

Growth of Finitely Presented Rees Quotients of Free Inverse Semigroups

Dedicated to Professor John Meakin

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We prove that a finitely presented Rees quotient of a free inverse semigroup has polynomial growth if and only if it has bounded height. This occurs if and only if the set of nonzero reduced words has bounded Shirshov height and all nonzero reduced but not cyclically reduced words are nilpotent. This occurs also if and only if the set of nonzero geodesic words have bounded Shirshov height. We also give a simple sufficient graphical condition for polynomial growth, which is necessary when all zero relators are reduced. As a final application of our results, we give an inverse semigroup analogue of a classical result that characterises polynomial growth of finitely presented Rees quotients of free semigroups in terms of primitive words that label loops of the Ufnarovsky graph of the presentation.

1. INTRODUCTION

In [21] the authors initiated the study of growth of finitely presented Rees quotients of free inverse semigroups and the relationship with satisfiability of identities. In [5] they considered satisfiability of identities in signature with involution in the one relator case. Lau considered rationality of growth series [9] and degrees of growth [10] for semigroups from this class (see also [20] by the first author, in the wider context of Gelfand-Kirillov dimension and Rees quotients using infinitely many relators). In [21], growth was shown to be polynomial or exponential for semigroups from this class and an algorithmic criterion given to recognise which type of growth occurs [21, Section 3]. The arguments relied on a graphical technique, the ingredients of which are outlined in the next section, which is a modification of an idea due to Ufnarovsky [23] [24] (see also [13, Chapter 24]). In Theorem 3.3 below a simplified graphical condition is given which is always sufficient for polynomial growth, and necessary when all zero relators are reduced. Shirshov [18] (see also [16, Chapter 4], [22]) introduced the notion of height in his celebrated study of rings which satisfy a polynomial identity. In Theorem 4.3 in the final section we give simple criteria for polynomial growth for semigroups from our class in terms of height, nilpotency and geodesic words. Some of the main results of the final section also appear in Chapter 4 of Lau's doctoral thesis [11], but with different proofs. The final result, Theorem 4.4, is an application of our results to yield an inverse semigroup analogue of a classical result characterising polynomial growth

of finitely presented Rees quotients of free semigroups in terms of primitive words that label loops of the Ufnarovsky graph.

2. PRELIMINARIES

We assume familiarity with the basic definitions and elementary results from the theory of semigroups, which can be found in any of [3], [7], [8] or [14]. Let S be a semigroup generated by a finite subset X . Recall that the *length* $l(t)$ of an element $t \in S$ (with respect to X) is the least number of factors in all representations of t as a product of elements of X , and

$$g_S(m) = |\{t \in S \mid l(t) \leq m\}|$$

is called the *growth function* of S . We say that S has *polynomial growth* if there exist natural numbers q and d such that $g_S(m) \leq qm^d$ for all natural numbers m , and *exponential growth* if there exists a real number $\alpha > 1$ such that $g_S(m) \geq \alpha^m$ for all sufficiently large m . These definitions and notions of growth apply also to subsets of S in an obvious way. Throughout this paper, we may implicitly assume that zero lies in the generating set of any semigroup with zero (so that zero always has length one). If $t \in S$ and the equation $t = w$ holds in S , where w is a product of $l(t)$ elements of X regarded as a word over the alphabet X , then we call the word w a *geodesic* for t (with respect to X).

Let A be a finite alphabet and put $B = A \cup A^{-1}$, where it is understood that the union is disjoint and any a in A may also be denoted by $(a^{-1})^{-1}$. The free semigroup and free inverse semigroup over A are denoted by F_A and FI_A respectively. Literal equality of words is denoted by the symbol $\underline{\quad}$, and $|w|$ denotes the number of letters in a word w . Recall that w is *reduced* if w does not contain xx^{-1} as a subword for any letter $x \in B$, and that w is *cyclically reduced* if w and w^2 are both reduced (whence all powers of w are reduced). A word is called *primitive* if it cannot be expressed as a power of a proper subword. Recall that elements of FI_A may be regarded as birooted word trees (introduced in [12]), the terminology and theory of which are explained in [2] (see also [21, Section 2]). As in [21], denote the word tree of a word w over B by $T(w)$. If u and v are elements of FI_A , then $T(u)$ is a subtree of $T(v)$ if and only if v may be expressed as a product of elements, one of whose factors is u , in which case we say that u *divides* v . Any reference to Green's relation \mathcal{J} will be with respect to FI_A . Note that two words over $A \cup A^{-1}$ are \mathcal{J} -related if and only if their word trees are identical. Recall that an element s of a semigroup S with zero is *nilpotent* if some power of s is zero.

