Complex Analysis (620-413):
Riemann mapping theorem and Riemann surfaces

Stephan Tillmann

These notes are compiled for an Honours course in complex analysis given by the author at the University of Melbourne in Semester 2, 2007. Covered are the Riemann mapping theorem as well as some basic facts about Riemann surfaces. The text is based on the books titled "Complex Analysis" by Ahlfors [1] and Gamelin [2].

1 The Riemann mapping theorem

1.1 Biholomorphic maps

A *domain* is an open, path connected subset of the complex plane.

Definition 1.1 (Biholomorphic) *Domains U and V are said to be* biholomorphic *if there is a holomorphic, bijective function $f\colon U \to V$ whose inverse is also holomorphic. Such a function f is also said to be* biholomorphic (onto its image).

Recall that a map $f\colon U \to \mathbb{C}$ is *conformal* (or angle-preserving) at z_0, if it preserves *oriented* angles between curves through z_0; oriented meaning that not only the size but also the sense of any angle is preserved. Roughly speaking, infinitesimally small figures are rotated or stretched but not reflected by a conformal mapping.

If U is an open subset of the complex plane, then a function $f\colon U \to \mathbb{C}$ is conformal if and only if it is holomorphic and its derivative is everywhere non-zero on U. If f is antiholomorphic (that is, the conjugate of a holomorphic function), it still preserves the size of angles, but it reverses their orientation.

Lemma 1.2 *If $f\colon U \to V$ is a biholomorphic map, then f is conformal. The domains U and V are accordingly also termed* conformally equivalent.

Proof Since $f\colon U \to V$ is holomorphic with holomorphic inverse, we have that $f \circ f^{-1}\colon V \to V$ is holomorphic. Taking the derivative of both sides of $f(f^{-1}(z)) = z$ gives $f'(f^{-1}(z))(f^{-1})'(z) = 1$. This implies that $f'(w) \neq 0$ for all $w \in U$ and hence f is conformal. ■

Lemma 1.3 *Let $U \subset \mathbb{C}$ be a domain and $f\colon U \to \mathbb{C}$ be a holomorphic, injective map. Then f is biholomorphic onto its image, $f(U)$.*

Proof First note that since U is open and f injective, f is not constant. Whence $V = f(U)$ is open. (Recall the *open mapping theorem*: If f is a non-constant holomorphic map on a domain U, then the image under f of any open set in U is open.) Denote the inverse of f by g. It remains to show that g is holomorphic.

First assume that $f'(z_0) = 0$ for some $z_0 \in U$, and let $w_0 = f(z_0)$. Then $f(z) - w_0$ has a zero of order $m \geq 2$ at z_0. So $f(z) = w_0 + (z - z_0)^m h(z)$, where h is holomorphic at z_0 with $h(z_0) \neq 0$. But this implies that f is not injective in a small neighbourhood of z_0: To see this, choose a holomorphic branch of $h(z)^{1/m}$ and let $g(z) = (z - z_0)h(z)^{1/m}$. The latter has a simple zero at z_0, so $g'(z_0) \neq 0$ and by the inverse function theorem g is injective near z_0. Whence $f(z)$ near z_0 is the composition of g followed by $z \to z^m$ followed by $z \to z + w_0$. It follows that $f'(z_0) \neq 0$ for all $z_0 \in U$.

Now one can use the definition of the derivative together with the fact that f is injective to show that g is differentiable. Moreover, if $f(z) = w$, then $g'(w) = (f'(z))^{-1}$. ■

It is shown in the above proof that f injective near z_0 implies that $f'(z_0) \neq 0$. The converse is also true:

Let f be holomorphic at z_0 and satisfy $f'(z_0) \neq 0$. Then

$$\det J_f(z) = \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = |f'(z)|^2 \neq 0,$$

where the second equality follows from the Cauchy–Riemann equations. The *inverse function theorem* from multi-variable calculus now implies that there is a small disc $D(z_0; r)$ such that the restriction of f to this disc is injective.

1.2 The Riemann mapping theorem

Denote the (open) unit disc, the unit circle and the closed unit disc respectively by

$$D^2 = \{z \in \mathbb{C} |\ ||z|| < 1\},\ S^1 = \partial D^2,\ \text{and}\ \overline{D}^2 = D^2 \cup S^1.$$

Exercise 1 *Define a continuous, bijective map $f\colon D^2 \to \mathbb{C}$ which has a continuous inverse. Is this map holomorphic? Is this map conformal?*

Theorem 1.4 (Riemann mapping theorem) *Let $U \subset \mathbb{C}$ be a simply connected, proper, open subset. Then U is biholomorphic to the interior of the unit disc. Moreover, if one fixes $z_0 \in U$ and $\varphi \in [0, 2\pi)$, then there is a unique such map f with $f(z_0) = 0$ and the argument of the derivative of f at z_0, $\arg f'(z_0)$, is equal to φ.*

The statement is not true in higher dimensions. For example, the open unit ball $D^{2n} = \{z \in \mathbb{C}^n | \, ||z|| < 1\}$ and the product of n–factors $D^2 \times ... \times D^2$ are not biholomorphically equivalent for $n > 1$.

Simply connected open sets in the plane can be highly complicated, for instance the boundary can be a nowhere differentiable fractal curve of infinite length, even if the set itself is bounded. The fact that such a set can be mapped in an angle-preserving manner to the nice and regular unit disc seems counter-intuitive.

Since the automorphisms of D^2 are the hyperbolic rigid motions, it follows that there exists a three-parameter family of injective, conformal mappings between any simply connected domain and D^2.

Corollary 1.5 *Two simply connected domains in the plane are homeomorphic.*

The Riemann mapping theorem can be generalized to the context of Riemann surfaces: If U is a simply-connected open subset of a Riemann surface, then U is biholomorphic to one of the following: the Riemann sphere, the complex plane or the open unit disk. This is known as the *Uniformization Theorem*, and will hopefully be addressed towards the end of these notes.

Exercise 2 *If z_0 is real and U is symmetric with respect to the real axis, prove by uniqueness that the function f in the statement of the Riemann Mapping Theorem satisfies*

$$f(\bar{z}) = \bar{f}(z)$$

if $\arg f'(z_0) = 0$. *Find a formula relating $f(\bar{z})$ and $\bar{f}(z)$ in the general case.*

1.3 Some examples

Round neighbourhood to unit disc: A conformal map from $D(z_0;r)$ onto D^2 as a composition of translation and dilation: First apply the translation $z \to z - z_0$. This is a biholomorphic map taking $D(z_0;r) \to D(0,r)$. Next apply the dilation $z \to \frac{z}{r}$. This is again a biholomorphic map taking $D(0,r) \to D(0,1) = D^2$. The composition $z \to \frac{1}{r}(z - z_0)$ is the desired map.

Upper halfspace to unit disc: The upper half space is:

$$\begin{aligned} \mathbf{H} &= \{z \in \mathbb{C} | \, \Im(z) > 0\} \\ &= \{z \in \mathbb{C} | \, 0 < \arg z < \pi\}. \end{aligned}$$

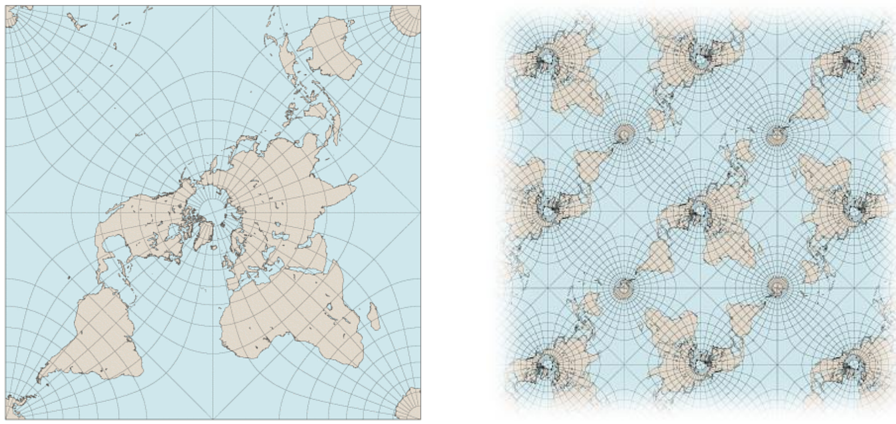

Figure 1: Conformal mapping of a hemisphere

The map

$$\varphi(z) = \frac{z-i}{z+i}$$

takes the upper half space onto D^2. Note that $\varphi(0) = -1$, $\varphi(-1) = i$, $\varphi(\infty) = 1$, $\varphi(1) = -i$. Also $\varphi(i) = 0$. Point out the inverse images of the coordinate axes.

An extension to the boundary can be obtained when the boundary is piecewise analytic (more later). Often it is easier to map a region first to the upper halfspace and then to use the above map.

Sector to upper half space: Any sector can first be mapped by a translation and rotation to a standard sector $S_\alpha = \{z \in \mathbb{C} |\ 0 < \arg z < \alpha\}$, where $0 < \alpha < 2\pi$. Let $\beta = \pi/\alpha$, then $z \to z^\beta$ maps S_α to the upper half space. Note that this map cannot be conformal at the corner of the sector!

1.4 Applications to cartography

The interior of a square can be mapped conformally onto D^2. In particular, there is a conformal map from a hemisphere to the unit square; first discovered by Charles Sanders Peirce in 1879. The pictures in Figure 1 are due to Furuti [3]. Shown in the figure is also a conformal map from the lower hemisphere, divided into four triangles. The whole map (centre square plus four triangles) is conformal everywhere except for the midpoints of the boundary edges; here the equator and four meridians make a right angle instead of a straight one. The whole map can be used to tesselate the plane.

