Integration and measurable functions

Convention: Suppose (X, Σ) is a measurable space, i.e. X is a set equipped with the σ–algebra Σ, and $f\colon X \to \mathbb{R}$ (or $\mathbb{R}^*$) is a function. Then f is *measurable* if it is $(\Sigma, \mathcal{B})$–measurable, where $\mathcal{B}$ is the Borel σ–algebra $\mathcal{B}(\mathbb{R})$ (or $\mathcal{B}(\mathbb{R}^*)$), and it is *Lebesgue measurable* if it is $(\Sigma, \mathcal{L})$–measurable, where $\mathcal{L}$ is the σ–algebra of all Lebesgue integrable sets.

Q13 If (X, Σ) is a measurable space and $f\colon X \to \mathbb{R}$ is a function, then the following are equivalent:

(a) f is measurable,
(b) $f^{-1}((a, \infty)) \in \Sigma$ for each $a \in \mathbb{R}$,
(c) $f^{-1}([a, \infty)) \in \Sigma$ for each $a \in \mathbb{R}$,
(d) $f^{-1}((-\infty, a)) \in \Sigma$ for each $a \in \mathbb{R}$,
(e) $f^{-1}((-\infty, a]) \in \Sigma$ for each $a \in \mathbb{R}$.

Q14 The Borel sets of $\mathbb{R}^* = [-\infty, \infty]$ are defined using the topology given in the lecture, resulting in

$$\mathcal{B}(\mathbb{R}^*) = \{A \subseteq \mathbb{R}^* \mid A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}.$$

Let (X, Σ) be a measurable space, and for each $n \in \mathbb{N}$, let $f_n\colon X \to \mathbb{R}^*$ be a measurable function. Prove:

(a) $x \to f_1(x) + f_2(x)$ is measurable;
(b) $x \to f_1(x) \times f_2(x)$ is measurable;
(c) $x \to |f_1(x)|$ is measurable;
(d) $x \to \max\{f_1(x), f_2(x)\}$ is measurable;
(e) $x \to \min\{f_1(x), f_2(x)\}$ is measurable;
(f) $x \to f(x)$ is measurable, where $f(x) = \sup\{f_n(x) \mid n \in \mathbb{N}\}$;
(g) $x \to f(x)$ is measurable, where $f(x) = \limsup_{n\to\infty} f_n(x)$.
(h) $x \to f(x)$ is measurable, where $f(x) = \inf\{f_n(x) \mid n \in \mathbb{N}\}$;
(i) $x \to f(x)$ is measurable, where $f(x) = \liminf_{n\to\infty} f_n(x)$.
(j) If $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in X$, then f is measurable.

Hint: Extend Q13 to $\mathbb{R}^*$ and prove the above 10 statements in an efficient order.

Q15 Suppose $f\colon (a, b) \to \mathbb{R}$ is differentiable. Show that its derivative f' is $(\mathcal{B}, \mathcal{B})$–measurable.

Q16 Let (X, Σ, μ) be a measure space. Show that $A \subseteq X$ is μ–measurable if and only if its characteristic function $x \to \chi_A(x)$ is measurable.

Q17 (Exercise 2.4)
Show that every simple function is measurable.

Q18 (Exercise 2.6)
Let (X, Σ, μ) be a measure space. Given a measurable function $f\colon X \to \mathbb{R}$, show that there are non-negative, measurable functions f^+ and f^- such that $f = f^+ - f^-$.