KOSTANT THEOREM FOR SPECIAL FILTERED ALGEBRAS

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Abstract. A famous result of Kostant states that the universal enveloping algebra of a semisimple complex Lie algebra is a free module over its center. We prove an analogue of this result for the class of so-called special filtered algebras and apply it to show the freeness over its center of the restricted Yangian and the universal enveloping algebra of the restricted current algebra, associated with the general linear Lie algebra.

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1. Introduction

A well-known theorem of Kostant [K] says that for a complex semisimple Lie algebra g the universal enveloping algebra U(g) is a free module over its center. For the Yangian $\mathrm{Y}(\mathrm{gl}_n)$ of the general linear Lie algebra gl_n the freeness over the center was shown in [MNO]. Another example of such situation gives the Gelfand-Tsetlin subalgebra Γ of $\mathrm{U}(\mathrm{gl}_n)$ (see [DFO], [O1]). It was shown in [O2] that $\mathrm{U}(\mathrm{gl}_n)$ is a free left (right) $\Gamma-$module. The knowledge of the freeness of a given algebra over its certain subalgebra often allows to establish other important results, e.g. to find the annihilators of Verma modules [Di], to prove the finiteness of the number of liftings to an irreducible module over $\mathrm{U}(\mathrm{gl}_n)$ from a given character of the Gelfand-Tsetlin subalgebra ([O2]), etc.

In [O2] was introduced a technique which generalizes Kostant methods ([K], see also [G1]) and which allows to study the universal enveloping algebras of Lie algebras as modules over its certain commutative subalgebras. In the present paper we develop a graded version of the technique from [O2] which can be applied to a large class of filtered associative algebras. In particular, we apply our result to the restricted Yangians $\mathrm{Y}_p(\mathrm{gl}_n)$ of any level p and to the universal enveloping algebra of restricted current algebras of any level m, associated with the general linear Lie algebra, and show that these algebras are free as modules over their centers. Particular case of this statement for $\mathrm{Y}_p(\mathrm{gl}_n)$ ($p = 2$) was considered by Geoffriau ([G1]) who showed that the universal enveloping algebra of a Takiff algebra is free over its center.

The structure of the paper is the following. In Section 2 we collect necessary known facts about Koszul complexes and complete intersections. In Section 3 we consider a filtered algebra U containing some commutative subalgebra Γ. We find sufficient conditions that guarantee the lifting of every character of Γ to a simple U-module. In Section 4 we establish the key result for special filtered algebras (Theorem 1) showing their freeness over commutative subalgebras generated by a sequence of elements whose graded images form a complete intersection for the associated graded algebra (equivalently, the corresponding characteristic variety is equidimensional of minimal possible dimension). This generalizes the Kostant theorem for semisimple Lie algebras. In Section 5 we apply Theorem 1 to the restricted Yangian of any level p for gl_n and to the universal enveloping algebras of the restricted current algebras of any level m for gl_n (Theorem 2 and 3).

Throughout the paper we fix an algebraically closed field $\Bbbk$ of characteristic 0. For an affine $\Bbbk$-algebra Λ (i.e. associative and commutative finitely generated $\Bbbk$-algebra) we denote by $\operatorname{Specm}\Lambda$ the variety of all maximal ideals of Λ. If Λ is a polynomial algebra in n variables then we identify $\operatorname{Specm}\Lambda$ and $\Bbbk^n$. $\dim\Lambda$ ($=\dim\operatorname{Specm}\Lambda$) means the Krull dimension. For an ideal $I\subset\Lambda$ denote by $\mathsf{V}(I)\subset\operatorname{Specm}\Lambda$ the set of all zeroes of I, $\mathsf{V}(I)=\{\mu\in\operatorname{Specm}\Lambda | I\subset\mu\}$. If I is generated by $f_1,\dots,f_s$ then we write $I=(f_1,\dots,f_s)$ and $\mathsf{V}(I)=\mathsf{V}(f_1,\dots,f_s)$. The word "graded" always means "positively graded".

2. Koszul complex and complete intersections

The main sources of references related this chapter are [Ei], [Ma] and [B].

Let U be an associative (not necessary commutative) algebra and M an $U-\Bbbk[X_1,\dots,X_t]$ bimodule. Recall ([B], §9) that the associated *Koszul complex* $\mathbf{K}_{\cdot}$ ($=\mathbf{K}_{\cdot}(M)$) of left U-modules is defined as follows. Let $e_1,\dots,e_t$ be a standard basis of $\Bbbk^t$. Set $\mathbf{K}_i=0$ for $i<0$ and $\mathbf{K}_i=M\otimes_{\Bbbk}\bigwedge\limits_{\Bbbk}^{i}\Bbbk^t$ for each $i\geqslant 0$. Define the differential $d_i:\mathbf{K}_i\to\mathbf{K}_{i-1}$ on the element $m\otimes(e_{j_1}\wedge\dots\wedge e_{j_i})$ of $\mathbf{K}_i$, $1\leqslant j_1<\dots<j_i\leqslant t$, $m\in M$ as:

$$d_i(m\otimes(e_{j_1}\wedge\dots\wedge e_{j_i}))=\sum_{k=1}^{i}(-1)^{k-1}m\cdot X_{j_k}\otimes(e_{j_1}\wedge\dots\widehat{e}_{j_k}\dots\wedge e_{j_i}),\tag{1}$$

where $\widehat{e}_{j_k}$ means the omission of e_{j_k}. Let $\mathsf{g}=\{g_1,\dots,g_t\}\subset U$ be a sequence of mutually commuting elements, $M=U$ with the natural structure of left U-module and $a\cdot X_i=ag_i$, $a\in U$, $i=1,\dots,t$. We denote the corresponding Koszul complex by $\mathbf{K}_{\cdot}(\mathsf{g},U)$.

The sequence g is called a *complete intersection for* U (we say simply *complete intersection* if U is fixed) provided for $\mathbf{K}_{\cdot}(\mathsf{g},U)$ holds $H_i(\mathsf{g},U)=0$ for $i\neq 0$ and $H_0(\mathsf{g},U)\neq 0$, where $H_i(\mathsf{g},U)$ is the i-th homology of $\mathbf{K}_{\cdot}(\mathsf{g},U)$. Note that $H_0(\mathsf{g},U)\simeq U/(Ug_1+\dots+Ug_t)$.

The sequence $g_1,\dots,g_t$ in an affine algebra Λ is called *regular*, provided the class of g_i is not a zero divisor and not invertible in $\Lambda/(g_1,\dots,g_{i-1})$ for any $i=1,\dots,t$. A regular sequence is a complete intersection for Λ. If Λ is graded and $g_1,\dots,g_t$ are homogeneous then the properties to be regular and to be a complete intersection are equivalent ([Ma], Theorem 16.5,(ii) or [B], §7, Corollary 2).

An affine algebra Λ is called a *complete intersection, shortly CI* , provided $\Lambda\simeq A/I$, where A is a polynomial algebra $\Bbbk[X_1,\dots,X_n]$ and the ideal I is generated by a regular sequence $\mathsf{g}=\{g_1,\dots,g_t\}$ for A.