Any given element w of FI_A may also be expressed as

$$w = u_1 u_1^{-1} u_2 u_2^{-1} \dots u_r u_r^{-1} \overline{w} \tag{1}$$

for some nonnegative integer r and reduced words u_1, \dots, u_r, \bar{w} . If r is as small as possible, so that no u_i can be an initial segment of u_j for $i \neq j$, then (as in [20]) we call (1) the *Schein (left) canonical form of w* , which is unique up to order of idempotents. Note that r is bounded by the number of leaf vertices on $T(w)$. We refer to the idempotents $u_1 u_1^{-1}, \dots, u_r u_r^{-1}$ as the *components* and \bar{w} as the *irreducible part of w* .

Let w be a nonempty word over B . A geodesic \tilde{w} for w in FI_A has the form

$$\tilde{w} \stackrel{\circ}{=} u_0 e_1 u_1 e_2 \dots e_n u_n$$

where $n \geq 0$ is the number of idempotents in the Schein canonical form,

$$\bar{w} \stackrel{\circ}{=} u_0 u_1 \dots u_n \quad \text{and} \quad \bar{e}_1 \stackrel{\circ}{=} \dots \stackrel{\circ}{=} \bar{e}_n \stackrel{\circ}{=} 1.$$

Geodesics in free inverse monoids were considered by Choffrut [4, Section 3.2], and also by Poliakova and Schein [15], where they gave a rewriting system that reduces an arbitrary word in the free inverse semigroup to its geodesic form (which is unique up to the representation of idempotents). To form each word e_i representing an idempotent, one systematically traces the labels of edges of the branches of the word tree that emanate from the geodesic chain of $T(w)$ labelled by \bar{w} at the vertex pointed to by the last letter of u_{i-1} (or the initial vertex if $i = 1$ and u_0 is empty). The word e_i is not unique unless there is exactly one branch emanating from that vertex and that branch is a chain, because one may arbitrarily choose the order of multiple subbranches to trace from any given vertex. The number of occurrences of a given letter in e_i however is unique.

As in [21], let \mathfrak{M}_{FI} denote the class of finitely presented inverse semigroups S with zero having a presentation of the form

$$S = \langle A \mid c_i = 0 \quad \text{for } i = 1, \dots, k \rangle$$

where A is some finite alphabet, k some nonnegative integer and $c_i \in F_{A \cup A^{-1}}$ for $i = 1$ to k . Then \mathfrak{M}_{FI} is precisely the class of Rees quotients of finitely generated free inverse semigroups by finitely generated ideals. We refer to the words c_1, \dots, c_k appearing in the above presentation as (*zero*) *relators*. Note that S is unaffected by replacing any relator by a word to which it is \mathcal{J} -equivalent.

We now redefine the *Ufnarovsky graph* $\Gamma = \Gamma_S$ of S (depending on the presentation of S). The illustrative example given below in Section 3 may assist the reader in digesting the following construction. Suppose that there is at least one relator which is \mathcal{J} -equivalent to reduced word of length ≥ 2 . (This is a very mild restriction for our purposes, and cases when it does not hold are catalogued in comments following the statement of Theorem 2.1.) Note that the definition of vertices given below is slightly different to that given in [21, Section 3]. This modification shortens calculations when long relators appear which are not \mathcal{J} -equivalent to reduced words. Put $d + 1 = \max\{l(c_i) \mid i = 1, \dots, k\}$ and

$$\bar{d} + 1 = \max\{l(c) \mid c \text{ is a reduced word } \mathcal{J}\text{-equivalent to some relator}\}.$$

Note that \bar{d} may be calculated easily by inspecting word trees of relators. Also $1 \leq \bar{d} \leq d$. Vertices of Γ are defined to be reduced words of length \bar{d} which are nonzero in S . If v_1 and v_2 are vertices then a directed edge from v_1 to v_2 is defined in Γ if there exist letters $g, h \in A \cup A^{-1}$ such that v_1g is a reduced word which is nonzero in S and $v_1g \bar{\subseteq} hv_2$. We regard the letter g as a label for this edge. Paths in Γ may then be labelled by reduced words which are nonzero in S . Conversely if $w \bar{\subseteq} vu \bar{\subseteq} u'v'$ is any nonzero reduced word where v and v' have length \bar{d} then u labels a path in Γ emanating from v and terminating at v' .

