BMW ALGEBRA, QUANTIZED COORDINATE ALGEBRA AND
TYPE C SCHUR–WEYL DUALITY

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Abstract. We prove an integral version of the Schur–Weyl duality between the specialized Birman–Murakami–Wenzl algebra $B_n(-q^{2m+1}, q)$ and the quantum algebra associated to the symplectic Lie algebra $sp_{2m}$. In particular, we deduce that this Schur–Weyl duality holds over arbitrary (commutative) ground rings, which answers a question of Lehrer and Zhang ([38]) in the symplectic case. As a byproduct, we show that, as a $\mathbb{Z}[q, q^{-1}]$-algebra, the quantized coordinate algebra defined by Kashiwara in [34] (which was denoted by $A_{\mathbb{Z}[q]}$ there) is isomorphic to the quantized coordinate algebra arising from a generalized Faddeev–Reshetikhin–Takhtajan construction (see [23], [29], [47]).

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1. INTRODUCTION

Let $m, n \in \mathbb{N}$. Let $U(\mathfrak{gl}_m)$ be the universal enveloping algebra of the general linear Lie algebra $\mathfrak{gl}_m(\mathbb{C})$ over $\mathbb{Q}$. Let $U_{\mathbb{Z}}(\mathfrak{gl}_m)$ be the Kostant $\mathbb{Z}$-form ([37]) in $U(\mathfrak{gl}_m)$. For any commutative $\mathbb{Z}$-algebra $K$, let $U_K(\mathfrak{gl}_m) := U_{\mathbb{Z}}(\mathfrak{gl}_m) \otimes_{\mathbb{Z}} K$. The natural left action of $U_K(\mathfrak{gl}_m)$ on $(K^m)^{\otimes n}$ commutes with the right place permutation action of the symmetric group algebra $K \mathfrak{S}_n$. Let $\varphi_A, \psi_A$ be the natural representations

$$\varphi_A : (K \mathfrak{S}_n)^{\text{op}} \to \text{End}_K((K^m)^{\otimes n}), \quad \psi_A : U_K(\mathfrak{gl}_m) \to \text{End}_K((K^m)^{\otimes n}),$$

respectively. The well-known type $A$ Schur–Weyl duality (see [7], [9], [14], [26], [48], [49]) says that

(a) $\varphi_A(K \mathfrak{S}_n) = \text{End}_{U_K(\mathfrak{gl}_m)}((K^m)^{\otimes n})$,

(b) $\psi_A(U_K(\mathfrak{gl}_m)) = \text{End}_{K \mathfrak{S}_n}((K^m)^{\otimes n})$;

(c) if $K$ is an infinite field, then

$$\text{End}_{U_K(\mathfrak{gl}_m)}((K^m)^{\otimes n}) = \text{End}_{KGL_m(K)}((K^m)^{\otimes n}),$$

and the image of the group algebra $KGL_m(K)$ in $\text{End}_K((K^m)^{\otimes n})$ also coincides with $\text{End}_{K \mathfrak{S}_n}((K^m)^{\otimes n})$;

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(d) if $K$ is a field of characteristic 0, then there is an irreducible $U_K(\mathfrak{g}_m)$-$K\mathfrak{S}_n$-bimodule decomposition

$$(K^m)^{\otimes n} = \bigoplus_{\lambda=(\lambda_1, \lambda_2, \ldots) ; \ell(\lambda) \leq m} \Delta_\lambda \otimes S^\lambda,$$

where $\Delta_\lambda$ (resp., $S^\lambda$) denotes the irreducible left $U_K(\mathfrak{g}_m)$-module (resp., irreducible right $K\mathfrak{S}_n$-module) associated to $\lambda$, $\lambda \vdash n$ means $\lambda$ is a partition of $n$, and $\ell(\lambda)$ denotes the largest integer $i$ such that $\lambda_i \neq 0$.

There is a quantized version of the above type $A$ Schur–Weyl duality. Let $q$ be an indeterminate over $\mathbb{Z}$. Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ be the Laurent polynomial ring in $q$. Let $\mathcal{U}(\mathfrak{g}_m)$ be the quantized enveloping algebra of $\mathfrak{g}_m$ over $\mathbb{Q}(q)$ ([19], [32], [33]), where $q$ is the quantum parameter. Let $\mathcal{U}(\mathfrak{g}_m)$ be the Lusztig’s $\mathcal{A}$-form of $\mathbb{Q}(q)(\mathfrak{g}_m)$. Let $\mathcal{H}_\mathcal{A}(\mathfrak{S}_n)$ be the Iwahori–Hecke algebra associated to the symmetric group $\mathfrak{S}_n$, defined over $\mathcal{A}$ and with parameter $q$. By definition, $\mathcal{H}_\mathcal{A}(\mathfrak{S}_n)$ is generated by $\{T_i\}_{1 \leq i \leq n - 1}$ which satisfy the well-known braid relations as well as the relation $(T_i - q)(T_i + q^{-1}) = 0$, for $i = 1, 2, \ldots, n - 1$. For any commutative $\mathcal{A}$-algebra $K$, we use $\zeta$ to denote the natural image of $q$ in $K$, and we define $U_K(\mathfrak{g}_m) := \mathcal{U}(\mathfrak{g}_m) \otimes_\mathcal{A} K$, $\mathcal{H}_\mathcal{A}(\mathfrak{S}_n) := \mathcal{H}(\mathfrak{S}_n) \otimes_\mathcal{A} K$. Then, there is a left action of $U_K(\mathfrak{g}_m)$ on $\mathcal{H}(\mathfrak{S}_n)$ which quantizes the natural representation of $\mathfrak{g}_m(\mathbb{C})$. Via the coproduct, we get an action of $U_K(\mathfrak{g}_m)$ on $(\mathfrak{Q}(q)^m)^{\otimes n}$. Furthermore, this action actually gives rise to an action of $\mathcal{U}(\mathfrak{g}_m)$ on $(\mathfrak{Q}(q)^m)^{\otimes n}$ ([20]). By base change, we get an action of $U_K(\mathfrak{g}_m)$ on $(K^m)^{\otimes n}$ for any commutative $\mathcal{A}$-algebra $K$. There is also a right action of $\mathcal{H}_\mathcal{A}(\mathfrak{S}_n)$ on $(K^m)^{\otimes n}$. Let $\varphi_A, \psi_A$ be the natural representations

$$\varphi_A : (\mathcal{H}_\mathcal{A}(\mathfrak{S}_n))^\text{op} \to \text{End}_K((K^m)^{\otimes n}), \quad \psi_A : U_K(\mathfrak{g}_m) \to \text{End}_K((K^m)^{\otimes n}),$$

respectively. Then by [2], [20], [21] and [33],

(a') $\varphi_A(\mathcal{H}_\mathcal{A}(\mathfrak{S}_n)) = \text{End}_{U_K(\mathfrak{g}_m)}((K^m)^{\otimes n});$

(b') $\psi_A(U_K(\mathfrak{g}_m)) = \text{End}_{\mathcal{U}(\mathfrak{g}_m)}((K^m)^{\otimes n});$

(c') if $K$ is a field of characteristic 0 and $\zeta$ is not a root of unity in $K$, then there is an irreducible $U_K(\mathfrak{g}_m)$-$\mathcal{H}_\mathcal{A}(\mathfrak{S}_n)$-bimodules decomposition

$$(K^m)^{\otimes n} = \bigoplus_{\lambda=(\lambda_1, \lambda_2, \ldots) ; \ell(\lambda) \leq m} \Delta_\lambda \otimes S^\lambda,$$

where $\Delta_\lambda$ (resp., $S^\lambda$) denotes the irreducible left $U_K(\mathfrak{g}_m)$-module (resp., irreducible right $\mathcal{H}_\mathcal{A}(\mathfrak{S}_n)$-module) associated to $\lambda$.

The algebra $\text{End}_{\mathcal{U}(\mathfrak{g}_m)}((K^m)^{\otimes n})$ is called “$q$-Schur algebra”, which forms an important class of quasi-hereditary algebra and has been extensively studied by Dipper–James and many other people. It plays an important role in the modular representation theory of finite groups of Lie type (cf. [11], [12], [24]). The significance of the above results lies in that it provide a bridge between the representation theory of type $A$ quantum groups and of type $A$ Hecke algebras at an integral level. Note that in the semisimple case, the above Schur–Weyl duality follows easily from the complete reducibility. The difficult part lies in the non-semisimple case, where the surjectivity of $\varphi_A$ was established in [21] by making use of Kazhdan–Lusztig bases of type $A$ Hecke algebra, while the proof of the surjectivity of $\psi_A$ relies heavily on the amazing work of [2], where the quantized enveloping algebra of $\mathfrak{g}_m$ is realized as certain “limit” of $q$-Schur algebras. To the best of our knowledge, there is no alternative approach for this part.
A natural question arises: how about the Schur–Weyl dualities in other types? The answer is: there do exist Schur–Weyl dualities in types $B, C, D$ in semisimple case (for both classical and quantized versions). However, it is an open question (see [29, Page 80, Line 1], [38, Abstract]) whether or not these Schur–Weyl dualities hold in an integral or characteristic free setting (like the type $A$ situation).

The purpose of this paper is to give an affirmative answer to the above open question in the quantized type $C$ case. That is, we shall prove an integral version of quantized type $C$ Schur–Weyl duality. Note that there is no counterpart in type $C$ of the work [2] in the literature. It turns out that our approach provides a new and general framework to prove integral Schur–Weyl dualities for all classical types. Before stating the main results in this paper, we first recall the known results for the classical type $C$ Schur–Weyl duality. Let $K$ be an infinite field. Let $V$ be a $2m$-dimensional $K$-linear space equipped with a skew bilinear form $(, )$. Let $GSp(V)$ (resp., $Sp(V)$) be the symplectic similitude group (resp., the symplectic group) on $V$ ([15], [28]). For any integer $i$ with $1 \leq i \leq 2m$, set $v' = 2m + 1 - i$. We fix an ordered basis $\{v_1, v_2, \ldots, v_{2m}\}$ of $V$ such that

$$(v_i, v_j) = 0 = (v_i, v_j'), \quad (v_i, v_j') = \delta_{ij} = -(v_j', v_i), \quad \forall \ 1 \leq i, j \leq m.$$ 

Let $\mathfrak{B}_n(-2m)$ be the specialized Brauer algebra over $K$. This algebra contains the group algebra $K\mathfrak{S}_n$ as a subalgebra. There is a right action of $\mathfrak{B}_n(-2m)$ on $V^\otimes n$ which extends the sign permutation action of $\mathfrak{S}_n$. We refer the reader to [10] for definitions of $\mathfrak{B}_n(-2m)$ and its action. Let $\varphi_C, \psi_C$ be the natural representations

$$\varphi_C : (\mathfrak{B}_n(-2m))^\text{op} \to \text{End}_K(V^\otimes n), \quad \psi_C : KGSp(V) \to \text{End}_K(V^\otimes n),$$

respectively.

**Theorem 1.1.** ([3], [4], [5])

1. The natural left action of $GSp(V)$ on $V^\otimes n$ commutes with the right action of $\mathfrak{B}_n(-2m)$. Moreover, if $K = \mathbb{C}$, then

$$\varphi_C(\mathfrak{B}_n(-2m)) = \text{End}_{C GSp(V)}(V^\otimes n) = \text{End}_{C Sp(V)}(V^\otimes n),$$

$$\psi_C(CGSp(V)) = \psi_C(CSp(V)) = \text{End}_{\mathfrak{B}_n(-2m)}(V^\otimes n).$$

2. If $K = \mathbb{C}$, then there is an irreducible $CGSp(V) - \mathfrak{B}_n(-2m)$-bimodule decomposition

$$V^\otimes n = \bigoplus_{f=0}^{[n/2]} \bigoplus_{\lambda : \ell(\lambda) \leq m} \Delta(\lambda) \otimes D(\lambda'),$$

where $\Delta(\lambda)$ (resp., $D(\lambda')$) denotes the irreducible left $CGSp(V)$-module (resp., the irreducible right $\mathfrak{B}_n(-2m)$-module) corresponding to $\lambda$ (resp., $\lambda'$, corresponding to $\lambda'$), $\lambda'$ denotes the transpose of $\lambda$.

By the work of [9], [10] and [46], the complex field $\mathbb{C}$ used in part (1) of the above theorem can be replaced by arbitrary infinite field. That is, we have a characteristic free version of type $C$ Schur–Weyl duality in group case.

**Theorem 1.2.** ([9], [10], [46]) Let $K$ be an arbitrary infinite field. Then

1. $\psi_C(KGSp(V)) = \text{End}_{\mathfrak{B}_n(-2m)}(V^\otimes n)$;
2. $\varphi_C(\mathfrak{B}_n(-2m)) = \text{End}_{KGSp(V)}(V^\otimes n) = \text{End}_{KSp(V)}(V^\otimes n)$.

For the quantized type $C$ Schur–Weyl duality, we require $V$ to be a $2m$ dimensional vector space over $\mathbb{Q}(q)$ equipped with a skew bilinear form $(,)$. We fix an ordered basis $\{v_i\}_{i=1}^{2m}$ as before. Let $U_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})$ be the quantized enveloping algebra of $\mathfrak{sp}_{2m}(\mathbb{C})$ over $\mathbb{Q}(q)$, where $q$ is the quantum parameter. Let $\mathfrak{B}_n(-q^{2m+1}, q)$
be the specialized Birman–Murakami–Wenzl algebra (specialized BMW algebra for short) over \( \mathbb{Q}(q) \). There is a right action of \( \mathcal{B}_n(-q^{2m+1}, q) \) on \( V^\otimes n \) which quantizes the right action of \( \mathcal{B}_n(-2m) \). We refer the reader to Section 3 for precise definitions of \( \mathcal{B}_n(-q^{2m+1}, q) \) and its action. Let \( \varphi_C, \psi_C \) be the natural representations

\[
\varphi_C : (\mathcal{B}_n(-q^{2m+1}, q))^{op} \to \text{End}_{\mathbb{Q}(q)}(V^\otimes n),
\]

\[
\psi_C : \mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m}) \to \text{End}_{\mathbb{Q}(q)}(V^\otimes n),
\]

respectively.

**Theorem 1.3.** ([8, 10.2], [38])

1. The natural left action of \( \mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m}) \) on \( V^\otimes n \) commutes with the right action of \( \mathcal{B}_n(-q^{2m+1}, q) \). Moreover,

\[
\varphi_C(\mathcal{B}_n(-q^{2m+1}, q)) = \text{End}_{\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})}(V^\otimes n),
\]

\[
\psi_C(\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})) = \text{End}_{\mathcal{B}_n(-q^{2m+1}, q)}(V^\otimes n);
\]

2. there is an irreducible \( \mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m}) \cdot \mathcal{B}_n(-q^{2m+1}, q) \)-bimodule decomposition

\[
V^\otimes n = \bigoplus_{f=0}^{[n/2]} \bigoplus_{\ell(\lambda) \leq m} \Delta(\lambda) \otimes D(\lambda'),
\]

where \( \Delta(\lambda) \) (respectively, \( D(\lambda') \)) denotes the irreducible left \( \mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m}) \)-module (respectively, the irreducible right \( \mathcal{B}_n(-q^{2m+1}, q) \)-module) corresponding to \( \lambda \) (resp., corresponding to \( \lambda' \)).

Let \( \mathbb{U}_{\mathfrak{A}}(\mathfrak{sp}_{2m}) \) be the Lusztig’s \( \mathfrak{A} \)-form in \( \mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m}) \). Let \( V_{\mathfrak{A}} \) be the free \( \mathfrak{A} \)-module spanned by \( \{v_i\}_{i=1}^{2m} \). Note that \( \mathcal{B}_n(-q^{2m+1}, q) \) has a natural \( \mathfrak{A} \)-form \( \mathcal{B}_n(-q^{2m+1}, q)_{\mathfrak{A}} \). For any commutative \( \mathfrak{A} \)-algebra \( K \), let \( \zeta \) be the natural image of \( q \) in \( K \), and we define \( \mathbb{U}_K(\mathfrak{sp}_{2m}) := \mathbb{U}_{\mathfrak{A}}(\mathfrak{sp}_{2m}) \otimes_{\mathfrak{A}} K \), \( \mathcal{B}_n(-q^{2m+1}, \zeta) := \mathcal{B}_n(-q^{2m+1}, q)_{\mathfrak{A}} \otimes_{\mathfrak{A}} K \). The representation \( \psi_C \) naturally gives rise to an action of \( \mathbb{U}_{\mathfrak{A}}(\mathfrak{sp}_{2m}) \) on \( V^\otimes K \) which commutes with the right action of \( \mathcal{B}_n(-q^{2m+1}, \zeta) \). By base change, for any commutative \( \mathfrak{A} \)-algebra \( K \), we get an action of \( \mathbb{U}_K(\mathfrak{sp}_{2m}) \) on \( V^\otimes K \) which commutes with the right action of \( \mathcal{B}_n(-\zeta^{2m+1}, \zeta) \).

The main results in this paper are the following two theorems.

**Theorem 1.4.** For any commutative \( \mathfrak{A} \)-algebra \( K \),

\[
\psi_C(\mathbb{U}_K(\mathfrak{sp}_{2m})) = \text{End}_{\mathcal{B}_n(-\zeta^{2m+1}, \zeta)}(V^\otimes K).
\]

**Theorem 1.5.** For any commutative \( \mathfrak{A} \)-algebra \( K \),

\[
\varphi_C(\mathcal{B}_n(-\zeta^{2m+1}, \zeta)) = \text{End}_{\mathbb{U}_K(\mathfrak{sp}_{2m})}(V^\otimes K).
\]

Note that if we specialize the parameter \( q \) to 1 \( K \in \mathbb{E} \), then the BMW algebra \( \mathcal{B}_n(-q^{2m+1}, q) \) becomes the specialized Brauer algebra \( \mathcal{B}_n(-2m) \), and the action of \( \mathcal{B}_n(-q^{2m+1}, q) \) on \( n \)-tensor space becomes the action of \( \mathcal{B}_n(-2m) \) used in [10]. Applying the above two theorem, we get the following corollary.