An affine algebra Λ is called *Cohen-Macauley, shortly CM,* provided for every ideal $I\subset\Lambda$ holds $\operatorname{depth}I=\operatorname{codim}I$, where $\operatorname{depth}I$ is a length of maximal regular sequence, contained in I and $\operatorname{codim}I$ is a minimal codimension of irreducible components of $\mathsf{V}(I)$ in $\operatorname{Specm}\Lambda$. If $g_1,\dots,g_t$ is a sequence of elements in a CM algebra Λ such that for $I=(g_1,\dots,g_t)$ $\mathsf{V}(I)$ is equidimensional and $\operatorname{codim}I=t$ then Λ/I

is CM ([Ei], Proposition 18.13). In particular, any CI algebra is a CM algebra. *We will assume that a CI algebra $\Lambda \simeq A/I$ is reduced, i.e. I is a radical ideal.*

Proposition 1. *Let Λ be an affine algebra of dimension n and $\mathsf{g} = \{g_1, \dots, g_t\}$, $0 \leqslant t \leqslant n$ a sequence of elements of Λ.*

(1) *([B], §9, Prop. 6) Let g be a complete intersection for Λ, $L \in GL_t(\Lambda)$ and $(h_1, \dots, h_t) = (g_1, \dots, g_t) \cdot L$. Then the sequence $h_1, \dots, h_t$ is a complete intersection for Λ.*
(2) *Let Λ be a graded CM algebra, g consists of homogeneous elements and $\operatorname{codim} I = t$, $I = (g_1, \dots, g_t)$. Then the sequence g is regular.*
(3) *Let Λ be a CM algebra. Then g is a complete intersection for Λ if and only if the variety $\mathsf{V}(g_1, \dots, g_t)$ is equidimensional of dimension $n - t$.*
(4) *Let Λ be a CM algebra. Then g is a regular sequence if and only if every variety $\mathsf{V}(g_1, \dots, g_i)$ is equidimensional of dimension $n - i$, $i = 1, \dots, t$.*
(5) *Let Λ be a graded CM algebra and g be a complete intersection for Λ, consisting of homogeneous elements. Then any subsequence of g is a complete intersection.*

Proof. It follows from [Ma], Theorem 16.8, that for a sequence $\mathsf{g} = \{g_1, \dots, g_t\}$ in Λ, such that $I = (g_1, \dots, g_t) \neq \Lambda$ holds $\operatorname{depth} I = t - q$, where $q = \max_i \{H_i(\mathrm{g}, \Lambda) \neq 0\}$. Hence g is a complete intersection if and only if $\operatorname{depth} I = t$ and, in assumption Λ is CM, if and only if $\operatorname{codim} I = t$. On other hand $\operatorname{codim} I \geqslant t$ and we have the equality if and only if $\mathsf{V}(I)$ is equidimensional of minimal possible dimension $n - t$. This implies immediately (2) and (3).In (4) the statement "only if" is obvious. The statement "if" we prove by induction on i. The base of induction $i = 0$ is obvious. Denote $\Lambda_i = \Lambda/(g_1, \dots, g_i)$ and denote by g the class of g_{i+1} in Λ_i. Obviously, g is not invertible. Let $J = \Lambda_i g$. Then in Λ_i holds $1 = \operatorname{codim} J = \operatorname{depth} J$, hence J contains an element which is not a zero divisor, hence g is not a zero divisor. Statement (5) follows from (3) and (4), Since every component of $\mathsf{V}(g_{i+1})$ intersects with every component of $\mathsf{V}(g_1, \dots, g_i)$, $i = 1, \dots, t$, if for some i $\dim \mathsf{V}(g_1, \dots, g_i) > n - i$, then $\dim \mathsf{V}(g_1, \dots, g_t) > n - t$. $\square$

The following result follows immediately from Proposition 1, (3).

Lemma 2.1. *Let $A = \Bbbk[X_1, \dots, X_n]$ be the polynomial algebra, $G_1, \dots, G_t \in A$. A sequence $X_1, \dots, X_r, G_1, \dots, G_t$ is a complete intersection for A if and only if the sequence $g_1, \dots, g_t$ is a complete intersection for $\Bbbk[X_{r+1}, \dots, X_n]$, where $g_i(X_{r+1}, \dots, X_n) = G_i(0, \dots, 0, X_{r+1}, \dots, X_n)$, $i = 1, \dots, t$.*

3. Filtered algebras

Let U be a *filtered* algebra, i.e an associative algebra over $\Bbbk$, endowed with an increasing exhausting filtration $\{U_i\}_{i \geqslant 0}$, $U_0 = \Bbbk$, $U_i U_j \subset U_{i+j}$, $U = \cup_{i \geqslant 0} U_i$. For $u \in U_i \setminus U_{i-1}$ set $\deg u = i$ ($U_{-1} = \{0\}$). Let $\overline{U} = \operatorname{gr} U$ be the associated graded algebra $\overline{U} = \bigoplus_{i=0}^{\infty} U_i/U_{i-1}$. For $u \in U$ denote by $\overline{u}$ its image in $\overline{U}$ and for a subset $S \subset U$ denote $S_i = S \cap U_i$, $\overline{S} = \{\overline{s} \mid s \in S\} \subset \overline{U}$. Set $\overline{U}_{(i)} = U_i/U_{i-1}$ and for any $T \subset \overline{U}$ denote $T_{(i)} = T \cap \overline{U}_{(i)}$. Given a graded algebra $U = \bigoplus_{i \geqslant 0} U_{(i)}$, $U_{(\Bbbk)} = 0$ we always assume in U the associated filtration $\{U_i = \bigoplus_{j \leqslant i} U_{(j)}, i \geqslant 0\}$ identifying U and $\overline{U}$.

Lemma 3.1. *Let* $\mathsf{g} = \{g_1, \ldots, g_t\}$ *be a sequence of mutually commuting elements of a filtered algebra* U*, such that* $\overline{\mathsf{g}} = \{\overline{g}_1, \ldots, \overline{g}_t\}$ *is a complete intersection for* $\overline{U}$*. Then*

(1) g *is a complete intersection for* U*;*
(2) *if* $I = Ug_1 + \cdots + Ug_t$ *then* $\overline{I} = \overline{U}\overline{g}_1 + \cdots + \overline{U}\overline{g}_t$.

Proof. Endow the complex $\mathbf{K}_{\cdot} = \mathbf{K}_{\cdot}(\mathsf{g}, U)$ with a filtration F:

$$\cdots \subset F^{p-1}\mathbf{K}_i \subset F^p\mathbf{K}_i \subset F^{p+1}\mathbf{K}_i \subset \cdots \subset \mathbf{K}_i, \text{ where } p, i \in \mathbb{Z},$$

by setting $\deg u \otimes (e_{j_1} \wedge \cdots \wedge e_{j_i}) = \deg u + \deg g_{j_1} + \cdots + \deg g_{j_i}$ $u \in U$, $1 \leqslant j_1 < \cdots < j_i \leqslant t$. This filtration is exhausting, bounded below and turns $\mathbf{K}_{\cdot}$ into a filtered complex. Analogous grading turns $\mathbf{K}_{\cdot}(\overline{\mathsf{g}}, \overline{U})$ into a complex of graded modules. By the conditions $H_i(\overline{\mathsf{g}}, \overline{U}) = 0$ for $i \neq 0$ and the p-th graded component of 0-homology is $H_0(\overline{\mathsf{g}}, \overline{U})_{(p)} = \overline{U}_{(p)} / \sum\limits_{i=1}^{t} \overline{U}_{(p-\deg g_i)}\overline{g}_i$.

