Problem Set 3

Q20 Let (X, d) be a metric space, and $(x_n)_{n\in\mathbb{N}}$ be a sequence that satisfies $d(x_n, x_{n+1}) \leq 2^{-n}$ for all $n \in \mathbb{N}$. Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Q21 Let (X, d_X) and (Y, d_Y) be metric spaces. Show that $f\colon X \to Y$ is continuous if and only if for each open $V \subseteq Y$, we have that $f^{-1}(V))$ is an open subset of X.

Q22 Let (X, d) be a metric space. Show that finite unions of closed sets are closed, and that arbitrary intersections of closed sets are closed.

Q23 Let $(X, \mathcal{O})$ be a topological space, and I be a finite index set. Show that if $U_\alpha \in \mathcal{O}$ for each $\alpha \in I$, then $\bigcup_{\alpha\in I} U_\alpha$ is open.

Q24 (a) Verify that the trivial, discrete and co-finite topologies are indeed topologies.
(b) Show that the discrete topology is generated by the discrete metric.
(c) Show that the co-finite topology equals the discrete topology if the underlying set is finite.

Q25 Prove parts (3)–(7) of Lemma 2.14.

Q26 Let $(X, \mathcal{O})$ be a topological space, and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $(x_n)_{n\in\mathbb{N}}$ *converges to* $x \in X$ *with respect to* $\mathcal{O}$, written $x_n \to x$, if for each neighbourhood N of x, there is $m \in \mathbb{N}$ such that $x_n \in N$ for all $n > m$.

(a) Suppose $\mathcal{O}$ is induced by the metric d on X. Show that $x_n \to x$ with respect to $\mathcal{O}$ if and only if $x_n \to x$ with respect to d.
(b) Suppose X is an infinite set and $\mathcal{O}$ is the co-finite topology on X. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of pairwise distinct points in X. Show that $x_n \to x$ for each $x \in X$.

Q27 Let $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq \frac{1}{2} \text{ or } x^2 + y^2 = 1\}$, and give X the subspace topology induced from $\mathbb{R}^2$ with the Euclidean topology. Determine the closure, boundary and interior of $B_1((0, 0)) \subseteq X$ with respect to the subspace topology.

Q28 Show that every metrizable space is normal.

Q29 Let $\mathcal{O} = \{E \subseteq \mathbb{R} \mid 0 \notin A \text{ or } A = \mathbb{R}\}$. Show that $\mathcal{O}$ is a topology on $\mathbb{R}$. What are the closed sets in $(\mathbb{R}, \mathcal{O})$? What is the closure of the set $\{1\}$? Is this topology Hausdorff?

Q30 Let $\mathcal{B} = \{\, [a, b) \mid a, b \in \mathbb{R}\}$.

(a) Show that $\mathcal{B}$ is the basis for a topology on $\mathbb{R}$. This is the *lower-limit topology* on $\mathbb{R}$.
(b) Show that the lower limit topology is finer than the Euclidean topology on $\mathbb{R}$.
(c) Decide whether the sets $[a, b), (a, b), (a, b], [a, b]$, where $a, b \in \mathbb{R}$, are closed with respect to the lower limit topology.
(d) For each set in the previous part, find its closure with respect to the lower limit topology.
(e) Is the lower limit topology Hausdorff, regular, normal?
(f) Is $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to the lower limit topology?
(g) Is the lower limit topology second countable?
(h) Is the lower limit topology metrizable?