ULTRADISCRETE HYPERGEOMETRIC SOLUTIONS TO PAINLEVÉ CELLULAR AUTOMATA

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Abstract. We propose new solutions to the Painlevé cellular automata that do not and cannot be derived using the ultradiscretization method. We consider a lift of these equations to a non-archimedean valuation field in which we may relax the subtraction free framework of previous explorations of the area. Using such a method, we derive a set of hypergeometric solutions to the Painlevé equations which has long been thought impossible to derive.

The area of integrable discrete mappings has blossomed within the last few years. In particular, the interest in discrete version of the Painlevé equations [2]. Such equations are thought to be integrable, and there are standard notions of integrability such as singularity confinement[16] and algebraic entropy [20]. Analogous to the continuous Painlevé equations, the discrete Painlevé equations admit rational solutions [10] and also hypergeometric solutions [11].

The ultradiscrete Painelevé equations are obtained via a limiting process called ultradiscretization [18]. The ultradiscretization process sends a rational function of some variables $f(a_1, \ldots, a_n)$ to a rational function over a semiring in a new set of ultradiscrete variables by the following limit

$$F(A_1, \ldots, A_n) = \lim_{\epsilon \to 0} \epsilon \log(f(a_1, \ldots, a_n)$$

where the new ultradiscrete variables are related to the old variables by the equations $a_i = e^{A_i/\epsilon}$. Such a process successfully related an integrable cellular automata known as the box-ball system [19] to integrable q-difference equations [18]. Roughly speaking it is a transformation bringing the following binary operations to their ultradiscrete equivalent given by

$$a + b \rightarrow \max(A, B) \tag{1a}$$

$$ab \rightarrow A + B \tag{1b}$$

$$a/b \rightarrow A - B \tag{1c}$$

where there is no analog of subtraction. We derive an ultradiscrete Painlevé equation by applying the ultradiscretization method to special subtraction free versions of the q-difference analogs of the Painlevé equations [6]. There are ultradiscrete analogs of all six of the Painlevé equations [5]. There is scant evidence that such equations are integrable, such as the existence of a Lax Pair [15, 8] and some vague analog of a version of singularity confinement[9]. There are known rational solutions to such equations [17]. These are the ultradiscretized subtraction free rational solutions to the q-difference equations. What is not known is the existence of the hypergeometric solutions of the Painlevé equations.

Using a construction of a field K with a non archimedean valuation ν, we lift the Painelvé equations over S to equations over K in which ν acts as "almost" a homomorphism. Under some set of conditions beyond the subtraction free nature of a function, the mapping ν is a homomorphism. We show an application of this is the derivation of the hypergeometric solutions of the third Painlevé cellular automata which we regard as a derivation of a solution which is not derived using the ultradiscretization method.

1. Lifting

As mentioned above, we consider the lift of an equation to a field. The choice of valuation field really depends on the application. One choice popular amongst tropical geometrists seems to be the field of algebraic functions with valuation defined to be the index of the pole or root at 0 [1]. In a previous paper, we used the lifting of a valuation field isomorphic to the inversible max-plus algebra [13] to derive some results regarding the solutions of linear systems over the tropical semiring S [14]. We choose to use the same construction.

Let $Q \subset \mathbb{R}$ be an additively closed and topologically subset of the reals. We let S be the set $Q \cup \{-\infty\}$ and equip S with the normal ordering where $-\infty$ is a minimal element. Letting $a, b \in S$, we equip S with operations $\oplus$ and $\otimes$ which are defined by

$$a \oplus b = \max(a, b) \tag{2a}$$

$$a \otimes b = a + b \tag{2b}$$

These operations are called topical addition and tropical multiplication respectively. Here the 0 element plays the role of the multiplicative identity while $-\infty$ plays the role of the additive identity.