Consider associated with the filtration F spectral sequence $(E^r,\ d^r)$ with a first term (E^1, d^1), $E^1_{p,q} = H_{p+q}(F^p\mathbf{K}_{\cdot}/F^{p+1}\mathbf{K}_{\cdot})$ ([W], 5.4). The graded complex $\{F^p\mathbf{K}_{\cdot}/F^{p+1}\mathbf{K}_{\cdot}\}_{p\in\mathbb{Z}}$ is isomorphic to the graded complex $\mathbf{K}_{\cdot}(\overline{\mathsf{g}}, \overline{U})$. Hence, the nonzero components of E^1 are among graded components of $H_0(\overline{\mathsf{g}}, \overline{U})$ $E^1_{p,-p}, p \in \mathbb{Z}$, besides all differentials in E^1 equal 0. It gives us $E^1 \simeq E^2 \simeq \cdots \simeq E^\infty$. Since (E^r, d^r) converges to $H(\mathsf{g}, U)$ it follows that all homologies $H_i(\mathsf{g}, U) = 0$ for $i \neq 0$ which proves the statement (1).

To prove the statement (2) we consider subcomplexe $\mathbf{K}'_{\cdot}(\mathsf{g}, U) \subset \mathbf{K}_{\cdot}(\mathsf{g}, U)$ with $\mathbf{K}'_i(\mathsf{g}, U) = \mathbf{K}_i(\mathsf{g}, U)$, $i \neq 0$, $\mathbf{K}'_0(\overline{g}, U) = 0$, and a subcomplex $\mathbf{K}'_{\cdot}(\overline{\mathsf{g}}, \overline{U}) \subset \mathbf{K}_{\cdot}(\overline{\mathsf{g}}, \overline{U})$ with $\mathbf{K}'_i(\overline{\mathsf{g}}, \overline{U}) = \mathbf{K}_i(\overline{\mathsf{g}}, \overline{U})$, $i \neq 0$, $\mathbf{K}'_0(\overline{\mathsf{g}}, \overline{U}) = 0$. The only nonzero homologies of these complexes are: $H_1(\mathbf{K}'_{\cdot}(\overline{\mathsf{g}}, \overline{U})) \simeq U\overline{g}_1 + \cdots + U\overline{g}_t$ and $H_1(\mathbf{K}'_{\cdot}(\mathsf{g}, U)) \simeq I$ respectively. The filtration on $\mathbf{K}_{\cdot}$ induces a filtration on $\mathbf{K}'_{\cdot}$. As above, considering the corresponding spectral sequences we conclude that $\overline{I} = U\overline{g}_1 + \cdots + U\overline{g}_t$. □

In commutative case we use more precise statement.

Lemma 3.2. *Let* Λ *be a filtered affine algebra* ($\Lambda = \overline{\Lambda}$) *and* $\mathsf{g} = \{g_1, \ldots, g_t\} \subset \Lambda$ *a sequence such that* $\overline{\mathsf{g}}$ *is a complete intersection for* $\overline{\Lambda}$*. Then* g *is a regular sequence.*

Proof. By Proposition 1, (2) the sequence $\overline{\mathsf{g}}$ is regular. Following Lemma 3.1, (2), for the ideal $I_k = (g_1, \ldots, g_k)$ holds $\overline{I}_k = (\overline{g}_1, \ldots, \overline{g}_k)$ for any $k = 1, \ldots, t$. Let $g \notin (g_1, \ldots, g_i)$ be an element of minimal degree, such that $g_{i+1}g \in (g_1, \ldots, g_i)$. Then $\overline{g}_{i+1}\overline{g} \in (\overline{g}_1, \ldots, \overline{g}_i)$. By regularity $\overline{\mathsf{g}}$ we obtain $\overline{g} \in (\overline{g}_1, \ldots, \overline{g}_i)$, hence there exists $h \in (g_1, \ldots, g_i)$, such that $\deg(g - h) < \deg g$, besides $g_{i+1}(g - h) \in (g_1, \ldots, g_i)$. That proves the lemma. □

Lemma 3.3. *Let* Λ *be a graded affine algebra,* $\{\Lambda_i, i \in \mathbb{Z}\}$ *the associated filtration,* $\mathsf{g} = \{g_1, \ldots, g_t\}$ *a complete intersection for* Λ *consisting of homogeneous elements,* $\Gamma = \Bbbk[g_1, \ldots, g_t]$, $\mu = (\mu_1, \ldots, \mu_t) \in \Lambda^t$ *such that* $\deg \mu_i < \deg g_i$, $i = 1, \ldots, t$, $I_\mu = (g_1 - \mu_1, \ldots, g_t - \mu_t)$*. Then*

(1) $I_\mu \cap \Lambda_m = \sum\limits_{i=1}^{t} \Lambda_{m-d_i}(g_i - \mu_i)$ *where* $d_i = \deg g_i$, $i = 1, \ldots, t$.
(2) *The sequence* $\{g_1 - \mu_1, \ldots, g_t - \mu_t\}$ *is a complete intersection for* Λ*, moreover, it is regular.*

(3) $\overline{I}_\mu$ *is an ideal generated by* $g_1, \dots, g_t$*, in particular* $I_\mu \neq \Lambda$.
(4) *If* Λ *is CM, then the regular map* $p_{\mathsf{g}} : \operatorname{Specm}\Lambda \longrightarrow \operatorname{Specm}\Gamma = \Bbbk^t$ *induced by the inclusion* $i_{\mathsf{g}} : \Gamma \hookrightarrow \Lambda$ *is an epimorphism and* $\dim p_{\mathsf{g}}^{-1}(\mu) = n - t$ *for any* $\mu \in \operatorname{Specm}\Gamma$.

Proof. Statements (1) and (3) follow from Lemma 3.1, (2), while the statement (2) follows from Lemma 3.2. Applying to $\mu = (\mu_1, \dots, \mu_t) \in \Bbbk^t$ (2) of this lemma and Proposition 1, (3) we obtain, that $p_{\mathsf{g}}^{-1}(\mu) = \mathsf{V}(g_1 - \mu_1, \dots, g_t - \mu_t)$ is equidimensional of dimension $n - t$. This implies (4). □

Proposition 2. *Let* U *be a filtered algebra,* $\mathsf{g} = \{g_1, \dots, g_t\}$ *be a sequence of mutually commuting elements from* U *such that* $\{\overline{g}_1, \dots, \overline{g}_t\}$ *is a complete intersection for* $\overline{U}$ *and* $\mu_i \in U$ *be such, that* $\deg g_i > \deg \mu_i$, $i = 1, \dots, t$. *Denote* $\mu = (\mu_1, \dots, \mu_t)$, $I_\mu = U(g_1 - \mu_1) + \dots + U(g_t - \mu_t)$ *and assume that the elements* $\{g_i - \mu_i | i = 1, \dots, t\}$ *are mutually commuting (e.g. it holds if* $\mu_i \in \Bbbk$ *for all* i*). Then*

(1) $I_\mu \cap U_m = \sum_{i=1}^{t} U_{m-d_i}(g_i - \mu_i)$ *where* $d_i = \deg g_i$.
(2) $\overline{I}_\mu = \overline{U}\,\overline{g}_1 + \dots + \overline{U}\,\overline{g}_t$*, in particular* $I_\mu \neq U$.
(3) *There exists a simple left* $U-$*module* M *generated by* $m \in M$ *such that for every* g_i *holds* $g_i m = \mu_i m$.
(4) *For any* $\nu \in \operatorname{Specm}\Gamma$ *there exists a simple left* $U-$*module* M *generated by* $m \in M$ *such that for every* $\gamma \in \Gamma$ *holds* $\gamma \cdot m = \nu(\gamma) m$ *(i.e.* ν *lifts to a simple* U*-module).*