By a *cycle* in Γ we mean a path which starts and finishes at the same vertex. By a *loop at a vertex* v we mean a cycle which begins at v using no other vertex more than once. Recall from [21, Section 3] that (z, P) is an *adjacent pair* if z is a reduced word which labels a loop in Γ at a vertex v and P is a letter labelling an edge which emanates from v and terminates outside the loop. The proof of [21, Theorem 1] requires only trivial modifications to yield

THEOREM 2.1. *Let $S = \langle A \mid c_i = 0 \text{ for } i = 1, \dots, k \rangle$ be an inverse semigroup from the class \mathfrak{M}_{FI} and suppose that $\bar{d} \geq 1$. Then the following conditions are equivalent:*

- (a) S has polynomial growth.
- (b) S does not contain any noncyclic free subsemigroup.
- (c) (i) Γ_S has no vertex contained in different cycles; and
(ii) if (z, P) is an adjacent pair in Γ_S then

$$z^{d+1}PP^{-1}z^{d+1} = 0 \quad \text{in } S.$$

Note that if $\bar{d} = 0$ then relators that are \mathcal{J} -related to reduced words are \mathcal{J} -related to generators, in which case such generators and any relators in which they or their inverses occur may be deleted without altering S (up to isomorphism). If \bar{d} does not exist then no relator is \mathcal{J} -related to a reduced word, so that the word tree of any relator must have at least one vertex of degree > 2 , in which case either (i) $A = \{a\}$ and there are no relators, so that S is a free monogenic inverse semigroup with zero, which has polynomial (in fact cubic) growth, or (ii) A contains distinct letters a and b that generate a nonmonogenic free subsemigroup of S , so that the growth of S is exponential. These observations and the previous theorem tell us (as in [21, Corollaries 1 and 2]) that any Rees quotient of a finitely generated free inverse semigroup by a finitely generated ideal has polynomial or exponential growth (and there is an algorithm to determine which type of growth occurs).

The proof of [21, Theorem 1] relied on a delicate argument that bounds the number of idempotents in the Schein canonical form when the semigroup S has polynomial growth. From that proof we extract the following result for use in the final section of this paper:

SCHOLIUM 2.2. Let S be an inverse semigroup from the class \mathfrak{M}_{FI} having polynomial growth where $\bar{d} \geq 1$. Form the Ufnarovsky graph Γ_S and let N be the number of paths in Γ_S in which no cycle is repeated, and T the maximum number of cycles in any such path. Put

$$\lambda = (2d + 2)^T + 1 \quad \text{and} \quad M = \lambda N + (d - 1)(2|A|)^{d-1}.$$

Then any nonzero element of S has at most M components in its Schein canonical form. In particular, the number of leaf vertices of $T(w)$ for any word w that is nonzero in S is bounded by $M + 2$.

Finally we recall the notion of height introduced by Shirshov [18]. Let S be any semigroup and X a subset of S . Denote by (X) the subsemigroup of S generated by X . If $s \in (X)$ can be expressed as a product

$$s = h_1^{\alpha_1} \dots h_k^{\alpha_k} \tag{2}$$

for some $h_1, \dots, h_k \in X$ and positive integers $\alpha_1, \dots, \alpha_k$, and k is as small as possible, then we say the *height of s with respect to X* is k . We say that a subset T of S has *bounded height $\leq k$* if there exists a finite subset X of S such that $T \subseteq (X)$ and the height of elements of T with respect to X is at most k . In the final section of this paper we prove that, for semigroups from the class \mathfrak{M}_{FI} , the properties of having polynomial growth and having bounded height are equivalent.

Semigroups with bounded height in general need not have polynomial growth (for example, the Baumslag-Solitar groups $BS(1, k)$ for $|k| > 1$ [2]), and so various authors have placed restrictions that guarantee polynomial growth (see, for example, [20, 1, 25]). For example, suppose in addition to the above there is an ambient generating set Y for S (so for semigroups from the class \mathfrak{M}_{FI} we would take Y to be B). If k can always be chosen minimally in (2) so that the word $h_1^{\alpha_1} \dots h_k^{\alpha_k}$ is a geodesic for s (with respect to Y) then we call k the *Shirshov height* of s . We say a subset T of S has *bounded Shirshov height $\leq k$* if each element of T has Shirshov height and this is at most k . For example, in the simplest nontrivial case, if S is the free monogenic inverse semigroup generated by a then its subset of nonzero reduced words has bounded height 1 (being the set of positive and negative powers of a), and S has bounded Shirshov height 3 using geodesic forms.