We now remind the reader of a few of the key terms and concepts we shall use. The first is a valuation ring which is defined in the following way

Definition 1. *A valuation ring is a ring R with a valuation $\nu : R \to \mathbb{R} \cup \{-\infty\}$ such that*

(1) $\nu(x) = -\infty$ *if and only if* $x = 0$.
(2) $\nu(xy) = \nu(x) + \nu(y)$.
(3) $\nu(x + y) \leq \nu(x) + \nu(y)$.

we call a valuation non archimedean if it has the property that

$$\nu(x + y) \leq \max(\nu(x), \nu(y))$$

We now construct the ring which is actually a field which we shall use. Let $\Phi = \mathbb{Z}[G]$ be the set of formal $\mathbb{Z}$-linear combinations of a group G equipped with the natural product and addition. We choose G to be the group of reals $\mathbb{R}$ under addition. We let Ω be the field of fractions of Φ. We define the valuation on Ω to be the mapping

$$\nu\left(\frac{\sum n_i x_i}{\sum m_i y_i}\right) = \max(x_i) - \max(y_i)$$

It is clear such a valuation is non archimedean. We have a lifting of any equation over S to Ω by the mapping

$$A \to 1(A)$$

which we call the standard lift. It also has the property that the operations commute via the standard lift commutes with ν. We also not that if we restrict our attention to a special sub-semiring given by

$$\Omega_0 = \left\{\frac{\sum n_i x_i}{\sum m_i y_i} | n_i, m_i \in \mathbb{N}\right\}$$

then ν acts as a homomorphism of semirings. It is clear that any discrete dynamical system over S has a corresponding lifted dynamical system over Ω_0 in which anything that can be said for Ω_0 can be said for S through the valuation.

2. Derivation of hypergeometric solutions

The continuous Painlevé equations possess hypergeometric solutions given by Gauss's hypergeometric function. In the case of discrete equations, we require q-hypergeometric functions that come as generalizations of Hienes hypergeometric functions[11]. Such equations then generalize to several variables giving an Askey-scheme of hypergeometric orthogonal polynomials and its q-analogs[12]. These are given by

$$ {}_r\phi_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \; | q : z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} (-1)^{(1+r-s)k} q^{(1+r-s)\binom{k}{2}} \frac{z^k}{(q;q)_k} $$

where

$$ (a_1, \ldots, a_n; q)_k = (a_1; q) \ldots (a_n; q) \quad \text{and} \quad (a; q) = (1-a)(1-aq)\ldots(1-aq^{k-1}) $$

These functions have the property that they limit to Gauss' hypergeometric function as $q \to 1$. We will predominantly using the form in which $r = s-1$ in which case the above form will simplify to the following

$$ {}_{s+1}\phi_s \left(\begin{matrix} a_1, \ldots, a_{s+1} \\ b_1, \ldots, b_s \end{matrix} \; | q : z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \frac{z^k}{(q;q)_k} $$

such a case is called well-posed. We begin by examining an equation in which the hypergeometric solutions are known. We start with the q-P$_{III}$ given by

$$ \overline{w}\underline{w} = \frac{a_3 a_4 (w + a_1 t)(w + a_2 t)}{(w + a_3)(w + a_4)} \tag{3} $$

which was introduced in [2]. This equation can also be derived as the necessary condition for a connection preserving deformation, and is related to q-P$_{VI}$ [7]. The equation is known to has hypergeometric solutions of type ${}_2\phi_1$. If $a_2 = \frac{q a_1 a_3}{a_4}$ then we find that (3) linearizes to give the following the Riccati equation

$$ \overline{w} = -\frac{q t a_1 a_3 + a_4 w}{a_3 + w} \tag{4} $$

We make the substitution $w = sx$ where $s = \sqrt{t}$ and $p = \sqrt{q}$ to derive the following equation

$$ \overline{x} = -\frac{q s a_1 a_3 + a_4 x}{a_3 + s x} \tag{5} $$

by using the Cole-Hopf transformation, $x = u/v$, we reduce this to the following linear system