Proof. Statements (1), (2) follow from Lemma 3.1, (2). To obtain from (2) the statement (3) we consider a maximal left ideal $\mathfrak{m}$ in U, containing I_μ, and set $M = U/\mathfrak{m}$, $m = 1 + \mathfrak{m}$. Statement (4) follows from (3) applied to $\mu_i = g_i(\nu)$, $i = 1, \dots, t$. □

Corollary 1. *Let* U *be a filtered algebra such that the associated graded algebra* $\overline{U}$ *is CM. If* $g_1, \dots, g_t$ *are mutually commuting elements of* U *such that* $\mathsf{V}(\overline{g}_1, \dots, \overline{g}_t) \subset \operatorname{Specm}\overline{U}$ *is equidimensional of codimension* t*, then every* $\mu \in \operatorname{Specm}\Bbbk[g_1, \dots, g_t]$ *lifts to a simple* U*-module.*

Proof. Follows from Proposition 2, (4) and Proposition 1, (3). □

4. An analogue of Kostant theorem

A filtered algebra U such that $\overline{U}$ is a reduced CI algebra will be called *special filtered.* Due to the Poincaré-Birkhoff-Witt theorem the universal enveloping algebra of a finite-dimensional Lie algebra is special filtered. In this section we prove the following analogue of Kostant theorem ([K]) for special filtered algebras.

Theorem 1. *Let* U *be a special filtered algebra,* $g_1, \dots, g_t \in U$ *mutually commuting elements such that* $\overline{g}_1, \dots, \overline{g}_t$ *is a complete intersection for* $\overline{U}$, $\Gamma = \Bbbk[g_1, \dots, g_t]$. *Then* U *is a free left (right)* $\Gamma-$*module.*

We start from the following known fact

Lemma 4.1. *Let V be a finite-dimensional space, $Y \subset V$ a subspace, $Gr_r(V)$ the Grassmanian of r-dimensional subspaces of V, Λ an affine algebra, $F : \operatorname{Specm}\Lambda \longrightarrow Gr_r(V)$ a regular map and*

$$d \;=\; \min_{\mu \in \operatorname{Specm}\Lambda} \dim Y \cap F(\mu).$$

Then the set $\mathsf{U} = \{\mu \in \operatorname{Specm}\Lambda \mid \dim Y \cap F(\mu) = d\}$ is nonempty and open in $\operatorname{Specm}\Lambda$.

Proof. The function $\varphi : Gr_r(V) \to \mathbb{Z}$, $W \longmapsto \dim W \cap Y$ is upper semi-continuous, i.e. for every $n \in \mathbb{Z}$ the set $G_n = \{W \in Gr_r(V) \mid \varphi(W) \leqslant n\}$ is open in $Gr_r(V)$. Then $\mathsf{U} = F^{-1}(G_d)$ is open and the statement of the lemma follows. $\square$

Let $A = \Bbbk[X_1, \dots, X_n]$ be the polynomial algebra. For $f \in A$ denote by $\mathrm{d}(f)$ the column $\begin{pmatrix} \frac{\partial f}{\partial X_1} \\ \vdots \\ \frac{\partial f}{\partial X_n} \end{pmatrix} \in A^n$ and for a sequence $\mathsf{f} = \{f_1, \dots, f_k\}$ denote by $J(\mathsf{f}) = J(f_1, \dots, f_k)$ the $n \times k$ *Jacobian matrix* $\left(\frac{\partial f_j}{\partial X_i}\right)$, $i = 1, \dots, n$; $j = 1, \dots, k$. If $x \in \Bbbk^n$ then $r_{\mathsf{f}}(x)$ will denote the rank of the matrix $J(\mathsf{f})(x)$. If f is a complete intersection then the ideal $I = (f_1, \dots, f_k) \subset A$ is radical if and only if for every component C of $\mathsf{V}(I)$ there exists an open dense $\mathsf{U} \subset \mathsf{C}$ such that $r_{\mathsf{f}}(x) = k$ for every $x \in \mathsf{U}$ (Theorem 18.15, [Ei]).

Lemma 4.2. *Let $\tilde{h} = \{h_1, \dots, h_s, g\} \subset A$, $s \geqslant 0$, be a complete intersection such that $I = (h_1, \dots, h_s)$ is radical ideal. Consider the set U consisting of $\mu \in \Bbbk$ such that the algebra A/I_μ is reduced CI, where $I_\mu = (h_1, \dots, h_s, g - \mu)$. Then U is open dense in $\Bbbk$.*

Proof. Set $\mathsf{X} = \mathsf{V}(I)$ and let $\mathsf{X} = \mathsf{C}_1 \cup \dots \cup \mathsf{C}_N$ be the decomposition of X into irreducible components, $e : \mathsf{X} \to \Bbbk$ the regular map, $e(x) = g(x)$, $x \in \mathsf{X}$ (hence $\mathsf{V}(I_\mu) = e^{-1}(\mu)$). For any C_i the image $e(\mathsf{C}_i) \subset \Bbbk$ is either a point or a nonempty open subset in $\Bbbk$. Since $\dim e^{-1}(0) = \dim \mathsf{V}(I_0) = n - (s+1)$ there exists C_i such that $e(\mathsf{C}_i)$ is dense in $\Bbbk$. If $\mathsf{U}' \subset \Bbbk \setminus \bigcup\limits_{i, \dim e(\mathsf{C}_i) = 0} e(\mathsf{C}_i)$ consists of points μ such that $\dim e^{-1}(\mu)$ is maximal possible, then for $\mu \in \mathsf{U}'$ the variety $\mathsf{V}(I_\mu)$ is equidimensional of dimension $n - (s+1)$ and hence $h_1, \dots, h_s, g - \mu$ is a complete intersection for A.

We can assume that the map $e|_{\mathsf{C}_i} : \mathsf{C}_i \to \Bbbk$ is dominant for every $i = 1, \dots, N$. Let $\mathsf{h} = \{h_1, \dots, h_s\}$. By the conditions for any $i = 1, \dots, N$ there exists an open dense set $\mathsf{U}_i \subset \mathsf{C}_i$ such that $r_{\mathsf{h}}(x) = s$ for any $x \in \mathsf{U}_i$, i.e. there exists a $s \times s$-minor Δ_i of $J(\mathsf{h})$ which is nonzero on an open subset of C_i.

Assume first that $\mathsf{X} = \mathsf{C}_1$ and $\Delta = \Delta_1$ is invertible in Λ. Let K be the field of fractions of Λ. If $r_{\tilde{\mathsf{h}}}|_{\mathsf{X}} \leqslant s$ then the column $\mathrm{d}(g)|_{\mathsf{X}} \in \Lambda^n$ is a linear combination of $\mathrm{d}(h_1)|_{\mathsf{X}}, \dots, \mathrm{d}(h_s)|_{\mathsf{X}} \in \Lambda^n$ with coefficients from K. Moreover due to the invertibility of Δ holds $\mathrm{d}(g)|_{\mathsf{X}} = \lambda_1 \mathrm{d}(h_1)|_{\mathsf{X}} + \dots + \lambda_s \mathrm{d}(h_s)|_{\mathsf{X}}$ for some $\lambda_1, \dots, \lambda_s \in \Lambda$. Lifting this equality in A^n we obtain that $v = \mathrm{d}(g) - \sum\limits_{i=1}^{s} f_i \,\mathrm{d}(h_i) \in I^n$ for $f_i \in A$, $f_i|_{\mathsf{X}} = \lambda_i$.