**Corollary 1.6.** For any commutative \( \mathbb{E} \)-algebra \( K \),

1. \( \psi_C(\mathbb{U}_K(\mathfrak{sp}_{2m})) = \text{End}_{\mathcal{B}_n(-2m)_K}(V^\otimes K) \);
2. \( \varphi_C(\mathcal{B}_n(-2m)_K) = \text{End}_{\mathbb{U}_K(\mathfrak{sp}_{2m})}(V^\otimes K) \).
Note that this corollary can also be deduced from the main result in [10] by using the equivalence between the category of rational $Sp_{2m}(K)$-modules and the category of locally finite $U_K(\mathfrak{sp}_{2m})$-modules.

The algebra $S_K^{sp}(2m, n) := \text{End}_{\mathbb{C}}(\psi_{2m+1, \zeta})(V_K^{\otimes n})$ is called “symplectic $\zeta$-Schur algebra” by Oehms ([47]). It is a cellular (in the sense of [25]) and quasi-hereditary $K$-algebra. The strategy that we use to prove Theorem 1.4 is to inspect the induced natural homomorphism $\psi_{C}$ from Lusztig’s modified quantum algebra (see [41]) to the symplectic $q$-Schur algebra $S_K^{sp}(2m, n)$, and (roughly speaking) to interpret $\psi_{C}$ as the dual of the natural map from the $n$th homogeneous component of the quantized coordinate algebra of $Sp_{2m}(K)$ (symplectic monoid) to the quantized coordinate algebra of $Sp_{2m}(K)$ (symplectic group). It turns out that the kernel of $\psi_{C}$ is spanned by the canonical basis elements it contains. As a consequence, we deduce the following result, which is announced in [16] without proof.

**Corollary 1.7.** For any commutative $\mathcal{S}$-algebra $K$, $S_K^{sp}(2m, n)$ is isomorphic to the generalized $q$-Schur algebra $K\mathcal{S}(\pi)$ defined in [16], where $\pi$ is the set of dominant weights occurring in $V^{\otimes n}$. In particular, if specializing $q$ to 1, then we recover the symplectic Schur algebra studied in [13] and [15].

The strategy that we use to prove Theorem 1.5 is similar to that used in [10]. We first prove the equality under the assumption that $m \geq n$. Then we reduce the case $m < n$ to the case $m = n$ via a commutative diagram. Finally, we convert the task of proving the equality concerning $\psi_{C}$ to a purely type $C$ quantum algebra representation theoretic problem which involves no BMW algebras. However, the direct generalization from [10] does not work here. In our quantized case the problem is much more difficult. We expect that our approach for both equalities can be applied to prove integral versions of various other Schur–Howe–Weyl dualities in Lie theory.

The paper is organized as follows. In Section 2, we collect some basic knowledge about the usual and the modified form of the quantized enveloping algebra of $\mathfrak{osp}_{2m}(C)$ as well as their actions on the $n$-tensor space $V^{\otimes n}$. The new result is Lemma 2.3, which enables us to reduce the proof of the equality concerning $\psi_{C}$ to the proof of an equality concerning $\tilde{\psi}_{C}$. In Section 3, we show that each finite truncation $A^{sp}_{\mathfrak{g}}(2m, \leq n)$ of the quantized coordinate algebra $A^{sp}_{\mathfrak{g}}(2m)$ of $Sp_{2m}(K)$ is a cellular coalgebra. The two-sided simple comodule decomposition of the quantized coordinate algebra $A^{sp}_{\mathfrak{g}}(2m)$ of $Sp_{2m}(K)$ is obtained, which actually coincides with Peter–Weyl’s decomposition proved by Kashiwara ([34]). In Section 4, after proving that the tensor product $\nabla_K^\mathfrak{g}(\lambda) \otimes \nabla_K(\lambda)$ of a cell left comodule and a cell right comodule of $A^{sp}_{\mathfrak{g}}(2m, \leq n)$ is actually a co-Weyl module of the quantum algebra $U_K(\mathfrak{g} \oplus \mathfrak{g})$ (Lemma 4.13), we are able to identify the type $C$ quantized coordinate algebra $A^{sp}_{\mathfrak{g}}(\mathfrak{g})$ defined by Kashiwara in [34] with the quantized coordinate algebra $A^{sp}_{\mathfrak{g}}(2m)$ arising from generalized FRT construction. The proof relies on some nice properties of the upper global crystal basis (i.e., the dual canonical basis) of the quantized coordinate algebra $A^{sp}_{\mathfrak{g}}(\mathfrak{g})$ introduced by Kashiwara. Then we give a proof of our first main result—Theorem 1.4. In Section 5, we give a proof of our second main result—Theorem 1.5 in the case where $m \geq n$, following a similar idea (but different and more difficult arguments than) in [10, Section 3]. The case where $m < n$ is dealt with in Section 6. We reduce the proof of Theorem 1.5 to the proof of the surjectivity of a map between coinvariants of two tensor spaces and the commutativity of a certain diagram of maps (Lemma 6.1). For the former, we use Lusztig’s theory on based modules ([41]) as in [10, Section 4]. The proof of the
latter turns out to be quite delicate and more interesting than in the classical case.

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2. TENSOR SPACE AND MODIFIED QUANTIZED ENVELOPING ALGEBRA OF TYPE C

In this section we shall recall some basic facts about the usual and the modified form of the quantized enveloping algebra of type C as well as their actions on the \( n \)-tensor space \( V \otimes^n \). We shall show that, for any commutative \( \mathcal{A} \)-algebra \( K \), the image of the quantum algebra \( \mathbb{U}_K(\mathfrak{sp}_{2m}) \) in the endomorphism algebra of the \( n \)-tensor space \( V^\otimes_K \) coincides with the image of the corresponding modified quantum algebra \( \mathbb{U}_K(\mathfrak{sp}_{2m}) \).

Recall the Dynkin diagram of \( \mathfrak{sp}_{2m} \):

![Dynkin diagram](image)

where each vertex labeled by \( i \) represents a simple root \( \alpha_i \). For each integer \( i \) with \( 1 \leq i \leq m \), let \( \alpha_i^\vee \) be the corresponding simple co-root. Let \( X \) be the weight lattice of \( \mathfrak{sp}_{2m} \). We realize \( X \) as a free \( \mathbb{Z} \)-module with basis \( \varepsilon_1, \ldots, \varepsilon_m \). Then \( \alpha_m = 2\varepsilon_m, \alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, 2, \ldots, m-1 \).

Throughout this paper, we shall identify the weight \( \lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_m \varepsilon_m \in X \) with the sequence \((\lambda_1, \ldots, \lambda_m)\) of integers. We shall also write \( \lambda = (\lambda_1, \cdots, \lambda_m) \) and \( |\lambda| := \lambda_1 + \cdots + \lambda_m \). Let

\[
X^+ := \{ \lambda \in X | (\lambda, \alpha_i) \geq 0, \forall 1 \leq i \leq m \},
\]

i.e., the set of dominant weights. Then \( \lambda = (\lambda_1, \cdots, \lambda_m) \in X^+ \) if and only if \( \lambda \) is a partition. Let \((,\) be the symmetric bilinear form \( X \) defined by \((\varepsilon_i, \varepsilon_j) = \delta_{i,j}\) for any \( i, j \). The Cartan matrix \( A = (a_{i,j}) \) (where \( a_{i,j} := 2(\alpha_j, \alpha_i)/(|\alpha_i|) \)) of \( \mathfrak{sp}_{2m} \) is:

\[
\begin{pmatrix}
2 & -1 & & & & \\
-1 & 2 & -1 & & & \\
& -1 & \ddots & \ddots & & \\
& & \ddots & 2 & -2 & \\
& & & -1 & 2 & \\
& & & & & \\
\end{pmatrix}
\]

with rows and columns indexed by \( \{1, \ldots, m\} \). Given a fixed indeterminate \( q \) set

\[
q_i = \begin{cases} q, & \text{if } i \neq m \\ q^2, & \text{if } i = m \end{cases},
\]

\[
[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}} \quad \text{and} \quad [k]_i^1 = [k]_i[k - 1]_i \cdots [1]_i.
\]
The quantized enveloping algebra \( U_Q(\mathfrak{g}) := U_Q(\mathfrak{sp}_{2m}) \) is the associative unital algebra over \( \mathbb{Q} \) generated by \( e_i, f_i, k_i, k_i^{-1} \) \( i = 1, \ldots, m \), subject to the relations:

\[
\begin{align*}
    k_i k_i^{-1} &= k_i^{-1} k_i = 1, \\
    k_i^\pm 1 k_j^\pm 1 &= k_j^\pm 1 k_i^\pm 1, \\
    k_i e_j k_i^{-1} &= q^{\delta_{i,j}} e_j, \\
    k_i f_j k_i^{-1} &= q^{-\delta_{i,j}} f_j,
\end{align*}
\]

where \( e_i = e_i^k / [k]^2 \) and \( f_i = f_i^k / [k]^2 \).

\( U_Q(\mathfrak{g}) \) is a Hopf algebra with coproduct \( \Delta \), counit \( \varepsilon \) and antipode \( S \) defined on generators by:

\[
\begin{align*}
    \Delta(e_i) &= e_i \otimes 1 + 1 \otimes e_i, \\
    \Delta(f_i) &= 1 \otimes f_i + f_i \otimes k_i^{-1}, \\
    \Delta(k_i) &= k_i \otimes k_i, \\
    \varepsilon(e_i) &= \varepsilon(f_i) = 0, \\
    \varepsilon(k_i) &= 1, \\
    S(e_i) &= -k_i^{-1} e_i, \\
    S(f_i) &= -f_i k_i, \\
    S(k_i) &= k_i^{-1}.
\end{align*}
\]

Recall our definition of \( V_{\mathfrak{g}} \) in the first paragraph below Theorem 1.3. For each \( 1 \leq i \leq 2m \), let \( \nu' := 2m + 1 - i \). The action of the generators of \( U_Q(\mathfrak{g}) \) on \( V_{\mathfrak{g}} := V_{\mathfrak{g}} \otimes U_Q(\mathfrak{g}) \) is as follows (cf. [29, (4.16)])^1:

\[
\begin{align*}
    e_i v_j &= \begin{cases} v_j, & \text{if } j = i + 1, \\
    -v_{(i+1)\nu'}, & \text{if } j = \nu', \\
    v_m, & \text{if } j = m', \\
    0, & \text{otherwise}, \end{cases} \\
    e_m v_j &= \begin{cases} v_j, & \text{if } j = \nu', \\
    0, & \text{otherwise}, \end{cases} \\
    f_i v_j &= \begin{cases} v_{i+1}, & \text{if } j = i, \\
    -v_{\nu'}, & \text{if } j = (i+1)', \\
    v_m, & \text{if } j = m, \\
    0, & \text{otherwise}, \end{cases} \\
    f_{m} v_j &= \begin{cases} v_{\nu'}, & \text{if } j = \nu', \\
    0, & \text{otherwise}, \end{cases} \\
    k_i v_j &= \begin{cases} q v_j, & \text{if } j = i \text{ or } j = (i+1)\nu', \\
    q^{-1} v_j, & \text{if } j = i + 1 \text{ or } j = \nu', \\
    v_j, & \text{otherwise}, \end{cases} \\
    k_m v_j &= \begin{cases} q v_j, & \text{if } j = m, \\
    q^{-1} v_j, & \text{if } j = m', \\
    v_j, & \text{otherwise}, \end{cases}
\end{align*}
\]

where \( 1 \leq i < m \), \( j \in \{1, \ldots, m\} \cup \{m', \ldots, \nu'\} \). Via the coproduct, we get an action of \( U_Q(\mathfrak{g}) \) on \( V_{\mathfrak{g}}^\otimes n \). Let \( U_{\mathfrak{g}} := U_{\mathfrak{g}}(\mathfrak{sp}_{2m}) \) be the Lusztig’s \( \mathfrak{g} \)-form in \( U_Q(\mathfrak{sp}_{2m}) \). As an \( \mathfrak{g} \)-algebra, \( U_{\mathfrak{g}} \) is generated by:

\[
\begin{align*}
    e_i^{(a)}(k), f_i^{(a)}(k), k_i, k_i^{-1}, a = 0, 1, 2, \ldots, 1 \leq i \leq m.
\end{align*}
\]

**Lemma 2.1.** The above action of \( U_Q(\mathfrak{g}) \) on \( V_{\mathfrak{g}}^\otimes n \) naturally gives rise to an action of \( U_{\mathfrak{g}} \) on \( V_{\mathfrak{g}}^\otimes n \).

**Proof.** It is well-known that \( U_{\mathfrak{g}} \) is an \( \mathfrak{g} \)-Hopf algebra. Hence it suffices to show that \( U_{\mathfrak{g}} \circ V_{\mathfrak{g}} \subseteq V_{\mathfrak{g}} \). However, this follows from direct verification. \( \square \)

---

^1Note that our \( \overline{k}_m = k_m^2 \) in this paper corresponds to \( k_m \) in the notation of [29].
For any commutative $\mathcal{A}$-algebra $K$, we define $U_K = U_K(\mathfrak{sp}_m) := U_{\mathcal{A}} \otimes_{\mathcal{A}} K$.

By base change, we see that there is a representation

$$\psi_C : U_K \to \text{End}_K(V_K^{\otimes n}).$$

In Lusztig’s book [41, Part IV], a “modified form” $\hat{U}_Q(q)$ of $U_Q(q)$ was introduced. The algebra $\hat{U}_Q(q)$ in general does not have a unit element. But it does have a family $\{1_\mu\}_{\mu \in X}$ of orthogonal idempotents such that $\hat{U}_Q(q) = \oplus_{\lambda, \mu \in X} \hat{U}_Q(q)_1 \lambda \mu$. In a sense, the family $\{1_\lambda\}$ serves as a replacement for the identity. Let $\pi_{\lambda, \mu}$ be the canonical projection from $\hat{U}_Q(q)$ onto $1_\lambda \hat{U}_Q(q)_1 \mu$ (see [41, (23.1.1)]). As a $Q(q)$-algebra, $\hat{U}_Q(q)$ is generated by the elements $e_i, f_i, 1_\lambda$ and $1_\lambda$ with $i \in \{1, 2, \cdots, m\}$ and $\lambda \in X$, where the following relations are satisfied:

$$1_\lambda 1_\mu = 1_\mu 1_\lambda = \delta_{\lambda, \mu} 1_\lambda, \quad e_i 1_\lambda = 1_\lambda + \alpha_i (e_i 1_\lambda), \quad f_i 1_\lambda = 1_\lambda - \alpha_i (f_i 1_\lambda),$$

$$(e_i 1_\lambda - \alpha_i) (f_j 1_\lambda) = (f_j 1_\lambda + \alpha_i) (e_i 1_\lambda) - \delta_{i,j} (\lambda, \alpha_j) 1_\lambda,$$

$$\sum_{k=0}^{1-\delta_{i,j}} (-1)^k e_i^{(1-\delta_{i,j}-k)} e_j^{(1-\delta_{i,j}-k)} = \sum_{k=0}^{1-\delta_{i,j}} (-1)^k f_i^{(k)} f_j^{(k)} = 0 \quad \text{if} \quad i \neq j,$$

where $(\lambda, \alpha_j) := 2(\lambda, \alpha_j)/\langle \alpha_j, \alpha_j \rangle$, and the last identity is understood as its canonical image under $\pi_{\lambda, \mu}$ for any $\lambda, \mu \in X$.

Let $\hat{U}_\lambda$ be the $\mathcal{A}$-subalgebra of $\hat{U}_Q(q)$ generated by $e_i^{(k)} 1_\lambda, f_i^{(k)} 1_\lambda$ for $i = 1, 2, \cdots, m, \ k = 0, 1, 2, \cdots, \lambda \in X$. Then by [41, (23.2)], $\hat{U}_\lambda$ is a free $\mathcal{A}$-module, and in fact $\hat{U}_\lambda$ is an $\mathcal{A}$-form of $\hat{U}_Q(q)$.

Since $V_Q^{\otimes n}$ is a finite dimensional integrable module over $U_Q(q)$, it follows that $V_Q^{\otimes n}$ naturally becomes a unital $U_Q(q)$-module in the sense of [41, (23.1.4)]. For each $\lambda \in X$, we define $p_\lambda$ to be the projection operator from $V_Q^{\otimes n}$ onto its $\lambda$-weight space (with respect to the subalgebra generated by $k^1_{i=1}, \cdots, k^1_{m-1}, k^2_{m}$).

**Lemma 2.2.** Let $\tilde{\psi}_C$ be the map

$$1_\lambda \mapsto p_\lambda, \quad e_i 1_\lambda \mapsto \psi_C(e_i) p_\lambda, \quad f_i 1_\lambda \mapsto \psi_C(f_i) p_\lambda, \quad i = 1, 2, \cdots, m, \ \lambda \in X.$$

Then $\tilde{\psi}_C$ can be naturally extended to a representation of $\hat{U}_Q(q)$ on $V_Q^{\otimes n}$ such that

$$\tilde{\psi}_C(P 1_\lambda) = \psi_C(P) p_\lambda \quad \text{for any} \quad P \in U_Q(q) \quad \text{and} \quad \lambda \in X.$$

**Proof.** This follows directly from the definition of $\hat{U}_Q(q)$ and the fact that $V_Q^{\otimes n}$ is a direct sum of its weight spaces.

By restriction and applying Lemma 2.1, we see that $V_Q^{\otimes n}$ naturally becomes an $\hat{U}_\lambda$-module. For any commutative $\mathcal{A}$-algebra $K$, we define $\hat{U}_K = \hat{U}_K(\mathfrak{sp}_m) := \hat{U}_{\mathcal{A}} \otimes_{\mathcal{A}} K$. By base change, we get a representation

$$\tilde{\psi}_C : \hat{U}_K \to \text{End}_K(V_K^{\otimes n}).$$

**Lemma 2.3.** With the above notation, for any commutative $\mathcal{A}$-algebra $K$,

$$\psi_C(U_K) = \tilde{\psi}_C(\hat{U}_K).$$

**Proof.** It suffices to show that $\psi_C(U_{\mathcal{A}}) = \tilde{\psi}_C(\hat{U}_{\mathcal{A}})$. Let $X_n$ be the set of weights (with respect to the Cartan part of $\mathfrak{sp}_m$) in $V_Q^{\otimes n}$. Obviously, $X_n$ is a finite set. As linear operators on $V_Q^{\otimes n}$, it is easy to check that
for $i = 1, 2, \cdots, m$ and $a = 0, 1, 2, \cdots$,

\[
\psi_C(e_i^{(a)}) = \tilde{\psi}_C \left( \sum_{\lambda \in \Lambda} e_i^{(a)} 1_\lambda \right), \quad \psi_C(f_i^{(a)}) = \tilde{\psi}_C \left( \sum_{\lambda \in \Lambda} f_i^{(a)} 1_\lambda \right), \quad \psi_C(k_i) = \tilde{\psi}_C \left( \sum_{\lambda \in \Lambda} q^{(\lambda, \alpha_i^\vee)} 1_\lambda \right).
\]

As a result, we deduce that $\tilde{\psi}_C \left( U_{\mathfrak{sl}_2} \right) \subseteq \tilde{\psi}_C \left( \hat{U}_{\mathfrak{sl}_2} \right)$. It remains to show that $\tilde{\psi}_C \left( \hat{U}_{\mathfrak{sl}_2} \right) \subseteq \psi_C \left( U_{\mathfrak{sl}_2} \right)$.