$$ \begin{aligned} \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} &= \left(\begin{pmatrix} a_4 & 0 \\ 0 & a_3 \end{pmatrix} + \begin{pmatrix} 0 & q a_1 a_3 \\ 1 & 0 \end{pmatrix} s \right) \begin{pmatrix} u \\ v \end{pmatrix} \\ \overline{\Psi} &= (A_0 + A_1 s)\Psi \end{aligned} $$

which can be explicitly solved in terms of the work on linear systems of q-difference equations in terms of ${}_2\phi_1$ as fundamental solutions Ψ_∞ and Ψ_0. Explicit solutions can be obtained via an application of the work of LeCain [3]. This gives is that the fundamental solution of Ψ is

$$ Y(x,t) = \Gamma_p \left(1 + \frac{t}{a_1 a_3 \sqrt{q}} \right) \begin{pmatrix} \Phi_1(s) & \Phi_2(s) \\ \frac{\Phi_1(qs)}{\sqrt{a_1 a_4}} & \frac{\Phi_2(ps)}{a_3}\sqrt{\frac{a_4}{q a_1}} \end{pmatrix} \operatorname{diag}\left((-a_4/p)^{\log_p s}, a_3^{\log_p s} \right) $$

where the Φ functions are given by

$$\Phi_1(s) ={}_2\phi_1\left(\begin{array}{c} \frac{1}{\sqrt{a_1a_4}}, a_3\sqrt{\frac{qa_1}{a_4}} \\ q\frac{a_3}{a_4} \end{array} \middle| \sqrt{q} : s\right)$$

$$\Phi_2(s) ={}_2\phi_1\left(\begin{array}{c} \frac{1}{a_3}\sqrt{\frac{a_4}{qa_2}}, \sqrt{a_1a_4} \\ \frac{a_4}{a_3} \end{array} \middle| \sqrt{q} : s\right)$$

and Γ_q is the q-gamma function that can be found in [12]. The question we wish to address, is whether something can be said for the correspondence with these explicit solutions and the ultradiscrete version of the same equation. Ideally, one would expect that if one has a solution, then the ultradiscretization of such a solution of yields a solution of the ultradiscrete equation. Yet since the evolution of the any solution is given by (4) and (5), it is apparent that we must leave the subtraction free framework dominant in the field. The ultradiscretization method gives us the discrete dynamical system

$$\begin{aligned} \overline{W} + \underline{W} &= \max(W, A_1 + T) + \max(W, A_2 + T) \\ &\quad - \max(W, A_3) - \max(W, A_4) \end{aligned} \tag{6}$$

The equation is known to possess rational solutions such as those obtained via Bäcklund transformations [5] of the solution $W = 0$ for $A_1 = A_2 = A_3 = A_4 = 0$. It is also known to admit the Affine Weyl group $A_1^{(1)} \times A_1^{(1)}$ as a set of Bäcklund transformations. By following such a procedure, one obtains solutions that are simply the rational solutions that one may obtain through the ultradiscretization of known rational solutions of (3).

We lift equation (6) to Ω via the standard lift. Although it is preferable to lift the dynamical variable W not to $1(W)$, but rather just W over Ω since the variable will often be an expression over Ω. The lifting gives the equation

$$\overline{W}\underline{W} = \frac{1(A_3)1(A_4)(W + 1(A_1 + T))(W + 1(A_2 + T))}{(W + 1(A_3))(W + 1(A_4))} \tag{7}$$

If W is an element of Ω_0, then we have that the valuation ν brings such an equation to (6). Here we note that any rational expression over Ω_0 is the manifestation of a subtraction free expression over the reals, making it apparent that the set of rational solutions can be expressed as the mappings of solutions of the lifted equation that exist in Ω_0. In general however, we have a set of inequalities given by breaking up the parts. Given $W = \sum n_i(W_i)$,we have the following list of inequalities

$$\nu(W + 1(A_1 + T)) \leq \max(W_i, A_1 + T) \tag{8a}$$
$$\nu(W + 1(A_2 + T)) \leq \max(W_i, A_2 + T) \tag{8b}$$
$$\nu(W + 1(A_3)) \leq \max(W_i, A_3) \tag{8c}$$
$$\nu(W + 1(A_4)) \leq \max(W_i, A_4) \tag{8d}$$

