DOMINATION BY GEOMETRIC 4-MANIFOLDS

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Abstract. We consider aspects of the question "when is a closed orientable 4-manifold Y dominated by another such manifold X?", focusing on the cases when X is geometric or fibres non-trivially over a closed orientable surface.

S.-C. Wang has considered in detail properties of maps of non-zero degree between 3-manifolds. In particular, he grouped aspherical 3-manifolds into 8 families, according to the nature of their JSJ decompositions, and determined which pairs allowed maps of non-zero degree between representative 3-manifolds [13]. Purely algebraic arguments for PD_n-groups with JSJ decompositions and all n were given in [6]. Sharper results for maps between aspherical geometric 4-manifolds were given in [10]. We shall complement this work in dimension 4 by considering cases where the domain is geometric but not aspherical, or fibres non-trivially over a surface. In the latter case the strongest results are when the range is aspherical.

We begin by reviewing the results of [6] and [10] for aspherical geometric 4-manifolds. (Some of these earlier results are recovered below in passing.) In §2 we make some basic observations and give four simple lemmas. The remaining sections are organized in terms of the dominating space X. The next two sections consider the five geometries $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^3 \times \mathbb{E}^1$ and $\mathbb{S}^2 \times \mathbb{E}^2$, in §3, and the six geometries of solvable Lie type, in §4. All total spaces of bundles with base and fibre S^2 or the torus T have such geometries, excepting only $S^2 \tilde{\times} S^2$. In §5 we consider domination by total spaces of F-bundles with base a hyperbolic surface, where $F = S^2$ or T. Among these are manifolds with geometry $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ or $\mathbb{S}^2 \times \mathbb{H}^2$, but there are also non-geometric T-bundle spaces. The next section, on S^1-bundle spaces (including $\mathbb{H}^3 \times \mathbb{E}^1$) is very short. In §8 we consider bundles with fibre a hyperbolic closed surface, and we show that if a bundle space Y is dominated by a product $B \times F$ then it is also a product. Most such bundle spaces are not geometric. In the final section we consider

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dominations of aspherical geometric 4-manifolds by bundles with hyperbolic fibre. These include reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds, which are finitely covered by products, and $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds. No such bundle space is an irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold or a $\mathbb{H}^2(\mathbb{C})$-manifold, and it seems unlikely that any are $\mathbb{H}^4$-manifolds.

This note was prompted by a query from R. İ. Baykur, arising from [1]. In that paper the authors consider the more specific question of which closed 4-manifolds have branched coverings by the total spaces of surface bundles. Their main results are that every 1-connected closed 4-manifold has a branched covering of degree $\leqslant 16$ by a product $B \times T$, with B a closed surface and T the torus, and every product $B \times S^1$ with B a closed orientable 3-manifold has a 2-fold branched cover by a symplectic 4-manifold which fibres over T. The maps considered below often have degree 1, and then are either homotopy equivalences or not homotopic to branched covers.

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1. DOMINATIONS BETWEEN ASPHERICAL GEOMETRIC 4-MANIFOLDS

In the aspherical case the underlying question is essentially one of group theory. In [6] it is shown that PD_n-groups with *max-c* may be partitioned into families, analogous to those of Wang, and that the pattern of possible maps of non-zero degree is very similar. A group has *max-c* if all chains of centralizers in the group are finite. A PD_n-pair of groups $(G, \partial G)$ is *atoroidal* if every polycyclic subgroup of Hirsch length $n-1$ is conjugate into a boundary component, and is of *Seifert type* if it has a normal polycyclic subgroup of Hirsch length $n-2$. Kropholler showed that all PD_n-groups with *max-c* have JSJ decompositions along virtually polycyclic subgroups of Hirsch length $n-1$ into pieces which are either atoroidal or of Seifert type [9].

When $n = 4$ the qualification "of Seifert type" reduces to "having a normal $\mathbb{Z}^2$ subgroup", and these families of groups are either

(1) atoroidal;
(2) have a non-trivial JSJ decomposition with at least one atoroidal piece;
(3) have a non-trivial JSJ decomposition with all pieces of Seifert type;
(4) (a) virtually polycyclic, but not virtually of Seifert type; or (b) virtually polycyclic and of Seifert type but not virtually nilpotent:

(5) virtually a product $G \times \mathbb{Z}^2$ with G a PD_2-group and $\chi(G) < 0$;
(6) Seifert type but not virtually a product nor virtually polycyclic;
(7) virtually nilpotent but not virtually abelian; or
(8) virtually abelian.

We have preserved Wang's enumeration, but in higher dimensions it is useful to subdivide the analogue of the class of $\mathbb{S}ol^3$-manifolds.

The fundamental groups of aspherical n-manifolds are PD_n-groups, and the groups of geometric 4-manifolds satisfy *max-c*. Not all aspherical 4-manifolds with groups of types (1) or (6) are geometric, and there are no geometric 4-manifolds with groups of type (2) or (3). The correspondence with geometries is (1) $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$, and $\mathbb{H}^2 \times \mathbb{H}^2$; (4.a) $\mathbb{S}ol^4_{m,n}$ (with $m \neq n$) and $\mathbb{S}ol^4_1$; (4.b) $\mathbb{S}ol^3 \times E^1$; (5) $\mathbb{H}^2 \times \mathbb{E}^2$; (6) $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$; (7) $\mathbb{N}il^3 \times \mathbb{E}^1$ and $\mathbb{N}il^4$; and (8) $\mathbb{E}^4$.