1.5 The hypotheses are necessary

Connected: Any two points $z, w \in D^2$ can be connected by a path γ. Since $f\colon U \to D^2$ is bijective, for any $u, v \in U$, take $z = f(u), w = f(v)$. Then $f^{-1} \circ \gamma\colon I \to U$ is a path with endpoints u, v.

Simply connected: Let $\gamma\colon S^1 \to U$ be a closed curve. Then $f \circ \gamma\colon S^1 \to D^2$ is contractible; denote a homotopy by $H\colon [0,1] \times S^1 \to D^2$. Then $f^{-1} \circ H$ gives a homotopy contracting γ.

Proper subset: Liouville's theorem (Every bounded entire function is constant. That is, every holomorphic function f for which there exists a positive number M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$ is constant.)

Open: Take the unit square $\{z \in \mathbb{C} | 0 \leq \Re(z) \leq 1, 0 \leq \Im(z) \leq 1\}$ and the closed unit disc; at corners have problem with angles! This, in particular, implies that derivative at corners goes to zero.

2 Proof of the Riemann mapping theorem

2.1 Proof of the uniqueness part

If f_1 and f_2 are two such maps, then the composition

$$g = f_1 \circ f_2^{-1}\colon D^2 \to D^2$$

is a map with $g(0) = 0$ and argument of $g'(0)$ is zero. By the *Schwarz Lemma*, an automorphism of the unit disc fixing the origin is of the form $z \to az$ for some $a \in S^1$. It follows that $a = 1$, so $g(z) = z$ and hence $f_1 = f_2$.

2.2 Outline of the remaining proof

It follows from the proof of the uniqueness part, that if a biholomorphic map f exists, then (up to a rotation) one may assume that the argument of $f'(z_0)$ is zero. Whence $f'(z_0)$ is a positive real number, written $f'(z_0) > 0$.

It remains to prove the existence of a map. The main idea is to consider the family

$$\mathscr{F} = \{f\colon U \to D^2 |\ f \text{ is holomorphic, injective, } f(z_0) = 0, f'(z_0) > 0\}.$$

Each element f of the family is biholomorphic onto its image $f(U) \subseteq D^2$ – the image of the desired map is D^2. It will be shown that the desired map is the unique element in $\mathscr{F}$ whose derivative at z_0 is maximal. The proof has three parts:

(1) Show that $\mathscr{F}$ is non-empty.

(2) Show that there is a unique element whose derivative is maximal.

(3) Show that this element has the required properties.

2.3 $\mathscr{F}$ is non-empty

This part introduces some nice techniques that are useful in other contexts, and are essentially applications of Cauchy's theorem, which is stated below in a fairly general form.

Definition (Homotopy) *Let X be a topological space and let $\gamma_i\colon [0,1] \to X$ be two paths in X, $i \in \{0,1\}$, with common endpoints, $\gamma_i(0) = x$ and $\gamma_i(1) = y$. A* homotopy *from γ_0 to γ_1 is a continuous map*

$$H\colon [0,1] \times [0,1] \to X,$$

such that $\gamma_i(t) = H(i,t)$, $H(s,0) = x$ and $H(s,1) = y$ for all $s,t \in [0,2]$.

So two paths are homotopic if one can be deformed into the other continuously.

Definition (Simply connected) *A topological space X is* simply connected *if any closed path in X is homotopic to a constant path.*

A *chain* is a formal sum of paths $\gamma = \sum n_i \gamma_i$. A *cycle* is a chain such that $\partial\gamma = 0$, where the boundary operator is the operator assigning to a chain the *formal* sum of endpoints, i.e. the sum of $\partial\gamma_i = [\gamma_i(0)] - [\gamma_i(1)]$, where the square brackets indicate that the usual arithmetic between points is not allowed; $[1] + [2] \neq [3]$. Define

$$\int_\gamma f(z)dz = \sum n_i \int_{\gamma_i} f(z)dz.$$

Then the cycle γ is *homologous to zero in U* if the winding number $n(\gamma;a) = 0$ for all $a \notin U$. Also recall that the *winding number* is the integer

$$n(\gamma;a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz.$$

If γ is a closed path homotopic to zero, then it is homologous to zero.

Theorem 2.1 (Cauchy's theorem) *Let $f\colon U \to \mathbb{C}$ be a holomorphic function on a domain U and let γ be a cycle homologous to zero. Then*

$$\int_\gamma f(z)dz = 0.$$

To paraphrase Cauchy's theorem: if the integral $\int_\gamma f(z)dz$ vanishes for every function of the form $\frac{1}{z-a}$, where $a \notin U$, then it vanishes for every holomorphic function on U.

Corollary 2.2 *Let U be a simply connected domain. Then*

$$\int_\gamma f(z)dz = 0,$$

for every holomorphic function f on U and every cycle γ.

Proof Every cycle is a linear combination of closed paths. Any closed path in a simply connected domain is homotopic to zero, hence homologous to zero (winding number argument). Thus, Cauchy's theorem applies. ■

Recall some basic theory of integration and differentiation. Assume that f is a holomorphic function on a domain U such that the integral

$$\int_\gamma f(z)dz$$

only depends on the endpoints of γ. Let $z_0 \in U$. Then we may define a function

$$F(z) = \int_{z_0}^{z} f(z)dz,$$

where we pick any path between z_0 and z. The function $F(z)$ is holomorphic, $F(z_0) = 0$ and $F'(z) = f(z)$. The verification of this is bookwork.

Corollary 2.3 *Let U be a simply connected domain and let $f(z)$ be a holomorphic function on U which is no-where zero on U. Then we may define holomorphic functions $\log f(z)$ and $\sqrt[n]{f(z)}$ on U.*

Proof The function

$$\frac{f'(z)}{f(z)}$$

is holomorphic on U since $f(z)$ is nowhere zero. Pick $z_0 \in U$. By the preceding discussion and Corollary 2.2 we may find a holomorphic function $F(z)$ on U such that

$$F'(z) = \frac{f'(z)}{f(z)} \quad \text{and} \quad F(z_0) = 0.$$

Consider the function

$$h(z) = e^{-F(z)-\log f(z_0)} f(z).$$

The derivative is

$$h'(z) = e^{-F(z)-\log f(z_0)}\Big(-\frac{f'(z)}{f(z)}\Big)f(z) + e^{-F(z)-\log f(z_0)}f'(z) = 0.$$

Whence h is constant. Since $h(z_0) = 1$, it follows that

$$f(z) = e^{F(z)+\log f(z_0)}. \tag{1}$$

Thus,

$$\log f(z) = F(z) + \log f(z_0). \tag{2}$$

This shows that the logarithm exists. The n–th roots are then defined by letting:

$$g(z) = e^{1/n \log f(z)}. \tag{3}$$

This completes the proof of the corollary. ■

Exercise 3 *Let*

$$f(z) = \frac{z-i}{z+i}.$$

For the functions $\log f$ *and* $\sqrt{f}$ *give an explicit definition of a single-valued branch in a suitable region using the method in the proof of Corollary 2.3. Use the equation* $f(z) = e^{F(z)+\log f(z_0)}$ *to deduce a known identity (and hence to verify your answer).*

Lemma 2.4 *The set* $\mathscr{F}$ *is non-empty.*

Proof Pick $a \notin U$. By Corollary 2.3 we may pick a holomorphic function $h(z)$ such that

$$(h(z))^2 = z - a.$$

This function is injective on U : $h(z_2) = h(z_1)$ implies $(h(z_2))^2 = (h(z_1))^2$ implies $z_2 - a = z_1 - a$ implies $z_2 = z_1$.

By the same chain of events, $h(z_2) = -h(z_1)$ also implies $z_2 = z_1$ whence $h(z_2) = h(z_1)$, so $h(z_2) = 0$ and $a = z_2 \in U$, contradicting the choice of a. Thus, at most one of $\pm b$ is in the image of h for any $b \in \mathbb{C}$.

Recall that z_0 denotes a specified point in U. By the open mapping theorem, there is a positive constant δ such that h surjects onto the disc of radius δ about $h(z_0)$. Therefore the image does not meet the disc of radius δ about $-h(z_0)$. Equivalently,

$$|\, h(z) + h(z_0)\, | \geq \delta$$

for all $z \in U$. In particular

$$2|\, h(z_0)\, | \geq \delta.$$

Consider the function

$$g(z) = \frac{\delta}{4} \cdot \frac{|h'(z_0)|}{|h(z_0)|^2} \cdot \frac{h(z_0)}{h'(z_0)} \cdot \frac{h(z) - h(z_0)}{h(z) + h(z_0)}.$$

Then g is the composition of h with a Möbius transformation. In particular, g is injective. Clearly $g(z_0) = 0$, and using the quotient rule $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$ and substituting, we have

$$g'(z_0) = \frac{\delta}{4} \cdot \frac{|h'(z_0)|}{|h(z_0)|^2} \cdot \frac{h(z_0)}{h'(z_0)} \cdot \frac{h'(z_0)}{2h(z_0)} = \frac{\delta}{8} \cdot \frac{|h'(z_0)|}{|h(z_0)|^2} > 0.$$

Finally we need to show that g maps into the unit ball.

$$\begin{aligned} &\Big| \frac{h(z) - h(z_0)}{h(z) + h(z_0)} \Big| \\ =& \Big| \frac{h(z) + h(z_0) - 2h(z_0)}{h(z) + h(z_0)} \Big| \\ =& |h(z_0)| \cdot \Big| \frac{1}{h(z_0)} - \frac{2}{h(z) + h(z_0)} \Big| \le |h(z_0)| \frac{4}{\delta}, \end{aligned}$$

using the triangle inequality and the above inequalities. Thus, $|g(z)| \le 1$. Since g is holomorphic, the open mapping theorem implies that $g(U) \subseteq D^2$ and so $g \in \mathscr{F}$. ■

2.4 Some classical results

Definition 2.5 *A sequence of functions (f_n) is said to* converge uniformly *on X to a function f if for each $\varepsilon > 0$ there is an index n_0 such that for all $n > n_0$ and all $x \in X$ we have:*

$$|\ f_n(x) - f(x)\ | < \varepsilon.$$

It is not difficult to check that the *Cauchy criterion* applies:

Cauchy criterion The sequence (f_n) converges uniformly on X if and only if for each $\varepsilon > 0$ there is an index n_0 such that for all $n, m > n_0$ and all $x \in X$ we have:

$$|\ f_m(x) - f_n(x)\ | < \varepsilon.$$

Lemma 2.6 *If $f_n \to f$ uniformly on X, then*

$$\lim_{n\to\infty} \lim_{x\to\xi} f_n(x) = \lim_{x\to\xi} \lim_{n\to\infty} f_n(x).$$

Proof Let $\varepsilon > 0$.

