{24}$$

Proof: The proof has four steps.

- Transform to the heat equation
- Solve the PDE using Laplace Transforms
- Transform back to original variables
- Evaluate at the maturity date, $t = T$

A.1 Transform to the heat equation

The Black Scholes PDE with the reflecting boundary

$$V_t = rV - (r-q)xV_x - \tfrac{1}{2}\sigma^2 x^2 V_{xx} \qquad \text{in } x < b,\ t \leq T \tag{25}$$

$$V(x,T) = f(x), \qquad V_x(b,t) = 0$$

can be transformed to the Heat Equation using

$$\tau = T - t, \qquad \xi = \log b/x, \qquad V = e^{\frac{1}{2}\alpha\xi - \beta\tau} u(\xi,\tau)$$

This gives the mixed boundary value problem

$$\begin{aligned} u_t &= \tfrac{1}{2}\sigma^2 u_{\xi\xi} \qquad \text{in} \qquad \xi > 0,\ \tau > 0 \\ u(\xi,0) &= e^{-\frac{1}{2}\alpha\xi} f(be^{-\xi}) \ = \ h(\xi) \\ u_\xi + \tfrac{1}{2}\alpha u &= 0 \qquad \text{when } \xi = 0 \end{aligned} \tag{26}$$

where $\alpha = 2(r-q)/\sigma^2 - 1$ and $\beta = r + \alpha^2\sigma^2/8$.

A.2 Solving the PDE

Using Laplace transforms and standard techniques this mixed boundary value problem can be solved. Let

$$\bar{u}(\xi,\tau) = \int_{-\infty}^{\infty} G(\xi - \eta, \tau) h^+(\eta) d\eta \tag{27}$$

where $h^+(\eta) = h(\eta)\mathbb{I}(\eta > 0)$ and $G(\xi,\tau)$ is the Greens function

$$G(\xi,\tau) = \frac{e^{-\xi^2/2\sigma^2\tau}}{\sqrt{2\pi\sigma^2\tau}}.$$

The solution can then be written in terms of $\bar{u}$ and its image $\bar{u}(-\xi,\tau)$, as:

$$u(\xi,\tau) = \bar{u}(\xi,\tau) + \bar{u}(-\xi,\tau) + \int_{\xi}^{\infty} \bar{u}(-\nu,\tau) e^{\frac{1}{2}\alpha(\nu-\xi)} d\nu \tag{28}$$

A.3 Conversion back to the Black-Scholes variables

Let $\xi = \log b/x$, $t = T - \tau$ and

$$\begin{aligned} V(x,t) &= e^{\frac{1}{2}\alpha\xi - \beta\tau} u(\xi,\tau) \\ V_b(x,t) &= e^{\frac{1}{2}\alpha\xi - \beta\tau} \bar{u}(\xi,\tau) \end{aligned}$$

Now $V_b(x,t)$ solves

$$\begin{aligned}\mathcal{L}V_b(x,t) &= 0 \qquad \text{in } t<T,\ x>0\\ V_b(x,T) &= f(x)\mathbb{I}(x<b)\end{aligned}$$

Further

$$\begin{aligned}\overset{*}{V}_b(x,t) &= (b/x)^{\alpha}V_b(b^2/x,t)\\ &= (b/x)^{\alpha}(x/b)^{\frac{1}{2}\alpha}e^{-\beta\tau}u(\log x/b,\tau)\\ &= (b/x)^{\frac{1}{2}\alpha}e^{-\beta\tau}u(-\log b/x,\tau)\\ &= e^{\frac{1}{2}\alpha\xi-\beta\tau}\bar{u}(-\xi,\tau)\end{aligned}$$

Multiplying equation (28) by $e^{\frac{1}{2}\alpha\xi-\beta\tau}$ gives

$$V(x,t)=V_b(x,t)+\overset{*}{V}_b(x,t)+\int_{\xi}^{\infty}\overset{*}{V}_b(be^{-\nu},t)d\nu$$

The last term can be simplified by using $\xi=\log b/x$ and $y=be^{-\nu}$ to change variables giving

$$V(x,t)=V_b(x,t)+\overset{*}{V}_b(x,t)+\alpha\int_0^x\overset{*}{V}_b(y,t)\frac{dy}{y} \tag{29}$$

A.4 The Equivalent European Payoff

To find the equivalent European payoff evaluate (29) at $t=T$ giving

$$\begin{aligned}V_{\mathrm{eq}}(x,T) &= f(x)\mathbb{I}(x<b)+\overset{*}{f}(x)\mathbb{I}(x>b)+\alpha\int_0^x\overset{*}{f}(y)\mathbb{I}(y>b)\frac{dy}{y}\\ &= f(x)\mathbb{I}(x<b)+\overset{*}{f}(x)\mathbb{I}(x>b)+\alpha\int_b^x\overset{*}{f}(y)\frac{dy}{y}\mathbb{I}(x>b)\end{aligned}$$

□

B The Power Binary

$$P_b^s(x,t;c) = e^{-r\tau}\mathbb{E}\left\{\left(\frac{X_T}{b}\right)^c \mathbb{I}(sX_T > sb)|X_t = x\right\} \tag{30}$$

where under the risk neutral measure

$$X_T = xe^{(r-q-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}z}$$

and $\tau = T - t$ and $z \sim \mathcal{N}(0,1)$. Therefore

$$P_b^s(x,t;c) = e^{-r\tau}(x/b)^c e^{c(r-q-\frac{1}{2}\sigma^2)\tau}\mathbb{E}\{e^{c\sigma\sqrt{\tau}z}\mathbb{I}(sz > -sd')\}$$

where

$$d' = [\log x/b + (r - q - \tfrac{1}{2}\sigma^2)\tau]/\sigma\sqrt{\tau}$$

and $\mathbb{E}$ is the expectation operator.

Using the result

$$\mathbb{E}\{e^{az}F(z)\} = e^{\frac{1}{2}a^2}\mathbb{E}\{F(z+a)\}$$

for any constant a and arbitrary function F, we obtain:

$$\begin{aligned}
P_b^s(x,t;c) &= e^{-r\tau}(x/b)^c e^{c(r-q-\frac{1}{2}\sigma^2)\tau+\frac{1}{2}\sigma^2c^2\tau}\mathbb{E}\{\mathbb{I}(s(z+c\sigma\sqrt{\tau}) > -sd')\} \\
&= (x/b)^c e^{-(r-c(r-q-\frac{1}{2}\sigma^2))\tau+\frac{1}{2}\sigma^2c^2\tau}\mathbb{E}\{\mathbb{I}(sz > s(-d' - c\sigma\sqrt{\tau}))\} \\
&= (x/b)^c e^{-(r-c(r-q-\frac{1}{2}\sigma^2))\tau+\frac{1}{2}\sigma^2c^2\tau}\mathcal{N}[s(d' + c\sigma\sqrt{\tau})]
\end{aligned}$$

This can be expressed as

$$P_b^s(x,t;c) = (x/b)^c\, e^{\mu(c)\tau}\,\mathcal{N}[sd(x,\tau;c)] \tag{31}$$

for all $t < T$, where $\tau = T - t$, $\mathcal{N}(d)$ is the cumulative normal distribution function and

$$\begin{aligned}
\mu(c) &= \tfrac{1}{2}\sigma^2c^2 + (r - q - \tfrac{1}{2}\sigma^2)c - r \\
d(x,\tau;c) &= d' + c\sigma\sqrt{\tau} \;=\; \frac{\log(x/b) + [r - q + (c - \frac{1}{2})\sigma^2]\tau}{\sigma\sqrt{\tau}}.
\end{aligned}$$