Problem Set 1

Ask for hints if you'd like some!

Q1 Let $||\cdot||_1$ and $||\cdot||_2$ be two norms on a vector space V.
Is $||x|| = \min\{||x||_1, ||x||_2\}$ necessarily a norm on V?

Q2 If $X = (V, ||\cdot||)$ is a normed space, show that

$$d(x, y) := ||x - y||$$

defines a metric on V.

Q3 If V is a vector space and d is a translation-invariant metric on V, show that

$$||x|| := d(0, x)$$

defines a norm on V.

Q4 Find a sequence in $\mathbb{R}$ which converges to 0 but is not contained in any of the spaces l_p with $1 \leq p < \infty$.

Q5 Find a sequence which is contained in every sequence space l_p except for l_1, i.e. a sequence x such that $x \in l_p$ if and only if $1 < p \leq \infty$.

Q6 Show that l_∞^n, l_∞ and l_p are Banach spaces.

Q7 Let $M \subset l_\infty$ be the subspace consisting of all sequences $x = (x_i)$ with at most finitely many non-zero terms. Find a Cauchy sequence in M which does not converge in M and hence conclude that M is not complete.

Q8 For $1 \leq p \leq \infty$, let $||\cdot||_p$ be the l_p–norm on $\mathbb{R}^n$ or $\mathbb{C}^n$. Show that if $1 \leq p < q \leq \infty$, then $||x||_p \geq ||x||_q$. For which points x do we have equality?
Prove that for every $\varepsilon > 0$, there is an N such that if $N < p < \infty$, then

$$||x||_\infty \leq ||x||_p \leq (1 + \varepsilon)||x||_\infty.$$

Q9 Show that if $x_n \to x$ in the normed space $(X, ||\cdot||)$, then $||x_n|| \to ||x||$ in $(\mathbb{R}, |\cdot|)$.

Q10 Let $\{x_n\}$ be a sequence of points in a Banach space X such that $\sum_{n=1}^\infty ||x_n|| < \infty$.
Prove that the series $\sum_{n=1}^\infty x_n$ converges in X.