For geometric 4-manifolds in the families (4–8) the conclusion of [6] is that all maps between groups of different types have degree 0, except for maps from groups of type (5) to groups of type (8) and from (6) to $\mathbb{N}il^3 \times \mathbb{E}^1$-groups in type (7). (The assertion there that there are maps of nonzero degree from groups of type (5) to groups of type (4c) is wrong.) On the other hand every $\mathbb{N}il^3 \times \mathbb{E}^1$-group and every $\mathbb{E}^4$-group is so dominated. Theorem 1.1 of [10] is slightly sharper, in that it shows that there are no such maps between groups of distinct geometries within types (4.a) and in (7).

2. SOME GENERAL OBSERVATIONS

If X and Y are closed orientable n-manifolds then X *d-dominates* Y if there is a map $f : X \to Y$ with nonzero degree d (for some choice of orientations). If so, then

(1) d-fold (branched) finite covers have degree d;
(2) the image of $\pi_1(X)$ in Y has finite index, and so f factors through a map $\widehat{f} : X \to \widehat{Y}$, where $\widehat{Y}$ covers Y;
(3) if R is a ring in which $d = deg(f)$ is invertible then $H^*(Y; R)$ is a subring of $H^*(X; R)$, and is a direct summand as an R-module:
(4) hence if $n = 4$ then $\chi(Y) \leqslant 2 + \beta_2(X; \mathbb{Q})$.

If X *1-dominates* Y then $\pi_1(f)$ is an epimorphism, and $H^*(Y; \mathbb{Z})$ is a direct summand of $H^*(X; \mathbb{Z})$. We say that X *essentially dominates* Y if f has non-zero degree and $\pi_1(f)$ is an epimorphism. (Such maps need not have degree 1, as is already clear when $X = S^2$. Self maps of S^2 of degree > 1 induce isomorphisms on π_1, but do not induce splittings of $H^2(S^2; \mathbb{Z})$.) Clearly X dominates Y if and only if X essentially dominates some finite cover of Y.

Our main interest is in the existence of maps of non-zero degree. For this purpose we may replace X and Y by more convenient covering spaces, with X essentially dominating Y, where appropriate. However in several places we ask whether a specific 4-manifold Y is essentially dominated by one of the representative geometric 4-manifolds under consideration.

Lemma 1. *Let $f : X \to Y$ be a map of non-zero degree between orientable closed 4-manifolds X and Y. Suppose that there is an integer $D \geqslant 0$ such that $\beta_2(\hat{X}) \leqslant D$, for all finite covering spaces $\hat{X}$ of X. If $\pi_1(Y)$ is infinite and has subgroups of arbitrarily large finite index then $\chi(Y) \leqslant 0$. If $\pi_1(Y)$ is finite then it has order at most $\frac{1}{2}(D+2)$.*

Proof. If : $\hat{Y} \to Y$ is a finite covering and : $\hat{X} \to X$ is the induced covering then f lifts to a dominating map $\hat{f} : \hat{X} \to \hat{Y}$. On the one hand $\chi(\hat{Y}) = [\pi_1(Y) : \pi_1(\hat{Y})]\chi(Y)$; on the other, $\chi(\hat{Y}) \leqslant D + 2$. The first assertion follows easily.

The second assertion has a similar proof. □

In conjunction with this lemma, note that if $\beta_i^{(2)}(Y) = 0$ for $i \leqslant 1$ (for instance, if $\pi_1(Y)$ is infinite and amenable, or has a finitely generated infinite normal subgroup of infinite index) then $\chi(Y) = \beta_2^{(2)}(Y) \geqslant 0$, by the L^2-Euler characteristic formula.

Lemma 2. *Let $f : X \to Y$ be a map of odd degree between closed orientable 4-manifolds X and Y. If $w_2(X) = 0$ then $w_2(Y) = 0$. If, moreover, Y is 1-connected then $\chi(Y)$ is even.*

Proof. The map f induces a monomorphism $H^*(f)$ from $H^*(Y;\mathbb{Z}/2\mathbb{Z})$ to $H^*(X;\mathbb{Z}/2\mathbb{Z})$. Since $\xi^2 = 0$ for all $\xi \in H^*(X;\mathbb{Z}/2\mathbb{Z})$ the same is true for $H^*(Y;\mathbb{Z}/2\mathbb{Z})$, and so $w_2(Y) = 0$.

If M is an orientable 4-manifold then $w_2(M)^2 = w_4(M)$, by the Wu formulae, and so $[M] \cap w_2(M)^2 \equiv \chi(M)$ *mod* (2). Hence if Y is 1-connected then $\chi(Y)$ is even. □

On the other hand, there is a degree-1 map from CP^2 to S^4, and so $w_2(Y) = 0$ does not imply that $w_2(X) = 0$.

If X is a cell complex of dimension $\leqslant 4$ then $[X, CP^2] = [X, K(\mathbb{Z}, 2)]$, by general position, since we may construct $K(\mathbb{Z}, 2) \simeq CP^\infty$ by adding cells of dimension $\geqslant 6$ to CP^2. Hence if u is a generator of $H^2(CP^2;\mathbb{Z})$ then $f \mapsto f^*u$ defines a bijection $[X, CP^2] \to H^2(X;\mathbb{Z})$. If X is a closed orientable 4-manifold the degree of f is given by $d = [X] \cap (f^*u)^2$.

An element $\xi \in H^2(X;\mathbb{Z})$ is in the image of $[X, S^2] = [X, CP^1]$ if and only if $\xi^2 = 0$ [12, Theorem 8.11].

There is a similarly defined surjection from $[X, S^3]$ to $H^3(X;\mathbb{Z})$.

Lemma 3. *Let M be the mapping torus of a self-homeomorphism φ of an n-manifold N. Then M 1-dominates $S^n \times S^1$.*

Proof. We may assume that φ fixes a disc $D^n \subset N$. Collapsing the image of $\overline{N \setminus D^n}$ to a point in each fibre induces a map from M to $S^n \times S^1$ which clearly has degree 1. □

The following simple lemma is based on [4, Lemma 2.3].