It suffices to show that $p_\lambda = \tilde{\psi}_C (1_\lambda) \in \psi_C \left( U_{\mathfrak{sl}_2} \right)$ for each $\lambda \in \Lambda$. For each integer $i$ with $1 \leq i \leq m$, we set (following Lusztig)

\[
\begin{bmatrix}
K_i; c \\ t
\end{bmatrix} = \prod_{s=1}^t \frac{k_i q_i^{c-s+1} - k_i^{-1} t_i q_i^{-c-s-1}}{q_i^s - q_i^{-s}}, \quad \begin{bmatrix}
K_i; c \\ 0
\end{bmatrix} = 1,
\]

for $t \geq 1$, $c \in \mathbb{Z}$. By [39, Lemma 4.4], we know that \[
\begin{bmatrix}
K_i; c \\ t
\end{bmatrix} \in \mathbb{U}_{\mathfrak{sl}_2}.
\]

Let $\lambda \in \Lambda$. We write $\hat{\lambda}_i := (\lambda, \alpha_i^\vee)$ for each $i$. Note that for each $i$,

\[
\hat{\lambda}_i \in \begin{cases}
\{-n, -n+1, \cdots, -1, 0, 1, \cdots, n-1, n\}, & \text{if } i \neq m, \\
\{-2n, -2n+2, \cdots, -2, 0, 2, \cdots, 2n-2, 2n\}, & \text{if } i = m.
\end{cases}
\]

We define

\[
p_\lambda' := \left( \prod_{i=1}^{m-1} \begin{bmatrix}
K_i; -\hat{\lambda}_i - 1 \\ 2n
\end{bmatrix} \begin{bmatrix}
K_i; -\hat{\lambda}_i + 1 \\ 2n
\end{bmatrix} \times 
\begin{bmatrix}
K_m; -\hat{\lambda}_m - 1 \\ 4n
\end{bmatrix} \begin{bmatrix}
K_m; -\hat{\lambda}_m + 4n \\ 4n
\end{bmatrix} \right).
\]

Clearly, $p_\lambda' \in \mathbb{U}_{\mathfrak{sl}_2}$. For each $\mu \in \Lambda$, we use $V_{\mathfrak{sl}_2}^{\otimes n}[\mu]$ to denote the $\mu$-weight space of $V_{\mathfrak{sl}_2}^{\otimes n}$. One can verify directly that

\[
\psi_C(p_\lambda')(x) = \begin{cases}
x, & \text{if } x \in V_{\mathfrak{sl}_2}^{\otimes n}[\lambda], \\
0, & \text{if } x \in V_{\mathfrak{sl}_2}^{\otimes n}[\mu] \text{ with } \mu \neq \lambda.
\end{cases}
\]

As a result, we deduce that $p_\lambda = \tilde{\psi}_C (1_\lambda) = \psi_C(p_\lambda') \in \psi_C \left( U_{\mathfrak{sl}_2} \right)$, as required. This completes the proof of the lemma.

\[\square\]

Remark 2.4. Lemma 2.3 enables use to reduce the proof of the equality concerning $\psi_C$ to the proof of the equality concerning $\tilde{\psi}_C$. Note that the arguments used in the proof of Lemma 2.3 actually work in all types.

3. BMW algebras and a generalized FRT construction

In this section we shall first recall the definitions of specialized BMW algebras and Oehms's results on a generalized Faddeev–Reshetikhin–Takhtajan (FRT for short) construction. Then we shall analyze the structure of each finite truncation $A_{\mathfrak{sl}_2}^n(2m \leq n)$ of the quantized coordinate algebra $A_{\mathfrak{sl}_2}^n(2m)$ of $\text{SpM}_{2m}$. Using Oehms's symplectic bideterminant basis for the quantized coordinate algebra of
symplectic monoid scheme $\text{SpM}_{2m}$, we shall conclude that $A^q_{\text{SpM}}(2m, \leq n)$ is a cellular coalgebra. The two-sided simple comodule decomposition of the quantized coordinate algebra $A^q_{\mathbb{Q}(q)}(2m)$ is obtained.

Let $x, r, z$ be three indeterminates over $\mathbb{Z}$. Let $R$ be the ring

$$R := \mathbb{Z}[r, r^{-1}, z, x]/(1 - x)z + (r - r^{-1})].$$

For simplicity, we shall use the same letters $r, r^{-1}, z, x$ to denote their images in $R$ respectively.

**Definition 3.1.** ([6], [43], [44]) The Birman–Murakami–Wenzl algebra (or BMW algebra for short) $\mathcal{B}_n(r, x, z)$ is a unital associative $R$-algebra with generators $T_i, E_i, 1 \leq i \leq n - 1$ and relations

1. $T_i - T_i^{-1} = x(1 - E_i)$, for $1 \leq i \leq n - 1$,
2. $E_i^2 = xE_i$, for $1 \leq i \leq n - 1$,
3. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, for $1 \leq i \leq n - 2$,
4. $T_i T_j = T_j T_i$, for $|i - j| > 1$,
5. $E_i E_{i+1} E_i = E_{i+1}$, $E_{i+1} E_i E_{i+1} = E_{i+1}$, for $1 \leq i \leq n - 2$,
6. $T_i T_{i+1} E_i = E_{i+1} E_i$, $T_{i+1} T_i E_{i+1} = E_i E_{i+1}$, for $1 \leq i \leq n - 2$,
7. $T_i E_i = T_i E_i - r^{-1} E_i$, for $1 \leq i \leq n - 1$.
8. $E_i T_{i+1} E_i = r E_i$, $E_{i+1} T_i E_{i+1} = r E_{i+1}$, for $1 \leq i \leq n - 2$.

In [43], Morton and Wassermann proved that $\mathcal{B}_n(r, x, z)$ is isomorphic to the Kauffman’s tangle algebra [36] whose $R$-basis is indexed by Brauer diagrams. As a consequence, they show that $\mathcal{B}_n(r, x, z)$ is a free $R$-module with rank $(2n - 1)!$. In fact, the same is still true if one replaces the ring $R$ by any commutative $R$-algebra $K$. Note that if we specialize $r$ to 1 and $z$ to 0, then $\mathcal{B}_n(r, x, z)$ will become the usual Brauer algebra with parameter $x$.

We regard $\mathcal{T}$ as an $R$-algebra by sending $r$ to $-q^{2m+1}$, $z$ to $q^{-1}$ and $x$ to $1 - \sum_{i=1}^m q^{2i}$. The resulting $\mathcal{T}$-algebra will be denoted by $\mathcal{B}_n(-q^{2m+1}, q)_\mathcal{T}$. We set $\mathcal{B}_n(-q^{2m+1}, q) = \mathcal{B}_n(-q^{2m+1}, q)_\mathcal{T}$ and we call it the specialized Birman–Murakami–Wenzl algebra (or specialized BMW algebra for short). Note that if we specialize further $q$ to 1, then $\mathcal{B}_n(-q^{2m+1}, q)$ will become the specialized Brauer algebra $\mathcal{B}_n(-2m)$ used in [10] and [31].

There is an action of the algebra $\mathcal{B}_n(-q^{2m+1}, q)_\mathcal{T}$ on the $n$-tensor space $V^{\otimes n}$ which we now recall. We set

$$(\rho_1, \ldots, \rho_{2m}) := (m, m - 1, \ldots, 1, -1, \ldots, -m, 1 - m),$$

and $\epsilon_i := \text{sign}(\rho_i)$. For any $i, j \in \{1, 2, \ldots, 2m\}$, we use $E_{i,j}$ to denote the corresponding basis of matrix units for $\text{End}_{\mathbb{Q}(q)}(V_q)$. Let “$-”$ be the ring involution defined on $\mathcal{T}$ by $q^{-1} = q_{\mathcal{T}}$, $k = k$ for any $k \in \mathbb{Z}$. The involution “$-”$ can be uniquely extended to an involution of $\text{End}_{\mathcal{T}}(V^{\otimes 2})$ such that

$$E_{i,j} \otimes E_{k,l} = E_{i,j} \otimes E_{k,l}, \quad r r = r r,$$

for any integers $1 \leq i, j, k, l \leq 2m$, any $r \in \mathcal{T}$ and any $x \in \text{End}_{\mathcal{T}}(V^{\otimes 2})$. Following [47, Section 2], we set

$$\beta := \sum_{1 \leq i, j \leq 2m} \left( q^2 E_{i,j} \otimes E_{i,j} + E_{i,j'} \otimes E_{i,j'} \right) + q \sum_{1 \leq i, j \leq 2m, i \neq j} E_{i,j} \otimes E_{j,i} +$$

$$+ (q^2 - 1) \sum_{1 \leq j, k \leq 2m} \left( E_{i,j} \otimes E_{j,k} - q^{\rho_j - \rho_i} \epsilon_i \epsilon_j E_{i,j} \otimes E_{j,k} \right),$$

$$\gamma := \sum_{1 \leq i, j \leq 2m} q^{\rho_i - \rho_j} \epsilon_i \epsilon_j E_{i,j} \otimes E_{j,k}.$$
We define $\beta' := q^{3/4}$, $\gamma' := \tau$. By direct verification, we have that
\[
\beta' := \sum_{1 \leq i \leq 2m} \left( qE_{i,i} \otimes E_{i,i} + q^{-1}E_{i,i'} \otimes E_{i,i'} \right) + \sum_{1 \leq i,j \leq 2m, i \neq j} E_{i,j} \otimes E_{j,i} + (q - q^{-1}) \sum_{1 \leq i < j \leq 2m} \left( E_{i,i} \otimes E_{j,j} - q^{i_j} \epsilon_i \epsilon_j E_{i,j} \otimes E_{i,j} \right),
\]
\[
\gamma' := \sum_{1 \leq i,j \leq 2m} q^{i_j} \epsilon_i \epsilon_j E_{i,j} \otimes E_{i,j}.
\]

Note that the operators\(^2\) $\beta', \gamma'$ are related to each other by the equation
\[
\beta' - (\beta')^{-1} = (q - q^{-1})(id_{V^\otimes 2} - \gamma').
\]

For $i = 1, 2, \ldots, n - 1$, we set
\[
\beta_i' := id_{V^\otimes i-1} \otimes \beta' \otimes id_{V^\otimes n-i-1}, \quad \gamma_i' := id_{V^\otimes i-1} \otimes \gamma' \otimes id_{V^\otimes n-i-1}.
\]

By [8, (10.2.5)] and [29, Section 4], the map $\varphi_C$ which sends each $T_i$ to $\beta_i'$ and each $E_{i,j}$ to $\gamma_i'$ for $i = 1, 2, \ldots, n - 1$ can be naturally extended to a right action of $B_n(-q^{2m+1}, q)$ on $V_{ad}^{\otimes n}$ such that all the statements in Theorem 1.3 hold. Note that if we specialize $q$ to 1, then $T_i$ degenerates to $-s_i$ in $B_n(-2m)$ for each $i$, and this action of $B_n(-q^{2m+1}, q)$ becomes the action studied in [10] of the specialized Brauer algebra $B_n(-2m)$ on $V_{ad}^{\otimes n}$.

Now we recall what Oehms called “generalized Faddeev–Reshetikhin–Takhtajan construction”. The basic references are [23], [29] and [47]. We concentrate on the quantized type $C$ case. Let $u_i, 1 \leq i \leq 2m$, (resp., $X_{i,j}$, $1 \leq i, j \leq 2m$) be a basis of $V^*$ (resp., of $V^* \otimes V$) satisfying $u_i(v_j) = \delta_{i,j}$ (resp., $X_{i,j} = u_i \otimes v_j$) for each $i, j$. Set
\[
I(2m, n) := \{(i_1, \ldots, i_n) \mid i_j \in \{1, 2, \ldots, 2m\} \text{ for each } j\}.
\]

For each $i = (i_1, \ldots, i_n) \in I(2m, n)$, we set $v_i := v_{i_1} \otimes \cdots \otimes v_{i_n}$. An endomorphism $\mu \in \text{End}(V_{ad}^{\otimes n})$ is uniquely determined by its coefficient $\mu_{i,j}$ with respect to the basis $\{v_i\}_{i \in I(2m, n)}$, that is,
\[
\mu(v_j) = \sum_{i \in I(2m, n)} \mu_{i,j} v_i.
\]

For any commutative $\mathcal{A}$-algebra $K$, we use $F_K(2m)$ to denote the tensor algebra over $V^* \otimes V$, which can be identified with the free $K$-algebra generated by the $(2m)^2$ symbols $X_{i,j}$ for $i,j \in \{1, 2, \ldots, 2m\}$. For each $i,j \in I(2m, n)$, we write
\[
X_{i,j} := X_{i_1,j_1} X_{i_2,j_2} \cdots X_{i_n,j_n}.
\]

Following [47, Section 2], for an endomorphism $\mu \in \text{End}(V_{ad}^{\otimes n})$, we write
\[
\mu \lambda X_{i,j} = \sum_{b \in I(2m, n)} \mu_{i,b} X_{b,j}, \quad X_{i,j} \lambda \mu = \sum_{b \in I(2m, n)} X_{i,b} \mu_{b,j}.
\]

Note that the algebra $F_K(2m)$ possesses a structure of bialgebra where comultiplication and augmentation on the generators $x_{i,j}$ are given by
\[
\Delta(X_{i,j}) = \sum_{b \in I(2m, n)} X_{i,b} \otimes X_{b,j}, \quad \varepsilon(X_{i,j}) = \delta_{i,j}.
\]

\(^2\)The reason we use the operators $\beta', \gamma'$ is because we want to let $B_n(-q^{2m+1}, q)_{ad}$ act on $V_{ad}^{\otimes n}$ from the right hand side instead of from the left hand side.
Following Oehms [47], we can apply the generalized FRT construction with respect to the subset \( \{ \beta, \gamma \} \subseteq \text{End}(V) \otimes \text{End}(V) \) to obtain a new bialgebra \( A^{sy}_K(2m) \). Precisely, we define
\[
A^{sy}_K(2m) := \mathcal{D}(2m)/(\beta \otimes X_{ij} - X_{ij} \beta) \text{ for each positive integer } j, \ i, j \in \mathbb{Z},
\]
which is necessarily a bialgebra as the ideal generated by \( \beta \otimes X_{ij} - X_{ij} \beta \) and \( \gamma \otimes X_{ij} - X_{ij} \gamma \) for \( i, j \in \mathbb{Z} \). Classically, \( A^{sy}_K(2m) \) is a \( \mathbb{Z} \)-graded algebra, that is,
\[
A^{sy}_K(2m) = \bigoplus_{0 \leq n \in \mathbb{Z}} A^{sy}_K(2m, n).
\]
For each \( i, j \in \mathbb{Z} \), let \( x_{ij} \) be the canonical image of \( X_{ij} \) in \( A^{sy}_K(2m) \). By the main result in [47], the natural map \( A^{sy}_K(2m) \otimes_{\mathcal{A}} K \rightarrow A^{sy}_K(2m, n) \) is an isomorphism for any commutative \( \mathcal{A} \)-algebra \( K \). The algebra \( A^{sy}_K(2m) \) is a quantization of the coordinate ring of the symplectic monoid scheme \( \text{SpM}_{2m} \), which is defined by (for any commutative \( \mathbb{Z} \)-algebra \( K \))
\[
\text{SpM}_{2m}(K) := \{ A \in \mathcal{M}_{2m}(K) \mid \exists \text{d}(A) \in K, A^2JA = A^2J, d(A) = d(A)J \},
\]
where \( J \) is the Gram matrix of the given skew bilinear form with respect to the basis \( \{v_i\}_{i=1}^{2m} \).

In [47], Oehms constructed a basis for \( A^{sy}_K(2m, n) \) for each \( n \). To describe that basis, we need some more notations. For \( i = 1, 2, \ldots, n-1 \), we set
\[
\beta_i := \text{id}_{V^m} \otimes \beta \otimes \text{id}_{V^{n-i-1}}, \quad \gamma_i := \text{id}_{V^m} \otimes \gamma \otimes \text{id}_{V^{n-i-1}}.
\]
Recall that for \( i = 1, 2, \ldots, n-2 \), we have the braid relation \( \beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1} \). Recall also that (see Section 1) we have used \( \mathcal{S}_n \) to denote the symmetric group on \( n \) letters. For each \( w \in \mathcal{S}_n \), there is a well-defined element \( \beta(w) \in \text{End}(V^m) \), where \( \beta(w) = \beta_{i_1} \ldots \beta_{i_k} \) whenever \( k \) is minimal such that \( w = (i_1, i_1+1) \ldots (i_k, i_k+1) \). For each partition \( \lambda \) of \( n \), let \( \lambda^t \) be the transpose of \( \lambda \), and let \( \mathcal{S}_{\lambda^t} \) be the corresponding Young subgroup of \( \mathcal{S}_n \) (which is the subgroup fixing the sets \( \{1, 2, \ldots, \lambda_1^t\}, \{\lambda_1^t + 1, 2, \ldots, \lambda_1^t + \lambda_2^t\}, \ldots \)). For each \( w \in \mathcal{S}_n \) and each pair of multi-indices \( i, j \in \mathbb{Z} \), we set (following Oehms)
\[
T^λ_\gamma(i : j) := \sum_{w \in \mathcal{S}_{\lambda^t}} (-q^2)^{-\ell(w)} \beta(w) \otimes x_{ij},
\]
and call \( T^λ_\gamma(i : j) \) a quantum symplectic bideterminant.

