Assignment 3

Due at the beginning of the lecture on Monday, 4 October 2010.

For all questions, you should not need to cite any literature or prove auxiliary facts. They follow from facts given in the lectures or stuff you've learned in previous courses.

Q1 Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x$ for all $x \in [0, 1]$ and $f(x)$ is zero everywhere else. We know that

$$\int_0^1 f(x)dx = \frac{1}{2}.$$

Show that f is Lebesgue integrable and that $\displaystyle\int_{\mathbb{R}} f \; dm = \frac{1}{2}$.

Q2 In each of the examples, the group operation is matrix multiplication and the topology is induced from the ambient space $\mathbb{R}^n$, where $n = 9$ in (a) and 4 in (b). You may assume that the groups are topological.

(a) The Heisenberg group H is the group of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. Show that $dxdydz$ is a left and right Haar measure on H.

(b) Let G be the group of 2×2 matrices of the form

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix},$$

where $x > 0$ and $y \in \mathbb{R}$. Show that $x^{-2}dxdy$ is a left Haar measure on G and $x^{-1}dxdy$ is a right Haar measure on G.

Q3 Let (X, Σ, μ) be a measure space and $f \in \mathcal{L}^1(\mu)$.
Suppose $(f_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ is a sequence with the properties that

(a) $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$ for almost every x, and

(b) $\int |f_n| \to \int |f|$.

Show that $[f_n] \to [f]$ in $L^1(\mu)$.

Remarks on Q2: You can use the following definitions and facts.

The group G is a **topological group** if it has a topology such that the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous maps, where $G \times G$ is given the product topology.

A Borel measure μ on G is **left (resp. right) invariant** if for all $x \in G$ and all $E \in \mathcal{B}(G)$, we have $\mu(xE) = \mu(E)$ (resp. $\mu(Ex) = \mu(E)$).

A left (resp. right) invariant Borel measure μ on G is a left (resp. right) **Haar measure** if $\mu(U) > 0$ for every open set $U \subseteq G$ and $\mu(K) < \infty$ for every compact set $K \subseteq G$.

You can also use the relationship between right and left measures that was stated in the lecture, as well as the usual results about integration in $\mathbb{R}^k$ from previous courses.