Lemma 4. *Let L be a lens space with $\pi_1(L) \cong \mathbb{Z}/n\mathbb{Z}$. If M is an orientable 3-manifold such that $H^1(M;\mathbb{Z}) \neq 0$ then there is a map $f : M \to L$ of degree n and such that $\pi_1(f)$ is an epimorphism.*

Proof. Since $H^1(M;\mathbb{Z}) \neq 0$ there is a map $g : M \to S^1 \to L$ which induces an epimorphism on fundamental groups. Let $p : M \to M \vee S^3$ be the pinch map, and let $c : S^3 \to L$ be the universal covering projection. Then $f = (g \vee c) \circ p$ has the desired properties. □

Lemma 5. *Let $f : X \to Y$ be a map between closed 4-manifolds. If Y is aspherical and $\pi_1(f)$ factors through a group G such that $H_4(G;\mathbb{Q}) = 0$ then f has degree 0.*

Proof. If Y is aspherical then f is determined by $\pi_1(f)$, and so factors through $K(G,1)$. Since $H_4(G;\mathbb{Q}) = 0$ the lemma follows. □

We shall henceforth assume that X and Y are closed orientable 4-manifolds and that $f : X \to Y$ has degree $d \neq 0$. All homology and cohomology groups have coefficients $\mathbb{Q}$, unless otherwise specified.

3. Domain compact geometric or mixed compact-solvable

Suppose that X has one of the compact or mixed compact-solvable geometries $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^3 \times \mathbb{E}^1$ or $\mathbb{S}^2 \times \mathbb{E}^2$. Then X is finitely covered by one of S^4, CP^2, $S^2 \times S^2$, $S^3 \times S^1$ or $S^2 \times T$, respectively. (See [5, Chapters 10–12].)

$\mathbb{S}^4$. We may assume that $X = S^4$. Then $\pi_1(Y) = 1$ and $\beta_2(Y) = 0$, and so $Y \simeq S^4$. Hence f is homotopic to a homeomorphism.

$\mathbb{CP}^2$. We may assume that $X = CP^2$. Then $\pi_1(Y) = 1$ and $\beta_2(Y) = 1$ or 0. Hence Y is homeomorphic to one of CP^2, Ch or S^4.

$\mathbb{S}^2 \times \mathbb{S}^2$. We may assume that $X = S^2 \times S^2$. Then $\pi_1(Y) = 1$ and $\beta_2(Y) = 2, 1$ or 0. If $\beta_2(Y) = 2$ then $Y \simeq X$. If also $d = 1$ then f is homotopic to a homeomorphism.

If $\beta_2(Y) = 1$ then $Y \simeq CP^2$. There are maps $f : S^2 \times S^2 \to CP^2$ of every even degree, but none of degree 1.

If $\beta_2(Y) = 0$ then Y is homeomorphic to S^4.

$\mathbb{S}^3 \times \mathbb{E}^1$. We may assume that $X = S^3 \times S^1$. Then $\pi_1(Y)$ is cyclic, and $\beta_2(Y) = 0$.

If $\pi_1(Y) \cong \mathbb{Z}$ then $Y \simeq S^3 \times S^1$ [5, Theorem 11.1]. If also $d = 1$ then f is homotopic to a homeomorphism.

If $\pi_1(Y) \cong \mathbb{Z}/n\mathbb{Z}$ then $n = 1$, by Lemma 1. Hence Y is homeomorphic to S^4.

$\mathbb{S}^2 \times \mathbb{E}^2$. We may assume that $X = S^2 \times T$. If $\beta_1(Y) = 2$ then $\pi_1(Y) \cong \mathbb{Z}^2$, and the classifying map $c_Y : Y \to T$ induces an embedding of rings $H^*(T) \to H^*(Y)$. Hence $\beta_2(Y)$ must be 2, by the non-singularity of Poincaré duality. If also $d = 1$ then f is homotopic to a homeomorphism.

If $\beta_1(Y) = 1$ then $\chi(Y) = 0$, by Lemma 1. If $\pi_1(Y) \cong \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ then $Y \simeq L \times S^1$, where L is a lens space [5, Theorem 11.1]. Since $S^2 \times S^1$ essentially dominates L, by Lemma 4, $S^2 \times T$ essentially dominates Y.

If $\beta_1(Y) = 0$ then $\pi_1(Y) = 1$, by Lemma 1. There are obvious degree-1 maps to $S^2 \times S^2$ (since T 1-dominates S^2) and S^4. There are maps $f : S^2 \times T \to CP^2$ of every even degree, but none of degree 1.

4. Domain a solvmanifold

There are six solvable Lie geometries. One is an infinite family $\mathbb{S}ol^4_{m,n}$ of closely related geometries, which includes the product geometry $\mathbb{S}ol^3 \times \mathbb{E}^1$ as the equal parameter case $m = n$. This product geometry needs separate consideration here. (See [5, Chapters 7–8] for details of these geometries and the associated fundamental groups. Note that it appears to be unknown in general when different pairs (m, n) and (m', n') with $m \neq n$ and $m' \neq n'$ determine the same geometry $\mathbb{S}ol^4_{m,n}$. See [5, page 137].)

Suppose that X has a solvable Lie geometry. Then $\pi_X = \pi_1(X)$ is polycyclic of Hirsch length 4 and $\beta_2(X) \leqslant 6$. Hence $\pi_Y = \pi_1(Y)$ is polycyclic of Hirsch length $h \leqslant 4$. If $h > 0$ then π_Y is infinite and $\chi(Y) = 0$, by Lemma 1, and the remark immediately following this lemma, while if $h = 0$ then π_Y is finite and $\chi(Y) \geqslant 2$. After passing to a covering space, if necessary, we may assume that either $X = T^4$ or that π_X is not virtually abelian. In the latter case X is a mapping torus $N \rtimes S^1$, where $N = T^3$ (and the monodromy has infinite order) or is a quotient of Nil^3.