Recall that for each partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) one can naturally associates a Young diagram reading row lengths out of the components \( \lambda_i \). For example,
\[
(3, 3, 2, 1) \leftrightarrow \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\]

For each positive integer \( k \), let \( \Lambda^+(m, k) \) be the set of partitions of \( k \) into not more than \( m \) parts. Let \( \lambda \in \Lambda^+(m, k) \) with \( 0 \leq k \leq n \). For each \( i \in \mathbb{Z} \), one can construct a \( \lambda \)-tableau \( T^λ_\gamma \) by inserting the components of \( i \) column by column into the boxes of the Young diagram of \( \lambda \). In the above example,
\[
T^λ_\gamma = \begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\lambda_4 & \lambda_5 & \lambda_6 \\
\lambda_7 & \lambda_8 & \lambda_9 \\
\end{array}
\]

Let \( i_\lambda \) be the unique multi-index in \( \mathbb{Z} \) such that in the corresponding \( \lambda \)-tableau \( T^λ_\gamma \), the \( j \)th row is filled with the number \( j \) for each integer \( j \) with \( 1 \leq j \leq m \).
Lemma 3.4. It in this paper, we include a proof here. The following result was used in [47, Section 15] without proof. Since we shall use $p$

For each integer $1 \leq i \leq \lambda_1 = \ell(\lambda')$, we use $w_{i,0}$ to denote the unique longest element in the symmetric group $\mathfrak{S}_{\lambda'}$. Let

$$w_{0,\lambda'} := \prod_{i=1}^{\lambda_1} w_{i,0}. \quad \text{Let } \tilde{t}_\lambda \in I(2m, k) \text{ be such that } T_{\tilde{t}_\lambda}^\lambda = T_{t_\lambda}^\lambda w_{0,\lambda'}. \quad \text{Following [47, Section 6], we put a new order } \prec \text{ on the set } \{1, 2, \ldots, 2m\}, \text{ namely }, m' < m' < (m - 1)' < (m - 2)' < \cdots < 1', \text{ and for each } 1 \leq i < m, i', i'' \text{ are limited to the first } m - i + 1 \text{ rows.}

Therefore, to prove the lemma, it suffices to consider the case where $\lambda$ is weakly increasing along rows and strictly increasing down columns according to the order $\prec$. We define $I_{\lambda}^{muits}$ to be the set of $i \in I(2m, k)$ such that the entries in $T_i^\lambda$ are weakly increasing along rows and strictly increasing down columns according to the order $\prec$, and for each $1 \leq i \leq m, i, i'$ are limited to the first $m - i + 1$ rows.

In the previous example, we have

$$T_i^\lambda = \begin{array}{ccc} 1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 1 \\
4 & 1 & 1 \end{array}, \quad T_{\tilde{t}_\lambda}^\lambda = \begin{array}{ccc} 4 & 3 & 2 \\
3 & 2 & 1 \\
2 & 1 & 1 \end{array}.$$ 

Lemma 3.3. For any $\lambda \in \Lambda^+(m, k)$, we have $\tilde{t}_\lambda \in I_{\lambda}^{muits}$.

Proof. This follows directly from the definition of $I_{\lambda}^{muits}$. \hfill \Box

For each $\lambda \in \Lambda^+(m, k)$, let $I_{\lambda}^{\mu}$ be the set of multi-indices $i \in I(2m, k)$ such that the entries in $T_i^\lambda$ are strictly increasing down columns with respect to the usual order $\prec_0$ on $\{1, 2, \ldots, 2m\}$. For each integer $j \geq 1$, we write

$$\Delta^{(j)} := (\Delta \otimes 1^{2j-1}) \circ \cdots \circ (\Delta \otimes 1) \circ \Delta.$$ 

The following result was used in [47, Section 15] without proof. Since we shall use it in this paper, we include a proof here.

Lemma 3.4. Let $\lambda \in \Lambda^+(m, k)$, $i, j \in I_{\lambda}^{muits}$ and $j \geq 1$ be an integer. We have that

$$\Delta^{(j)}\left(T_i^\lambda(\mu, j)\right) = \sum_{h^{(1)}, \ldots, h^{(j)} \in I_{\lambda}^{\mu}} T_i^\lambda(1, h^{(1)}) \otimes T_i^\lambda(h^{(1)}, h^{(2)}) \otimes \cdots \otimes T_i^\lambda(h^{(j)}, j).$$ 

Proof. We only prove the special case where $j = 1$. The general case follows from the same argument. For each positive integer $r$, we set $\omega_r := (1, 1, \ldots, 1)$. Let $\lambda' = (\mu_1, \ldots, \mu_p)$ be the transpose of $\lambda$, where $p = \lambda_1$. We split $j$ into $p$ multi-indices $j^l \in I(2m, \mu_l)$, where for each $l \in \{1, 2, \ldots, p\}$, the entries of $j^l$ are taken from the $l$th column of $T_i^\lambda$. The same thing can be done with $i$. Then

$$T_i^\lambda(1, j) = T_q^{\lambda^{(1)}}(1, j^1) T_q^{\lambda^{(2)}}(j^2, j^2) \cdots T_q^{\lambda^{(p)}}(j^p, j^p),$$ 

hence

$$\Delta\left(T_i^\lambda(1, j)\right) = \Delta\left(T_q^{\lambda^{(1)}}(1, j^1)\right) \Delta\left(T_q^{\lambda^{(2)}}(j^2, j^2)\right) \cdots \Delta\left(T_q^{\lambda^{(p)}}(j^p, j^p)\right).$$ 

Therefore, to prove the lemma, it suffices to consider the case where $p = 1$.

Now assume that $p = 1$. Then $\lambda = \omega_k$. If $i, j \in I_{\lambda}^{muits}$, then the lemma follows from [47, (20)]. In general, by the definition of $I_{\lambda}^{muits}$ ([47, Section 6]), we can find some simple reflections $s_{q_1}, \ldots, s_{q_n}, s_{t_1}, \ldots, s_{t_k} \in \mathfrak{S}_{\lambda'}$ such that

$$\begin{align*}
(1) \quad \tilde{i} & := s_{q_1} s_{q_2} \cdots s_{q_n} \in I_{\lambda'}^{muits}, \\
\tilde{j} & := s_{t_1} s_{t_2} \cdots s_{t_k} \in I_{\lambda'}^{muits}.
\end{align*}$$
(2) For each integer $1 \leq c \leq a$, the action of $s_{qc}$ on $s_{q_1}s_{q_2}\cdots s_{q_{c-1}}$ does not exchange the indices $i, i'$ for any $i$;
(3) For each integer $1 \leq c \leq b$, the action of $s_{tc}$ on $j_1s_{j_2}s_{j_3}\cdots s_{j_{c-1}}$ does not exchange the indices $i, i'$ for any $i$.

Now using [47, (13), (20), Corollary 11.9], we deduce that there exist integers $a(i), b(j)$, such that

$$T_q^\lambda(i, j) = (-q)^{a(i)+b(j)}T_q^\lambda(\tilde{i}, \tilde{j}).$$

Therefore, applying [47, (20)], we get that

$$\Delta(T_q^\lambda(i, j)) = (-q)^{a(i)+b(j)}\Delta(T_q^\lambda(\tilde{i}, \tilde{j}))$$

$$= \sum_{h \in I^\lambda_\sigma} (-q)^{a(i)}T_q^\lambda(\tilde{i}, h) \otimes (-q)^{b(j)}T_q^\lambda(h, \tilde{j})$$

$$= \sum_{h \in I^\lambda_\sigma} T_q^\lambda(i, h) \otimes T_q^\lambda(h, j),$$

as required. \qed

For each integer $n \geq 0$, set

$$\Lambda_n := \{ \lambda := (\lambda, l) \mid l \in \mathbb{Z}, 0 \leq l \leq n/2, \lambda \in \Lambda^+(m, n-2l) \}.$$ 

Let $d_\lambda \in A_{\mathfrak{A}}^{\mathfrak{A}}(2m)$ be the central group-like element as defined in [47, (7)] and [29, Corollary 6.3]. By definition,

$$d_\lambda = -q^{-\rho_\lambda-\rho_l} \xi_{k, l} x_{(k, k'), (l, l')} \gamma \in A_{\mathfrak{A}}^{\mathfrak{A}}(2m, 2),$$

which is independent of the choices of $k, l \in \{1, 2, \cdots, 2m\}$. For each $\lambda := (\lambda, l) \in \Lambda_n$ and each $i, j \in I^\lambda_\mathfrak{A}$, we set $D^\lambda_{ij} := d_\lambda T^\lambda(i, j)$. Oehms ([47, (7.1)]) proved that $A_{\mathfrak{A}}^{\mathfrak{A}}(2m, n) \otimes_{\mathfrak{A}} K \cong A_{\mathfrak{A}}^{\mathfrak{A}}(2m, n)$ for any commutative $\mathfrak{A}$-algebra $K$. Indeed, $A_{\mathfrak{A}}^{\mathfrak{A}}(2m, n)$ is a free $\mathfrak{A}$-module and the elements in the following set

$$(3.5) \quad \left\{ D^\lambda_{ij} \mid \lambda = (\lambda, l) \in \Lambda_n, i, j \in I^\lambda_\mathfrak{A} \right\}$$

form an $\mathfrak{A}$-basis of $A_{\mathfrak{A}}^{\mathfrak{A}}(2m, n)$.

For each integer $k \geq 0$, we put an order on the set $\Lambda^+(m, k)$, write $\lambda \prec \mu$ if $\lambda'_i = \mu'_i$ for $i = 1, 2, \cdots, s - 1$ and $\lambda'_s < \mu'_s$ for some $s$. For $\lambda = (2, 2, 1), \mu = (3, 1, 1) \in \Lambda^+(3, 5)$, we have $\mu \prec \lambda$. Next, we put an order on $\Lambda_n$. For any $\lambda := (\lambda, l), \mu = (\mu, b) \in \Lambda_n$, write $\lambda \prec \mu$ if $l < b$ or $l = b$ and $\lambda \prec \mu$. For each $\lambda := (\lambda, l) \in \Lambda_n$, we set $M(\lambda) = I^\lambda_\mathfrak{A}$. Let $\sim$ be the $\mathfrak{A}$-linear involution of $A_{\mathfrak{A}}^{\mathfrak{A}}(2m, n)$ which is defined on generators by $\left( D^\lambda_{ij} \right)^\sim = D^\lambda_{ji}$ for all $\lambda \in \Lambda_n, i, j \in \Lambda_n$.

**Lemma 3.6.** ([47, Theorem 7.1]) With respect to the ordered set $(\Lambda_n, \prec)$, the finite set $M(\lambda)$ and the $\mathfrak{A}$-linear involution $\sim$, the coalgebra $A_{\mathfrak{A}}^{\mathfrak{A}}(2m, n)$ is a cellular coalgebra in the sense of [47, Section 5, page 860] with cellular basis given by

$$\left\{ D^\lambda_{ij} \mid \lambda = (\lambda, l) \in \Lambda_n, i, j \in I^\lambda_\mathfrak{A} \right\}.$$

For each $\lambda := (\lambda, l) \in \Lambda_n$, we set

$$A_{\mathfrak{A}}^{\mathfrak{A}}(2m, n)^\lambda := \mathfrak{A}\text{-Span}\left\{ D^\lambda_{ij} \mid \lambda \prec \mu = (\mu, b) \in \Lambda_n, i, j \in I^\mu_\mathfrak{A} \right\},$$

$$A_{\mathfrak{A}}^{\mathfrak{A}}(2m, n)^\lambda :\mathfrak{A}\text{-Span}\left\{ D^\lambda_{ij} \mid \lambda \preceq \mu = (\mu, b) \in \Lambda_n, i, j \in I^\mu_\mathfrak{A} \right\}.$$
By general theory of cellular coalgebra ([47, Section 5, page 860]), we know that both $A^{sy}_C(2m,n)^{-\Delta}$ and $A^{sy}_C(2m,n)^{-\Delta}$ are two-sided coideal of $A^{sy}_C(2m,n)$, and we have a two-sided $A^{sy}_C(2m,n)$ comodule isomorphism

$$A^{sy}_C(2m,n)^{-\Delta}/A^{sy}_C(2m,n)^{+\Delta} \cong \nabla(\Lambda) \otimes \nabla(\Lambda),$$

where $\nabla(\Lambda)$ (resp., $\nabla^*(\Lambda)$) is the cell right (resp., left) comodule corresponding to $\Lambda$. Note that $\nabla(\Lambda) = \nabla^*(\Lambda)$ as $\mathcal{A}$-module, while the left $A^{sy}_C(2m,n)$-coaction is obtained by twisting the right $A^{sy}_C(2m,n)$-coaction using "*". If we extend the base ring $\mathcal{A}$ to the rational function field $\mathbb{Q}(q)$, then $\nabla(\Lambda)_{\mathbb{Q}(q)}$ is an irreducible right $A^{sy}_{\mathbb{Q}(q)}(2m,n)$-comodule.

For any commutative $\mathcal{A}$-algebra $K$, we set

$$S^{sy}_K(2m,n) := \text{End}_{\mathcal{B}_n(-\zeta^{2m+1},\zeta)}(V^{\otimes n}_K),$$

where $\zeta$ is the image of $q$ in $K$. The algebra $S^{sy}_K(m,n)$ is called "symplectic $\zeta$-Schur algebra" by Oehms ([47]). By [29, Proposition 2.1], there is a non-degenerate pairing between $F_{\mathbb{Q}(q)}(2m)$ and $\text{End}_{\mathbb{Q}(q)}(V^{\otimes n}_K)$ such that for any $x_{ij} \in F_{\mathbb{Q}(q)}(2m)$, $f \in \text{End}_{\mathbb{Q}(q)}(V^{\otimes n}_K)$, where $i,j \in I(2m,n)$, $n, \tilde{n} \in \mathbb{Z}^{\geq 0}$,

$$\langle x_{ij}, f \rangle_0 := \begin{cases} u_1(f(v_j)), & \text{if } n = \tilde{n}; \\ 0, & \text{otherwise}. \end{cases}$$

where $u_1 := u_{i_1} \otimes \cdots \otimes u_{i_n}$, $\{u_i\}$ is the basis of $V^*$ dual to $\{v_j\}$. By [29, Proposition 2.1], it induces a non-degenerate pairing $\langle \cdot, \cdot \rangle_0$ between $A^{sy}_{\mathbb{Q}(q)}(2m)$ and

$$\bigoplus_{0 \leq n \in \mathbb{Z}} S^{sy}_{\mathbb{Q}(q)}(2m,n).$$

Furthermore, it induces a pairing $\langle \cdot, \cdot \rangle_0$ between $A^{sy}_{\mathbb{Q}(q)}(2m)$ and $\mathbb{U}_{\mathbb{Q}(q)}$. Precisely, for any $x_{ij} \in A^{sy}_{\mathbb{Q}(q)}(2m,n)$, $u \in \mathbb{U}_{\mathbb{Q}(q)}$, where $i,j \in I(2m,n)$, $n \in \mathbb{Z}^{\geq 0},$

$$\langle x_{ij}, u \rangle_0 := \langle x_{ij}, \psi_C(u) \rangle_0,$$

where $\psi_C$ is the canonical homomorphism

$$\psi_C : \mathbb{U}_{\mathbb{Q}(q)} \to S^{sy}_{\mathbb{Q}(q)}(2m,n) := \text{End}_{\mathcal{B}_n(-q^{2m+1},q)}(V^{\otimes n}_{\mathbb{Q}(q)}).$$

In [46] and [47], Oehms proved the pairing $\langle \cdot, \cdot \rangle_0$ actually induces an $\mathcal{A}$-algebra isomorphism $S^{sy}_C(2m,n) \cong (A^{sy}_C(2m,n))^\dagger$ for each $n \in \mathbb{Z}^{\geq 0}$. Note that our $F(2m)$ is just $T(E)$ in the notation of [29, Section 2], while the ideal generated by $\beta l X_{ij} - X_{ij} l \beta$ for $i,j \in I(2m,2)$ in our paper is the same as the ideal generated by $\text{Im}(id_{E \otimes E} - \beta_E)$ in [29, Section 2]. Note also that over $\mathcal{A}$, the algebra $A^{sy}_C(2m)$ is only a homomorphic image of $S(E)$ in [29, Proposition 2.1]. However, if we work over $\mathbb{Q}(q)$, then $A^{sy}_{\mathbb{Q}(q)}(2m)$ coincides with $S(E)$ in the notation of [29, Proposition 2.1] because of the relation (3.2). Therefore, for each $\Lambda = (\lambda, l) \in \Lambda_n$, $\nabla(\Lambda)_{\mathbb{Q}(q)}$ is also an irreducible left $S^{sy}_{\mathbb{Q}(q)}(2m,n)$-module, and hence an irreducible left $\mathbb{U}_{\mathbb{Q}(q)}$-module.

For any commutative $\mathcal{A}$-algebra $K$, let $\zeta$ be the natural image of $q$ in $K$, we define $\tilde{A}^{sy}_K(2m) := A^{sy}_K(2m)/(d_{\Delta} - 1)$, where $d_{\Delta}$ is the natural image of $d_{\Delta}$ in $A^{sy}_K(2m)$. Note that $\tilde{A}^{sy}_K(2m)$ is a quantized version of the coordinate algebra of the symplectic monoid $\text{SpM}_{2m}(K)$, while $\tilde{A}^{sy}_K(2m)$ is a quantized version of the coordinate algebra of the symplectic group $\text{Sp}_{2m}(K)$. The algebra $\tilde{A}^{sy}_K(2m)$ will play a key role in the proof of Theorem 1.4 in the next section. We use $\pi_C$ to denote the natural projection

$$\pi_C : A^{sy}_K(2m) \to \tilde{A}^{sy}_K(2m).$$
For each integer \( n \geq 0 \) and each commutative \( \mathcal{A} \)-algebra \( K \), we define
\[
\Lambda^+(m, \leq n) := \bigcup_{0 \leq k \leq n} \Lambda^+(m, k), \quad A^K_n(2m, \leq n) := \bigoplus_{0 \leq k \leq n} A^K_{\pi}(2m, k).
\]

We shall call \( A^K_n(2m, \leq n) \) a finite truncation of the quantized coordinate algebra \( A^K_n(2m) \) of \( \text{SpM}_{2m} \). Clearly, \( A^K_n(2m, \leq n) \) is a sub-coalgebra of \( A^K_n(2m) \) as well as a free \( \mathcal{A} \)-submodule with basis
\[
\{ T^\lambda_q(i, j) \mid \lambda \in \Lambda^+(m, \leq n), \ 1, j \in I^K_{\lambda, q} \}.
\]

The author is grateful to Professor S. Doty and the referee for the first part of the following corollary.