Although this does not provide upper or even lower bounds for $\overline{W}$ or $\underline{W}$, imposing equality is a relaxation of the subtraction free framework. We note that imposing some equality of the differences is also a valid relaxation. It is also rather artificial way of regaining singularities in ultradiscrete systems where any lift of an ultradiscrete system can possess singularities in $-\Omega_0$. If it is possible to show the equality or equality of the difference of arguments in (8) under certain specific conditions, we have an expression for some solution W of (7) over Ω but not Ω_0 in which equality of each of the statements for all time, then $\nu(W)$ is indeed a solution or (6). Furthermore, $\nu(W)$ is a solution that is in no way analogous to an expression derivable via standard ultradiscretization.

(a) Hypergeometric solution for parameters $A_1 = -2, A_2 = -1, A_3 = 0, A_4 = 0, Q = 1$ and $W_0(0) = 0$.

(b) Hypergeometric solution for parameters $A_1 = 0, A_2 = 2, A_3 = 1, A_4 = 0, Q = 1$ and $W_0(0) = 1$.

FIGURE 1. 2 Hypergeometric solutions to (6).

As an example, let us consider the linearization of (7). We should be able to derive solutions in a similar manner to the work above for (3). We substitute the following lifted ultradiscretized Riccati equation over Ω

$$\overline{W} = \frac{1(\alpha) + 1(\beta)W}{1(\gamma) + 1(\delta)W} \tag{9}$$

We find conditions for the (9) to describe the evolution for (7). These conditions are that $A_2 = Q + A_1 + A_3 - A_4$ in which we derive following linear system over Ω

$$\overline{W} = \frac{-1(A_1 + A_3 + T + Q) - 1(A_4)W}{1(A_3) + W} \tag{10a}$$

$$\underline{W} = \frac{-1(A_1 + A_3 + T) - 1(A_3)W}{1(A_4) + W} \tag{10b}$$

In which we may derive inequalities for (10a) given by

$$\nu(-1(A_1 + A_3 + T + Q) - 1(A_4)W) \leq A_4 + \max(A_2 + T, \nu(W)) \tag{11a}$$

$$\nu(1(A_3) + W) \leq \max(A_3, \nu(W)) \tag{11b}$$

and for (10b) we have

$$\nu(-1(A_1 + A_3 + T) - 1(A_3)W) \leq A_3 + \max(A_1 + T, \nu(W)) \tag{12a}$$

$$\nu(1(A_4) + W) \leq \max(A_4, \nu(W)) \tag{12b}$$

where if each case if (11) and (12), the equality holds or even the equality of the difference in inequality holds, then $\nu(W)$ is a solution of (6). Given appropriate initial conditions, it is possible to make it so that equality hold for all time. If equality does hold, then the positive and negative evolution should be given simply by two ultradiscrete Riccati type equations for each direction

$$\overline{W} = \max(A_1 + A_3 + T + Q, A_4 + W) - \max(A_3, W) \tag{13a}$$

$$\underline{W} = \max(A_1 + A_3 + T, A_3 + W) - \max(A_4, W) \tag{13b}$$

Any resultant solution is not any solution that can come from the ultradiscretization of a solution of (3) but rather a hypergeometric solution of (6) which is unique to the semiring setting S. One may take the view that this is the mapping of something that is not a solution (3), but rather some function in which the way it is not a solution is hidden in the loss of information in the ultradiscretization method. Figure 2 shows two examples of evolutions in which the functions defined by (13) and (6) coincide.