Consider the element $h = g - \sum\limits_{i=1}^{s} f_i h_i \in A$. Then $\mathrm{d}(h) = v - \sum\limits_{i=1}^{s} \mathrm{d}(f_i) h_i \in I^n$ and hence $\mathrm{d}(h)|_{\mathsf{X}} = 0$ implying that h, and therefore g is a constant on X. It follows that $r_{\tilde{\mathsf{h}}}(x) = s+1$ for an open subset in X.

Now consider the general case and fix i, $1 \leqslant i \leqslant N$. Take a polynomial f which is nonzero on $\mathsf{C} = \mathsf{C}_i$ and vanishes on all other components of X. Set $\Delta = \Delta_i$. Consider a sequence $\mathsf{h}' = \{h_1, \ldots, h_s, 1 + f\Delta X_{n+1}\}$ in $A' = \Bbbk[X_1, \ldots, X_n, X_{n+1}]$ and the ideal $I' = (h_1, \ldots, h_s, 1 + f\Delta X_{n+1})$. Then the projection of $\mathsf{C}' = \mathsf{V}(h_1, \ldots, h_s, 1 + f\Delta X_{n+1})$ on the first n coordinates identifies C' with an open subset $\mathsf{C} \setminus \mathsf{V}(f\Delta)$, i.e. Δ is invertible on C'. Proposition 1, (4) implies that h' is a regular sequence. Moreover, the ring $\Lambda' = A'/I'$ is isomorphic to the localization of Λ by $f\Delta|_{\mathsf{X}}$, hence is reduced. Then as above we can prove that the rank of

$$J(h_1, \ldots, h_s, 1 + f\Delta X_{n+1}, g - \mu) = \begin{pmatrix} \frac{\partial h_1}{\partial X_1} & \cdots & \frac{\partial h_s}{\partial X_1} & \frac{\partial (f\Delta)}{\partial X_1} X_{n+1} & \frac{\partial g}{\partial X_1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{\partial h_1}{\partial X_n} & \cdots & \frac{\partial h_s}{\partial X_n} & \frac{\partial (f\Delta)}{\partial X_n} X_{n+1} & \frac{\partial g}{\partial X_n} \\ 0 & \cdots & 0 & f\Delta & 0 \end{pmatrix}$$

equals $s+2$ on an open dense set $\mathsf{U} \subset \mathsf{C}'$. Since $f\Delta$ is invertible on C' we conclude that the rank of $J(h_1, \ldots, h_s, g)$ equals $s+1$ on U, hence the ideal I_μ is radical. The statement follows. □

Lemma 4.3. *Let Λ be a reduced graded CI algebra, $\mathsf{g} = \{g_1, \ldots, g_t\} \subset \Lambda$ a complete intersection for Λ consisting of homogeneous elements, $\Gamma = \Bbbk[g_1, \ldots, g_t]$. Then there exists an open dense $\mathsf{U}_{\mathsf{g}} \subset \operatorname{Specm}\Gamma$ such that for any $\mu \in \mathsf{U}_{\mathsf{g}}$ the ideal $I_\mu = (g_1 - g_1(\mu), \ldots, g_t - g_t(\mu)) \subset \Lambda$ is radical.*

Proof. Denote $\mathsf{X}_{\mathsf{g}} = \{\mu \in \operatorname{Specm}\Gamma \,|\, I_\mu \text{ is radical}\}$. The criterion of radicality of I_μ before Lemma 4.2 implies that X_{g} is a constructible set. Hence it is enough to show that X_{g} is dense in $\operatorname{Specm}\Gamma$. We proceed by induction on t. The base of induction $t = 1$ follows immediately from Lemma 4.2. Suppose that there exists an open dense $\mathsf{U}' \subset \Bbbk^{t-1}$ such that for any $\mu' = (\mu_1, \ldots, \mu_{t-1}) \in \mathsf{U}'$ the ideal $I_{\mu'} = (g_1 - \mu_1, \ldots, g_{t-1} - \mu_{t-1})$ is radical. By Lemma 3.3, (2) the sequence $g_1 - \mu_1, \ldots, g_{t-1} - \mu_{t-1}$ is regular and therefore the algebra $\Lambda_{\mu'} = \Lambda/(g_1 - \mu_1, \ldots, g_{t-1} - \mu_{t-1})$ is a reduced CI. By Lemma 4.2 for any such μ' there exists an open dense $\mathsf{U}_{\mu'} \subset \Bbbk$ such that for $\mu_t \in \mathsf{U}_{\mu'}$ the algebra $\Lambda_{\mu'}/(g_t - \mu_t) \simeq \Lambda/(g_1 - \mu_1, \ldots, g_{t-1} - \mu_{t-1}, g_t - \mu_t)$ is a reduced CI, therefore the ideal $(g_1 - \mu_1, \ldots, g_t - \mu_t)$ is radical. Since $\mathsf{U} = \bigcup_{\mu' \in \mathsf{U}'} \bigcup_{\mu_t \in \mathsf{U}_{\mu'}} (\mu', \mu_t) \subset \mathsf{X}_{\mathsf{g}}$ is dense in $\operatorname{Specm}\Gamma$ the proof follows. □

Till the end of this section we follow closely the techniques from [Di], 8.2.3 and [O2]. In particular Lemmas 4.4 and 4.5 below are analogous to [Di], 8.2.1, 8.2.2. Let Λ be an affine algebra. Following [Di] we say that $h_1, \ldots, h_k \in \Lambda$ are *linearly independent over an ideal $I \subset \Lambda$*, provided $h_1 + I, \ldots, h_k + I \in \Lambda/I$ are linearly independent over $\Bbbk$.

Lemma 4.4. *Let $\mathsf{g} = \{g_1, \ldots, g_t\}$ be a complete intersection of homogeneous elements for a reduced graded CI algebra Λ, $\Gamma = \Bbbk[g_1, \ldots, g_t]$ and let $h_1, \ldots, h_k \in \Lambda$ be linearly independent over $I = (g_1, \ldots, g_t)$. Then*

(1) *There exists an open dense set $\mathsf{U}_1 \subset \operatorname{Specm}\Gamma$ such that for every $\mu \in \mathsf{U}_1$ $h_1, \ldots, h_k$ are linearly independent over $I_\mu = (g_1 - g_1(\mu), \ldots, g_t - g_t(\mu))$.*
(2) *There exists an open dense $\mathsf{U}_2 \subset \operatorname{Specm}\Gamma$ such that the restrictions of $h_1, \ldots, h_k$ on $\mathsf{V}(I_\mu)$ are linearly independent over $\Bbbk$ for each $\mu \in \mathsf{U}_2$.*