**Corollary 3.7.** Let \( n \geq 0 \) be an integer and \( K \) be a commutative \( \mathcal{A} \)-algebra.

1) The maps
\[
\pi_C \downarrow A^K_n(2m, n): A^K_n(2m, n) \to \pi_C(A^K_n(2m, n))
\]
\[
\pi_C \downarrow A^K_n(2m, \leq n): A^K_n(2m, \leq n) \to \pi_C(A^K_n(2m, \leq n))
\]
are both isomorphisms.

2) \( \tilde{A}^K_n(2m) \) is a free \( \mathcal{A} \)-module and the elements in the following set
\[
\left\{ \pi_C(T^\lambda_q(i, j)) \mid \lambda \in \Lambda^+(m, n - 2l) \text{ for some integer } n, l \right\}
\]
form an \( \mathcal{A} \)-basis of \( \tilde{A}^K_n(2m) \). Moreover, the canonical map
\[
\tilde{A}^K_n(2m) \otimes_{\mathcal{A}} K \to \tilde{A}^K_n(2m)
\]
is an isomorphism.

**Proof.** Suppose that \( 0 \neq x \in \text{Ker}(\pi_C \downarrow A^K_m(2m, n)) \). Since \( d_0 \) is central, it follows that \( x = (d_0 - 1)y \) for some \( y \in A^K_n(2m) \). Note that \( d_0 \neq 0 \) is homogeneous of degree 2, while \( x \) is homogeneous of degree \( n \). By Lemma 3.6, the elements in the following set
\[
\left\{ T^\lambda_q(i, j) \mid \lambda = (\lambda, l) \in A_k \text{ for some } k \geq 0 \text{ and } 0 \leq l \leq k/2, \ i, j \in I^K_{\lambda, q} \right\}
\]
form a homogeneous basis of \( A^K_n(2m) \). Expressing \( y \) into a linear combination of this basis and comparing the degree of each homogeneous component, we get a contradiction. This proves that \( \pi_C \downarrow A^K_n(2m, n) \) is injective and hence an isomorphism. As a result, we deduce that \( \pi_C \downarrow A^K_n(2m, \leq n) \) is also an isomorphism. This proves 1). The statement 2) is an immediate consequence of the statement 1). \( \square \)

Henceforth, we shall use Corollary 3.7 to identify \( A^K_n(2m, n) \) and \( A^K_n(2m, \leq n) \) as subspaces of \( \tilde{A}^K_n(2m) \) without further comments. The involution “*” gives rise to an \( \mathcal{A} \)-linear involution of \( A^K_n(2m, \leq n) \), which will be still denoted by “*”. Recall that in the paragraph below (3.5) we have introduced an order “<” on the set of partitions of a fixed integer. Now we generalize it to the case of partitions of possibly different integers. For any \( \lambda, \mu \in \Lambda^+(m, \leq n) \), write \( \lambda < \mu \) if
\[
|\lambda| - |\mu| \in 2\mathbb{N} \quad \text{or} \quad |\lambda| = |\mu| \quad \text{and} \quad \lambda < \mu.
\]

**Corollary 3.8.** With respect to the ordered set \( (\Lambda^+(m, \leq n), <) \), the finite set \( I^K_{\lambda, q} \)
(for each \( \lambda \in \Lambda^+(m, \leq n) \)) and the \( \mathcal{A} \)-linear involution “*”, the coalgebra \( A^K_n(2m, \leq n) \) is a cellular coalgebra with cellular basis given by
\[
\left\{ T^\lambda_q(i, j) \mid \lambda \in \Lambda^+(m, \leq n), \ 1, j \in I^K_{\lambda, q} \right\}.
\]

Furthermore, for each commutative \( \mathcal{A} \)-algebra \( K \), the canonical map
\[
A^K_n(2m, \leq n) \otimes_{\mathcal{A}} K \to A^K_n(2m, \leq n)
\]
is an isomorphism.

Let $\Lambda^+(m) := \bigcup_{k \geq 0} \Lambda^+(m, k)$. For each $\lambda \in \Lambda^+(m, \leq n)$, we set

$$A^\emph{sy}_K(2m, \leq n)^{\lambda} := \mathcal{A} - \text{Span} \{ T^\emph{sy}_i(i, j) \mid \lambda < \mu \in \Lambda^+(m, \leq n), i, j \in I^{\emph{sy}}_\mu \},$$

$$A^\emph{sy}_A(2m, \leq n)^{\lambda} := \mathcal{A} - \text{Span} \{ T^\emph{sy}_i(i, j) \mid \lambda \leq \mu \in \Lambda^+(m, \leq n), i, j \in I^{\emph{sy}}_\mu \}.$$  

By general theory of cellular coalgebra, we have that both $A^\emph{sy}_K(2m, \leq n)^{-\lambda}$ and $A^\emph{sy}_A(2m, \leq n)_{-\lambda}$ are two-sided coalgebra of $A^\emph{sy}_K(2m, \leq n)$, and we have a two-sided $A^\emph{sy}_K(2m, \leq n)$-comodule isomorphism

$$A^\emph{sy}_K(2m, \leq n)_{-\lambda}/A^\emph{sy}_A(2m, \leq n)^{-\lambda} \cong \Delta^\emph{r}(-\lambda) \otimes \Delta^\emph{l}(\lambda),$$

where $\Delta^\emph{r}(\lambda)$ (resp., $\Delta^\emph{l}(\lambda)$) is the cell right (resp., left) comodule corresponding to $\lambda$. Note that $\Delta^\emph{r}(\lambda) = \Delta^\emph{l}(\lambda)$ as $\mathcal{A}$-module, while the left $A^\emph{sy}_K(2m, \leq n)$-coaction is obtained by twisting the right $A^\emph{sy}_K(2m, \leq n)$-coaction using “$*$”. If we extend the base ring $\mathcal{A}$ to the rational function field $\mathbb{Q}(q)$, then $\Delta^\emph{r}(\lambda)\mathbb{Q}(q)$ is an irreducible right $A^\emph{sy}_K(2m, \leq n)$-comodule.

Suppose that $\lambda \in \Lambda^+(m, k)$ with $0 \leq k \leq n$. We set $\underline{\lambda} = (\lambda, 0) \in \Lambda_k$. Note that the $A^\emph{sy}_K(2m, \leq n)$-comodule $\Delta^\emph{r}(\lambda)\mathcal{A}$ is isomorphic to the restriction of the $A^\emph{sy}_K(2m, k)$-comodule $\Delta^\emph{r}(\lambda)\mathbb{Q}(q)$. In particular, every simple $A^\emph{sy}_K(2m, \leq n)$-comodule comes from the restriction of a simple $A^\emph{sy}_K(2m, k)$-comodule, or equivalently, of a simple $S^\emph{sy}_K(2m, k)$-module for some $0 \leq k \leq n$. Therefore, by the surjection from $\mathbb{U}_{\Q(q)}$ onto $S^\emph{sy}_K(2m, k)$, we see that every simple $A^\emph{sy}_K(2m, \leq n)$-comodule comes from the restriction of a simple $\mathbb{U}_{\Q(q)}$-module. For any field $K$ which is an $\mathcal{A}$-algebra, we define

$$\nabla_K(\lambda) := \Delta^\emph{r}(\lambda) \otimes_{\mathcal{A}} K, \quad S^\emph{sy}_K(2m, k) := S^\emph{sy}_K(2m, k) \otimes_{\mathcal{A}} K.$$  

Then $\nabla_K(\lambda) \cong \nabla_K(\lambda, 0)$ can be regarded as a $S^\emph{sy}_K(2m, k)$-module.

Corollary 3.9. Let $K$ be a field which is an $\mathcal{A}$-algebra. Let $k \geq 0$ be an integer and $\lambda \in \Lambda^+(m, k)$. Then as a $S^\emph{sy}_K(2m, k)$-module, $\nabla_K(\lambda)$ has a unique simple socle.

Proof. By [47], $S^\emph{sy}_K(2m, k)$ is a cellular quasi-hereditary algebra. By definition, the dual of $\nabla_K(\lambda)$ is a cell module of $S^\emph{sy}_K(2m, k)$. Thus, the corollary follows from the general theory of cellular quasi-hereditary algebra. □

Recall that $\psi_C$ induces a natural morphism from $U_K(g)$ to $S^\emph{sy}_K(2m, k)$, via which $S^\emph{sy}_K(2m, k)$-module $\nabla_K(\lambda)$ can be regarded as an $U_K$-module. Note that the above corollary does not immediately imply that $\nabla_K(\lambda)$ has a unique simple socle as an $U_K$-module because at this moment we did not know if that natural morphism from $U_K$ to $S^\emph{sy}_K(2m, k)$ is surjective or not. However, in the next section we shall show that this is indeed the case.

Corollary 3.10. For each integer $n \geq 0$, we have a two-sided $A^\emph{sy}_K(2m, \leq n)$-comodule isomorphism

$$\theta_n : A^\emph{sy}_K(2m, \leq n) \cong \bigoplus_{\lambda \in \Lambda^+(m, \leq n)} \nabla^\emph{r}(\lambda)\Q(q) \otimes \nabla^\emph{l}(\lambda)\Q(q).$$  

Furthermore, we have a two-sided $\tilde{A}^\emph{sy}_K(2m)$-comodule isomorphism

$$\theta : \tilde{A}^\emph{sy}_K(2m) \cong \bigoplus_{k \geq 0, \lambda \in \Lambda^+(m, k)} \nabla^\emph{r}(\lambda)\Q(q) \otimes \nabla^\emph{l}(\lambda)\Q(q).$$
where the two vertical maps are natural embedding.

**Proof.** This follows from the cellular structure of $A_{sy}^y(2m, \leq n)$ given in Corollary 3.8 and the fact

$$A_{sy}^y(2m, \leq n) = \bigcup_{n \geq 0} A_{sy}^y(2m, \leq n),$$

and the following commutative diagram

$$\begin{array}{ccc}
A_{sy}^y(2m, \leq n) & \xrightarrow{\theta_n} & \bigoplus_{\lambda \in \Lambda^+(m, \leq n)} \nabla^r(\lambda)_{Q(q)} \otimes \nabla(\lambda)_{Q(q)} \\
\downarrow & & \downarrow \\
\tilde{A}_{sy}^y(2m) & \xrightarrow{\theta} & \bigoplus_{k \geq 0, \lambda \in \Lambda^+(m, k)} \nabla^r(\lambda)_{Q(q)} \otimes \nabla(\lambda)_{Q(q)}
\end{array}$$

where $\tilde{n} \geq n \geq 0$ and the two vertical maps are natural embedding.

**Corollary 3.11.** For each integer $n \geq 0$ and each $\lambda \in \Lambda^+(m, n)$, let $\text{pr}_\lambda$ be the natural projection from $\tilde{A}_{sy}^y(2m)$ onto $\nabla^r(\lambda)_{Q(q)} \otimes \nabla(\lambda)_{Q(q)}$. Then the elements in the following set

$$\left\{ \text{pr}_\lambda \left( \pi_C(T_q^\lambda(i,j)) \right) \mid i, j \in I_{\lambda}^{\text{sym}} \right\}$$

form a $\mathbb{Q}(q)$-basis of $\nabla^r(\lambda)_{Q(q)} \otimes \nabla(\lambda)_{Q(q)}$.

We end this section with the following lemma. Recall the definition of $i_\lambda$, $\hat{i}_\lambda$ given in the paragraph above Lemma 3.3.

**Lemma 3.12.** For each $\lambda \in \Lambda^+(m, n)$, we have

$$\pi_C \left( T_q^\lambda(i_\lambda, i_\lambda) \right) \equiv q^{a_\lambda} \pi_C \left( T_q^\lambda(\hat{i}_\lambda, \hat{i}_\lambda) \right) + \sum_{j, k \in I_{\lambda}^{\text{sym}}} C^\lambda_{jk} \pi_C \left( T_q^\lambda(j, k) \right) \pmod{A_{sy}^y(2m, \leq n)^{r=\lambda}},$$

where $a_\lambda \in \mathbb{Z}$, $C^\lambda_{jk} \in \mathcal{A}$, and “$a$” is the same as defined in [47, Proposition 8.4]. In particular, we have that

$$0 \neq \text{pr}_\lambda \left( \pi_C(T_q^\lambda(i_\lambda, i_\lambda)) \right) \in \nabla^r(\lambda) \otimes \nabla(\lambda).$$

**Proof.** This follows directly from [47, (8), Proposition 8.4, Corollary 9.12] and the facts that any quantum symplectic bideterminant in $A_{sy}^y(2m)$ is homogeneous and the central group-like element $d_q$ is homogeneous of degree 2. \qed
As a result, we have two Hopf algebra injections
\[ U_q(\mathfrak{g}) \hookrightarrow \mathcal{H}(\mathfrak{g}) \]
where for any Hopf algebra
\[ \mathcal{H}(\mathfrak{g}) \]

Lemma 4.2. ([29, Theorem 6.4(2)]) We have that
\[ \langle d_\lambda - 1, y \rangle_0 = 0, \quad \text{for any } y \in U_q(\mathfrak{g}), \]
and the pairing \( \langle \cdot, \cdot \rangle_0 \) induces a non-degenerate Hopf pairing \( \langle \cdot, \cdot \rangle_0 \) between \( \tilde{A}_Q^{sy}(2m) \) and the quantized enveloping algebra \( \tilde{U}_Q(\mathfrak{g}) \)
\[ \tilde{A}_Q^{sy}(2m) \times \tilde{U}_Q(\mathfrak{g}) \rightarrow \tilde{Q}(q). \]
As a result, we have two Hopf algebra injections
\[ \iota_A : \tilde{A}_Q^{sy}(2m) \hookrightarrow \left( \tilde{U}_Q(\mathfrak{g}) \right)^\circ, \quad \iota_U : \tilde{U}_Q(\mathfrak{g}) \hookrightarrow \left( \tilde{A}_Q^{sy}(2m) \right)^\circ, \]
where for any Hopf algebra \( H, H^\circ \) denotes the Hopf dual of \( H \).

Lemma 4.3. With the above notations, the pairing \( \langle \cdot, \cdot \rangle_0 \) naturally induces a non-degenerate Hopf pairing \( \langle \cdot, \cdot \rangle_0 \) between \( \tilde{A}_Q^{sy}(2m) \) and the modified quantized enveloping algebra \( \tilde{U}_Q(\mathfrak{g}) \)
\[ \tilde{A}_Q^{sy}(2m) \times \tilde{U}_Q(\mathfrak{g}) \rightarrow \tilde{Q}(q). \]
As a result, we have two Hopf algebra injections
\[ \tilde{\iota}_A : \tilde{A}_Q^{sy}(2m) \hookrightarrow \left( \tilde{U}_Q(\mathfrak{g}) \right)^\circ, \quad \tilde{\iota}_U : \tilde{U}_Q(\mathfrak{g}) \hookrightarrow \left( \tilde{A}_Q^{sy}(2m) \right)^\circ, \]
Proof. This is an easy consequence of Lemma 2.3, Lemma 4.2 as well as the following two standard facts:

(a) any simple \( \mathcal{U}_{Q(q)} \)-module is a submodule of \( V_{n,q}^{\mathbb{Z}_n} \) for some \( n \in \mathbb{Z}^\geq \);
(b) if \( u \in \mathcal{U}_{Q(q)} \) acts as 0 on every simple \( \mathcal{U}_{Q(q)} \)-module, then \( u = 0 \). \( \square \)

For each \( i \in I(2m, n) \), we define \( \text{wt}(i) = (\mu_1, \cdots, \mu_m) \), where

\[
\mu_s := \# \{ 1 \leq j \leq n \mid i_j = s \} - \# \{ 1 \leq j \leq n \mid i_j = s' \}, \quad s = 1, 2, \cdots, m.
\]

We identify \( \text{wt}(i) = (\mu_1, \cdots, \mu_m) \) with the weight \( \mu_1 \varepsilon_1 + \cdots + \mu_m \varepsilon_m \in X \).

**Lemma 4.4.** Let \( i, j \in I(2m, n) \). Set \( \mu = \text{wt}(j) \). Then for any integer \( a \) with \( 1 \leq a \leq m \),

\[
\langle \pi_C(T_q^\lambda(i, j), k_a)_0 = q^{(\mu, \alpha_i^\vee)} \epsilon(\pi_C(T_q^\lambda(i, j))) \rangle.
\]

**Proof.** This follows from direct verification. \( \square \)

Note that \( \mathcal{U}_{Q(q)}(\mathfrak{g}) \cong \mathcal{U}_{Q(q)}(\mathfrak{g}^{opp}) \) via the anti-automorphism \( \phi \) defined on generators by:

\[
e_i \mapsto f_i, \quad f_i \mapsto e_i, \quad k_i \mapsto k_i, \quad i = 1, 2, \cdots, m.
\]

We identify \( \mathcal{U}_{Q(q)}(\mathfrak{g} \oplus \mathfrak{g}) \) with \( \mathcal{U}_{Q(q)}(\mathfrak{g}) \otimes \mathcal{U}_{Q(q)}(\mathfrak{g}) \). Using \( \phi \), the bi-\( \mathcal{U}_{Q(q)}(\mathfrak{g}) \) structure on \( (\mathcal{U}_{Q(q)})^* \) can be interpreted as a left \( \mathcal{U}_{Q(q)}(\mathfrak{g} \oplus \mathfrak{g}) \) structure, i.e.,

\[(a \otimes b)(x) := f(\phi(b)xa), \quad \forall a, b, x \in \mathcal{U}_{Q(q)}(\mathfrak{g}), f \in (\mathcal{U}_{Q(q)}(\mathfrak{g}))^*.
\]

Let \( W_m = W(C_m) \) be the Weyl group of type \( C_m \). Let \( w_0 \) be the longest element in \( W_m \). If \( \lambda = (\lambda_1, \cdots, \lambda_m) \in X \), then \( w^0 \lambda = (-\lambda_1, \cdots, -\lambda_m) \). Let \( k \in \mathbb{Z}^\geq \) and \( \lambda \in \Lambda^+(m, k) \). Recall our definitions of \( \iota_A \) given above Lemma 3.3. We have the following observation.