It is very easy to see that if S has bounded Shirshov height k then S has polynomial growth (with respect to the ambient generating set Y) of degree at most k . In all the cases considered below in this paper where we establish bounded Shirshov height, if s is also geodesic in (2) then the number of occurrences of any given letter in B is the same on both sides of (2). Thus, for the cases we consider, our notion in fact coincides with the original classical notion [18].

The following facts are used in the final section. The first is routine and well-known. For the second, the following notation is convenient: denote by $\|w\|$ the number of vertices of $T(w)$ where w is any word over B .

LEMMA 2.3. *If T is a subset of a free semigroup F_A where A is a finite alphabet and T has bounded height, then T has polynomial growth.*

LEMMA 2.4 *Suppose that T is a set of words over $B = A \cup A^{-1}$ where A is a finite alphabet. If T has bounded height in the free inverse semigroup FI_A then T has polynomial growth in FI_A . In particular, if a Rees quotient of a finitely generated free inverse semigroup has bounded height then it has polynomial growth.*

PROOF. Suppose that T has height $\leq K$ with respect to some finite set X of words over B . Consider $w \in T$. Then $w = v$ in FI_A where

$$v \stackrel{\bar{\square}}{=} h_1^{\alpha_1} \dots h_k^{\alpha_k}$$

for some $k \leq K$, words $h_1, \dots, h_k \in X$ and positive integers $\alpha_1, \dots, \alpha_k$. If h_i is not idempotent then the cyclically reduced part of $\overline{h_i}$ is nonempty, and it follows quickly that $\alpha_i < \|\overline{h_i^{\alpha_i}}\|$. If h_i is idempotent then we may assume $\alpha_i = 1$. Observe that $h_i^{\alpha_i}$ divides w so that in all cases

$$\alpha_i \leq \|\overline{h_i^{\alpha_i}}\| - 1 \leq \|w\| - 1 \leq |\tilde{w}| = \ell(w),$$

where $\ell(w)$ is calculated in FI_A with respect to B . This produces the constraint

$$\alpha_1 + \dots + \alpha_k \leq K \ell(w).$$

from which it follows readily that the growth of T in FI_A with respect to B is bounded above by some polynomial of degree $\leq K$, completing the proof of the lemma.

3. GROWTH

In this section we give a useful sufficient condition for polynomial growth in terms of the Ufnarovsky graph. This condition is also necessary when all the relators are reduced. We begin with two observations concerning the existence of loops and adjacent pairs. We assume for the time being that S has a presentation of the form given in the previous section and $\bar{d} \geq 1$.

LEMMA 3.1. *Any word labelling a loop in Γ_S is primitive.*

PROOF. Suppose that z^α labels a loop emanating from a vertex v . Then $z^{\alpha\bar{d}}$ labels a cycle ending at v , so v must be a suffix of $z^{\alpha\bar{d}}$. If $v \stackrel{\bar{\square}}{=} uz^\beta$ for some $\beta \geq 1$ and words u, u' such that $z \stackrel{\bar{\square}}{=} u'u$ then $vz \stackrel{\bar{\square}}{=} uu'v$. If $z \stackrel{\bar{\square}}{=} v'v$ for some v' then $vz \stackrel{\bar{\square}}{=} vv'v$. In both cases z labels a path beginning and ending at v . Hence, since z^α labels a loop, $\alpha = 1$, completing the proof.

LEMMA 3.2. *Suppose that Γ_S contains no vertex in different cycles, that v is a nonempty word, P is a letter that is not the first letter of v , and $v^{d+1}P$ is reduced and nonzero in S . Then $v \bar{\subseteq} z^\alpha$ for some word z and positive integer α such that (z, P) is an adjacent pair in Γ_S .*

PROOF. Certainly v^{d+1} is nonzero in S . This implies that all divisors of powers of v are nonzero in S , since all relators have length $\leq d+1$. Write $v^{\bar{d}} \bar{\subseteq} v_1 v_2$ where v_2 has length \bar{d} . Then $v_2 v^{\bar{d}+1} \bar{\subseteq} v_2 v v^{\bar{d}} \bar{\subseteq} (v_2 v v_1) v_2$. But $v_2 v^{\bar{d}+1}$ is reduced and nonzero in S , since it divides a power of v . Hence $v^{\bar{d}+1}$ labels a cycle beginning and ending at v_2 . Let z be the smallest suffix of $v^{\bar{d}+1}$ which labels a loop at v_2 . Since Γ_S contains no vertex in different cycles, $v^{\bar{d}+1}$ is a positive power of z . Hence both v and z must be positive powers of some word w . By Lemma 3.1, $z \bar{\subseteq} w$. Further, since $v^{\bar{d}+1}P$ is reduced and nonzero in S and again since no vertex is contained in different cycles, P must label an edge emanating from v_2 which terminates outside the loop labelled by z . Hence (z, P) is an adjacent pair in Γ_S and the lemma is proved.

