Problem Set 7

Q48 Let $y = (y_n)_{n=1}^{\infty}$ be a sequence in Λ with the property that $\sum_{n=1}^{\infty} x_n y_n$ converges for each $x = (x_n)_{n=1}^{\infty} \in c_0$, where $c_0 \subset l_\infty$ is the subspace of all null-sequences. Then $y \in l_1$.

Q49 Let X and Y be Banach spaces and let $(T_n)_{n=1}^{\infty}$ be a sequence in $\mathfrak{B}(X,Y)$ such that $\lim T_n(x)$ exists for all $x \in X$. Let $T(x) = \lim T_n(x)$. Show that $T \in \mathfrak{B}(X,Y)$.

Q50 (a) Let X and Y be Banach spaces and suppose $T \in \mathfrak{B}(X,Y)$ is open. Show that T is surjective.

(b) How much can you relax the hypotheses on the normed spaces X, Y or the linear operator T so that "open" still implies "surjective"?

Q51 Let X be a normed space.

(a) Show that $(x_n)_{n=1}^{\infty} \subseteq X$ converges in the weak topology to $x \in X$ if and only if

$$f(x_n) \to f(x)$$

for each $f \in X^*$. Decide whether a weak limit (if it exists) is unique.

(b) Show that $(f_n)_{n=1}^{\infty} \subseteq X^*$ converges in the weak-star topology to $f \in X^*$ if and only if

$$f_n(x) \to f(x)$$

for each $x \in X$. Decide whether a weak-star limit (if it exists) is unique.

Q52 Show that the weak topology on X and the weak-star topology on X^* are Hausdorff. Verify your conclusions about the uniqueness of limits from the previous question.

Q53 Let X be a Banach space.

(a) If $E \subset X$ is bounded, then so is its weak closure.

(b) If $F \subset X^*$ is bounded, then so is its weak-star closure.

Q54 Let X be a normed space and $A \subset X$. Show that A is bounded if and only if A is *weakly bounded*, i.e. for each $f \in X^*$, there is a constant $M(f) > 0$ such that $f(a) \leq M(f)$ for all $a \in A$.

Q55 Suppose that X is a Banach space. Show that every weak-star Cauchy sequence in X^* converges.

Q56 Recall that if $X = c_0$, then $X^* \cong l_1$ and $X^{**} \cong l_\infty$. Let $e_n = (\delta_{kn})_{k=1}^{\infty}$ in either space.

(a) Show that $(e_n)_{n=1}^{\infty}$ converges weakly in X to zero, but that it doesn't converge (strongly) in X.

(b) Show that $(e_n)_{n=1}^{\infty}$ converges in the weak-star topology on X^* to zero, but that it doesn't converge weakly or strongly in X^*.

(c) Show that $(\sum_{m=n}^{\infty} e_m)_{n=1}^{\infty}$ converges in the weak-star topology on X^{**} to zero, but it doesn't converge weakly or strongly in X^{**}.

Q57 Let X be a normed space. Give a direct proof to show that $B(X)$ is w–closed, and also show that $B(X^*)$ is w^*–closed.

Q58 Let X be a normed space and $Y \subset X$ be a subspace. Show that if Y is closed in the norm topology, then Y is also weakly closed. Is the converse also true?

Q59 Let X be a normed space and $f_1, \dots, f_n \in X^*$ such that

$$\bigcap_{k=1}^{n} \ker f_k = \{0\}.$$

Show that $\dim X \leq n$.

Q60 Let X be an infinite dimensional Banach space.

(a) Every non-empty w–open set in X is unbounded.
Every non-empty w^*–open set in X^* is unbounded.

(b) Every bounded subset of X is nowhere dense in the weak topology.
Every bounded subset of X^* is nowhere dense in the weak-star topology.

(c) X is meagre in itself with respect to the weak topology.
X^* is meagre in itself with respect to the weak-star topology.

(d) The weak-star topology on X^* is not defined by any translation invariant metric.

Q61 Let X be a reflexive normed space.

(a) Show that X is a Banach space.

(b) Show that X^* is reflexive.

(c) Show that $B(X)$ is weakly compact.

fin