**Corollary 4.5.** Let \( k \in \mathbb{Z}^\geq \) and \( \lambda \in \Lambda^+(m, k) \). Then \( \pi_C(T_q^\lambda(i, \lambda, i)) \) is a weight vector of weight \( (\lambda, \lambda) \) satisfying

\[
e_i \pi_C(T_q^\lambda(i, \lambda, i)) = 0 = \pi_C(T_q^\lambda(i, \lambda, i)) f_i, \quad \forall 1 \leq i \leq m.
\]

**Proof.** Note that \( \iota_A \in I^*_A \). We identify \( \pi_C(T_q^\lambda(i, \lambda, i)) \) as an element in \( (\mathcal{U}_{Q(q)})^* \) via \( \iota_A \). Recall that the \( \mathcal{U}_{Q(q)}(\mathfrak{g} \oplus \mathfrak{g}) \) structure on \( (\mathcal{U}_{Q(q)})^* \) comes from its bi-\( \mathcal{U}_{Q(q)}(\mathfrak{g}) \) structure.

We first look at the left \( \mathcal{U}_{Q(q)} \) action on \( (\mathcal{U}_{Q(q)})^* \). Recall that \( \pi_C \) is a bialgebra map. For each integer \( j \geq 1 \), by Lemma 3.4 and [47, (20)], we know that

\[
\Delta^{(j)}(\pi_C(T_q^\lambda(i, \lambda, i))) = \sum_{\mathbf{h} \in I^*_A} \pi_C(T_q^\lambda(i, \lambda, h^{(1)}), \pi_C(T_q^\lambda(h^{(1)}, h^{(2)})) \otimes \cdots \otimes \pi_C(T_q^\lambda(h^{(j)}, i, i)).
\]

In particular, we have that

\[
\pi_C(T_q^\lambda(i, \lambda, i)) = \sum_{\mathbf{h} \in I^*_A} \epsilon(\pi_C(T_q^\lambda(h, i, i)))_{\pi_C(T_q^\lambda(i, \lambda, h))}.
\]
With these in mind and using Lemma 4.4, for any \( f \in \mathcal{U}_{q}(q) \) and any integer \( a \) with \( 1 \leq a \leq m \), we get that

\[
\langle k_{a} \pi_{C}(T_{q}^{\lambda}(i_{\lambda}, i_{\lambda})), f \rangle_{0} = \langle \pi_{C}(T_{q}^{\lambda}(i_{\lambda}, i_{\lambda})), f k_{a} \rangle_{0} = \sum_{h \in I_{\lambda}^{c}} \langle \Delta \left( \pi_{C}(T_{q}^{\lambda}(i_{\lambda}, i_{\lambda})) \right), f \otimes k_{a} \rangle_{0} = \sum_{h \in I_{\lambda}^{c}} \langle \pi_{C}(T_{q}^{\lambda}(i_{\lambda}, h)), f \rangle_{0} \langle \pi_{C}(T_{q}^{\lambda}(h, i_{\lambda})), k_{a} \rangle_{0} = q^{(\lambda, \alpha_{r})} \sum_{h \in I_{\lambda}^{c}} \langle \varepsilon \left( \pi_{C}(T_{q}^{\lambda}(h, i_{\lambda})) \right) \pi_{C}(T_{q}^{\lambda}(h, i_{\lambda})), f \rangle_{0} = q^{(\lambda, \alpha_{r})} \langle \pi_{C}(T_{q}^{\lambda}(i_{\lambda}, i_{\lambda})), f \rangle_{0},
\]

which implies that \( k_{a} \pi_{C}(T_{q}^{\lambda}(i_{\lambda}, i_{\lambda})) = q^{(\lambda, \alpha_{r})} \pi_{C}(T_{q}^{\lambda}(i_{\lambda}, i_{\lambda})) \). In a similar way, one can prove that if we regard \( (\mathcal{U}_{q}(q))^{*} \) as a right \( \mathcal{U}_{q}(q) \)-module, then \( \pi_{C}(T_{q}^{\lambda}(i_{\lambda}, i_{\lambda})) \) is a weight vector of weight \( \lambda \).

By Corollary 3.11, Lemma 3.12 and Lemma 4.5, it is easy to see that

\[
\langle \pi_{C}(T_{q}^{\lambda}(i_{\lambda}, i_{\lambda})), f_{i} \rangle_{0} = 0 = \langle \pi_{C}(T_{q}^{\lambda}(i_{\lambda}, i_{\lambda})), f_{i} \rangle_{0},
\]

from which the equality (4.6) follows immediately. This completes the proof of the corollary.

By Corollary 3.10 and the discussion above it, we see that every simple \( \tilde{\mathcal{A}}_{q}(2m) \) comodule comes from the restriction of a simple \( \mathcal{U}_{q}(q) \)-module. For each \( \lambda \in X^{+} \), let \( V(\lambda) \) (resp., \( V^r(\lambda) \)) denotes the left (resp., the right) simple module with highest weight \( \lambda \). By Corollary 3.11, Lemma 3.12 and Lemma 4.5, it is easy to see that \( V(\lambda)_{q}(q) \) is identified with \( V(\lambda) \) as left \( \mathcal{U}_{q}(q) \)-module, and \( V^{r}(\lambda)_{q}(q) \) is identified with \( V^{r}(\lambda) \) as right \( \mathcal{U}_{q}(q) \)-module. By [34, Proposition 7.2.2], we have a Peter–Weyl decomposition

\[
A_{q}(\mathfrak{g}) \cong \bigoplus_{\lambda \in X^{+}} V^{r}(\lambda) \otimes V(\lambda),
\]

from which the following result follows easily.

**Lemma 4.8.** With the above notations, we have that

\[
\iota_{A}\left( \tilde{\mathcal{A}}_{q}(2m) \right) = A_{q}(\mathfrak{g}).
\]

For later use, we denote by \( \Phi_{A} \) the canonical embedding induced from the isomorphism (4.7) from \( V^{r}(\lambda) \otimes V(\lambda) \) into \( A_{q}(\mathfrak{g}) \) for each \( \lambda \in X^{+} \).
In [34], Kashiwara introduced a crystal basis $B(A_q(\mathfrak{g}))$ of $A_q(\mathfrak{g})$. He proved that $B(A_q(\mathfrak{g}))$ has the crystal structure

$$B(A_q(\mathfrak{g})) = \bigsqcup_{\lambda \in X^+} \tilde{B}(\lambda),$$

where $\tilde{B}(\lambda) := B'(\lambda) \otimes B(\lambda)$, and $B'(\lambda)$ (resp., $B(\lambda)$) denotes the crystal basis of $V'(\lambda)$ (resp., of $V(\lambda)$). For each $b \in \tilde{B}(\lambda)$, let $\tilde{G}(b)$ be the corresponding upper global crystal base of $A_q(\mathfrak{g})$. Recall that $\mathfrak{g} = \mathfrak{sp}_{2m}$ in this paper. By the results in [42], $\hat{U}_{\mathfrak{g}(q)}$ has also a crystal base (or lower global crystal base) $\{\tilde{G}(b) | b \in \tilde{B}(\lambda), \lambda \in X^+\}$. This canonical base is an $\mathcal{A}$-basis of $\hat{U}_{\mathfrak{g}(q)}$. There exists a canonical coupling $\langle \cdot, \cdot \rangle_1$ between $A_q(\mathfrak{g})$ and $\hat{U}_{\mathfrak{g}(q)}$ (cf. [35, (10.1.1)]) defined by

$$\langle \tilde{G}(b'), \tilde{G}(b) \rangle_1 = \delta_{b,b'}, \quad \forall b, b' \in \tilde{B}(\lambda).$$

In [34], Kashiwara proved that the upper global crystal bases $\{\tilde{G}(b) | b \in \tilde{B}(\lambda), \lambda \in X^+\}$ form a $\mathbb{Q}[q, q^{-1}]$ basis of $A_q^Z(\mathfrak{g})$, and he remarked that it is actually an $\mathcal{A}$-basis of $A_q^Z(\mathfrak{g})$. For the reader’s convenience, we include a proof here.

**Lemma 4.10.** With the above notations, we have that the elements in the following set

$$\left\{ \tilde{G}(b) \mid b \in \tilde{B}(\lambda), \lambda \in X^+ \right\}$$

form an $\mathcal{A}$-basis of $A_q^Z(\mathfrak{g})$.

**Proof.** First we show that for each $\lambda \in X^+$, and each $b \in \tilde{B}(\lambda)$,

$$\tilde{G}(b) \in A_q^Z(\mathfrak{g}).$$

Let $P \in U_{\mathcal{A}}$. By definition, for any $\mu \in X$, $\lambda \in X^+$, $u \in V'(\lambda), v \in V(\lambda)$,

$$\langle \Phi_\lambda(u \otimes v), P1_{1\mu} \rangle_1 = (u, P1_{1\mu}v).$$

It follows that for any given $\lambda \in X^+, u \in V'(\lambda), v \in V(\lambda)$, there are only finitely many $\mu \in X$ such that

$$\langle \Phi_\lambda(u \otimes v), P1_{1\mu} \rangle_1 \neq 0.$$

Therefore, applying [34, Proposition 7.2.2] and (4.9), we get that

$$\langle \tilde{G}(b), P \rangle = \sum_{\mu \in X} \langle \tilde{G}(b), P1_{1\mu} \rangle_1 \in \mathcal{A},$$

which implies that $\tilde{G}(b) \in A_q^Z(\mathfrak{g})$.

Second, we want to show that

$$A_q^Z(\mathfrak{g}) \subseteq \left\{ x \in A_q(\mathfrak{g}) \mid \langle x, \hat{U}_{\mathcal{A}} \rangle_1 \in \mathcal{A} \right\},$$

from which and together with (4.9) the lemma would follow immediately.

Let $f = \sum_{i=1}^t x_i \otimes y_i \in A_q^Z(\mathfrak{g})$, where for each $1 \leq i \leq t$, $x_i \in V'(\lambda^{(i)})$, $y_i \in V(\lambda^{(i)})$, and $\lambda^{(i)} \in X^+$. Let $P \in U_{\mathcal{A}}, \mu \in X$. Our purpose is to show that

$$\langle f, P1_{1\mu} \rangle_1 \in \mathcal{A}. \text{ Let } n \text{ be an integer which is large enough such that}$$

1. $|\mu| \leq n, |\mu| \equiv n \pmod{2Z}$, and
2. for each $1 \leq i \leq t$, either $V(\lambda^{(i)}) \not\subseteq V^{\otimes n-1}$ or $V(\lambda^{(i)}) \not\subseteq V^{\otimes n}$. 

In [34], Kashiwara proved that the upper global crystal bases $\{\tilde{G}(b) | b \in \tilde{B}(\lambda), \lambda \in X^+\}$ form a $\mathbb{Q}[q, q^{-1}]$ basis of $A_q^Z(\mathfrak{g})$, and he remarked that it is actually an $\mathcal{A}$-basis of $A_q^Z(\mathfrak{g})$. For the reader’s convenience, we include a proof here.

**Lemma 4.10.** With the above notations, we have that the elements in the following set

$$\left\{ \tilde{G}(b) \mid b \in \tilde{B}(\lambda), \lambda \in X^+ \right\}$$

form an $\mathcal{A}$-basis of $A_q^Z(\mathfrak{g})$.

**Proof.** First we show that for each $\lambda \in X^+$, and each $b \in \tilde{B}(\lambda)$,

$$\tilde{G}(b) \in A_q^Z(\mathfrak{g}).$$

Let $P \in U_{\mathcal{A}}$. By definition, for any $\mu \in X$, $\lambda \in X^+$, $u \in V'(\lambda), v \in V(\lambda)$,

$$\langle \Phi_\lambda(u \otimes v), P1_{1\mu} \rangle_1 = (u, P1_{1\mu}v).$$

It follows that for any given $\lambda \in X^+, u \in V'(\lambda), v \in V(\lambda)$, there are only finitely many $\mu \in X$ such that

$$\langle \Phi_\lambda(u \otimes v), P1_{1\mu} \rangle_1 \neq 0.$$
We write \([1, 2, \cdots, t]\) = \(I_1 \cup I_2\), where
\[
I_1 := \{1 \leq i \leq t \mid V(\lambda^{(i)}) \subseteq V^{\otimes n}\}, \quad I_2 := \{1 \leq i \leq t \mid V(\lambda^{(i)}) \subseteq V^{\otimes n-1}\}.
\]
Let \(p'_\mu \in U_{\mathfrak{sl}}\) be as defined in the proof of Lemma 2.3 with respect to our fixed \(n\) and \(\mu\). By the definition of \(A^r_q(\mathfrak{g})\), we have that \(\langle f, p'_\mu \rangle \in \mathcal{A}\). Note that \(p'_\mu V^{\otimes n-1} = 0 = 1_{\mathcal{A}} V^{\otimes n-1}\). It follows that
\[
\langle f, 1_{\mathcal{A}} \rangle = \sum_{i \in I_1} \langle x_i, 1_{\mathcal{A}} \rangle = \sum_{i \in I_1} \langle x_i, p'_\mu \rangle = \sum_{i \in I_1 \cup I_2} \langle x_i, p'_\mu \rangle = \langle f, p'_\mu \rangle \in \mathcal{A},
\]
as required. This completes the proof of the lemma.

Let \(\lambda \in X^+\). Let \(\Delta_{\mathcal{A}}(\lambda)\) denote the standard \(\mathcal{A}\)-form of \(V(\lambda)\), i.e., the \(U_{\mathfrak{sl}}(\mathfrak{g})\)-submodule generated by the highest weight vector \(v_\lambda\). Then \(\Delta_{\mathcal{A}}(\lambda)\) is spanned by Lusztig’s canonical basis as in [41, §14.4]. Note that the upper global crystal basis \(\{G(b) \mid b \in B(\lambda)\}\) is Lusztig’s dual canonical basis. The dual basis to the upper global crystal basis under the canonical contravariant form \(\langle \cdot, \cdot \rangle\) on \(V(\lambda)\) is the lower global crystal basis, i.e., Lusztig’s canonical basis, cf. [27] and [34, §3.3, 4.2.1]. Let
\[
V_{\mathcal{A}}(\lambda) := \{v \in V(\lambda) \mid (v, w) \in \mathcal{A} \text{ for all } w \in \Delta_{\mathcal{A}}(\lambda)\}.
\]
Then \(V_{\mathcal{A}}(\lambda) \cong \text{Hom}_{\mathcal{A}}(V_{\mathcal{A}}(\lambda), \mathcal{A})\) is \(U_{\mathfrak{sl}}(\mathfrak{g})\)-stable and spanned by the upper global crystal basis of \(V(\lambda)\). Similar statement is true for \(V'_{\mathcal{A}}(\lambda), V''_{\mathcal{A}}(\lambda)\). For any field \(K\) which is an \(\mathcal{A}\)-algebra, we define
\[
V'_K(\lambda) \otimes V_K(\lambda) := (V'_{\mathcal{A}}(\lambda) \otimes V_{\mathcal{A}}(\lambda)) \otimes_{\mathcal{A}} K,
\]
\[
\Delta_K(\lambda) \otimes \Delta_K(\lambda) := (\Delta_{\mathcal{A}}(\lambda) \otimes \Delta_{\mathcal{A}}(\lambda)) \otimes_{\mathcal{A}} K.
\]
Since \(\Delta_K(\lambda) \otimes \Delta_K(\lambda)\) is a highest weight module generated by its highest weight vector (cf. [41, Theorem 14.4.11]) and has the same dimension as \(V'_{\mathcal{A}}(\lambda) \otimes V_{\mathcal{A}}(\lambda)\), it follows that \(\Delta_K(\lambda) \otimes \Delta_K(\lambda)\) is isomorphic to the Weyl module of \(U_K(\mathfrak{g} \otimes \mathfrak{g})\) associated to \((\lambda, \lambda)\). Note that \(V'_K(\lambda) \otimes V_K(\lambda) \cong (\Delta_K(\lambda) \otimes \Delta_K(\lambda))^{\otimes 2}\). Therefore, we have

**Lemma 4.11.** Let \(\lambda \in X^+\). For each field \(K\) which is an \(\mathcal{A}\)-algebra, \(V'_K(\lambda) \otimes V_K(\lambda)\) is isomorphic to the co-Weyl module of \(U_K(\mathfrak{g} \otimes \mathfrak{g})\) associated to \((\lambda, \lambda)\).

We recall the Bruhat order “\(<\)" on the set \(X^+\) of dominant weights. Namely, \(\lambda \prec \mu\) if and only if \(\mu - \lambda \in \sum_{i=1}^m \mathbb{Z}_{\geq 0} \alpha_i\). Note that \(\lambda \prec \mu\) implies that \(\lambda \prec \mu\), where “\(<\)" is the usual dominance order defined on the set of partitions (cf. [11]). In particular, \(|\lambda| \leq |\mu|\). If \(|\lambda| < |\mu|\), then \(\lambda \prec \mu\) implies that \(|\mu| - |\lambda| = 2N\) and hence \(\lambda \prec \mu\); if \(|\lambda| = |\mu|\), then \(\lambda \prec \mu\) implies that \(\lambda^t > \mu^t\), which also implies that \(\lambda^t\) is bigger than \(\mu^t\) under the lexicographical order, hence we still have \(\lambda \succ \mu\). For each \(\lambda \in X^+\), we define
\[
A^r_q(\mathfrak{g})_{\lambda \leq \lambda} := \mathcal{A}\text{-Span}\{ \mathring{G}(b) \mid \lambda \geq \mu \in X^+, b \in \mathring{B}(\mu)\},
\]
\[
A^r_q(\mathfrak{g})_{\lambda < \lambda} := \mathcal{A}\text{-Span}\{ \mathring{G}(b) \mid \lambda > \mu \in X^+, b \in \mathring{B}(\mu)\},
\]
and \(A_q(\mathfrak{g})_{\lambda \leq \lambda} := A^r_q(\mathfrak{g})_{\lambda \leq \lambda} \otimes_{\mathcal{A}} \mathbb{Q}(q)\), \(A_q(\mathfrak{g})_{\lambda < \lambda} := A^r_q(\mathfrak{g})_{\lambda < \lambda} \otimes_{\mathcal{A}} \mathbb{Q}(q)\). For each \(b \in \mathring{B}(\lambda)\), let \(G(b)\) be the corresponding upper global crystal base of \(V^r(\lambda) \otimes V(\lambda)\). By construction, we know that \(V^r(\lambda) \otimes V(\lambda)\) is spanned by \(\{ G(b) \mid b \in \mathring{B}(\lambda)\}\). The following result is implicit in the proof of [34, Section 5.6]. The author is indebted to Professor Kashiwara for pointing out this to him.