THEOREM 3.3. *Let $S = \langle A \mid c_i = 0 \text{ for } i = 1, \dots, k \rangle$ be an inverse semigroup from the class \mathfrak{M}_{FI} where $\bar{d} \geq 1$. Then S has polynomial growth if*

- (i) Γ_S has no vertex contained in different cycles; and
- (ii) if (z, P) is any adjacent pair then (z^{-1}, P) is not an adjacent pair.

If S has polynomial growth and each relator is \mathcal{J} -equivalent to a reduced word, then (i) and (ii) hold.

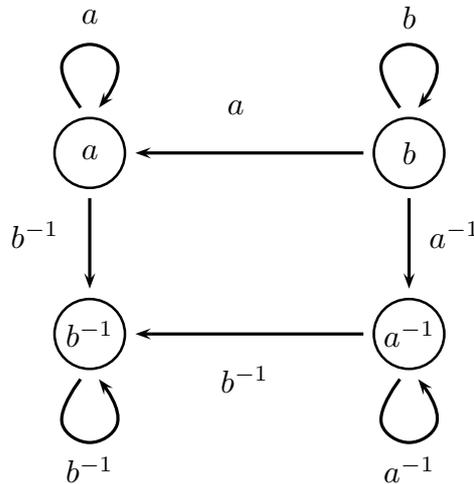
PROOF. Suppose that conditions (i) and (ii) hold and that (z, P) is an adjacent pair. Suppose that $z^{d+1}PP^{-1}z^{d+1}$ is nonzero in S . Then $(z^{-1})^{d+1}P$ is reduced and nonzero in S . Hence by Lemma 3.2, $z^{-1} \bar{\subseteq} w^\alpha$ for some word w and positive integer α such that (w, P) is an adjacent pair. But z^{-1} labels a loop in Γ_S , since z labels a loop and inversion induces an edge-reversing bijection of the graph. Hence, by Lemma 3.1, $\alpha = 0$. Thus (z^{-1}, P) is an adjacent pair, contradicting (ii). Hence $z^{d+1}PP^{-1}z^{d+1} = 0$ in S . By Theorem 2.1, S has polynomial growth.

Suppose now that S has polynomial growth and all relators are \mathcal{J} -equivalent to reduced words. Certainly (i) holds by Theorem 2.1. Let (z, P) be an adjacent pair. By Theorem 2.1, $z^{d+1}PP^{-1}z^{d+1} = 0$ in S . Hence some relator c_i divides $z^{d+1}PP^{-1}z^{d+1}$. Since c_i is \mathcal{J} -equivalent to a reduced word, its word tree is a chain of $\leq d+1$ edges, and so, by inspection, must be a subtree of the word tree of $z^{d+1}P$ or $P^{-1}z^{d+1}$. But $z^{d+1}P \neq 0$ in S , since (z, P) is an adjacent pair. Hence $P^{-1}z^{d+1} = 0$ in S , so (z^{-1}, P) cannot be an adjacent pair. This shows that (ii) holds, and the proof of the theorem is complete.

The condition given in the previous theorem is not in general necessary for polynomial growth. For example, consider the semigroup

$$S = \langle a, b \mid ab = b^{-1}a = ab^{-1}ba = 0 \rangle .$$

Then $\bar{d} = 1$, $d = 3$ and Γ_S is the following graph:



The adjacent pairs are (a, b^{-1}) , (a^{-1}, b^{-1}) , (b, a^{-1}) and (b, a) , and

$$a^4 b^{-1} b a^4 = a^{-4} b^{-1} b a^{-4} = b^4 a^{-1} a b^4 = b^4 a a^{-1} b^4 = 0$$

in S . There are no vertices in different cycles, so S has polynomial growth by Theorem 2.1. However, both (a, b^{-1}) and (a^{-1}, b^{-1}) are adjacent pairs of Γ_S , so that (ii) of Theorem 3.3 fails.

4. BOUNDED HEIGHT

This section begins with a lemma that reformulates condition (c)(ii) of Theorem 2.1 in terms of nilpotent elements. The paper culminates in several equivalences relating polynomial growth to height, the main result being that the growth of a finitely presented Rees quotient of a free inverse semigroup is polynomial if and only if this quotient has bounded height.