If $h > 2$ then $\chi(Y) = 0$ and $H^i(\pi_Y; \mathbb{Z}[\pi_Y]) = 0$ for $i \leqslant 2$, so Y is aspherical [5, Corollary 3.5.2]. In this case $\pi_1(f)$ is an isomorphism, for otherwise π_Y would have Hirsch length $\leqslant 3$, so $H_4(Y) = 0$ and f would have degree 0. The map f is homotopic to a homeomorphism

[5, Theorem 8.1]. In particular, there are no maps of non-zero degree between manifolds having distinct solvable Lie geometries.

If $h = 2$ then $\pi_Y \cong \mathbb{Z}^2$ or $\pi_1(Kb)$, and Y has a covering of degree dividing 4 by $S^2 \times T$ [5, Chapter 10.§5]. In fact $\pi_1(Kb)$ is not a quotient of π_X, with our assumptions on X above, and so $\pi_Y \cong \mathbb{Z}^2$. Hence either $Y \cong S^2 \times T$ or Y is the total space of the orientable S^2-bundle over T with $w_2(Y) \neq 0$ [5, Chapter 10].

If $h = 1$ then Y is finitely covered by $S^3 \times S^1$, and the maximal finite normal subgroup of π_Y has cohomology of period 4 [5, Theorem 11.1].

We shall consider the possibilities for X in decreasing order of $\beta_1(X)$.

$\mathbb{E}^4$. We may assume that $X = T^4$. Then π_Y is abelian, with at most 4 generators. Hence $\beta = \beta_1(Y) \leqslant 4$. Since π_Y is abelian $\beta_2(Y) \geqslant \binom{\beta}{2}$. On the other hand, $\beta_2(Y) \leqslant 6$.

There are obvious degree-1 maps to T^4, $S^2 \times T$, $S^3 \times S^1$, $S^2 \times S^2$ and S^4. (Hence there also maps to CP^2 of every even degree.) Are these the only possibilities? (Note that $H_*(Y;\mathbb{Z})$ must be torsion-free, and $w_2(Y) = 0$, since $d = 1$ and $w_2(T^4) = 0$.) If $\pi_Y \cong \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ then Y is essentially n-dominated by $S^2 \times T$, by Lemma 4 (as in §3). Hence there is an essential domination $f : X \to Y$ of degree n.

If $\pi_Y = 1$ then $2 \leqslant \chi(Y) \leqslant 8$, so $\beta_2(Y) \leqslant 6$. If also $d = 1$ then $\chi(Y)$ must be even, by Lemma 2.

Can we have $\beta_2(Y) = 4$ or 6?

Can we have π_Y of order $2, 3$ or 4?

$\mathbb{N}il^3 \times \mathbb{E}^1$. We may assume that $X = N \times S^1$, with N a closed $\mathbb{N}il^3$-manifold. If π_Y is infinite then $\chi(Y) = 0$. Let x and z be generators of $H^1(N;\mathbb{Z})$ and $H^1(S^1;\mathbb{Z})$, respectively, and let $\xi \in H^2(X;\mathbb{Z})$ be a class such that $[X] \cap \xi xz = 1$. Since $w_2(X) = 0$ we may assume that $\xi^2 = 0$. The maps to S^1 corresponding to x and z and the map to S^2 corresponding to ξ together define a degree-1 map from X to $S^2 \times T$. There are also such maps to $S^3 \times S^1$, by Lemma 3.

If $\pi_Y = 1$ then $2 \leqslant \chi(Y) \leqslant 6$. There are degree-1 maps from X to $S^2 \times S^2$ and to S^4, and hence also maps to CP^2 of every even degree. If $d = 1$ and $\pi_Y = 1$ then $\chi(Y)$ must be even.

Can we have $\beta_2(Y) = 4$?

Can we have π_Y a nilpotent extension of $\mathbb{Z}$ by a non-trivial finite group with cohomology of period 4, or of order 2 or 3?

$\mathbb{N}il^4$ and $\mathbb{S}ol^3 \times \mathbb{E}^1$. We may assume that X is the total space of a T-bundle over T and is also the mapping torus of a self-homeomorphism of T^3, and that $\beta_1(X) = \beta_2(X) = 2$.

If π_Y is infinite then $\chi(Y) = 0$. As above, there are degree-1 maps to $S^2 \times T$ and to $S^3 \times S^1$.

If $\pi_Y = 1$ then $2 \leqslant \chi(Y) \leqslant 4$. Since $H^2(X;\mathbb{Z})$ has a basis η, ξ such that $\eta^2 = \xi^2 = 0$ and $[X] \cap \eta\xi = 1$, there are degree-1 maps to $S^2 \times S^2$ and to S^4, and hence also maps to CP^2 of every even degree.

If M is a closed orientable 4-manifold with $\pi_1(M)$ finite and $\chi(\widetilde{M}) \leqslant 4$ then either $\pi_1(M) = 1$ or $\widetilde{M} \simeq S^2 \times S^2$ or $S^2 \tilde{\times} S^2$ [5, Chapter 12.§5].)

Can we have π_Y an extension of $\mathbb{Z}$ by a non-trivial finite group with cohomology of period 4, or of order 2?

$\mathbb{S}ol^4_{m,n}$ with $m \neq n$, $\mathbb{S}ol^4_0$ and $\mathbb{S}ol^4_1$. We may assume that X is a mapping torus $F \rtimes S^1$, where $F = T^3$ or is a closed $\mathbb{N}il^3$-manifold, and that $\beta_1(X) = 1$ and $\beta_2(X) = 0$. No quotient of π_X has Hirsch length 2, and so either f is a homotopy equivalence, or Y is finitely covered by $S^3 \times S^1$ or is S^4.