**Lemma 4.12.** ([34]) With the decomposition (4.7) in mind, we have that
In particular, the co-Weyl module of $h$ means that we divide the proof into four steps:

1. Let $i$ generality that the same argument used in the proof of Lemma 3.4, we can assume without loss of

2. $\Lambda_{\varphi}(\mu) \otimes V(\mu)$.

In particular, $A_{\varphi}(g)^{\leq \lambda}/A_{\varphi}(g)^{< \lambda} \cong V_{\varphi}(\lambda) \otimes V(\lambda)$;

3. both $A_{\varphi}^{\geq \lambda}(g)$ and $A_{\varphi}^{< \lambda}(g)$ are $\varphi(g \oplus g)$-stable.

For each integer $k \geq 0$, we define

$$A_{\varphi}^{< \lambda}(g)^{\leq k} := \varphi(\text{Span}\{G(b) \mid b \in B(\mu), \mu \in \Lambda^{+}(m, \leq k)\}).$$

Then it is clear that

$$A_{\varphi}^{\geq \lambda}(g)^{\leq k} = \sum_{\lambda \in \Lambda^{+}(m, \leq k)} A_{\varphi}^{\geq \lambda}(g)^{\leq \lambda}.$$
It suffices to show that \((T_q^λ(h,j), e_i^{(t)})_0 \neq 0\) only if \(h \overset{(i,t)}{\sim} j\), and in that case it is equal to \((-1)^{\tilde{e}(h,j)} q^{\tilde{a}(h,j)}\) for some \(\tilde{a}(h,j), a(h,j) \in \mathbb{Z}\).

By definition,
\[
(T_q^λ(h,j), e_i^{(t)})_0 = \sum_{b \in I(2m,n)} \sum_{w \in \Theta_X} (-q^2)^{-\ell(w)} \beta(w)_{h,b} (e_i^{(t)} v_j).
\]

Note that \(h,j \in I^{κ}_λ\). Recall the definition of \(e_i^{(t)}\) (cf. [41, 3.1.5]) and the action of \(e_i\) given in Section 2. To calculate the above sum, it suffices to consider only those \(b \in I(2m,n)\) such that \(b \overset{(i,t)}{\sim} j\) and the entries in each column of \(T_q^λ\) are weakly increasing from top to bottom. By the definition of \(\beta(w)\), we know that \(\beta(w)_{h,b} \neq 0\) only if each column of \(h\) has the same set of entries as the corresponding column of \(b\). It follows that we can further restrict ourselves to those \(b \in I^{κ}_λ\). Now applying [47, Lemma 11.8], we deduce that only the case when \(w = 1, b = h\) can make contribution to the above sum, from which our claim follows immediately.

**Step 2.** We define \(P(N) := \{\{\mu, \nu\} \in X \times X | N_{\mu,\nu} \neq 0\}\). We claim that for some \(k_1, \cdots, k_{m-1} \in \mathbb{Z}\) and some \(\nu \in X, (\nu, \lambda_1 ε_1 + k_2 ε_2 + \cdots + k_m ε_m) \in P(N)\). For simplicity, for each \(i, j \in I^{\mu \nu}_λ\), we write
\[
v(i,j) := \text{pr}_λ(\pi_C(T_q^λ(i,j))).
\]

Then the elements in \(\{v(i,j) | i, j \in I^{\mu \nu}_λ\}\) form a \(K\)-basis of \(\nabla^λ_\nu(\lambda) \otimes \nabla_K(\lambda)\).

Since \(N\) is a submodule of \(\nabla^λ_\nu(\lambda) \otimes \nabla_K(\lambda)\), any weight \((\nu, \mu)\) of \(N\) satisfies \(\lambda \geq \nu, \lambda \geq \mu\). In particular, \(\lambda_1 \geq \nu_1, \lambda_1 \geq \mu_1\). By (4.14), we can assume without loss of generality that for some integers \(k_1, \cdots, k_m\) with \(0 < k_m \leq \lambda_1\) and some \(\nu \in X, (\nu, k_1 ε_1 + \cdots + k_m ε_m) \in P(N)\). Furthermore, we assume that our \(k_m\) is chosen such that \(k_m\) is as big as possible. For each weight vector \(0 \neq x \in \lambda\) with weight \((\nu, k_1 ε_1 + \cdots + k_m ε_m)\), we can write
\[
x = \sum_{i,j \in I^{\nu \mu}_{\nu \mu}} C_{i,j} v(i,j),
\]
for some \(C_{i,j} \in K\). Set
\[
J(x) = \{j \in I^{\nu \mu}_{\nu \mu} | C_{i,j} \neq 0, \text{ for some } i \in I^{\nu \mu}_{\nu \mu}\}.
\]
For each \(j \in J(x)\), the assumption \(k_m > 0\) and the fact that \(j \in I^{\nu \mu}_{\nu \mu}\) imply that \(j_1 = m\). Let \(t_k\) be the standard \(λ\)-tableau such that the numbers \(1, 2, \cdots, k\) entered in usual order along the successive columns of \(λ\). We define
\[
c_j := \#\{1 \leq t \leq k \mid j_t = j_1 \text{ and } t \text{ sits in the first row of } t_λ\},
\]
\[
c_\lambda := \#\{1 \leq t \leq k \mid j_t = j'_1 \text{ and } t \text{ sits in the first row of } t_λ\}.
\]
We assert that \(c_j = k_m\) and \(c_\lambda = 0\), \forall \(j \in J(x)\). In fact, if this is not true, then we can find some \(j_0 \in J(x)\) such that \(0 < c_\lambda \geq c_j\) for any \(j \in J(x)\). It is easy to see that \(e_i^{(t_k)} x\) is a nonzero weight vector with weight \((\nu, \sum_{t=1}^{m-1} k_t ε_t + (k_m + 2c_λ) ε_m)\), a contradiction to the maximality of \(k_m\).

If \(k_m = \lambda_1\), then we are done. Henceforth we assume \(0 < k_m < \lambda_1\). For each \(j \in J(x)\), we define \(b_j \in \{1, 2, \cdots, 2m\}\) to be the least integer (with respect to the
order \(\prec\)” in the first row of \(T^\lambda_\mu\) such that \(b_j \geq m\). In particular, \(b_j \succ m\) (because \(\hat{c}_j = 0\)). We set
\[
J_1(x) := \{j \in J(x) \mid b_j \leq b_h, \forall h \in J(x)\}.
\]
Let \(b = b_j\) for any \(j \in J_1(x)\). For any \(j \in J_1(x)\), we define
\[
c''_j := \#\{1 \leq t \leq n \mid j_t = b \text{ and } t \text{ sits in the first row of } t_\lambda\}.
\]
Let
\[
J_2(x) := \{j \in J_1(x) \mid c''_j \geq c''_h, \forall h \in J_1(x)\}.
\]
Let \(c'' = c''_j\) for any \(j \in J_2(x)\). Then \(c'' > 0\). We define
\[
z := \begin{cases}
f_m(c''+k_m) f_m(c'') \cdots f_{m-1}(c''+1) b_j & \text{if } 1 \leq b < m; \\
f_m(c''+1 \cdots c''+c''+1) f_m(k_m) & \text{if } m < b < 2m.
\end{cases}
\]
Using the first two formulae we have given in Step 1 and the fact that \(T^\lambda_\mu(i,h) = 0\) whenever there are two identical indices appearing in an adjacent position in a column of \(T^\lambda_\mu\) ([47, Corollary 9.2]) as well as the definition of \(T^\lambda_\mu\) and the action of each \(e_i, f_i\) given in Section 2, we deduce that \(zx \in N\) is a nonzero weight vector with weight equal to either
\[
\left(\nu', \sum_{1 \leq s \leq m \atop s \neq h} k_s e_s + (k_h - c'') e_b - (k_m + c'') e_m\right),
\]
or
\[
\left(\nu', \sum_{1 \leq s \leq m \atop s \neq h'} k_s e_s + (k_{h'} + c'') e_{b'} + (k_m + c'') e_m\right).
\]
In both case, applying (4.14) if necessary, we get a contradiction to the maximality of \(k_m\). This proves our claim.

**Step 3.** We claim that for any integer \(1 \leq t \leq m\), there exist some integers \(k''_{t+1}, \ldots, k''_m \in \mathbb{Z}\) and some \(\nu'' \in X\), such that \((\nu'', \sum_{j=1}^t \lambda_j e_j + \sum_{s=t+1}^m k''_s e_s) \in P(N)\).

We make induction on \(t\). If \(t = 1\), this is true by the result obtained in Step 2. Suppose that the claim is true for \(t - 1\). That is, for some integers \(k_1, \ldots, k_m \in \mathbb{Z}\) and some \(\nu \in X\), we have that
\[
(\nu, \tilde{\lambda}) := \left(\nu, \sum_{j=1}^{t-1} \lambda_j e_j + \sum_{s=t}^m k_s e_s\right) \in P(N).
\]
Using (4.14), we can further assume that the weight \(\tilde{\lambda}\) is chosen such that
\[
k_m = \max\{|k_s| \mid t \leq s \leq m\} \geq 0, \text{ and } k_m \text{ is as big as possible.}
\]
We are going to show that for some integers \(\hat{k}_1, \ldots, \hat{k}_{m-1} \in \mathbb{Z}\),
\[
\left(\nu, \sum_{j=1}^{t-1} \lambda_j e_j + \lambda_t e_m + \sum_{s=t}^{m-1} \hat{k}_s e_s\right) \in P(N).
\]
If this is true, then we can apply (4.14) again to get our claim.

If \(\ell(\lambda) \leq t - 1\), then there is nothing to prove. Henceforth, we assume that \(\ell(\lambda) \geq t\). In particular, \(\lambda_t > 0\). By (4.14), we know that \((\nu, \sum_{j=1}^{t-1} \lambda_j e_j + k_m e_t + \sum_{s=t+1}^m k_{s-1} e_s) \in P(N)\), which implies that \(\sum_{j=1}^{t-1} \lambda_j e_j + k_m e_t + \sum_{s=t+1}^m k_{s-1} e_s \leq \lambda\).

It follows that \(0 \leq k_m \leq \lambda\). We assert that \(k_m = \lambda_t\), from which our claim will follows immediately.
Suppose $k_m < \lambda_t$. For each nonzero weight vector $x$ of weight $(\nu, \lambda)$ in $N$, we write

$$x = \sum_{I_j} C_{I_j}(x) u(I_j, j).$$

for some $C_{I_j}(x) \in K$. We define $J(x), c_3, \hat{b}_3, j_1, J_1(x), b, c''_j, J_2(x), c''$ as in Step 2. We first show that $k_m > 0$. In fact, if $k_m = 0$, then by (4.15) we know that $k_i = 0$ for any $t \leq i \leq m$. Using the definition of $I^K_{\text{mys}}$, it is easy to see that for each $j \in J(x)$,

$$\forall 1 \leq l \leq t - 1, l \text{ appears } \lambda_l \text{ times in } j \text{ and } l' \text{ does not appear in } j.$$

Since $\lambda_t > 0$, it follows that for any $j \in J(x), j_1 \in \{m, m', m - 1, (m - 1)', \cdots, t', t\}$.

If there exists $j \in J(x)$ such that $j_1 = m$, then (as $k_m = 0$) $b_j = m'$. In this case $b = m'$ and it is easy to see that $e^{c''(x)} \in N$ is a nonzero weight vector of weight $(\nu, \sum_{j=1}^{t-1} \lambda_j \epsilon_j + 2c'' \epsilon_m)$. Since $c'' > 0$, we get a contradiction to (4.15).

Henceforth we assume that $j_1 \neq m$ for any $j \in J(x)$. In particular, $j_1 > m'$. Note that $j_1 \geq t'$. We define $j$ to be the unique integer such that $j < j_1$ for some $j \in J(x)$ and $j > h_1, \forall h \in J(x)$. Set $c = \max(c[j \in J(x), j_1 > j])$. Then $c > 0$. If $t < j \leq m$ then it is easy to see that $f_m^{(x)} \cdots f_{j+1}^{(x)} \in N$ is a nonzero weight vector of weight $(\nu, \sum_{s=1}^{t-1} \lambda_s \epsilon_s + c \epsilon_j - c \epsilon_m)$; while if $m' < j < t'$ then it is easy to see that $e^{c(x)} \cdots e^{t_k(x)} x \in N$ is a nonzero weight vector of weight $(\nu, \sum_{s=1}^{t-1} \lambda_s \epsilon_s + c \epsilon_j + c \epsilon_m)$. In both cases using (4.14), we get a contradiction to (4.15). Therefore, we must have $k_m > 0$. In particular, for any $j \in J(x), j_1 = m$.

The remaining argument is similar to that used in Step 2 with some slight modification. First, by the same argument used in Step 2, we can show that $j_1 = m, c_3 = k_m$ and $\hat{c}_3 = 0$. In particular, $b \geq m'$. We claim that $b \leq t'$. In fact, if $b \geq t - 1$, then for any $j \in J(x)$, the first row of the remaining tableau after deleting all the entries of $T_j^\lambda$ in $\{t - 1, (t - 1)', \cdots, 2', 1'\}$ has length $k_m$. On the other hand, we know that for any $j \in I^K_{\text{mys}}$ satisfying (4.16), the first row of the remaining tableau after deleting all of the entries of $T_j^\lambda$ in $\{t - 1, (t - 1)', \cdots, 2', 1'\}$ must have length $\lambda_1 \geq \lambda_t$. It follows that $k_m = \lambda_t$, a contradiction to our assumption. This proves that $m' < b \leq t'$.

Now we follow exactly the same argument used in Step 2 to define an element $z$. Note that the condition $m' < b \leq t'$ ensures that $xz \neq 0$ is a weight vector in $N$ with weight

$$\left(\nu, \sum_{j=1}^{t-1} \lambda_j \epsilon_j + \sum_{s=t}^{m} \hat{k}_s \epsilon_s\right)$$

such that $|\hat{k}_m| > k_m$. Applying (4.14) if necessary, we get a contradiction to (4.15). This proves our assertion that $k_m = \lambda_t$.

By induction and set $t = m$, we get that for some $\nu \in X, (\nu, \lambda) \in P(N)$.

Step 4. Starting from a nonzero weight vector $x \in N$ with weight $(\nu, \lambda)$ and Using a similar argument as used in Step 3, we can prove that $(\nu, \lambda) \in P(N)$. Note that the $(\nu, \lambda)$-weight space of $\nabla^\lambda_K(\lambda) \otimes \nabla^\lambda_K(\lambda)$ is one-dimensional and is spanned by $pr_\lambda \left(\pi_C(T_q^\lambda(1, 1'))\right) \otimes 1_K$. As $N$ is a submodule of $\nabla^\lambda_K(\lambda) \otimes \nabla^\lambda_K(\lambda)$, we can deduce that

$$pr_\lambda \left(\pi_C(T_q^\lambda(1, 1'))\right) \otimes 1_K \in N.$$

This completes the proof of the lemma. □
Corollary 4.17. Let $K$ be a field which is an $\mathcal{A}$-algebra and $n \geq 0$ be an integer. Let $\lambda \in \Lambda^+(m, n)$. Then $\nabla_K(\lambda)$ is isomorphic to the co-Weyl module of $U_K(\mathfrak{g})$ associated to $\lambda$.

Proof. If $L$ is a simple $U_K(\mathfrak{g})$-submodule of $\nabla_K(\lambda)$, then $L \otimes L$ is a simple $U_K(\mathfrak{g} \otimes \mathfrak{g})$-submodule of $\nabla_K(\lambda) \otimes \nabla_K(\lambda)$. By Lemma 4.13, $\nabla_K(\lambda) \otimes \nabla_K(\lambda)$ is isomorphic to the co-Weyl module of $U_K(\mathfrak{g} \otimes \mathfrak{g})$ associated to $(\lambda, \lambda)$. So it must have a unique simple $U_K(\mathfrak{g} \otimes \mathfrak{g})$-socle. This implies that $\nabla_K(\lambda)$ also has a unique simple $U_K(\mathfrak{g})$-socle. By the universal property of co-Weyl module, we see that there exists an embedding from $\nabla_K(\lambda)$ into the co-Weyl module $V_K(\lambda)$. Comparing their dimensions, we deduce that this embedding must be an isomorphism. □

Theorem 4.18. With the above notations, we have that

$$\iota_A\left(\tilde{A}_\mathcal{A}^{sy}(2m)\right) = A^\mathcal{E}_\mathcal{A}(g), \quad \iota_A\left(\tilde{A}_\mathcal{A}^{sy}(2m, \leq k)\right) = A^\mathcal{E}_\mathcal{A}(g)_{\leq k}, \quad \forall k \geq 0.$$  

In other words, the quantized coordinate algebra defined by Kashiwara and the quantized coordinate algebra $A^{sy}_\mathcal{A}(2m)$ arising from a generalized FRT construction are isomorphic to each other as $\mathcal{A}$-algebras. Furthermore, we have the following commutative diagram

$$\begin{array}{ccc}
\hat{U}_{\mathbb{Q}(q)} & \xrightarrow{\text{id}} & \hat{U}_{\mathbb{Q}(q)} \\
\downarrow & & \downarrow \\
A_q(\mathfrak{g})^* & \xrightarrow{\iota_A^*} & (\tilde{A}_\mathcal{A}^{sy}(2m))^*
\end{array}$$

where the two vertical maps are induced by the two natural pairings $\langle , \rangle_0, \langle , \rangle_1$ respectively.

