Assignment 1

Your solutions should be submitted by the beginning of the lecture on
Monday, 10 August 2009.

Q1 Show that if $x_n \to x$ in the normed space $(X, || \cdot ||)$, then $||x_n|| \to ||x||$ in $(\mathbb{R}, | \cdot |)$.

Q2 Let $\{x_n\}$ be a sequence of points in a Banach space X such that $\sum_{n=1}^{\infty} ||x_n|| < \infty$. Prove that the series $\sum_{n=1}^{\infty} x_n$ converges in X.

Q3 For each $x = (a, b) \in \mathbb{R}^2$, let $||x||_1 = |a| + |b|$ and $||x||_2 = \sqrt{a^2 + b^2}$. You may assume that $|| \cdot ||_1$ and $|| \cdot ||_2$ are norms on $\mathbb{R}^2$.

(a) For which vectors $x, y \in \mathbb{R}^2$ do we have $||x||_1 + ||y||_1 = ||x + y||_1$?

(b) For which vectors $x, y \in \mathbb{R}^2$ do we have $||x||_2 + ||y||_2 = ||x + y||_2$?

(c) Formulate a general hypothesis about normed spaces X containing linearly independent elements $x, y \in X$ such that $||x|| + ||y|| = ||x + y||$. Can you prove this hypothesis?

Q4 Let V be a real vector space and $\emptyset \neq D \subset V$ a subset satisfying:

(a) If $x, y \in D$ and $\alpha, \beta \in \mathbb{R}$ such that $|\alpha| + |\beta| \leq 1$, then $\alpha x + \beta y \in D$.

(b) If $x \in D$, then there is $\varepsilon > 0$ such that $x + \varepsilon D \subset D$.

(c) For each non-zero $x \in V$, there exist non-zero $\alpha, \beta \in \mathbb{R}$ such that $\alpha x \in D$ and $\beta x \notin D$.

Define

$$||x|| = \inf\{ \ t \mid t > 0, \ x \in tD \ \}.$$

Show that this defines a norm on V and, moreover, that D is the open unit ball with respect to this norm. (Hint: First observe that if $x \in D$, then $\alpha x \in D$ for all $\alpha \in [-1, 1]$.)