LEMMA 4.1. *Let S belong to the class \mathfrak{M}_{FI} and suppose that no vertex of Γ_S is contained in different cycles. Then $z^{d+1} P P^{-1} z^{d+1} = 0$ in S for all adjacent pairs (z, P) in Γ_S if and only if all reduced words that are not cyclically reduced are nilpotent in S .*

PROOF. Suppose that $w \sqsubseteq u^{-1} v u$ is reduced, u is nonempty, v is cyclically reduced and w is not nilpotent in S . Let P be the first letter of u . Since w^{d+1}

is nonzero in S , and since $v^{d+1}P$ is a reduced word dividing w^{d+1} , $v^{d+1}P$ is nonzero in S . Hence, by Lemma 3.2, $v \underline{\bar{\alpha}} z^\alpha$ for some word z and positive integer α such that (z, P) is an adjacent pair. But then $z^{d+1}PP^{-1}z^{d+1}$ is nonzero in S since it divides w^{2d+2} and w is not nilpotent.

Conversely let (z, P) be an adjacent pair such that $z^{d+1}PP^{-1}z^{d+1}$ is nonzero in S . Put $w \underline{\bar{\alpha}} P^{-1}z^{d+1}P$. Observe that w is a reduced but not cyclically reduced word. If w is nilpotent then, by inspection, because the length of each relator is $\leq d+1$ some relator must divide $z^{d+1}PP^{-1}z^{d+1}$, a contradiction. Hence w is nilpotent and the proof of the lemma is complete.

The if part of the following theorem is a standard height argument, but is included for completeness and the convenience of the reader. The only if part relies on Lemma 2.4. The result also appears as Theorem 4.3.4 of [9] with a different proof.

THEOREM 4.2. *Let S belong to the class \mathfrak{M}_{FI} and let T be the set of reduced words that are nonzero in S . Suppose also that $\bar{d} \geq 1$. Then T has bounded height [bounded Shirshov height with respect to a finite set of reduced words over B] if and only if no vertex of Γ_S is contained in different cycles.*

PROOF. Suppose firstly that no vertex of Γ_S is contained in different cycles. Let Y be the set of labels of loops in Γ_S and put $X = Y \cup A \cup A^{-1}$, which is clearly finite. Let l_1 be the number of edges in the longest path in Γ_S which contains no loops and let l_2 be the number of loops in Γ_S . Put $k = \bar{d} + l_1 + l_2$. Let w be a reduced word which is nonzero in S . If $l(w) \leq \bar{d}$ then certainly w is a product of $\leq k$ elements of X . Suppose that $l(w) > \bar{d}$. Then $w \underline{\bar{\alpha}} vw'$ where v is a vertex of Γ_S and w' is the label of a path π in Γ_S emanating from v . Suppose that π has m loops. No vertex of π is contained in two different loops. Hence

$$w \underline{\bar{\alpha}} vP_1(x_1y_1)^{\alpha_1}x_1P_2(x_2y_2)^{\alpha_2}x_2 \dots P_m(x_my_m)^{\alpha_m}x_mP_{m+1} \quad (3)$$

where, for $i = 1, \dots, m$, x_i labels the portion of the i th loop from the vertex where π enters to the vertex where π exits, y_i labels the remainder of the i th loop (so that x_iy_i labels the entire i th loop at the vertex of entry), P_1 and P_{m+1} are labels of the portions of π which lead up to the first and away from the last loop respectively, and for $i = 2, \dots, m$, P_i labels the portion of π connecting the $(i-1)$ th to the i th loop. It is understood here that any of $P_1, P_{m+1}, y_1, \dots, y_m$ could be empty, but certainly all of $P_2, \dots, P_m, x_1, \dots, x_m$ are nonempty. Observe that $l(P_1x_1P_2x_2 \dots P_mx_mP_{m+1}) \leq l_1$ and $m \leq l_2$, so that w is a product of $\leq k$ powers of elements of X . Because we have graphical equality above of reduced words, the right hand side is automatically a geodesic for the left hand side, and this shows that T has bounded Shirshov height $\leq k$ with respect to a finite set of reduced words over B .

Conversely, if Γ_S has a vertex in two different cycles, then the growth of S is exponential, so that T does not have bounded height, by Lemma 2.4, and the proof of the theorem is complete.

THEOREM 4.3. *Let S belong to the class \mathfrak{M}_{FI} . The following conditions are equivalent:*

- (a) S has polynomial growth.
- (b) The set of reduced words that are nonzero in S has bounded height [bounded Shirshov height with respect to some finite set of reduced words over B] and all reduced words that are not cyclically reduced are nilpotent.
- (c) The set of geodesic words that are nonzero in S has bounded Shirshov height with respect to some finite set of reduced words over B .
- (d) S has bounded height.