As above, there are degree-1 maps to $S^3 \times S^1$ and to S^4. If $\pi_Y = 1$ then $\chi(Y) = 2$, so $Y \cong S^4$, which has no orientable proper quotients.

Can we have π_Y an extension of $\mathbb{Z}$ by a non-trivial finite group with cohomology of period 4?

5. DOMAIN FIBRED OVER A HYPERBOLIC BASE SURFACE

In this section we assume that the domain X fibres over a closed hyperbolic surface B, with fibre $F = S^2$ or T.

If $F = S^2$ then X is finitely covered by $S^2 \times \hat{B}$, where $\hat{B}$ is a closed hyperbolic surface. Any map from $S^2 \times \hat{B}$ to an aspherical 4-manifold factors through the projection to $\hat{B}$, and so has degree 0. Every $\mathbb{S}^2 \times \mathbb{E}^2$- or $\mathbb{S}^2 \times \mathbb{S}^2$-manifold is dominated by such a product. Since there is a degree-1 map from $\hat{B}$ to T, there is a degree-1 map from $S^2 \times \hat{B}$ to $S^2 \times T$, and hence there are such maps to $S^3 \times S^1$ and S^4.

If $F = T$ then $\chi(X) = 0$, and X is geometric if and only if the *monodromy* $\theta : \pi_1(B) \to GL(2,\mathbb{Z})$ has finite image. If X is finitely covered by $\hat{B} \times T$, where $\hat{B}$ is a closed hyperbolic surface, then the geometry is $\mathbb{H}^2 \times \mathbb{E}^2$. The other possible geometry is $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, and then X is finitely covered by $M \times S^1$, where M is a closed $\widetilde{\mathbb{SL}}$-manifold. [5, Corollary 7.3.1 and Theorem 9.3]. In the geometric cases we can extend the the argument for Lemma 1 partially.

Let A be the image of $\pi_1(T)$ in π_Y. If X is geometric then either $A \cong \mathbb{Z}^2$ (i.e., $\pi_1(f|T)$ is injective), or A has rank 1 or it is finite. If X is not geometric then π_X has no $\mathbb{Z}$ normal subgroup, and so either $A \cong \mathbb{Z}^2$ or A is finite.

Lemma 6. *Let $f : X \to Y$ be an essentially dominating map, where X is the total space of a T-bundle over a closed hyperbolic surface B, and* *suppose that $A \cong \mathbb{Z}^2$. If X is geometric or if the projection $p : X \to B$ has a section then $\chi(Y) = 0$. If X is not geometric then $[\pi_Y : A] = \infty$.*

Proof. Let $A \cong \mathbb{Z}^2$ be the image of $\pi_1(T)$ in π_Y. If X is geometric then $\pi_X \cong \rho \times \mathbb{Z}$ and $\pi_Y \cong GA$ is the product of two finitely presentable proper normal subgroups. If p has a section then π_X is a semidirect product $\mathbb{Z}^2 \rtimes \beta$, and so π_Y is also such a semidirect product. In each case we may apply the argument of Lemma 1 to the covering spaces associated to the subgroups of the form $\rho \times n\mathbb{Z}$ and $G.nA$, and $n\mathbb{Z}^2 \rtimes \beta$ and $nA \rtimes (\pi_Y/A)$, respectively, to conclude that $\chi(Y) \leqslant 0$. Since π_Y has an infinite abelian normal subgroup $\beta_i^{(2)}(\pi_Y) = 0$ for $i \leqslant 1$, and so $\chi(Y) = 0$.

If A has finite index in π_Y then, after passing to finite covering spaces, if necessary, we may assume that $A = \pi_Y$. But then the inclusion of $\pi_1(T)$ into π_X splits, and so $\pi_X \cong \mathbb{Z}^2 \times \pi_1(B)$, and X is a $\mathbb{H}^2 \times \mathbb{E}^2$-manifold. □

In general, T-bundles ξ over B with monodromy θ are parametrized by classes $[\xi] \in H^2(B; \mathcal{T})$, where $\mathcal{T}$ is the $\pi_1(B)$ module determined by θ, and the bundle has a section if and only if $[\xi] = 0$.

If $A \cong \mathbb{Z}^2$ and $[\pi_Y : A] = \infty$ must $\chi(Y) = 0$ (without some further hypothesis)?

Lemma 7. *Let $f : X \to Y$ be an essentially dominating map, where X is the total space of a T-bundle over a closed hyperbolic surface B. If the image of $H_1(T)$ in $H_1(Y)$ has rank 2 then $Y \simeq C \times T$, where C is a closed surface, while if it has rank 1 then $Y \simeq P \times S^1$, where P is a PD_3-complex.*

Proof. If the image of $H_1(T)$ in $H_1(Y)$ is infinite then the image of $H_1(T)$ in $H_1(X)$ is infinite, and so X is geometric. Thus we may assume that $X = M \times S^1$, where M is an aspherical closed 3-manifold. Then π_Y is a product $\sigma \times \mathbb{Z}$, for some group σ, and $\chi(Y) = 0$, by Lemma 6. Hence $Y \simeq P \times S^1$, where P is a PD_3-complex [5, Theorem 4.5]. If the image of $H_1(T)$ in $H_1(Y)$ has rank 2 then $P \simeq C \times S^1$, where C is a closed surface. □

If $A \cong \mathbb{Z}^2$, has infinite index in π_Y and $\chi(Y) = 0$ then Y is aspherical. Conversely, if Y is aspherical then $A \cong \mathbb{Z}^2$ and $G = \pi_Y/A$ is infinite. (For otherwise f would have degree 0, by Lemma 5.) A spectral sequence argument then shows that $H^2(G; \mathbb{Z}[G]) \cong \mathbb{Z}$, and so G is virtually a PD_2-group. Hence Y is Seifert fibred and $\chi(Y) = 0$. (See [5, Corollary 7.3.1].) If $A \cong \mathbb{Z}^2$ and has finite index in π_Y then $\chi(Y) = 0$, and so Y is finitely covered by $S^2 \times T$.