First consider the left hand side. Let $\alpha_n = \lim_{x\to\xi} f_n(x)$ for each n. By Cauchy's criterion there is n_0 such that $\mid f_m(x) - f_n(x) \mid < \varepsilon$ for all $n, m > n_0$ and all $x \in X$. Letting $x \to \xi$, we get $\mid \alpha_m - \alpha_n \mid \leq \varepsilon$. Whence (α_n) is a Cauchy sequence. So the limit $\lim \alpha_n$ exists, denote it by α.

We now need to show that $\lim_{x\to\xi} f(x)$ exists and equals α. Using the triangle inequaliity gives

$$\mid f(x) - \alpha \mid \leq \mid f(x) - f_m(x) \mid + \mid f_m(x) - \alpha_m \mid + \mid \alpha_m - \alpha \mid.$$

Now fix m such that $\mid f(x) - f_m(x) \mid < \frac{\varepsilon}{3}$ for all $x \in X$, and $\mid \alpha_m - \alpha \mid < \frac{\varepsilon}{3}$. From continuity of f_m, we can get $\delta > 0$ such that

$$\mid f_m(x) - \alpha_m \mid < \frac{\varepsilon}{3} \quad \forall x \in X \text{ s.t. } 0 < \mid x - \xi \mid < \delta.$$

For these x, we have $\mid f(x) - \alpha \mid < \varepsilon$, and the lemma follows. ■

Lemma 2.7 *If $f_n \to f$ uniformly on X and each f_n is continuous, then f is continuous.*

Proof Let $\xi \in X$. Applying the previous lemma gives:

$$\lim_{x\to\xi} f(x) = \lim_{x\to\xi} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to\xi} f_n(x) = \lim_{n\to\infty} f_n(\xi) = f(\xi).$$

Whence f is continuous at ξ. ■

Theorem 2.8 (Weierstrass) *Let*

$$U_1 \subset U_2 \subset U_3 \subset \dots$$

be an infinite sequence of domains whose union is U. Suppose that $f_n \colon U_n \to \mathbb{C}$ is a sequence of holomorphic functions which tends to a limit function $f \colon U \to \mathbb{C}$, uniformly on compact subsets. Then f is holomorphic. Moreover, (f_n') converges uniformly on compact subsets to f'.

Proof Let $z \in U$ and D be a closed disc with centre z contained in U. Recall that the Heine–Borel theorem states that any open cover of a compact subset has a finite subcover. It follows that $D \subset U_k$ for some k. Let $\gamma = \partial D$. The *Cauchy integral formula* implies that for each $n \geq k$, we have

$$f_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(w)}{w - z} dw.$$

Lemma 2.6 implies that letting $n \to \infty$ yields:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw,$$

whence f is analytic in D and in particular at z.

For the second part, note that

$$f_n'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(w)}{(w-z)^2} dw,$$

which gives

$$\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^2} dw = f'(z).$$

So $f_n' \to f'$ pointwise. We need to show that the convergence is uniform on *any compact subset*.

Let $C \subset U$ be compact. For each point $z \in C$ there is a radius $r_z > 0$ such that the closed disc $\overline{D}(z; r_z)$ is contained in U. Now consider all the discs $D(z; \frac{r_z}{2})$. This is an open cover of C and hence has a finite sub-cover. Now let r denote the minimum radius of a disc in this sub-cover. Then for every point z in C the disc $D(z; r)$ is contained in C.

Let $\varepsilon > 0$. Since $f_n \to f$ uniformly on compact subsets, there is n_0 such that for all $n > n_0$ and all $z \in C$ we have:

$$| f_n(z) - f(z) | < \varepsilon r.$$

This implies that whenever $n > n_0$ and $z \in C$:

$$\begin{aligned} | f_n'(z) - f'(z) | &= \left| \frac{1}{2\pi i} \int_\gamma \frac{f_n(w) - f(w)}{(w-z)^2} dw \right| \\ &\le \frac{1}{2\pi} \frac{\varepsilon r}{r^2} L(\gamma) \\ &= \varepsilon, \end{aligned}$$

where $\gamma = \partial D(z; r)$ and hence $L(\gamma) = 2\pi r$ — using $| \int_\gamma g | \le \max_\gamma\{|g|\} \cdot L(\gamma)$. ■

Remark 2.9 *Repeated application of the above shows that, under the hypothesis,* $f_n^{(k)} \to f^{(k)}$ *uniformly on compact subsets.*

The following is an application of Taylor's formula:

Proposition 2.10 *Let $f\colon U \to \mathbb{C}$ be a holomorphic function which is not identically zero. Enumerate the zeros of f by z_j, where each zero is counted with multiplicity. Let D be a disc contained in U and γ a closed curve in D.*

If γ does not pass through a zero of f, then

$$\sum_j n(\gamma; z_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz.$$

Proof If $z_j \notin D$, then clearly $n(\gamma; z_j) = 0$. Moreover, D contains at most finitely many zeros of f since otherwise the set of zeros has an accumulation point in the closed disc due to the Bolzano–Weierstrass theorem; this violates the fact that zeros are isolated. It follows that the sum has at most a finite number of non-zero terms. To prove equality recall that Taylor's theorem implies that:

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \ldots + \frac{f^{(k-1)}(a)}{(k-1)!}(z-a)^{k-1} + g_k(z)(z-a)^k,$$

where g_k is analytic in U. Successively applying this at all the zeros gives

$$f(z) = (z-z_1)\cdots(z-z_n)g(z),$$

where $z_1,\ldots,z_n$ are the zeros of f in D and g is a no-where zero, holomorphic function on D. Forming the *logarithmic derivative* gives:

$$\frac{f'(z)}{f(z)} = \frac{1}{z-z_1} + \ldots + \frac{1}{z-z_n} + \frac{g'(z)}{g(z)}$$

whenever $z \neq z_j$ and, in particular, on γ. Since $g(z) \neq 0$ on D, Cauchy's theorem yields

$$\int_\gamma \frac{g'(z)}{g(z)}dz = 0.$$

Recalling the definition of $n(\gamma; z_j)$ one obtains the desired conclusion. ■

For a given function f which is holomorphic in the neighbourhood of disc D and not zero on ∂D. the above propositions enables us to count the number of zeros of f (with multiplicity) contained in D by taking γ to be the boundary circle of D. Then each number $n(\gamma; z_j)$ is either zero (when $z_j \notin D$) or one (when $z_j \in D$).

Theorem 2.11 (Hurwitz) *Suppose that the sequence of holomorphic functions f_n converges uniformly to a function f on U. If the functions f_n are nowhere zero, then either f is identically zero or nowhere zero.*

Proof Suppose that f is not identically zero. The zeros of f are then isolated. Given $z_0 \in U$, there is $r > 0$ such that f is nowhere zero on the punctured disc $0 < |z - z_0| \leq r$. Let m be the minimum of $|f(z)|$ on the circle γ defined by $|z - z_0| = r$.

We first claim that $\frac{1}{f_n} \to \frac{1}{f}$ uniformly on γ:

Note that $\min_\gamma\{|f_n(z)|\} \to m$ as $n \to \infty$. There is n_1 such that $\min_\gamma |f_n(z)| > \frac{m}{2}$ for all $n > n_1$. In particular, these f_n don't vanish on γ. Given $\varepsilon > 0$, since $f_n \to f$ uniformly, there is n_0 such that

$$| \frac{1}{f_n(z)} - \frac{1}{f(x)} | = | \frac{f(z) - f_n(z)}{f(z) f_n(z)} | \leq \frac{2\varepsilon}{m^2},$$

for all $n > \max\{n_0, n_1\}$. This proves the claim.

Using this and the Weierstrass theorem, one has:

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{f_n'(z)}{f_n(z)} dz = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz.$$

The integrals count the number of zeros of f_n and f respectively in the disc bounded by γ. Since the functions f_n are nowhere zero, it follows that f is nowhere zero. ■

Definition 2.12 (Normal family) *A family $\mathscr{F}$ is said to be normal in U if every sequence of functions $(f_n) \subset \mathscr{F}$ contains a subsequence which converges uniformly on every compact subset of U.*

Note that the definition does not require the limiting function to be contained in $\mathscr{F}$.