Proof. Let $l = \max_i \deg h_i$, $V = \Lambda_l$, $Y = \Bbbk h_1 + \ldots + \Bbbk h_k \subset V$ and $r = \dim(I_\mu)_l$. Note that by Lemma 3.3, (3), r does not depend on $\mu \in \operatorname{Specm}\Gamma$ and hence there is a well defined map $F : \operatorname{Specm}\Gamma \to Gr_r(\Lambda_l)$ such that $F(\mu) = (I_\mu)_l$. It follows from Lemma 3.3, (1) that F is a regular map. Then by linear independency of $h_1, \ldots h_k$ over I it follows that $d = \min\limits_{\mu \in \operatorname{Specm}\Gamma} \dim Y \cap (I_\mu)_l = 0$. Applying Lemma 4.1 we conclude that the set U_1 consisting of μ's, such that $\dim Y \cap (I_\mu)_l = 0$, is open dense in $\operatorname{Specm}\Gamma$ proving the statement (1). By Lemma 4.3 there exists an open dense set $\mathsf{U}_{\mathsf{g}} \subset \operatorname{Specm}\Gamma$ such that I_μ is radical for every $\mu \in \mathsf{U}_{\mathsf{g}}$. For such μ's the linear independence of $h_1, \ldots, h_k$ over I_μ is equivalent to the linear independence of the restrictions $h_1|_{\mathsf{V}(I_\mu)}, \ldots, h_k|_{\mathsf{V}(I_\mu)}$ as regular functions. Thus the statement (2) follows from (1) and Lemma 4.3 if we set $\mathsf{U}_2 = \mathsf{U}_1 \cap \mathsf{U}_{\mathsf{g}}$. □

Lemma 4.5. *Let Λ be a graded CI algebra, $\mathsf{g} = \{g_1, \ldots, g_t\}$ a complete intersection of homogeneous elements in Λ, $I = (g_1, \ldots, g_t)$ and $\Gamma = \Bbbk[g_1, \ldots, g_t]$. Suppose $I = \sum_{i=0}^{\infty} I_{(i)}$ is a graded decomposition of I and $H = \sum_{i=0}^{\infty} H_{(i)}$ is a graded complement of I in Λ as a $\Bbbk-$vector space. Then the mapping $\pi : \Gamma \otimes_{\Bbbk} H \longrightarrow \Lambda$ defined by $\pi(\gamma \otimes h) = \gamma h$, $\gamma \in \Gamma$, $h \in H$ is an isomorphism of Γ-modules. In particular, Λ is a free module over Γ.*

Proof. Note that $H_{(0)} = \Bbbk$ and $\operatorname{Im}\pi$ is a $\Gamma-$submodule in Λ containing Γ and H. We prove by induction on i that $\Lambda_{(i)} = I_{(i)} + H_{(i)} \subset \operatorname{Im}\pi$. If $f \in \Lambda_{(i)}$ then $f = f_I + f_H$, $f_I \in I_{(i)}$, $f_H \in H_{(i)} \subset \operatorname{Im}\pi$. We have that $f_I = \sum_{j=1}^{t} f_j g_j$, where $\deg f_j < i$, $j = 1, \ldots, t$. By induction, $f_j \in \operatorname{Im}\pi$ for all j and, since $\operatorname{Im}\pi$ is a Γ-module, $f_I \in \operatorname{Im}\pi$, therefore $f = f_I + f_H \in \operatorname{Im}\pi$. It is left to show that π is a monomorphism. Suppose $h_1, \ldots, h_k \in H$ are linearly independent over $\Bbbk$, hence over I, and for some $\gamma_1, \ldots, \gamma_k \in \Gamma$ holds $\gamma_1 h_1 + \ldots + \gamma_k h_k = 0$. By Lemma 4.4, (2) there exists an open dense set $\mathsf{U}_2 \subset \operatorname{Specm}\Gamma$ such that for each $\mu \in \mathsf{U}_2$ the restrictions of functions $h_1|_{\mathsf{V}(I_\mu)}, \ldots, h_k|_{\mathsf{V}(I_\mu)}$ are linearly independent over $\Bbbk$, where $I_\mu = (g_1 - g_1(\mu), \ldots, g_t - g_t(\mu))$. Since $\gamma_i|_{\mathsf{V}(I_\mu)} = \gamma_i(\mu) \in \Bbbk$, $i = 1, \ldots, k$, we get that $\gamma_1|_{\mathsf{V}(I_\mu)} = \ldots = \gamma_k|_{\mathsf{V}(I_\mu)} = 0$ for every $\mu \in \mathsf{U}_2$. This implies $\gamma_1|_{p_{\mathsf{g}}^{-1}(\mathsf{U}_2)} = \ldots = \gamma_k|_{p_{\mathsf{g}}^{-1}(\mathsf{U}_2)} = 0$ where p_{g} is defined as in Lemma 3.3, (4). Since $p_{\mathsf{g}}^{-1}(\mathsf{U}_2)$ is dense in $\operatorname{Specm}\Lambda$ we conclude that $\gamma_1 = \ldots = \gamma_k = 0$. □

Now we are in the position to prove our main result.

Proof of Theorem 1. We prove the statement for U as a left module (right module structure is treated analogously). We apply Lemma 4.5 to $\Lambda = \overline{U}$ and the sequence $\{\overline{g}_1, \ldots, \overline{g}_t\}$. Let $\overline{I}$ be the left ideal of $\overline{U}$ generated by $\overline{g}_1, \ldots, \overline{g}_t$, $\overline{\Gamma} = \Bbbk[\overline{g}_1, \ldots, \overline{g}_t]$ and $\overline{H} = \sum_{i=0}^{\infty} \overline{H}_i$ be a graded complement to $\overline{I}$ in Λ. Then the map $\overline{\pi} : \overline{\Gamma} \otimes_{\Bbbk} \overline{H} \longrightarrow \Lambda$ which sends $\overline{\gamma} \otimes \overline{h} \longmapsto \overline{\gamma}\overline{h}$ is an isomorphism of vector spaces. Choose for all $i \geqslant 0$ the $\Bbbk$-vector spaces $H_i \subset U_i$ such that $\operatorname{gr} : U \longrightarrow \overline{U}$ induces a $\Bbbk-$linear isomorphism H_i onto $\overline{H}_i$ and $H = \sum_{i=0}^{\infty} H_i$. The filtrations on Γ and H

induce a filtration on $\Gamma \otimes_{\Bbbk} H$ by

$$(\Gamma \otimes_{\Bbbk} H)_k = \sum_{i=0}^{n} \Gamma_i \otimes_{\Bbbk} H_{k-i}, \text{ and in particular } \operatorname{gr}(\Gamma \otimes_{\Bbbk} H)_{(k)} = \sum_{i=0}^{n} \overline{\Gamma}_i \otimes_{\Bbbk} \overline{H}_{k-i}.$$

The map $\pi : \Gamma \otimes_{\Bbbk} H \longrightarrow U$, $\gamma \otimes h \longmapsto \gamma h$ preserves the filtrations and the induced graded map coincides with $\overline{\pi}$. Since $\overline{\pi}$ is an isomorphism, we see that π is a $\Bbbk$-isomorphism which completes the proof.

5. Applications

5.1. **Semisimple Lie algebras.** Let $U = \mathrm{U}(\mathrm{g})$ be the universal enveloping algebra of a semisimple Lie algebra of rank r, $I(\mathrm{g})$ the algebra of g-invariants in $S(\mathrm{g})$ ([Di], 2.4.11). Then in $I(\mathrm{g})$ there exists a system of homogeneous algebraically independent generators $f_1, \ldots, f_r$ ([Di], Theorem 7.3.8) and the variety $\mathsf{V}(f_1, \ldots, f_r)$ is irreducible of dimension $\dim \mathrm{g} - r$ ([Di], Theorem 8.1.3). If $\varphi : S(\mathrm{g}) \to \mathrm{U}(\mathrm{g})$ is the symmetrization map then it maps $I(\mathrm{g})$ isomorphically onto the center $Z(\mathrm{g})$ of $\mathrm{U}(\mathrm{g})$ and $\mathsf{f} = \{\varphi(f_1), \ldots, \varphi(f_r)\}$ are the generators of $Z(\mathrm{g})$. Since $\operatorname{gr}(\mathrm{U}(\mathrm{g})) \simeq S(\mathrm{g})$ and $\operatorname{gr}(\varphi(f_i)) = f_i$, $i = 1, \ldots, r$ the classical Kostant theorem ([Di], 8.2.4) follows from Theorem 1 applied to $U = \mathrm{U}(\mathrm{g})$ and the family f of generators of $Z(\mathrm{g})$ and Proposition 1, (3).