If A has rank 1 and infinite index in π_Y then A is central in π_Y, and so π_Y has one end. If A has rank 1 and finite index in π_Y then Y is finitely covered by $S^3 \times S^1$.

Every solvmanifold of type $\mathbb{E}^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{N}il^4$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$ is the total space of a T-bundle over T, and so is 1-dominated by T-bundles over hyperbolic bases. Of these, only T^4 is dominated by $\mathbb{H}^2 \times \mathbb{E}^2$-manifolds. (However, $\mathbb{N}il^3 \times \mathbb{E}^1$-manifolds are dominated by $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds.)

On the other hand, no T-bundle space can dominate a solvmanifold of type $\mathbb{S}ol^4_{m,n}$ with $m \neq n$, $\mathbb{S}ol^4_0$ or $\mathbb{S}ol^4_1$.

The case with hyperbolic base and fibre is considered in §6 below.

6. S^1-bundles

If X is the total space of an S^1-bundle over a 3-manifold base B and Y is aspherical then π_Y must have an infinite cyclic normal subgroup, and so $\chi(Y) = 0$. For otherwise f would have degree 0, by Lemma 5.

Theorem 8. *An aspherical 4-manifold Y is dominated by a $\mathbb{H}^3 \times \mathbb{E}^1$-manifold if and only if π_Y is virtually a product $\mathbb{Z} \times G$, where G is a PD_3-group.*

Proof. Every $\mathbb{H}^3 \times \mathbb{E}^1$-manifold is finitely covered by a product $B \times S^1$, where B is a closed $\mathbb{H}^3$-manifold. If X is such a product and $f : X \to Y$ is an essentially dominating map then $\pi_Y \cong \mathbb{Z} \times G$, and the second factor must be a PD_3-group.

Conversely, if Y is aspherical and $\pi_Y \cong \mathbb{Z} \times G$ then G is a finitely presentable PD_3-group, and so $P = K(G, 1)$ is a PD_3-complex. An application of the Atiyah-Hirzebruch spectral sequence for oriented bordism $\Omega_*(P)$ (or direct desingularization of a geometric 3-cycle representing a fundamental class) shows that P is 1-dominated by a closed orientable 3-manifold. This is in turn 1-dominated by a closed $\mathbb{H}^3$-manifold [13]. Hence $Y \simeq P \times S^1$ is 1-dominated by a $\mathbb{H}^3 \times \mathbb{E}^1$-manifold. □

Hence solvmanifolds of type $\mathbb{E}^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$ and $\mathbb{S}ol^3 \times \mathbb{E}^1$ are 1-dominated by $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds. Similarly, $\mathbb{H}^2 \times \mathbb{E}^2$- and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds are so dominated. However no $\mathbb{N}il^4$-manifold or a solvmanifold of type $\mathbb{S}ol^4_{m,n}$ with $m \neq n$, $\mathbb{S}ol^4_0$ or $\mathbb{S}ol^4_1$ can be so dominated.

7. Surface bundles

A *Surface bundle* is a bundle projection $p : X \to B$ with fibre F, where B and F are closed surfaces, $\chi(B) \leqslant 0$ and $\chi(F) < 0$. A *Surface bundle space* is the total space of a Surface bundle, and a *Surface bundle group* is the fundamental group of a Surface bundle space. Such Surface bundles may be partitioned into three types. Type

I consists of such bundles for which the monodromy has infinite image, but is not injective, type II are those which are virtually products and type III have injective monodromy [7]. The fibration is essentially unique when the bundle is of type I; product bundles have only the two obvious bundle projections, and a surface bundle space X with a fibration of type II may have many inequivalent fibrations. (See the "Johnson trichotomy" in [5, Chapter 5.2].)

Pullback along a degree-1 map of bases induces a degree-1 map of Surface bundle spaces. If the original bundle is of type I or III and the base change map is not a homotopy equivalence then the induced bundle is of type I. If the monodromy fixes a separating curve in the fibre F then we may construct a degree-1 map by crushing one of the two complementary regions of this curve in F to a point. In this way we may show that every Surface bundle space of type I or II is 1-dominated by Surface bundle spaces of types I and III. On the other hand, we shall see that Surface bundle spaces of type II can only dominate bundle spaces of the same type.

Lemma 9. *Let $f : \pi \to G$ be an epimorphism of PD_4-groups, where π has a normal subgroup κ such that κ and π/κ are PD_2-groups. If $f(\kappa)$ or $G/f(\kappa)$ is virtually free then f has degree 0.*

Proof. If $f(\phi) = 1$ then f factors though the PD_2-group $\rho = \pi/\kappa$, and so f has degree 0, by Lemma 5. In general, f induces a map between the LHS spectral sequences for $\mathbb{Q}$-homology of π and G as extensions or ρ by κ and of $G/f(\kappa)$ by $f(\kappa)$, respectively. If $f(\kappa)$ or $G/f(\kappa)$ is virtually free then the homomorphisms between $E^2_{p,q}$ terms with $p+q = 4$ either have domain 0 or codomain 0. Hence they are all trivial, and so $H_4(f) = 0$. □

We have a nearly complete understanding of the possibilities when X is of type II and Y is aspherical.