Exercise 4 *Show that the family of functions $f_n(z) = z^n$, where n is a positive integer, is a normal family in $D^2 = \{z \in \mathbb{C} : |z| < 1\}$, but not in any domain that contains a point in its complement. What can be said about the family (f_n')?*

Exercise 5 *Let $\mathscr{F}$ be a family of holomorphic functions which is not normal in U. Show that there is a point $z_0 \in U$ such that $\mathfrak{F}$ is not normal in any neighbourhood of z_0. Hint: A compactness argument.*

2.5 Necessary and sufficient conditions for normal families

Definition 2.13 (Equicontinuous) *Let U be a domain and $\mathscr{F}$ be a family of continuous functions $f\colon U \to \mathbb{C}$. The functions in $\mathscr{F}$ are said to be* equicontinuous *on a set $E \subset U$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z - w| < \delta$, $z, w \in E$ and $f \in \mathscr{F}$.*

Note that each element in an equicontinuous family is uniformly continuous on E. The following result was proved in Metric Spaces, and will be assumed without proof:

Theorem 2.14 (Arzela–Ascoli) *A family $\mathscr{F}$ of continuous functions with values in a metric space X is normal in the region U of the complex plane if and only if*

(1) *$\mathscr{F}$ is equicontinuous on every compact subset $C \subset U$;*

(2) *for any $z \in U$, the values $f(z)$, $f \in \mathscr{F}$, lie in a compact subset of X.*

Assuming this theorem, the main technical result needed to prove the Riemann mapping theorem is the following:

Proposition 2.15 *A family of holomorphic functions $\mathscr{F}$ is normal if and only if the functions are uniformly bounded on compact subsets.*

Proof First assume that $\mathscr{F}$ is normal. Let $C \subset U$ be a compact subset. The Arzela–Ascoli Theorem, Part 1, implies that given $\varepsilon > 0$ there is $\delta > 0$ such that $| f(z) - f(w) | < \varepsilon$ whenever $| z - w | < \delta$, $z, w \in C$ and $f \in \mathscr{F}$.

Now C is compact, and hence can be covered by a finite number of δ–balls. Part 2 of the theorem implies that for each centre z there is a compact subset C_z containing $f(z)$ for each $f \in \mathscr{F}$. Take the union of all these sets C_z and take the closed ε–neighbourhood of that. This is a compact set and we have $f(C)$ is contained in that. Since any compact subset is contained in some ball, the functions are uniformly bounded.

Conversely, assume that the functions are uniformly bounded. Clearly Part 2 of the Arzela–Ascoli Theorem is satisfied. It remains to show that $\mathscr{F}$ is equicontinuous on compact subsets.

Let γ be the boundary of a closed ball of radius r contained in U. Pick z and w in the interior of this ball. Then Cauchy's formula implies that

$$\begin{aligned} f(z) - f(w) &= \frac{1}{2\pi i} \int_\gamma \Big(\frac{1}{\xi - z} - \frac{1}{\xi - w} \Big) f(\xi) d\xi \\ &= \frac{z - w}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)(\xi - w)} d\xi. \end{aligned}$$

The functions are uniformly bounded, say by K, so in particular $|f| \le K$ on γ. Assume that z and w belong to the ball of radius $\frac{r}{2}$ with the same centre, then the distance $|\xi - z| > \frac{r}{2}$ for $\xi \in \gamma$ and so:

$$| f(z) - f(w) | < \frac{4K|z - w|}{r}.$$

Now let C be a compact subset of U. For each point $z \in C$ there is a radius $r_z > 0$ such that the closed disc $\overline{D}(z, r_z)$ is contained in U. Now consider all the discs $D(z, \frac{r_z}{4})$. This is an open cover of C and hence has a finite sub-cover. Denote the respective centres of discs by z_i. Let r be the minimum over all radii r_{z_i}, and write $r_{z_i} = r_i$.

Given $\varepsilon > 0$ let

$$\delta = \min\{\frac{r}{4}, \frac{\varepsilon r}{4K}\}.$$

Suppose that $\mid z - w \mid < \delta$. Then there is z_k such that $\mid w - z_k \mid < \frac{r_k}{4}$. Then

$$\mid z - z_k \mid \leq \mid z - w \mid + \mid w - z_k \mid < \delta + \frac{r_k}{4} \leq \frac{r_k}{2}.$$

But then we can use the above estimate and get:

$$\mid f(z) - f(w) \mid < \frac{4K|z-w|}{r_k} < \frac{4K\delta}{r} \leq \varepsilon.$$

Thus, $\mathscr{F}$ is equicontinuous on C. ■

Proof of the Riemann mapping theorem Let B be a least upper bound for the derivatives $f'(z_0)$ as f ranges over $\mathscr{F}$. Then $B > 0$, but it may be the case that $B = \infty$. Pick a sequence $(g_i)_n \subset \mathscr{F}$ such that $g_i'(z_0)$ approaches B. It follows from Proposition 2.15 that $\mathscr{F}$ is a normal family. Whence $(g_i)_n$ contains a subsequence (also denoted by $(g_i)_n$) which converges to a function f, uniformly on every compact subset. The function f is holomorphic due to Weierstrass' theorem, and $(g_i')_n$ converges to f', again uniformly on compact subsets. Whence $f'(z_0) = B$ and B is a positive real number, not infinity.

We now show that $f \in \mathscr{F}$. We have $f(z_0) = 0$ since $g_i(z_0) = 0$ for each i. Similarly, $|f(z)| \leq 1$ for all $z \in U$. The function f is not constant since $f'(z_0) = B > 0$. Since f is holomorphic, the open mapping theorem implies $|f(z)| < 1$ for all $z \in U$. It remains to show that f is injective. Let $w \in U$. Consider the functions $g_w(z) = g(z) - g(w)$, where $g \in \mathscr{F}$. They are nowhere zero on $U \setminus \{w\}$ as each g is injective. Then $f(z) - f(w)$ is a limit of such functions; it is nowhere zero due to Hurwitz's theorem. Since w was chosen arbitrarily, it follows that f is injective. Thus, $f \in \mathscr{F}$, and f is an element of $\mathscr{F}$ with maximal derivative at z_0.

Suppose f is not surjective. We will now use the fact that any branch of the square–root function defined on a region in the interior of the unit disc is expanding. Since f is not surjective, we may choose $w \in D^2 \setminus f(U)$. As U is simply connected, we may find a holomorphic branch of

$$F(z) = \sqrt{\frac{f(z) - w}{1 - \overline{w} f(z)}}.$$

Note that F is injective and $|F(z)| < 1$. Let

$$G(z) = \frac{|F(z_0)|}{F'(z_0)} \cdot \frac{F(z) - F(z_0)}{1 - \overline{F}(z_0)F(z)}.$$

Then $G(z_0) = 0,\ G$ is injective and $G'(z_0) > 0$. So $G \in \mathscr{F}$. In fact

$$G'(z_0) = \frac{|F(z_0|}{1 - |F(z_0)|^2} = \frac{1 + |w|}{2\sqrt{|w|}} B > B,$$

which contradicts the choice of B. Hence f is surjective, and this completes the proof of the Riemann mapping theorem.

The fact that f is the *unique* map with maximal derivative at z_0 now follows (as in the proof of the uniqueness part) from the Schwarz lemma. For the proof, however, this fact is irrelevant. ■

3 The Schwarz–Christoffel formula

The Riemann mapping theorem only asserts existence, but does not give any hints on how to find a map. We'll now look at a method which applies to certain domains with boundary consisting of straight line segments.

3.1 Boundary behaviour and the reflection principle

The following (non-trivial) observation will just be stated; a more general treatment can be found in [1], Sections 6.1.2 – 2.1.4.

Let P be a simply connected region which is bounded by a finite-sided polygon, and $f\colon\ P \to D^2$ be a biholomorphic map. Then f extends holomorphically to the boundary of P minus the vertices, it extends continuously to all of ∂P, and the extension maps ∂P bijectively onto S^1. Composition with the map φ^{-1} thus gives a map $P \to \mathbb{H}$, where one may assume that all vertices are mapped to real numbers. We will now try to understand the inverse of such a map explicitly.

3.2 The Schwarz–Christoffel formula

Recall the description of a map from $\mathbb{H}$ to a sector. The map $f(z) = (z-a)^{\alpha}$ maps the upper half-plane to a sector with angle $\pi\alpha$ at 0. It extends analytically to $\partial\mathbb{H} = \mathbb{R}$ where it is one-to-one.