5.2. **Restricted Yangians.** Let p be a positive integer. The *level p Yangian* $Y_p(\mathrm{gl}_n)$ for the Lie algebra gl_n ([D1], [C]) can be defined as the associative algebra with generators $t_{ij}^{(1)}, \ldots, t_{ij}^{(p)}$, $i, j = 1, \ldots, n$, subject to the relations

$$[T_{ij}(u), T_{kl}(v)] = \frac{1}{u - v}(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)), \tag{2}$$

where u, v are formal variables and

$$T_{ij}(u) = \delta_{ij}\, u^p + \sum_{k=1}^{p} t_{ij}^{(k)}\, u^{p-k} \in \mathrm{Y}_p(\mathrm{gl}_n)[u]. \tag{3}$$

These relations are equivalent to the following

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min(r,s)} \left(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)}\right), \tag{4}$$

where $t_{ij}^{(0)} = \delta_{ij}$ and $t_{ij}^{(r)} = 0$ for $r \geqslant p + 1$.

The importance of the restricted Yangian $Y_p(\mathrm{gl}_n)$ is motivated by the fact that any irreducible finite-dimensional representation of the full Yangian $Y(\mathrm{gl}_n)$ is a representation of the restricted Yangian for some p [D2]. Note that the level 1 Yangian $\mathrm{Y}_1(\mathrm{gl}_n)$ coincides with the universal enveloping algebra $\mathrm{U}(\mathrm{gl}_n)$.

Set $\deg t_{ij}^{(k)} = k$. This defines a filtration on $\mathrm{Y}_p(\mathrm{gl}_n)$. The following analogue of the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}_p(\mathrm{gl}_n)$ ([C], [Mo]), shows that $\mathrm{Y}_p(\mathrm{gl}_n)$ is a special filtered algebra.

Proposition 3. *The associated graded algebra $\overline{\mathrm{Y}}_p(\mathrm{gl}_n) = \operatorname{gr} \mathrm{Y}_p(\mathrm{gl}_n)$ is a polynomial algebra in variables $\overline{t}_{ij}^{(k)}$, $i, j = 1, \ldots, n$, $k = 1, \ldots, p$.*

Note that the grading on $\overline{\mathrm{Y}}_p(\mathrm{gl}_n)$ induced by the grading on $\mathrm{Y}_p(\mathrm{gl}_n)$, $\deg \bar{t}_{ij}^{(k)} = k$, does not coincide with the standard polynomial grading for $p > 1$. Set $\mathrm{T}(u) = (T_{ij}(u))_{i,j=1}^n$ and consider the following element in $\mathrm{Y}_p(\mathrm{gl}_n)[u]$, called *quantum determinant*

$$\operatorname{qdet} \mathrm{T}(u) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, T_{1\sigma(1)}(u) T_{2\sigma(2)}(u-1) \ldots T_{n\sigma(n)}(u-n+1). \tag{5}$$

The coefficients d_s by the powers u^{np-s}, $s = 1, \ldots, np$ of $\operatorname{qdet} \mathrm{T}(u)$ are algebraically independent generators of the center of $\mathrm{Y}_p(\mathrm{gl}_n)$ ([C], [Mo]).

For $F = \sum_i f_i u^i \in \mathrm{Y}_p(\mathrm{gl}_n)[u]$ denote $\overline{F} = \sum_i \overline{f}_i u^i \in \overline{\mathrm{Y}}_p(\mathrm{gl}_n)[u]$. Also we denote $X_{ij}^{(k)} = \bar{t}_{ij}^{(k)}$, $X_{ij}(u) = \overline{T}_{ij}(u)$ and $X(u) = (X_{ij}(u))_{i,j=1}^n$. Since $\overline{T_{ij}(u-\lambda)} = X_{ij}(u)$ for any $\lambda \in \Bbbk$, one can easily check that $\operatorname{gr} \operatorname{qdet} T(u) = \det X(u)$.

Lemma 5.1. *The sequence $\overline{d}_1, \ldots, \overline{d}_{np}$ is a complete intersection for $\overline{\mathrm{Y}}_p(\mathrm{gl}_n)$.*

Proof. Due to Proposition 1, (5) it is enough to prove that the sequence

$$X_{ij}^{(k)}, i \neq j, i, j = 1, \ldots, n, k = 1, \ldots p;\ \overline{d}_1, \ldots, \overline{d}_{np}$$

is complete intersection for $\overline{\mathrm{Y}}_p(\mathrm{gl}_n)$. Let c_s is a polynomial in variables $X_{11}^{(1)}$, $X_{11}^{(2)}$, …, $X_{11}^{(p)}$, $X_{22}^{(1)}$, …, $X_{nn}^{(p)}$ obtained from $\overline{d}_s$ by substituting $X_{ij}^{(k)} = 0$ for all $i \neq j$ and all k. Due to Lemma 2.1 we only need to show the regularity of the sequence $c_1, \ldots, c_{np}$ in the polynomial ring $\Bbbk[X_{ii}^{(k)}, i = 1, \ldots, n, k = 1, \ldots, p]$, hence by Proposition 1, (3), we should show, that the variety $Z = \mathsf{V}(c_1, \ldots, c_{np})$ is equidimensional of dimension 0.

Substituting $X_{ij}^{(k)} = 0$ into the matrix $X(u)$ we obtain that c_s is the coefficient by u^{np-s} in $\det \operatorname{diag}\{X_{11}(u), \ldots, X_{nn}(u)\} = \prod_{i=1}^n X_{ii}(u)$. Consider the regular map $\varphi : \Bbbk^{np} \to \Bbbk^{np}$ which sends $(a_{ii}^{(k)}) \in \Bbbk^{np}$ to the coefficients of the following monic polynomial $\prod_{i=1}^n (u^p + a_{ii}^{(1)} u^{p-1} + \ldots + a_{ii}^{(p)})$. Obviously, $Z = \varphi^{-1}(0)$. Since $\Bbbk[u]$ is a factorial domain, $u^{np} = \underbrace{u^p \cdot u^p \cdot \ldots \cdot u^p}_{n}$ is a unique decomposition of u^{np} in the products of monic polynomials of degree p and hence $Z = \{0\}$. This completes the proof. $\square$

Applying Theorem 1 and Lemma 5.1 we obtain the following analogue of the Kostant theorem for the restricted Yangians.

Theorem 2. *For all $n, p \geqslant 1$ the restricted Yangian $\mathrm{Y}_p(\mathrm{gl}_n)$ is a free module over its center.*

5.3. **Current algebras.** In this section we consider the polynomial current Lie algebra $\mathrm{g} = \mathrm{gl}_n(\mathbb{C}) \otimes \mathbb{C}[x]$ and its restricted quotient algebra $\mathrm{g}_m = \mathrm{g}_m(n)$, $m > 0$, by the ideal $\sum_{k \geqslant m} \mathrm{gl}_n \otimes x^k$ ([RT]). We will show that the universal enveloping algebra $\mathrm{U}(\mathrm{g}_m)$ is a free module over its center for any $m > 0$. This generalizes in the case of gl_n the result of Geoffriau for Takiff algebra g_2 ([G1],[G2]).