Theorem 10. *Let $X = B \times F$, where B and F are hyperbolic surfaces, and let $f : X \to Y$ be an essentially dominating map with aspherical codomain Y. If $\chi(Y) \neq 0$ or if Y is the total space of a bundle with base or fibre T then Y is homotopy equivalent to a product of aspherical surfaces.*

Proof. Let G and N be the images of $\pi_1(B)$ and $\pi_1(F)$, respectively. Then G and N are finitely generated normal subgroups of π_Y such that $gn = ng$ for all $g \in G$ and $n \in N$, and $\pi_Y = GN$. Hence $gh = hg$ for all $g \in G \cap N$ and all $h \in \pi_Y$, and so $G \cap N$ is a subgroup of the centre of π_Y. Let $\overline{\pi} = \pi_Y/(G \cap N)$. Then $\overline{\pi} \cong \pi/G \times \pi/N$.

If $G \cap N \neq 1$ then $\beta_i^{(2)}(Y) = 0$ for all i, and so $\chi(Y) = 0$. Since we are assuming in this case that Y is a bundle space, $G \cap N \cong \mathbb{Z}^r$, for some $r > 0$. Then $\overline{\pi}$ is FP, and hence so are π/G and π/N. Consideration of the LHS spectral sequence for π_Y as an extension of $\overline{\pi}$ by $\mathbb{Z}^r$ then shows that $H^{4-r}(\overline{\pi}; \mathbb{Z}[\overline{\pi}]) \cong \mathbb{Z}$. (Compare [5, Theorem 3.10].) It then follows from the Künneth Theorem for $\overline{\pi} \cong \pi/G \times \pi/N$ that either π/G or π/N two ends or one is finite. This contradicts Lemma 9.

Therefore $G \cap N = 1$, so $\pi_Y \cong G \times N$, and so G and N are finitely presentable. Hence they are in fact FP_3 (each being a quotient of an FP group by a finitely presentable group). Neither G nor N is trivial or $\mathbb{Z}$, by Lemma 9, and so each is a PD_2-group [5, theorem 3.10]. □

We can push the argument for this theorem a little further if $\chi(Y) = 0$ and $G \cap N \neq 1$. We note first that $c.d.G \leqslant 3$ and $c.d.N \leqslant 3$, since G and N each have infinite index in π_Y. Hence $G \cap N$ has rank $r \leqslant 3$. The argument of the theorem shows that $G \cap N$ is not finitely generated Hence it has rank $r < c.d.G \cap N < 4$ and so $r = 1$ or 2.

Suppose that $r = 2$. Then $c.d.G \cap N = 3$, since $G \cap N$ is not finitely generated. But $c.d.G = 3$ also, and so G is abelian [2, Theorem 8.8]. Since G is finitely generated, $G \cap N$ must be finitely generated also. This is a contradiction, and so $r = 1$. It remains an open question whether the centre of a PD_n-group must be finitely generated, if $n > 3$.

General results on essential dominations of aspherical 4-manifolds by Surface bundle spaces of types I or III appear to be hard to find. When Y is also a bundle space we should ask whether such a domination need be compatible with a fibration, in the sense that the normal subgroup $N = f_*\pi_1(F)$ of π_Y should be a PD_2-group such that π_Y/N is also a PD_2-group.

Lemma 11. *Let E be a Surface bundle group. If N is a finitely generated infinite normal subgroup of infinite index in E then N is either a PD_2-group or is an extension of a PD_2-group by a free group of infinite rank. Moreover either N or E/N has one end.*

Proof. Let $p : E \to G$ be the projection of E onto $G = \pi_1(B)$, with kernel $K = \pi_1(F)$. If $p(N) \neq 1$ then $[G : p(N)] < \infty$, and $\mathrm{Ker}(p|_N)$ has infinite index in K. Hence either $p|_N$ is a monomorphism or $\ker(p|_N)$ is free of infinite rank. (This uses the fact that F is hyperbolic!)

The second assertion follows from consideration of the LHS spectral sequence for E as an extension of E/N by N. Since E/N and N are finitely generated and infinite, $H^p(E/N; H^q(N; \mathbb{Z}[E])) = 0$ if $q = 0$ or if $p = 0$ and $q = 1$, while

$$H^1(E/N; H^1(N; \mathbb{Z}[E])) \cong H^1(E/N; \mathbb{Z}[E/N]) \otimes H^1(N; \mathbb{Z}[N]).$$

Since $H^i(E;\mathbb{Z}[E]) = 0$ for $i \leqslant 3$, a spectral sequence corner argument shows that $H^1(E/N; H^1(N;\mathbb{Z}[E])) = 0$. Since the terms in the tensor product are free abelian groups, at least one must be 0. □

If $\beta_1(E) > \beta_1(B)$ then there are finitely generated normal subgroups N such that $E/N \cong \mathbb{Z}$ and $\ker(p|_N)$ is nontrivial, and if, moreover, $\chi(E) \neq 0$ then such a subgroup N cannot be FP_2 [3].

8. Bundle spaces and geometries

If a Surface bundle space X is geometric but is not a $\mathbb{H}^4$-manifold then either $\chi(X) = 0$ (so $B = T$) and the geometry is $\mathbb{H}^2 \times \mathbb{E}^2$ or $\mathbb{H}^3 \times \mathbb{E}^1$ or X is a reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold [5, Theorems 9.10, 13.5 and 13.6]. Every reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold is finitely covered by a product $B \times F$, while every $\mathbb{H}^3 \times \mathbb{E}^1$-manifold is finitely covered by a Surface bundle space of type I, with base T. No Surface bundle space has the geometry $\mathbb{H}^2(\mathbb{C})$ [5, Corollary 13.7.2], and it is believed that $\mathbb{H}^4$ is also impossible.