Choose points $a_1 < \cdots < a_n$ on the real line and choose a polygon with vertices $w_1, \ldots, w_n$ (in anti-clockwise order) in the plane. Denote the interior angle at w_i by $\pi\alpha_i$. Note that $0 < \alpha_i < 2$ and

$$\sum \pi\alpha_i = (n-2)\pi \quad \Longrightarrow \quad \sum \alpha_i = n-2.$$

A biholomorphic map taking $\mathbf{H}$ to the region bounded by the polygon looks locally near a_k like:

$$f(z) = w_k + e^{i\vartheta}(z-a_k)^{\alpha_k} h_k(z),$$

where $h_k(z)$ is analytic and non-zero in a small neighbourhood of a_k. Thus,

$$\begin{aligned} f'(z) &= e^{i\vartheta}\alpha_k(z-a_k)^{\alpha_k-1}h_k(z) + e^{i\vartheta}(z-a_k)^{\alpha_k}h_k'(z) \\ &= (z-a_k)^{\alpha_k-1}e^{i\vartheta}(\alpha_k h_k(z) + (z-a_k)h_k'(z)) \\ &= (z-a_k)^{\alpha_k-1}H_k(z), \end{aligned}$$

where $H_k(z)$ is analytic and non-zero in a small neighbourhood of a_k. It follows inductively, that the derivative of f has the expression

$$f'(z) = h(z)\prod_{k=1}^{n}(z-a_k)^{\alpha_k-1},$$

where $h(z)$ is a holomorphic map defined on $\mathbf{H}$, and $h(z) \neq 0$ for all $z \in \mathbf{H}$ since f is conformal. It will now be shown that $h(z)$ is in fact constant. We have:

$$h(z) = f'(z)\prod_{k=1}^{n}(z-a_k)^{1-\alpha_k}.$$

The right hand side extends continuously to $\mathbb{R}$ since the exponents are contained in $(-1,1)$, hence h extends continuously to $\mathbb{R}$. Let $a \in (a_i, a_{i+1})$. We have

$$\begin{aligned} \arg h(a) &= \arg f'(a) + \sum_{k=1}^{n}(1-\alpha_k)\arg(a-a_k) \\ &= \arg f'(a) + \sum_{k=1}^{i}(1-\alpha_k)\arg(a-a_k) + \sum_{k=i+1}^{n}(1-\alpha_k)\arg(a-a_k) \\ &= \arg f'(a) + 0 + \sum_{k=i+1}^{n}(1-\alpha_k)\pi. \end{aligned}$$

Note that $\arg f'(a)$ is the slope of the side $[w_i, w_{i+1}]$ of the polygon, and hence constant in (a_i, a_{i+1}). So $\arg h(a)$ is constant as a ranges over (a_i, a_{i+1}). Since h is continuous, it follows that $\arg h(z)$ is constant on all of $\mathbb{R}$. It now follows from the Maximum principle that h is constant:

Consider the function $h \circ \varphi^{-1} \colon D^2 \to \mathbb{C}$. The map φ^{-1} is non-constant and extends to a continuous map which takes the unit circle to the real line. The function $\log(h(\varphi(z)))$,

$z \in \mathbf{H}$, is well–defined (since h is no-where zero) and has constant imaginary part when restricted to S^1. Since the imaginary part of $\log(h(\varphi(z)))$ is a harmonic function, it must be constant on the whole disc (since the maximum and minimum of a non-constant harmonic function are attained on the boundary). But a holomorphic function with constant imaginary part is constant (this follows, for instance, from the Cauchy–Riemann equations and the Mean Value Theorem). Hence $h(z)$ is constant, and we have proved the following:

Proposition 3.1 (Schwarz–Christoffel formula) *The functions f which map $\mathbf{H}$ onto regions bounded by polygons with angles $\alpha_k\pi, 0 < \alpha_k < 2,\ k = 1,\dots,n,$ have derivative*

$$f'(z) = A\prod_{k=1}^{n}(z-a_k)^{\alpha_k-1},$$

and hence are of the form

$$f(z) = A\int_{z_0}^{z}\prod_{k=1}^{n}(w-a_k)^{\alpha_k-1}dw + B,$$

where $a_k \in \mathbb{R}$, and A, B are complex constants.

3.3 Examples and generalisations

Consider the triangle $\Delta(0,1,p)$, where $\Im(p) > 0$. Denote the angles at $0,1,p$ by $\alpha\pi, \beta\pi, \gamma\pi$ respectively. Then a map $f\colon\ \mathbf{H} \to \Delta(0,1,w)$ with $f(0) = 0, f(1) = 1$ and $f(a) = p$, where $a > 1$, is determined as follows:

The formula gives

$$f(z) = A\int_{z_0}^{z}(w-0)^{\alpha-1}(w-1)^{\beta-1}(w-a)^{\gamma-1}dw + B.$$

We may choose $z_0 = 0$, and the condition $f(0) = 0$ implies $B = 0$. The constant A is determined by $f(1) = 1$, and we can also use the relation $\alpha+\beta+\gamma = 1$ to simplify the integral.

The situation becomes more tractable when we include ∞ as a point on $\partial\mathbf{H}$ mapping to a vertex of the polygon. In this case, the term corresponding to $a_n = \infty$ is left out in the above formula, as can be verified using a limiting argument in which $a_n \to \infty$. For instance, a biholomorphic map $f\colon\ \mathbf{H} \to \Delta(0,1,p)$ with $f(0) = 0, f(1) = 1$ and $f(\infty) = p$ is determined by

$$f(z) = A\int_{z_0}^{z}(w-0)^{\alpha-1}(w-1)^{\beta-1}dw + B.$$

Another possibility is to include ∞ as a vertex of the polygon, P. There are various cases:

(1) As a vertex of P with *negative* angle.
    - *Example 1:* A sector can be viewed as a bigon with angles $\pi\alpha$ at 0 and $-\pi\alpha$ at ∞.
    - *Example 2:* A second example.

(2) As a vertex of P with angle 0 if it is the end of an infinite strip.
    - *Example 3:* Mapping the upper half plane to an infinite strip.

Exercise 6 *Determine a biholomorphic mapping from the upper half plane to the region* $\{z \in \mathbb{C} | \, \Im(z) > 0, \Re(z) > 0, \min(\Im(z), \Re(z)) < 1\}$.

4 Riemann surfaces

A function $f\colon \mathbb{R} \to \mathbb{R}$ can be understood geometrically through its graph

$$G(f) = \{(x, f(x)) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2.$$

By analogy, the graph of a complex valued function is a subset of $\mathbb{R}^4$, which is difficult to visualise. If a holomorphic function $f\colon U \to f(U) \subseteq \mathbb{C}$ is injective, its geometry can be understood quite well by determining the images of lines or circles. The geometry of a function that its not injective, or of a multi-valued function, is best understood using a construction that on the one hand captures the whole function but on the other hand makes it injective. This leads to the concept of a Riemann surface.

4.1 Riemann surfaces from multi-valued functions

To construct the Riemann surface S_f associated with a multi-valued function f, begin with one sheet S_i for each branch f_i of the function, make branch cuts so that the branches are defined continuously on each sheet, and identify each edge of a cut on one sheet to an appropriate edge on another sheet so that the limiting function values match up continuously.

Then $S_f = \cup S_i$ with identifications along the branch cuts, but with the branch points removed. (Note that Ahlfors includes these points, but Gamelin doesn't.) The surface S_f is independent of the chosen branch cuts, i.e. choosing different cuts will result in the same surface.

There is a function $\hat{f}\colon S_f \to \mathbb{C}$ which is one-to-one onto its image; it is defined by $\hat{f}(z) = f_i(z)$ if $z \in S_i$. If z is contained on a branch cut along which the sheets S_i and S_j have been joined, then

$$\hat{f}(z) = \lim_{w \to z} f_i(z) = \lim_{w \to z} f_j(z),$$

where w ranges over a small neighbourhood of z on S_f. This is well–defined due to the gluing rule. We will see later, that there is a suitable notion which allows us to say that $\hat{f}$ is holomorphic.

Let S'_f be the surface obtained from S_f by adding the branch points. The branch points can be characterised as follows: Let $z_0 \in \mathbb{C}$ be a point such that f does not have an analytic extension over the punctured disc $D(z_0) \setminus \{z_0\}$ but has an analytic extension along any path in $D(z_0) \setminus \{z_0\}$. Then the pre-images of z_0 in S'_f are termed *branch points of* f. If such a branch point is contained in exactly n sheets, then it is termed an *algebraic branch point of order* $n-1$. For example, $z_0 = 0$ gives rise to such a branch point for $f(z) = \sqrt[n]{z}$. If it is contained in infinitely many sheets, then it is termed a *logarithmic branch point*. For instance $z_0 = 0$ for $f(z) = \log(z)$.

In class, I described the Riemann surfaces associated to the square root function and the function arcsin. Similar worked examples can be found in Gamelin, I.4 and I.5, and Ahlfors, III.4.3.

Exercise 7 *Let* $f(z) = \sqrt{z-1}\sqrt[3]{z-i}$.

(1) *What are the branch points and what are their orders?*

(2) *Why is it not possible to define branches of* f *using a single branch cut connecting the branch points?*

(3) *How many values does* $f(z)$ *have at a generic point* $z \in \mathbb{C}$*?*

Exercise 8 *(Gamelin I.7.7) Describe the Riemann surface associated with the function*

$$f(z) = \sqrt{(z-x_1)\cdots(z-x_n)},$$

where $x_i \in \mathbb{R}$, $x_1 < \ldots < x_n$ *and* $n \geq 1$. *(Hint: Consider* n *even or odd separately.)*

4.2 Riemann surfaces: The formal approach

Definition 4.1 *An* n–manifold *is a second countable Hausdorff space which is locally homeomorphic to* $\mathbb{R}^n$. *A 1–manifold is called a curve and a 2–manifold is called a surface.*

A Hausdorff space is a separable topological space. Second countable means that there is a countable base for the topology. For a Hausdorff space, being second countable is equivalent to being metrizable.

Definition 4.2 *A* Riemann surface *is a connected surface S together with a* holomorphic atlas. *That is, an open cover $\{U_\alpha\}$ by* charts $h_\alpha\colon U_\alpha \to V_\alpha \subset \mathbb{C}$, *where h_α is a homeomorphism and V_α a domain. Moreover, the transition functions*

$$h_\beta \circ h_\alpha^{-1}\colon\ h_\alpha(U_\alpha \cap U_\beta) \to h_\beta(U_\alpha \cap U_\beta)$$

are holomorphic.

Example 1 (Sphere) The *Riemann sphere* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The charts are $U_1 = \mathbb{C}$ with $h_1\colon U_1 \to \mathbb{C}$ defined by $h_1(z) = z$. And $U_2 = \hat{\mathbb{C}} \setminus \{0\}$ with $h_2\colon U_2 \to \mathbb{C}$ defined by $h_2(z) = \frac{1}{z}$ for $z \neq \infty$ and $h_2(\infty) = 0$. Then $U_1 \cap U_2 = \mathbb{C} \setminus \{0\}$ and the change of coordinates

$$(h_2 \circ h_1^{-1})(z) = \frac{1}{z}$$

is holomorphic in this intersection.