PROOF. The equivalence of (a), (b), (c) and (d) is immediate if $\bar{d} = 0$ or \bar{d} does not exist, by the observations following Theorem 2.1 and by Lemma 2.3. We may suppose therefore that $\bar{d} \geq 1$.

(a) \iff (b): This follows by Theorem 2.1, Lemma 4.1 and Theorem 4.2.

(a) \implies (d): Suppose S has polynomial growth. Let T denote the set of reduced words that are nonzero in S . Because (a) implies (b), there exists a finite set X such that T has bounded height k with respect to X . Using the Schein canonical form and the bound M in Scholium 2.2, it is immediate then that S has height bounded by $(2M + 1)k$ with respect to X . Thus (d) holds.

(d) \implies (a): This follows by Lemma 2.4.

(a) \iff (c): As we noted it is easy to see that bounded Shirshov height implies polynomial growth (or alternatively, it is obvious that (c) implies (d), and we have just observed that (d) implies (a)). Suppose then S has polynomial growth, and as before let T denote the set of reduced words that are nonzero in S . Let \mathcal{U} denote the set of all geodesics that are nonzero in S . For $n \geq 2$, let \mathcal{U}_n denote the subset of geodesics whose word trees have at most n leaf vertices. Since a word tree has at most two more vertices than the number of idempotents in the Schein canonical form, we have $\mathcal{U} = \mathcal{U}_{M+2}$, by Scholium 2.2. We prove that \mathcal{U}_n has bounded Shirshov height by induction, for $n = 2, \dots, M+2$. By Theorem 4.2, there exists a set X of reduced words such that T has bounded Shirshov height k with respect to X and the alphabet B . Observe first that

$$\mathcal{U}_2 = \{g^{-1}gwhh^{-1} \mid g, h, gwh \in T\}$$

so that the Shirshov height of \mathcal{U}_2 is bounded by $3k$, which starts an induction. Note the height is Shirshov because each reduced word in the factorisation is expressed

as a product of powers of elements of X using usual concatenation of words in the free semigroup F_B . This same observation will hold in the inductive step below (without comment). Suppose $n > 2$ and, as inductive hypothesis, that \mathcal{U}_{n-1} has Shirshov height bounded by k' with respect to X . We claim that \mathcal{U}_n has Shirshov height bounded by

$$K = (M + 1)k' + 2k$$

where M is the constant in Scholium 2.2. Consider a geodesic word of the form

$$w \bar{\sqsubseteq} u_0 e_1 u_1 e_2 \dots e_m u_m$$

that is not in \mathcal{U}_2 . If u_0 is nonempty then $T(e_1 u_1 e_2 \dots e_m u_m)$ has one less leaf vertex than $T(w)$, so by the inductive hypothesis, w has height bounded by

$$k + k' \leq K .$$

Suppose then that

$$w \bar{\sqsubseteq} e_1 u_1 e_2 \dots e_m u_m$$

where e_1 is the (nonempty) label of a tree emanating from the initial vertex of $T(w)$. If $T(e_1)$ is a chain then u_1 and e_2 are nonempty (since $w \notin \mathcal{U}_2$), so $e_1 \bar{\sqsubseteq} g^{-1}g$ for some reduced word g such that gu_1 is reduced, and $T(e_2 u_2 \dots e_m u_m)$ has one fewer leaf vertex than $T(w)$, so, by the inductive hypothesis, w has height bounded by

$$2k + k' \leq K .$$

Suppose then that $T(e_1)$ is not a chain. There are two subcases. Suppose first that the degree of the initial vertex of the word tree $T(w)$ is $\delta > 2$, so that $e_1 \bar{\sqsubseteq} f_1 \dots f_{\delta-1}$ where $f_1, \dots, f_{\delta-1}$ label idempotents corresponding to branches emanating from the initial vertex. But $T(f_1), \dots, T(f_{\delta-1})$ and $T(u_1 e_2 \dots e_m u_m)$ have fewer than n leaf vertices, and $\delta - 1 \leq M$ by Scholium 2.2. Hence w has height bounded by

$$(\delta - 1)k' + k' \leq (M + 1)k' \leq K .$$

Finally, suppose the degree of the initial vertex is 2. Then

$$e_1 \bar{\sqsubseteq} z g_1 \dots g_{\ell-1} z^{-1}$$

for some reduced word z labelling a path from the initial vertex to a vertex of degree ℓ say, from which there are $\ell - 1$ branches corresponding to idempotents labelled by $g_1, \dots, g_{\ell-1}$. Now $T(g_1), \dots, T(g_{\ell-1})$ and $T(u_1 e_2 \dots e_m u_m)$ have fewer than n leaf vertices, so $w \bar{\sqsubseteq} z g_1 \dots g_{\ell-1} z^{-1} u_1 e_2 \dots e_m u_m$ has height bounded by

$$k + (\ell - 1)k' + k + k' \leq (M + 1)k' + 2k = K ,$$

and the proof of the theorem is complete.