In [Mo] were constructed the families of algebraically independent generators of the center of $\mathrm{U}(\mathrm{g}_m)$. Following [Mo] let E_{ij}, $i, j = 1, \ldots, n$ be the standard basis of gl_n, $E_{ij}^{(k)} = E_{ij} \otimes x^k$ with $1 \leqslant i, j \leqslant n$, $0 \leqslant k \leqslant m-1$ a basis of g_m. Set

$$F_{ij}^{(r)} = E_{ij}^{(r-1)},\ 1 < r \leqslant m \text{ and } F_{ij}^{(1)} = E_{ij}^{(0)} - m(j-1)\delta_{ij}.$$

For each $k \in \{1, \ldots, mn\}$ let $r \in \{1, \ldots, m\}$ and $s \in \{1, \ldots, n\}$ be such that $k = m(s-1) + r$. Then the elements

$$\xi_k = \sum_{\substack{i_1 < \ldots < i_s \\ j_1 + \ldots + j_s = k}} \sum_{\sigma \in S_s} \operatorname{sgn}(\sigma) F^{(j_1)}_{i_{\sigma(1)} i_1} \ldots F^{(j_s)}_{i_{\sigma(s)} i_s}, \tag{6}$$

are algebraically independent generators of the center of $\mathrm{U}(\mathrm{g}_m)$ ([Mo]). Note that $\mathrm{U}(\mathrm{g}_m)$ is a special filtered algebra with respect to the standard grading, $\deg E^{(k)}_{ij} = 1$ for all i, j, k.

We will show that the sequence $\overline{\xi}_1, \ldots, \overline{\xi}_{mn}$ is complete intersection for $\overline{U}(\mathrm{g}_m)$ where

$$\overline{\xi}_k = \sum_{\substack{i_1 < \ldots < i_s \\ j_1 + \ldots + j_s = k}} \sum_{\sigma \in S_s} \operatorname{sgn}(\sigma) \overline{F}^{(j_1)}_{i_{\sigma(1)} i_1} \ldots \overline{F}^{(j_s)}_{i_{\sigma(s)} i_s}. \tag{7}$$

As in the case of the Yangian $\mathrm{Y}_p(\mathrm{gl}_n)$ we complete this sequence by the elements $\overline{F}^{(l)}_{ij}$ for all $i, j = 1, \ldots, n$, $i \neq j$ and $l = 1, \ldots, m$ and apply Lemma 2.1 by substituting $\overline{F}^{(l)}_{ij} = 0$. Hence it is enough to prove the regularity of the sequence $\gamma^m_1, \ldots, \gamma^m_{mn}$, where $\gamma^m_k = \sum_{\substack{i_1 < \ldots < i_s \\ j_1 + \ldots + j_s = k}} \overline{F}^{(j_1)}_{i_1 i_1} \ldots \overline{F}^{(j_s)}_{i_s i_s}$ (compare with (3.2) in [Mo]). We will show that $\mathsf{X} = \mathsf{V}(\gamma^m_1, \ldots, \gamma^m_{mn}) = \{0\}$ by induction on m. Suppose that $m = 1$. Since for each $k = 1, \ldots, n$, γ^1_k is the elementary symmetric polynomial of degree k in variables $\overline{F}^{(1)}_{11}, \ldots, \overline{F}^{(1)}_{nn}$ we have that $\mathsf{V}(\gamma^1_1, \ldots, \gamma^1_n) = \{0\}$. Suppose now that the sequence $\gamma^{m-1}_1, \ldots, \gamma^{m-1}_{(m-1)n}$ is a complete intersection for $\overline{U}(\mathrm{g}_m)$ and $\mathsf{V}(\gamma^{m-1}_1, \ldots, \gamma^{m-1}_{(m-1)n}) = \{0\}$. Note that γ^m_{ms} is the elementary symmetric polynomial of degree s in variables $\overline{F}^{(m)}_{11}, \ldots, \overline{F}^{(m)}_{nn}$, $s = 1, \ldots, n$. Hence $\overline{F}^{(m)}_{11}(x) = \ldots = \overline{F}^{(m)}_{nn}(x) = 0$ for every $x \in \mathsf{X}$. Substituting these values in γ^m_k for every k that does not divide m we obtain the sequence $\gamma^{m-1}_1, \ldots, \gamma^{m-1}_{(m-1)n}$ which is a complete intersection by our assumption and therefore $\mathsf{X} = 0$. By induction on m we conclude that the sequence $\overline{\xi}_1, \ldots, \overline{\xi}_{mn}$ is a complete intersection for $\overline{U}(\mathrm{g}_m)$.

Applying Theorem 1 we immediately obtain the following analogue of the Kostant theorem for restricted current algebras:

Theorem 3. *For all $m, n \geqslant 1$ the algebra $\mathrm{U}(\mathrm{g}_m(n))$ is a free module over its center.*

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References

[B] Bourbaki, N., *Algébre homologique, Algébre chap. X*, Masson, 1980.

[C] Cherednik I.V., *Quantum groups as hidden symmetries of classic representation theory*, In: "Differential Geometric Methods in Physics" (A. I. Solomon, Ed.), World Scientific, Singapore, 1989, 47–54.

[Di] Dixmier J., *Algebras enveloppantes*, Gauthier-Villarrd, Paris, 1974.

[D1] Drinfeld V.G., *Hopf algebras and the quantum Yang–Baxter equation*, Soviet Math. Dokl. **32** (1985), 254–258.

[D2] Drinfeld V.G., *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** (1988), 212–216.

[DFO] Drozd Yu.A., Ovsienko S.A., Futorny V.M., *Harish - Chandra subalgebras and Gelfand Zetlin modules,* in: "Finite dimensional algebras and related topics", NATO ASI Ser. C., Math. and Phys. Sci., **424**, (1994), 79-93.

[Ei] Eisenbud D., *Commutative algebra*, Graduate Texts in Math., **150**, Springer, 1994.

[G1] Geoffriau F., *Une propriété des algèbres de Takiff*, C. R. Acad. Sci. Paris **319** (1994), Série I, 11–14.

[G2] Geoffriau F., *On the center of the enveloping algebra of a Takiff algebra*, Ann. Math. Blaise Pascal, **1** (1995), 15–31.

[K] Kostant B., *Lie groups representations on polynomial rings.* Amer.J.Math. **85**, (1963), 327-404.

[Ma] Matsumura H., *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press, 1997.

[Mo] Molev A.I., *Casimir elements for certain polynomial current Lie algebras*, In: "Group 21, Physical Applications and Mathematical Aspects of Geometry, Groups, and Algebras," Vol. 1, (H.-D. Doebner, W. Scherer, P. Nattermann, Eds). World Scientific, Singapore, 1997, 172–176.

[MNO] Molev A., Nazarov M, Olshanski G., *Yangians and classical Lie algebras*, Russian Math. Surveys **51**:2 (1996), 205-282.

[O1] Ovsienko S., *Strongly nilpotent matrices and Gelfand-Tsetlin modules*, J.Linear Algebra and Appl., (2003), 19pp. to appear.

[O2] Ovsienko S., *Finiteness statements for Gelfand-Tsetlin modules*, In: Proceedings of International Algebraic Conference, Sumi, 2002.

[RT] Rais M., Tauvel P., *Indice et polynomes invariants pour certaines algebres de Lie*, J. Reine Agnew. Math. **425** (1992), 123-140.

[W] Ch. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press (1994).

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