Example 2 (Torus) Let $\Lambda \subset \mathbb{C}$ be a lattice. That is $\Lambda \cong \mathbb{Z}^2$ as groups and the span of Λ as a vector space over $\mathbb{R}$ is all of $\mathbb{C}$. Then the quotient

$$E = \frac{\mathbb{C}}{\Lambda}$$

is a Riemann surface, which is known as an *elliptic curve*. In fact, the quotient is homeomorphic to $S^1 \times S^1$, so that E is compact. A holomorphic atlas is described explicitly as follows. Denote generators of Λ by ω_1 and ω_2, so

$$\Lambda = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\}.$$

Points of E are then congruence classes $z + \Lambda$, where $z \in \mathbb{C}$. Choose $\varepsilon > 0$ so small that all non-zero lattice points satisfy $|n\omega_1 + m\omega_2| > \varepsilon$. For each $e \in \mathbb{C}$, let

$$U_e = \{w + \Lambda : w \in \mathbb{C}, |w - e| < \varepsilon\} \subset E.$$

The coordinate map $h_e\colon U_e \to \mathbb{C}$ is defined by $h_e(w + \Lambda) = w'$, where $w' \in w + \Lambda$ is the unique element satisfying $|w' - e| < \varepsilon$. This maps U_e injectively onto the disc $\{z \in \mathbb{C} : |z - e| < \varepsilon\}$. Any change of coordinates is the identity.

Remark 4.3 *A Riemann surface is oriented. This comes from the fact that $\mathbb{C}$ has a natural orientation (multiplication by i) and holomorphic maps preserve this orientation. Hence orienting all coordinate patches gives a consistent orientation. Thus, a Riemann surface is a conformal structure on an oriented surface.*

It turns out that there is a unique conformal structure on the sphere, whilst on the torus there are infinitely many pairwise inequivalent conformal structures, and they can be parametrised (the parameter space is called a moduli space). The following result is stated without proof:

Theorem 4.4 *(Classification of orientable surfaces) Any orientable, compact surface is a sphere with g handles, where g is a non-negative integer. The number g is termed the genus of the surface. Any other orientable surface is obtained from a closed surface by deleting a finite number of points.*

The following examples provides a good starting point to understanding the construction of a holomorphic atlas.

Example 3: Polyhedral surfaces To show that there are in fact infinitely many compact, pairwise topologically distinct Riemann surfaces, we will associate to a polyhedral surface a Riemann surface S. A polyhedral surface P is a surface in $\mathbb{R}^3$ which is the union of finitely many faces. Each face is a closed subset of a plane in $\mathbb{R}^3$, with boundary finitely many line segments. Two line segments are either disjoint or meet in a vertex. Every line segment is contained in exactly two faces. After possibly subdividing, we may assume that each face is simply connected.

Assume that for every plane we have chosen a normal direction $\vec{N}$ so that the face is oriented using the right hand rule. To every line segment in a face, let $\vec{n}$ be the vector orthogonal to the line segment pointing into the face. Then orient the line segment by $\vec{n} \times \vec{N}$. Assume that for each line segment, the orientations coming from the two adjacent faces are opposite.

We now construct a holomorphic atlas. Identify $\mathbb{C}$ with the xy–plane with orientation induced from the normal vector $(0,0,1)$. For every face, take its interior and consider any orientation preserving isometry to the complex plane. To every interior point x on a line segment, choose two half circles of the same radius, centred at x and contained in the adjacent faces. Identify this with a circle of the same radius in the plane, again such that the orientation is preserved. Note that the transition functions are of the form $z \to az + b$, which are holomorphic.

Now choose r such that the length of each line segment in P is greater than $2r$. Given a vertex v, let α be the sum of the internal angles at v. Consider the set of all points at distance r from v. We may identify this set with a circle of radius r in the plane centred at the origin and the chart for v by the open disc bounded by this circle. Then any transition function is of the form $z \to z^{2\pi/\alpha}$, composed with a rotation and a translation. Since each vertex is contained in exactly one coordinate chart, we don't need to check anything near them. On the other hand, the map $z \to z^{2\pi/\alpha}$ is holomorphic away from zero. This shows that every orientable polyhedral surface has a holomorphic atlas (and hence can be given a conformal structure). Clearly, for each positive integer g, there is a polyhedral surfaces of genus g.

5 Maps between Riemann surfaces

Having defined Riemann surfaces using coordinate charts, it is natural to define *holomorphic* maps between Riemann surfaces:

Definition 5.1 *Let $f\colon S \to R$ be a continuous map of Riemann surfaces. We say that f is* holomorphic *if the induced maps on charts are holomorphic. That is, given* any *chart (U_α, h_α) of S containing s, and* any *chart (W_α, g_α) of R containing $f(s)$, we have that*

$$g_\alpha \circ f \circ h_\alpha^{-1}$$

is holomorphic in a small neighbourhood of $h_\alpha(s) \in \mathbb{C}$.

5.1 Conformal equivalence

Definition 5.2 *Riemann surfaces S and R are* conformally equivalent *if there is a holomorphic map $f\colon S \to R$ which is one-to-one and onto. In this case, the inverse is also holomorphic.*

Exercise 9 *Show that no two of the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$ and the open unit disc D^2 are conformally equivalent.*

Exercise 10 *Let $\omega \in \mathbb{C} \setminus \{0\}$, and let $\mathbb{Z}\omega$ be the set of all integral multiples of ω. Let S be the set of all congruence classes $z + \mathbb{Z}\omega$, $z \in \mathbb{C}$. Show that S is a Riemann surface which is conformally equivalent to the punctured plane $\mathbb{C} \setminus \{0\}$.*

Definition 5.3 *A Riemann surface is termed an* elliptic curve *if as a topological surface it is connected, compact and of genus one.*

Examples of elliptic curves have been given earlier using lattices in $\mathbb{C}$. It turns out that all elliptic curves arise in this way:

Theorem 5.4 *(Classification of elliptic curves) Let E be an elliptic curve. Then E is conformally equivalent to $\mathbb{C}/\Lambda$, where Λ is the lattice generated by 1 and a complex number ω whose imaginary part is positive. Moreover, two elliptic curves Y_i corresponding to lattices Λ_i generated by 1 and ω_i are conformally equivalent if and only if*

$$\omega_2 = \frac{a\omega_1 + b}{c\omega_1 + d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

i.e. $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

The above theorem implies that a conformal structure on the torus can be described by a single complex number with positive imaginary part, uniquely up to the action of the *modular group* $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. A fundamental domain for this action is given by:

$$\{z \in \mathbb{H} : |\Re(z)| \leq \frac{1}{2}, |z| \geq 1\}.$$

This is a hyperbolic triangle with one ideal vertex and two vertices with angle $\frac{\pi}{3}$. Since $T(z) = z+1$ and $S(z) = -\frac{1}{z}$ are in the modular group, we see that the set of conformal structures can be identified with a sphere with three distinguished points (corresponding to ∞, i and the third root of unity with positive imaginary part) at which the angles are 0, π, and $\frac{2\pi}{3}$) respectively. Note that this parameter space itself is a Riemann surface (and conformally equivalent to the Riemann sphere after adding the point ∞).

The conformal structures at the points corresponding to the parameters i (giving a square as a fundamental domain for the torus) and the third root of unity (giving the union of two regular triangles as a fundamental domain for the torus) are distinguished from all others due to their symmetries. As the point ∞ is approached, the conformal structures on the torus become longer and longer in one direction and the limiting surface is the interior of an annulus (equivalently a sphere with two points removed). So this point is distinguished since it corresponds to a change of the topological type (which is why it is not part of the parameter space but rather of its compactification).

5.2 Covering maps

Definition 5.5 *(Covering map) Let S and R be Riemann surfaces. A holomorphic map $p\colon S \to R$ is termed a* covering map *if each $p \in R$ belongs to an open subset $V \subset R$ whose inverse image $p^{-1}(V)$ is a union of pairwise disjoint open subsets of S, each of which is mapped injectively by p to V. One says that V is* evenly covered.

Note that if (U,h) is a coordinate patch on R, then each connected component of $p^{-1}(U)$ is a coordinate patch on S with coordinate map $h \circ p$.

Example: The Riemann surface of $\sqrt{z}$ comes with a natural two-to-one covering map onto $\mathbb{C} \setminus \{0\}$. An elliptic curve $E = \mathbb{C}/\Lambda$ has a natural covering $p\colon \mathbb{C} \to E$ defined by $p(z) = z + \Lambda$; this map is infinite-to-one.

Lemma 5.6 *Let $p\colon S \to R$ be a covering map. Then the cardinality of the fibres $p^{-1}(q)$ is the same for all fibres.*

Proof If $V \subset R$ is an evenly covered, connected open set, then for any $v_0, v_1 \in V$, there are bijective correspndences

$$p^{-1}(v_0) \leftrightarrow \{\text{components of } p^{-1}(V)\} \leftrightarrow p^{-1}(v_1).$$

This shows that the cardinality of fibres is constant on V. Since R is connected, given any two points $q_0, q_1 \in R$, there is a path joining them. For each point on the path, there is a neighbourhood which is evenly covered. These neighbourhoods restrict to an open cover of the (compact) path, and we may choose a finite subcover. It then follows that $p^{-1}(q_0)$ and $p^{-1}(q_1)$ have the same cardinality. ■

Definition 5.7 *(Covering transformation) Let $p\colon S \to R$ be a covering map. A* covering transformation *is an bijective, holomorphic map $\psi\colon S \to S$ such that $p(\psi(s)) = p(s)$ for all $s \in S$.*

The set of all covering transformations forms a group since the composition of two covering transformation is a covering transformation, and the inverse of a covering transformation is a covering transformation. A covering transformation acts as a permutation on $p^{-1}(q)$ for each $q \in R$. For a Riemann surface S, we call the set of all bijective, holomorphic maps $S \to S$ the *automorphism group of* S, $Aut(S)$. Given any covering map $S \to R$, the group of covering transformations is a subgroup of $Aut(S)$.