Over Ω however, given a solution in which the evolution governed by (6) and (13) coincide, we consider (10) as having a solution defined over Ω which may be considered an analog of the hypergeometric function over a field. From which we shall be able to derive explicit expressions for our new ultradiscrete hypergeometric functions. Since the expression for the evolution is given by (10), we can use a Cole-Hopf transformation over Ω, by substituting $W = U/V$, we derive the linear system for specific examples that satisfy the prerequisites for the existence of the hypergeometric solution. One is the system defined by the conditions $A_1 < -2Q, A_3 = 0, A_4 = 0$ where $W(0) = 0$. Under these assumptions, we let $A_1 = -(r+1)Q$ for some $r \in \mathbb{R}$, we have the linear system

$$\begin{pmatrix} \overline{U} \\ \overline{V} \end{pmatrix} = \begin{pmatrix} -1(0) & -1(T - rQ) \\ 1(0) & 1(0) \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = Y(1(T)1(Q)) = A(1(T))Y(1(T))$$

This is a linear system in which one can substitute the expression in the framework of []. We may solve this explicitly in such a case

$$Y(1(T)) = (I + Y_1 1(T) + Y_2 1(2T) + \ldots)\mathrm{diag}(-1(0)^{\frac{T}{Q}}, 1(0))$$

Giving the expansion

$$Y_k = \begin{pmatrix} 0 & \frac{1(Q)^{-kr}(1(Q)^k - 1(0))}{(1(Q):1(Q))_{k+1}(-1(Q);1(Q))} \\ 0 & \frac{1(Q)^{-kr}}{(1(Q):1(Q))_k(-1(Q);1(Q))} \end{pmatrix} \tag{14}$$

where we have the Pochammer symbol just working the same way over Ω in which

$$(A;B) = (1 - A)(1 - AB^2)\ldots(1 - AB^{k-1})$$

which if we follow through to then the hypergeometric representation of these objects, we may express the solution as

$$Y(t) = \begin{pmatrix} 1(0) & {}_2\phi_1\left(\begin{matrix} 0,0 \\ -1(0) \end{matrix} |1(Q); 1(T - rQ) \right) - 1(0) \\ 0 & {}_2\phi_1\left(\begin{matrix} 0,0 \\ -1(Q) \end{matrix} |1(Q); 1(T - rQ) \right) \end{pmatrix}$$

By stating explicitly the initial condition of $(1(0), 1(0))$ the solution is simply the ratio of these hypergeometric functions. We calculate the valuation of this solution from (14), and the valuation of hypergeometric solution results in the following formula for the hypergeometric over S

$$\nu \left(\frac{{}_2\phi_1\left(\begin{matrix} 0,0 \\ -1(0) \end{matrix} |1(Q); 1(T - rQ) \right)}{{}_2\phi_1\left(\begin{matrix} 0,0 \\ -1(Q) \end{matrix} |1(Q); 1(T - rQ) \right)} \right) = \begin{matrix} \max_{k\in\mathbb{N}}(kQ - krQ - (k+1)kQ - kX) - \\ \max_{k\in\mathbb{N}}(-krQ - (k+1)kQ - kX) \end{matrix}$$

which for all $r > 1$ then coincides precisely with the calculation of the solution given by calculating the evolution given the appropriate initial conditions. This shows that not only does such a hypergeometric solution have an evolution defined by some Riccati equation, but also can be derived as a valuation of a hypergeometric equation in a higher space.

3. Conclusion

Despite the existence of hypergeometric type solutions, the conditions in which the valuation yields equality of both sides is unclear. It is quite clear that some versions of u-P$_{II}$ should but do not contain any hypergeometric solutions. This is not however very surprising, since there exists versions of q-P$_{II}$ which do not even have any apparent rational solutions. Further investigation is required in order to determine precise conditions for when an ultradiscrete equation has what we consider a Hypergeometric solution. It seems that any specific example of a hypergeometric solution is very special. Overall such methods allow us to consider solutions of various ultradiscrete equations in which the solution does not come as an ultradiscretized solution of the q-difference equation it was derived from.

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