Suppose that X is a Surface bundle space which dominates a $\mathbb{H}^2 \times \mathbb{E}^2$-manifold Y. After passing to finite covers we may assume that Y is a product $C \times T$, where C is a hyperbolic surface. If N is a non-trivial finitely generated normal subgroup of infinite index in π_Y then either N has finite index in $\pi_1(C)$ or $\pi_1(T)$, or $N \cong \mathbb{Z} < \pi_1(T)$ or $N \cap \pi_1(C)$ has finite index in $\pi_1(C)$ and π_Y/N is virtually $\mathbb{Z}$. The latter two possibilities are ruled out by Lemma 9, and so f must be compatible with one of the two projections from Y to C or T.

If Y is a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold then after passing to finite covers we may assume that Y is the total space of a T-bundle over a hyperbolic surface. A similar argument applies, except that in this case the bundle fibration is unique and the only normal PD_2-groups in π_Y are subgroups of finite index in $\pi_1(T)$. Hence f must be compatible with the fibration of Y. In particular, no $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold is dominated by a product of surfaces.

Suppose now that Y is a solvmanifold. The image of $\pi_1(F)$ in π_Y is a finitely generated normal subgroup, and is torsion-free polycyclic. It must have Hirsch length 2, by Lemma 9, and so $\pi_1(f)$ factors though the group of a T-bundle over B. Every solvmanifold of type $\mathbb{E}^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{N}il^4$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$ is the total space of a T-bundle over T. It follows easily that such manifolds are 1-dominated by Surface bundle spaces.

There are epimorphisms from Surface bundle groups to semidirect products $\mathbb{Z}^3 \rtimes \mathbb{Z}$, with $\pi_1(F)$ mapping onto $\mathbb{Z}^3$. However all such maps have degree 0, and no solvmanifold of type $\mathbb{S}ol^4_{m,n}$ or $\mathbb{S}ol^4_0$ is dominated by a Surface bundle space. Similarly, no $\mathbb{S}ol^4_1$-manifold is dominated by a Surface bundle space.

If Y is aspherical then the image of $\pi_1(F)$ in π_Y is a finitely generated normal subgroup, and $G = \pi_Y/K$ is finitely presentable. Lemma 9 implies that K is not a free group and that G is not virtually free, so $[\pi_Y : K] = \infty$. Hence $c.d.K = 2$ or 3. Are there any such examples which are not bundle spaces? There are $\mathbb{H}^2(\mathbb{C})$-manifolds whose fundamental groups are extensions of hyperbolic surface groups, by finitely generated normal subgroups which are not FP_2 [8]. Are any such manifolds essentially dominated by Surface bundle spaces?

See [10] for observations on degree-1 maps from $\mathbb{H}^4$- and $\mathbb{H}^2(\mathbb{C})$-manifolds and irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds, and further references.

9. Appendix. Some remarks on the Kapovich-Livne groups

Let $H_d = \pi^{orb}S^2(2,2,2,2d)$, with presentation

$$\langle A_1, A_2, A_3 \mid A_1^2 = A_2^2 = A_3^2 = (A_1A_2A_3)^{2d} = 1\rangle.$$

The commutator subgroup H_d' is torsion-free. If $d = 1$ then $H_d' \cong \mathbb{Z}^2$, while if $d > 1$ then H_d' is the fundamental group of a hyperbolic surface.

Let $c_h(g) = hgh^{-1}$ for $g, h \in H$ and let σ and θ be the automorphisms defined by $\sigma(A_1) = A_2$, $\sigma(A_2) = A_3$ and $\sigma(A_3) = 1$, and $\theta(A_1) = A_2A_3A_2^{-1}$, $\theta(A_2) = A_2$ and $\theta(A_3) = A_1$. Let $\phi = c_{A_1} \circ \sigma$ and $\psi = c_{A_1} \circ \theta$. Then ϕ and ψ each fix $A_1A_2A_3$, and $\phi^3 = \psi^2 = c_{A_1A_2A_3}$.

Let G be the extension of $PSL(2,\mathbb{Z})$ by H_d with presentation

$$\langle H_d, x, y \mid x^3 = y^2 = A_1A_2A_3,\ xhx^{-1} = \phi(h),\ yhy^{-1} = \psi(h), \forall\, h \in H_d\rangle.$$

Then G is virtually a semidirect product $H' \rtimes F(2)$, since $PSL(2,\mathbb{Z})' = F(2)$ is free of rank 2, and so $H_4(G) = 0$.

If we add the relation $(yx^{-1})^N = 1$ then we get a group $\Gamma_{d,N}$ which is an extension of the triangle group $\pi^{orb}S^2(2,3,N)$ by a quotient H_d. Let $p : G \to \Gamma_{d,N}$ be the associated epimorphism. Livne showed in his 1981 Harvard thesis that if (n,d) is one of the pairs (7,7), (8,4), (9,3) or (12,2) then $\Gamma_{d,N}$ is a uniform lattice in $PU(2,1)$. It has a torsion free subgroup of finite index which maps onto a hyperbolic surface group (since $N \geqslant 7$). Kapovich observed that the kernel is finitely generated but cannot be finitely presentable, since $\mathbb{H}^2(\mathbb{C})$-manifolds do not fibre over complex curves.

If $h \neq 1$ then using the relation $(yx^{-1})^N = h$ instead gives an intermediate group. Such groups are also extensions of $\pi^{orb}S^2(2,3,N)$ by quotients of H_d. No proper quotient of H_d is an orbifold group. Can we nevertheless find an intermediate quotient of a subgroup of finite index in G which is a Surface bundle group, and for which the induced homomorphism to $\Gamma_{d,N}$ has non-zero degree?

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