For some Riemann surfaces, there is a complete classification of the automorphism group in each case:

(1) $Aut(\hat{\mathbb{C}}) = PGL(2)$, the group of Möbius transformations

$$z \to \frac{az+b}{cz+d},$$

where $ad - bc \neq 0,\ a,b,c,d \in \mathbb{C}$.

(2) $Aut(\mathbb{C})$ is the subgroup of the Möbius transformations which fix infinity:

$$z \to az + b,$$

where $a \neq 0$. These are also called *affine transformations*.

(3) $Aut(D^2)$ is the following subgroup of the Möbius transformations. Every automorphism has the form

$$z \to e^{i\vartheta}\frac{z-a}{1-\bar{z}a},$$

where $a \in D^2$ and $\vartheta \in [0, 2\pi)$.

Sketch of proof The group $Aut(D^2)$ has already been described in the first part of the course. Let $f \in Aut(\mathbb{C})$. Then f has a simple zero at some $w \in \mathbb{C}$ and a simple pole at ∞. It follows that $f(z) = az + b$ for some $a, b \in \mathbb{C}$ with $a \neq 0$. Similarly, every $f \in Aut(\hat{\mathbb{C}})$ has a simple zero and a simple pole, and it follows that it must be a Möbius transformation. ■

Let $p\colon S \to R$ be a covering map. Let $r \in R$ and V be an evenly covered neighbourhood of r. Denote the elements of $p^{-1}(r)$ by s_i, and $p^{-1}(V)$ accordingly by U_i. Let $\psi\colon S \to S$ be a covering transformation. If $\psi(s_i) = s_j$, then the restriction $\psi|_{U_i}\colon U_i \to U_j$ is an injective, holomorphic map. Moreover, denoting the restriction $p|_{U_i}\colon U_i \to V$ by p_i, it follows that p_i is bijective. We then have $\psi|_{U_i} = p_j^{-1} \circ p_i$. In particular, if $i = j$, then $\psi|_{U_i}$ is the identity map.

Exercise 11 *Let S be a Riemann surface and $f, g\colon S \to S$ be two holomorphic functions such that $f(s) = g(s)$ for all $s \in U$, where U is a non-empty, open subset of S. Show that $f(s) = g(s)$ for all $s \in S$.*

The above discussion together with the exercise imply the:

Proposition 5.8 *Let $p\colon S \to R$ be a covering map and $\psi\colon S \to S$ be a covering transformation. Then either ψ is the identity transformation or it has no fixed points.*

It follows that not every automorphism can be a covering transformation.

Example: The proposition allows us to determine all covering transformations of the cover $p\colon \mathbb{C} \to E$, where E is an elliptic curve. We may assume that E is defined by a lattice generated by ω_1 and ω_2. Then any translation $T_{m,n}\colon \mathbb{C} \to \mathbb{C}$ of the form

$$T_{m,n}(z) = z + m\omega_1 + n\omega_2,$$

where n and m are fixed integers, is a covering transformation. Now let $\psi\colon \mathbb{C} \to \mathbb{C}$ be an arbitrary covering translation. Since $\psi(0) \in p^{-1}(0 + \Lambda) = \Lambda$, there are integers m and n such that $\psi(0) = m\omega_1 + n\omega_2$. Then the covering transformation $T_{-m,-n} \circ \psi$ fixes 0 which implies that it is the identity. This shows that all covering transformations are of the form $T_{m,n}$. Note that this discussion did not use the fact that we know $Aut(\mathbb{C})$.

Example: The map $f\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ defined by $f(z) = e^z$ is a covering map (in fact, the *universal covering* introduced in the next section). Let us determine the covering transformations using the description of $Aut(\mathbb{C})$. Any automorphism of $\mathbb{C}$ without fixed points is of the form $\psi(z) = z + b$. We then consider $e^z = e^{\psi(z)} = e^{z+b}$. This forces $b = 2k\pi i$ for $k \in \mathbb{Z}$. Hence any covering transformation is of the form $z \to z + 2k\pi i$ for some $k \in \mathbb{Z}$. So the group G of covering transformations is generated by the translation $z \to z + 2\pi i$. Note that $\mathbb{C}/G$ is an infinite cylinder, and this is in fact conformally equivalent to $\mathbb{C} \setminus \{0\}$ via the function $f(z) = e^z$.

5.3 Uniformisation of Riemann surfaces

Recall that a topological space is simply connected if each closed path in X is homotopic to a constant path.

Theorem 5.9 (Universal covering exists) *For each Riemann surface R there is a simply connected Riemann surface S and a covering map $p\colon S \to R$. In this case, $p\colon S \to R$ is called a* universal covering *of R. Moreover, given $q \in R$ and $s_0, s_1 \in p^{-1}(q)$, there is a unique covering transformation ψ of S such that $\psi(s_0) = s_1$.*

Sketch of proof (Gamelin, Exercise XVI.7.10) Fix a point $x_0 \in R$. Let S be the set of pairs $(x, [\gamma])$, where $x \in R$, γ is a path from x_0 to x, and $[\gamma]$ denotes its homotopy class. We need to define a suitable topology on S as well as the charts which turn it into a Riemann surface. This can be done at once:

Let $x \in R$ and γ be a path from x_0 to x. There is a coordinate chart U_α containing x. Then define $V_{\alpha,[\gamma]}$ to be the set of pairs $(y, [\delta]) \in S$, where $y \in U_\alpha$ and δ is the path obtained from concatenating γ with any path in U_α from x to y. Then define $p\colon S \to R$ to be the map $p(x, [\delta]) = x$.

It needs to be checked that (1) S can be turned into a Riemann surface such that each $V_{\alpha,[\gamma]}$ is a coordinate chart and p is holomorphic, (2) $p\colon S \to R$ is a covering map, (3) S is simply connected. ■

Theorem 5.10 (Universal covering is unique) *Any two universal coverings of a Riemann surface are conformally equivalent. Moreover, if $f\colon S \to R$ is the universal cover of a Riemann surface S, and G is the associated group of covering transformations for f, then R is conformally equivalent to S/G.*

The proof makes use of *path lifting properties* which we don't have time to discuss.

Theorem 5.11 (Uniformization theorem for Riemann surfaces) *A simply connected Riemann surface is conformally equivalent to one of*

(1) *the Riemann sphere $\hat{\mathbb{C}}$,*

(2) *the complex plane $\mathbb{C}$,*

(3) *the open unit disc D^2.*

A proof can be given which is analogous to the proof of the Riemann mapping theorem. No two of these three Riemann surfaces are isomorphic, and we have described the automorphism groups earlier.

Corollary 5.12 *(Geometrisation of Riemann surfaces)*

(1) *The only Riemann surface having the Riemann sphere $\hat{\mathbb{C}}$ as a universal covering surface is the sphere itself. This has a spherical metric (i.e. a metric of constant curvature $+1$).*

(2) *The only Riemann surfaces having the complex plane $\mathbb{C}$ as universal covering surface are the complex plane, the punctured complex plane $\mathbb{C}\setminus\{0\}$, and tori. All of these surfaces admit a complete Euclidean metric (i.e. a metric of constant curvature 0).*

(3) *All other Riemann surfaces have the open unit disc as universal covering, and admit a complete hyperbolic metric (i.e. a metric of constant curvature -1).*

Proof Let $\psi\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a covering transformation for some covering. Then ψ is a fractional linear transformation and has a fixed point. Hence it must be the identity. The second part follows from the fact that the only covering transformations acting on the plane must be of the form $z \to z+b$, since they have no fixed points. They therefore are translations and hence isometries of the standard Euclidean metric on the plane. The third part follows from the fact that every element of $Aut(D^2)$ is an isometry with respect to the hyperbolic metric on D^2. Recall that the element of hyperbolic arc-length is defined by $\frac{2|dz|}{1-|z|^2}$, and that this fact follows from Pick's lemma. ■

5.4 Ramified covering maps

We now turn to the general study of holomorphic maps between Riemann surfaces. Covering maps are especially nice maps; the next best thing is the following:

Definition 5.13 *(Ramified covering maps) Let $f\colon S \to R$ be a holomorphic map of Riemann surfaces. We say that f is a* ramified covering map *if there is a (possibly empty) discrete set of points on S, whose images in R also form a discrete set, such that removing these points and their images makes f a covering map. The set of points on S where f is not locally a covering map is the set of* ramification points *and its image in R is the set of* branch points.

Example: Let $S = \mathbb{C} = R$, and $f\colon S \to R$ be the map $f(z) = z^2$. Then f is a ramified covering map. The set of branch points is $\{0\} \subset R$ and the set of ramification points is $\{0\} \subset S$. The same holds for any of the maps $z \to z^n$, where $n \geq 2$.

Every covering map is a ramified covering map; for emphasis, a covering map is also called an *unramified covering map*.