We remark that the equivalence of (a) and (d) is also proved, by different means, in [9, Theorem 4.3.6].

As an application of the equivalence of (a) and (b) of the previous theorem, we prove Theorem 4.4 below, which may be regarded as an inverse semigroup analogue of the following classical result: consider a finitely presented semigroup Π with zero such that all relations have zero on the right hand side (that is, Π is a finitely presented Rees quotient of a free semigroup). Then Π has polynomial growth if and only if each nonzero element of Π that is not nilpotent can be represented by a word whose primitive root labels some loop of the Ufnarovsky graph of the presentation. This follows from [6] where the syntactic structure of the Jacobson radical was described (also see this description in [24, Section 7.6, Theorem 3]) and the trivial fact that Π embeds in an associative monomial algebra with the same set of generators and defining relations (see also [13, Chapter 24]).

THEOREM 4.4. *Let S belong to the class \mathfrak{M}_{FI} . The following conditions are equivalent:*

- (a) *S has polynomial growth.*
- (b) *If u is a word over B such that u is not nilpotent in S , then either u is idempotent, or $u \underline{\cong} x^{-1}xwyy^{-1}$ where $w \underline{\cong} \bar{u}$ is cyclically reduced, x is a suffix of w , y is a prefix of w and $w \underline{\cong} z^\alpha$ ($\alpha \geq 1$) for some primitive word z that labels a loop of the Ufnarovsky graph.*

PROOF. As before, the equivalence of the conditions is immediate if \bar{d} is zero or does not exist, so we may suppose $\bar{d} \geq 1$.

Suppose first that (a) holds and u is a word over B that is neither idempotent nor nilpotent in S . By replacing u by a sufficiently high power, if necessary, we may suppose $\ell(w) > \bar{d}$ where $w \underline{\cong} \bar{u}$. By [21, Lemma 5.2], w is a cyclically reduced word such that u lies in the closure (in the sense of Schein [17], see also [8] or [14]) of the inverse subsemigroup of FI_A generated by w . Replacing w by a sufficiently high power, if necessary, we may suppose that $u \underline{\cong} x^{-1}xwyy^{-1}$ where x is a suffix and y is a prefix of w . But there is no vertex of Γ_S contained in different cycles, since S has polynomial growth, so an equation of the form (3) holds:

$$w \underline{\cong} vP_1(x_1y_1)^{\alpha_1}x_1P_2(x_2y_2)^{\alpha_2}x_2 \dots P_m(x_my_m)^{\alpha_m}x_mP_{m+1},$$

where $c_j \underline{\cong} x_jy_j$ are distinct words labelling loops of Γ_S for $j = 1, \dots, m$. By Lemma 3.1, each c_j is primitive. But w is not nilpotent and no vertex of Γ_S is contained in different cycles, so it follows quickly that $m = 1$ and

$$w \underline{\cong} vP_1c_1^{\alpha_1} \underline{\cong} z^\alpha$$

for some positive integer $\alpha > \alpha_1$, where $z \underline{\cong} c_1$, completing the proof that (b) holds.

Suppose conversely that (b) holds. We first prove that no vertex of Γ_S lies in distinct cycles. Suppose to the contrary that Γ_S has a vertex v in two distinct cycles, which we may assume are labelled by words ux and wx respectively where x is the last letter of v and u and w begin with different letters. For any positive integers β and γ , consider the word

$$D_{\beta,\gamma} \stackrel{\text{def}}{=} (ux)^\beta (wx)^\gamma .$$

Then $D_{\beta,\gamma}$ labels a path in Γ_S , so certainly $D_{\beta,\gamma}$ is not nilpotent in S . But we may take one of β or γ sufficiently large so that $D_{\beta,\gamma}$ is not a power of a word of length less than or equal to the maximum length of all reduced words labelling loops of Γ_S . This violates condition (b). Hence no vertex of Γ_S lies in distinct cycles. By Theorem 4.2, the set of all reduced words that are nonzero in S has bounded height. But also by (b), every reduced but not cyclically reduced nonzero word in S must be nilpotent. Hence condition (b) of Theorem 4.3 holds, so that (a) holds, completing the proof of the theorem.

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