Proposition 5.14 *Let $f\colon S \to R$ be a ramified covering map between Riemann surfaces. Then locally about a ramification point $s \in S$, the map is equivalent to the map $D^2 \to D^2$ given by $z \to z^n$ for a unique positive integer n. The number n is termed the* ramification index *of f at s.*

Sketch of proof Locally near a ramification point one has a topological map between unit discs which is a covering between the punctured discs. The set of such maps is classified by winding numbers, and (up to equivalence) the map $z \to z^n$ pops out uniquely. ■

Proposition 5.15 *Let $f\colon S \to R$ be a ramified covering map map between compact Riemann surfaces. Then there is a positive integer d such that for every $x \in R$ which is not a branch point, the set $f^{-1}(x)$ contains d elements. The number d is called the* degree *of f.*

Moreover, if $x \in R$ is a branch point, and $s_1, \ldots s_k$ are the ramification points lying over x with ramification indices $n_1, \ldots, n_k$, then we have

$$d = \sum_{i=1}^{k} n_i.$$

Proof Since the restriction of f to the complement of the set of ramification points is a covering map onto its image, the cardinality of $f^{-1}(q)$ for every $q \in R$ which is not a branch point is independent of q. It is finite since S is compact; denote it by d. The second statement follows from a counting argument taking into account the local model $z \to z^{n_i}$ of f near s_i. ■

Let g be the genus of a compact surface S. The quantity $2-2g$ is called the *Euler characteristic* of S and denoted by $\chi(S)$. If S is given a triangulation with v vertices, e edges and t triangles, then

$$2-2g = \chi(S) = v - e + t.$$

Theorem 5.16 *(Riemann–Hurwitz) Let $f\colon S \to R$ be a ramified covering map between compact Riemann surfaces, and let g and h be the genera of S and R respectively. Then*

$$\chi(S) = d\chi(R) - \sum_{s \in S}(n_s - 1), \tag{4}$$

where d is the degree of f and n_s is the ramification index of f at s. (If s is not a ramification point, then define $n_s = 1$.)

Proof We may triangulate R and, after possibly subdividing the triangulation, we may assume that every branch point of f is a vertex of the triangulation. Denote the number of vertices, edges and triangles by v, e and t respectively. Lift the triangulation of R to a triangulation of S. Since every point other than a branch point has d pre-images, there are de edges and dt triangles in this triangulation of S.

However, there may be fewer than dv vertices, since each branch point has fewer than d pre-images. We can count exactly how many pre-images there are. If $s \in S$ is a ramification point with ramification index n_s, then there is one vertex at s and not n_s many vertices. Hence there are

$$dv - \sum_{s \in S} (n_s - 1),$$

vertices in the triangulation of S. It follows that

$$\begin{aligned} \chi(S) &= d(v - e + t) - \sum_{s \in S} (n_s - 1) \\ &= d\chi(R) - \sum_{s \in S} (n_s - 1). \end{aligned}$$

This gives the desired equation. ■

5.5 Holomorphic maps between Riemann surfaces

We now turn to the study of holomorphic maps between Riemann surfaces in general. Let $f \colon S \to R$ be a holomorphic map between Riemann surfaces. Is there a well-defined derivative of f? Recall the definition of being holomorphic at $s \in S$:

Given *any* chart (U_α, h_α) in S containing s, and *any* chart (W_α, g_α) containing $f(s)$, we have that

$$g_\alpha \circ f \circ h_\alpha^{-1}$$

is holomorphic in a small neighbourhood of $h_\alpha(s) \in \mathbb{C}$.

If we choose different charts containing s and $f(s)$ respectively, say (U_β, h_β) and (W_β, g_β), then

$$g_\beta \circ f \circ h_\beta^{-1}$$

is holomorphic in a small neighbourhood of $h_\beta(s) \in \mathbb{C}$. These maps are related by the holomorphic changes of coordinates $h_\beta \circ h_\alpha^{-1}$ and $g_\alpha \circ g_\beta^{-1}$:

$$g_\alpha \circ f \circ h_\alpha^{-1} = (g_\alpha \circ g_\beta^{-1}) \circ (g_\beta \circ f \circ h_\beta^{-1}) \circ (h_\beta \circ h_\alpha^{-1}).$$

Since a change of coordinates is biholomorphic, it follows that even though the *value* $f'(s)$ is not well-defined in general, it does make sense to talk about it *being zero or not.*

Moreover, if $f'(s) = 0$ for all s in an open set $U \subset S$, then f is *locally constant* (and hence constant by Exercise 11), and if $f'(s) \neq 0$, then f is a *local homeomorphism* at s due to the open mapping theorem.

Remark 5.17 *In some texts, a Riemann surface, S, is not assumed to be connected. In this case* locally constant *implies constant on each connected component of S.*

Definition 5.18 *Let $f\colon S \to R$ be a holomorphic map between Riemann surfaces. If the image of every closed subset of S is a closed (respectively open) subset of R, then f is termed a* closed (respectively open) map.

Proposition 5.19 *Let $f\colon S \to R$ be a non-constant, holomorphic map between Riemann surfaces which is closed. Then f is a ramified covering map.*

Sketch of proof First note that the set of points where the derivative of f is zero is discrete. Indeed, this follows from the fact that this is true for any non-zero holomorphic map from $\mathbb{C}$ to $\mathbb{C}$. Since f is closed, the image of a discrete set is a discrete set. Hence delete this set of points and the set of its images, and denote the resulting map by $f^*\colon S^* \to R^*$. We need to show that this is a covering map.

Let $q \in R$ and choose a simply connected neighbourhood V of q. Let U be a connected component of $f^{-1}(V)$. Then $f|_U\colon U \to V$ is a local homeomorphism at each point $s \in U$ since $f'(s) \neq 0$. The proposition follows if $f|_U$ is in fact a homeomorphism onto its image. This can be shown using the so-called *homotopy lifting lemma*. The basic idea is as follows. If $f(s) = q$ and $f(s_0) = f(s_1) = y \in V$, pick paths in U connecting s_i to s. Map them down to V. Then in V we have two paths with identical end-points; they hence form a closed path. Since V is simply connected, there is a homotopy in V which takes one path to the other and fixes y. This homotopy can be lifted to a homotopy between the paths in U, which implies $s_0 = s_1$. Thus, $f|_U$ is injective (and hence bijective). ■

The following is a situation where a holomorphic map is automatically closed:

Proposition 5.20 *Let $f\colon S \to R$ be a non-constant, holomorphic map between Riemann surfaces, and assume that S is compact. Then f is a ramified covering map.*

Proof Since S is compact, a subset of S is closed if and only if it is compact, and the image of a compact set in S is a compact set in R, and hence closed. Thus, the previous proposition applies. ■

Corollary 5.21 *Let $f\colon S \to R$ be a holomorphic map between compact Riemann surfaces which is not constant, and let g and h be the genera of S and R respectively.*

If $g = 0$ then $h = 0$ and if $g = 1$ then $h \leq 1$.

Proof We know that f is a ramified covering map. The result now follows directly from the Riemann–Hurwitz Theorem since the sum on the right hand side of equation (4) is non-negative. ∎

6 The classification of elliptic curves

It is now interesting to classify all Riemann surfaces of a given topological type up to conformal equivalence. We have seen such a result earlier, Theorem 5.4, and are now in a position to outline its proof using the uniformisation theorem for Riemann surfaces. Details are omitted at this stage.

Lemma 6.1 *Let $G \subset \mathbb{C}$ be a discrete group. Then G is one of*

(1) $\{0\}$

(2) $\mathbb{Z} \cong \langle \omega \rangle$

(3) $\mathbb{Z}^2 \cong \langle \omega_1, \omega_2 \rangle$, *where $\frac{\omega_1}{\omega_2}$ is not real.*

Lemma 6.2 *Let φ be an automorphism of the upper half plane $\mathbb{H}$. Then*

$$\varphi(z) = \frac{az+b}{cz+d}, \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R}).$$

Moreover, up to conjugacy, $\varphi(z) = z$ or $\varphi(z) = 2z$ or $\varphi(z) = z+1$ or $\varphi(z) = -\frac{1}{z+1}$.

Lemma 6.3 *Let $G \subset Aut(D^2)$ be a subgroup. Suppose that*

(1) *G is free abelian, and*

(2) *G acts freely, and*

(3) *G acts properly discontinuously.*

Then the rank of G is at most one.

Lemma 6.4 *Every elliptic curve is isomorphic to $\mathbb{C}/\Lambda$, where Λ is a lattice generated by 1 and a complex number $\omega \in \mathbb{H}$.*

Lemma 6.5 *Let Λ_i be lattices in $\mathbb{C}$ with bases $\{v_i, w_i\}$, $i \in \{1,2\}$. Then $\Lambda_1 = \Lambda_2$ if and only if there is an element*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

such that

$$\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}.$$

Definition 6.6 *A map φ of the form $z \to az$, where $a \in \mathbb{C} \setminus \{0\}$, is called a* homothety. *Two lattices Λ_1 are termed* homothetic *if $\Lambda_2 = \varphi(\Lambda_1)$.*

Lemma 6.7 *Two lattices Λ_i generated by $\{1, \omega_i\}$ are homothetic if and only if*

$$\omega_2 = \frac{a\omega_1 + b}{c\omega_1 + d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Lemma 6.8 *Two elliptic curves are conformally equivalent if and only if they have homothetic lattices.*

The last lemma concludes the proof of Theorem 5.4.

References

[1] Lars Ahlfors: *Complex analysis*, McGraw-Hill, New York-Toronto-London, 1953.

[2] Theodore Gamelin: *Complex analysis*, Springer-Verlag, New York, 2001.

[3] Carlos Furuti, http://www.progonos.com/furuti/MapProj/Normal/ProjConf/projConf.html

Department of Mathematics and Statistics
The University of Melbourne
VIC 3010 Australia

Email: `tillmann@ms.unimelb.edu.au`