Proof. We first show that $\iota_A\left(A^{sy}_\mathcal{A}(2m, \leq k)\right) = A^\mathcal{E}_\mathcal{A}(g)_{\leq k}$ for any $k \geq 0$, from which the equality $\iota_A\left(A^{sy}_\mathcal{A}(2m)\right) = A^\mathcal{E}_\mathcal{A}(g)$ follows at once. We divide the proof into two steps:

Step 1. We claim that $\iota_A\left(A^{sy}_\mathcal{A}(2m, \leq k)\right) \subseteq A^\mathcal{E}_\mathcal{A}(g)_{\leq k}$.

By Lemma 4.8 and the bimodules decomposition we have discussed before, we have

$$\iota_A\left(A^{sy}_\mathcal{A}(2m, \leq k)\right) \subseteq \iota_A\left(A^{sy}_\mathcal{A}(\mathbb{Q}(q), 2m, \leq k)\right) \subseteq A_q(\mathfrak{g})_{\leq k}.$$  

For any integer $\lambda \in \Lambda^+(m, \leq k)$, and any $i, j \in f^{sy}_\lambda$, it is easy to verify directly that

$$\langle \pi_C(T^\lambda_q(i, j)), U_{\mathcal{A}} \rangle_0 \in \mathcal{A}.$$  

It follows that $\iota_A\left(A^{sy}_\mathcal{A}(2m, \leq k)\right) \subseteq A^\mathcal{E}_\mathcal{A}(g)$. Hence by Lemma 4.12,

$$\iota_A\left(A^{sy}_\mathcal{A}(2m, \leq k)\right) \subseteq A_q(\mathfrak{g})_{\leq k} \cap A^\mathcal{E}_\mathcal{A}(g) = A^\mathcal{E}_\mathcal{A}(g)_{\leq k},$$

as required.

Step 2. We now show that $\iota_A\left(A^{sy}_\mathcal{A}(2m, \leq k)\right) = A^\mathcal{E}_\mathcal{A}(g)_{\leq k}$. Our strategy is to show that for any field $K$ which is an $\mathcal{A}$-algebra, $\iota_K := \iota_A \otimes_{\mathcal{A}} 1_K$ is an injection from $A^{sy}_\mathcal{A}(2m, \leq k)$ into $A^\mathcal{E}_\mathcal{A}(g)_{\leq k}$.

For each $\lambda \in \Lambda^+(m, k)$, let $b_\lambda$ be the unique element in $\tilde{B}(\lambda)$ such that $G(b_\lambda) \in V^r(\lambda) \otimes V(\lambda)$ is a highest weight vector of weight $(\lambda, \lambda)$. Note that

$$e_i G(b_\lambda) = 0, \quad \forall e_i \in U_{\mathbb{Q}(q)}(\mathfrak{g} \oplus \mathfrak{g}).$$
We use induction on dominant weights with respect to the order \( \preceq \). If \( \lambda \in X^+ \) is maximal with respect to the order \( \preceq \), then \( \lambda \) is minimal with respect to the order \( \preceq \). Then
\[
\nabla^r(\lambda) \otimes \nabla(\lambda) \subseteq A_q^{2m}(2m, \leq k), \quad V^r(\lambda) \otimes V(\lambda) \subseteq A_q^{2}(g)^{\leq k}.
\]
By Lemma 4.12, we also know that \( \tilde{G}(b_\lambda) = G(b_\lambda) \). In that case, as both \( \tilde{G}(b_\lambda) \) and \( \iota_A(\pi_C(T_q^\lambda(i_1, i_\lambda))) \) are the highest weight vectors of weight \( (\lambda, \lambda) \) in \( A_q^2(g) \), we deduce (by Corollary 4.5 and Lemma 3.12) that
\[
\tilde{G}(b_\lambda) = \pm q^a \iota_A(\pi_C(T_q^\lambda(i_1, i_\lambda))),
\]
for some \( a \in \mathbb{Z} \). Since
\[
\iota_A \left( \nabla^r(\lambda) \otimes \nabla(\lambda) \right) \subseteq A_q^{2}(g)^{\leq k} \bigcap (V^r(\lambda) \otimes V(\lambda)) = V^r(\lambda) \otimes V(\lambda),
\]
we deduce that for any field \( K \) which is an \( \mathcal{A} \)-algebra, \( \iota \) induces a nonzero \( \mathcal{U}_K(g \oplus g) \)-homomorphism \( \iota_K := \iota \otimes_{\mathcal{A} \otimes K} 1_K \) from \( \nabla^r(\lambda) \otimes \nabla(\lambda) \) to \( V^r(\lambda) \otimes V(\lambda) \). By Lemma 4.11 and Lemma 4.13, we know that both modules are co-Weyl modules of \( \mathcal{U}_K(g \oplus g) \) associated to \( (\lambda, \lambda) \). It follows that \( \iota_K \) must always be an isomorphism. Hence \( \iota_A \) must be an isomorphism as well. In particular,
\[
\iota_A \left( \nabla^r(\lambda) \otimes \nabla(\lambda) \right) = V^r(\lambda) \otimes V(\lambda).
\]
In general, let \( \lambda \in X^+ \), assume that for any field \( K \) which is an \( \mathcal{A} \)-algebra, \( \iota_K \) is an injection from \( A_q^{2m}(2m, \leq k) \) into
\[
A_q^K(g)^{\leq k}(\not\geq \lambda) := \sum_{\mu \in \Lambda^\gamma(m, \leq k) \setminus \mu \not\geq \lambda} A_q^K(g)^{\leq \mu}.
\]
We want to prove that \( \iota_K \) is also an injection from \( A_q^{2m}(2m, \leq k) \) into
\[
A_q^K(g)^{\leq k}(\not\leq \lambda) := \sum_{\mu \in \Lambda^\gamma(m, \leq k) \setminus \mu \not\leq \lambda} A_q^K(g)^{\leq \mu}.
\]
By bimodules decomposition, Lemma 4.12 and definition, we know that
\[
\iota_{\mathcal{A}} \left( A_q^{2m}(2m, \leq k)^{\geq \lambda} \right) \subseteq A_q^2(g) \bigcap A_q^1(g)^{\leq k}(\not\leq \lambda) := A_q^K(g)^{\leq k}(\not\leq \lambda).
\]
By the same argument as before, we know that
\[
\tilde{G}(b_\lambda) = \pm q^a \iota_A \left( \pi_C(T_q^\lambda(i_1, i_\lambda)) \right),
\]
for some \( a \in \mathbb{Z} \). Therefore, \( \iota_{\mathcal{A}} \) induces a nonzero \( \mathcal{U}_K(g \oplus g) \)-homomorphism \( \tilde{\tau}_{\mathcal{A}} \) from
\[
\nabla^r(\lambda) \otimes \nabla(\lambda) \cong A_q^{2m}(2m, \leq k)^{\geq \lambda}/A_q^{2m}(2m, \leq k)^{\not\leq \lambda}
\]
to
\[
V^r(\lambda) \otimes V(\lambda) \cong A_q^{2}(g)^{\leq k}(\not\geq \lambda)/A_q^{2}(g)^{\leq k}(\not\leq \lambda).
\]
For any field \( K \) which is an \( \mathcal{A} \)-algebra, we get by base change a nonzero \( \mathcal{U}_K(g \oplus g) \)-homomorphism \( \tilde{\tau}_K \) from
\[
\nabla_K(\lambda) \otimes \nabla_K(\lambda) \cong A_q^{2m}(2m, \leq k)^{\geq \lambda}/A_q^{2m}(2m, \leq k)^{\not\leq \lambda}
\]
to
\[
V^r_K(\lambda) \otimes V_K(\lambda) \cong A_q^K(g)^{\leq k}(\not\geq \lambda)/A_q^K(g)^{\leq k}(\not\leq \lambda).
\]
By Lemma 4.11 and Lemma 4.13, we know that both modules are co-Weyl modules of \( \mathcal{U}_K(g \oplus g) \) associated to \( (\lambda, \lambda) \). It follows that \( \tilde{\tau}_K \) must always be an isomorphism. It follows that \( \iota_K \) must always be an injection, as required.
By induction, we know that for any field $K$ which is an $\mathcal{A}$-algebra, $\iota_K$ is an injection from $A^{sy}_{\mathcal{A}}(2m, \leq k)$ into $A^K_q(\mathfrak{g})$. Comparing their dimensions, we can deduce that it must be an isomorphism, from which we can deduce that $\iota_A$ must be an isomorphism as well. This proves the first part of this Theorem.

It remains to prove the commutativity of the diagram. Note that as $\hat{U}_q(\mathfrak{g} \oplus \mathfrak{g})$-module, $\hat{A}^{sy}_{\mathcal{A}}(2m)$ is generated by the elements $\pi_C(T_q^\lambda(i_\lambda, i_\lambda)), \lambda \in \Lambda^+(m)$. Therefore, it suffices to show that for any $\lambda \in \Lambda^+(m, k)$, $k \geq 0$ and any $P \in U_{\mathcal{A}}, \mu \in X$, $\langle \pi_C(T_q^\lambda(i_\lambda, i_\lambda)), P \rangle_0 = \langle \pi_C(T_q^\lambda(i_\lambda, i_\lambda)), P_{1\mu} \rangle_1$.

Note that there exists also a canonical coupling $\langle , \rangle_1$ between $A_q(\mathfrak{g})$ and $U_{Q(q)}$ (cf. [35]) defined by

$$
\langle \Phi_\lambda(u \otimes v), P \rangle_1 := (u, P v), \forall \lambda \in X^+, u \in V^r(\lambda), v \in V(\lambda), P \in U_{Q(q)},
$$

where $\langle , \rangle$ is the pairing between $V^r(\lambda)$ and $V(\lambda)$ introduced in [34, (7.1.2)]. By direct verification, we see that

$$
\langle \pi_C(T_q^\lambda(i_\lambda, i_\lambda)), P \rangle_0 = \delta_{\lambda, \mu}\langle \pi_C(T_q^\lambda(i_\lambda, i_\lambda)), P \rangle_0,
\langle \iota_A(\pi_C(T_q^\lambda(i_\lambda, i_\lambda))), P \rangle_1 = \delta_{\lambda, \mu}\langle \iota_A(\pi_C(T_q^\lambda(i_\lambda, i_\lambda))), P \rangle_1.
$$

Therefore, it suffices to show that

$$
\pi_C(T_q^\lambda(i_\lambda, i_\lambda)), P \rangle_0 = \langle \iota_A(\pi_C(T_q^\lambda(i_\lambda, i_\lambda))), P \rangle_1.
$$

Using the PBW basis of $U_{Q(q)}$ and Lemma 4.5, we can reduce to the proof of the case where $P$ is generated by $k^{\pm 1}_1, \cdots, k^{\pm 1}_m$. In that case, the proof follows from an easy verification. This completes the proof of the Theorem.

We remark that each integer $k \geq 0$, the dual of the $\hat{U}_q(\mathfrak{g} \oplus \mathfrak{g})$-module $A_q(\mathfrak{g})^{\leq k}$ together with the dual basis of its universal crystal basis actually forms a based module in the sense of [41, 27.1.2].

Henceforth, we shall identify $A^{sy}_{\mathcal{A}}(\mathfrak{g})$ with $\hat{A}^{sy}_{\mathcal{A}}(2m)$ via $\iota_A$. By Theorem 4.18, the quantized coordinate algebra $A^{sy}_{\mathcal{A}}(\mathfrak{g})$ was equipped with two bases. One is $\{\tilde{G}(b)\}_{\lambda \in X^+, b \in \hat{B}(b)}^\star$ another is $\{\pi_C(T_q^\lambda(i, j))\}_{\lambda \in X^+, i, j \in I^{\mathcal{A}}}.\tilde{G}(b)$ is the transition matrix between these two bases must be invertible as a matrix over $\mathcal{A}$. Combining this with (4.9), we can find an $\mathcal{A}$-basis $\{\tilde{G}_\lambda\}_{\lambda \in X^+, i, j \in I^{\mathcal{A}}}$ of $\hat{U}_{\mathcal{A}},$ such that

$$
\langle \pi_C(T_q^\mu(b, l), \tilde{G}_\lambda) = \begin{cases} 1, & \text{if } \lambda = \mu, \lambda = b, j = l, \\ 0, & \text{otherwise.} \end{cases}
$$

for any $\lambda, \mu \in X^+, i, j \in I^{\mathcal{A}}$ and $b, l \in I^{\mathcal{A}}$.

Proof of Theorem 1.4: For each integer $0 \leq l \leq [n/2]$ and each $\lambda \in \Lambda^+(m, n-2l)$, $i, j \in I^{\mathcal{A}}$, we use $(D_{ij}^{\lambda})^\star$ to denote the base element of $S^{sy}_{ij}(2m, n)$ dual to the base element $D_{ij}^{\lambda}$ of $A^{sy}_{\mathcal{A}}(2m, n)$. We have the following commutative diagram:

$$
\begin{array}{c}
\hat{U}_{Q(q)} \xrightarrow{\pi_C} \hat{A}^{sy}_{Q(q)}(2m) \xrightarrow{id} S^{sy}_{Q(q)}(2m, n) \\
\pi_{1\mu} \downarrow \quad \downarrow \\
\hat{A}^{sy}_{Q(q)}(2m) \xrightarrow{\pi_{1\mu}^\star} \left(A^{sy}_{Q(q)}(2m)\right)^\star \\
\end{array}
$$

By Theorem 4.18, (4.19) and the fact

$$
\langle d_q f, \pi_C \bar{\iota}_U(u) \rangle_0 = \langle \pi_C(f), \bar{\iota}_U(u) \rangle_0, \forall u \in \hat{U}_{Q(q)}, f \in A^{sy}_{Q(q)}(2m),
$$
we deduce that
\[
\tilde{\psi}_C^*(\tilde{G}_{ij}^\lambda) = \begin{cases} (D_{ij}^\lambda)^*, & \text{if } |\lambda| \leq n \text{ and } 2l := n - |\lambda| \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}
\]
In particular, this shows that \(\tilde{\psi}_C(\hat{U}_\mathcal{A}) = S^\mathcal{A}_{\mathcal{A}}(2m, n)\). By base change, we know that for any commutative \(\mathcal{A}\)-algebra \(K\),
\[
(\tilde{\psi}_C \downarrow_{\mathcal{A}} \otimes K)(\hat{U}_K) = S^\mathcal{A}_K(2m, n).
\]
This completes the proof of Theorem 1.4.

**Corollary 4.20.** With the above notations, we have that
\[
\ker(\tilde{\psi}_C \downarrow_{\mathcal{A}}) = \mathcal{A}\text{-Span}\{\tilde{G}_{ij}^\lambda \mid i, j \in I^\mathcal{A}_\max, \lambda \in \Lambda^+(m, k), k > n \text{ or } k < n \text{ and } n - k \text{ is odd}\},
\]
\[
\supseteq \mathcal{A}\text{-Span}\{\tilde{G}(b) \mid b \in \hat{B}(\lambda), \lambda \in \Lambda^+(m, k), k \leq n \text{ or } k < n \text{ and } n - k \text{ is odd}\}.
\]
As a result, this is still true if we replace \(\mathcal{A}\) by any commutative \(\mathcal{A}\)-algebra \(K\).

**Proof.** It suffices to prove the second statement. Let \(\pi\) be the set of dominant weights in \(V^\otimes n\). Let \(\mathbb{Q}(\mathcal{A})S(\pi)\) be the generalized Schur algebra associated to \(\pi\) defined by Doty [16]. Then it is easy to check that the homomorphism \(\tilde{\psi}_C\) factors through \(\mathbb{Q}(\mathcal{A})S(\pi)\). Let \(\mathcal{A}\text{-Span}(\pi)\) be the \(\mathcal{A}\)-form of \(\mathbb{Q}(\mathcal{A})S(\pi)\) defined in [16]. For any field \(K\) which is an \(\mathcal{A}\)-algebra, let \(K_S(\pi) := \mathcal{A}\text{-Span}(\pi) \otimes K\). Applying Theorem 1.4 and comparing dimensions we deduce that the natural homomorphism from \(K_S(\pi)\) to \(\text{End}_{\mathcal{Q}(\mathcal{A})(2m+1)}(V^\otimes n_K)\) is an isomorphism. So the same is true if we replace \(K\) by \(\mathcal{A}\). Now the second statement follows directly from the definition of \(\mathcal{A}\text{-Span}(\pi)\) given in [16].

Note that in the proof of the above corollary, we have also given a proof of Corollary 1.7.

5. **Proof of Theorem 1.5 in the case where \(m \geq n\)**

The purpose of this and the next section is to give a proof of Theorem 1.5. Before starting the proof, we make some reduction. By the results in [47], we know that the symplectic \(q\)-Schur algebra is stable under base change. That is, for any commutative \(\mathcal{A}\)-algebra \(K\), there is a canonical isomorphism
\[
S^\mathcal{A}_{\mathcal{A}}(2m, n) \otimes K \cong S^\mathcal{A}_K(2m, n).
\]
Furthermore, \(S^\mathcal{A}_{\mathcal{A}}(2m, n)\) is an integral quasi-hereditary algebra. For any field \(K\) which is an \(\mathcal{A}\)-algebra, \(V_K \cong \Delta_K(\varepsilon_1) \cong \nabla_K(\varepsilon_1) \cong L_K(\varepsilon_1)\) is a tilting module over \(S^\mathcal{A}_K(2m, n)\). It follows that \(V^\otimes n_K\) is also a tilting module over \(S^\mathcal{A}_K(2m, n)\). Applying Theorem 1.4 and using [21, Lemma 4.4 (c)], we get that
\[
\text{End}_{\mathcal{A}_\mathcal{A}}(V^\otimes n_K) \otimes K = \text{End}_{\mathcal{A}_\mathcal{A}}(V^\otimes n_K) \otimes K 
\cong \text{End}_{S^\mathcal{A}_K}(V^\otimes n_K) = \text{End}_{\mathcal{A}_\mathcal{A}}(V^\otimes n_K).
\]
In other words, the endomorphism algebra \(\text{End}_{\mathcal{A}_K}(V^\otimes n_K)\) is stable under base change. Therefore, to prove Theorem 1.5, it suffices to show that the natural
homomorphism from \((\mathfrak{S}_n(-q^{2m+1}, q))_{\mathrm{op}}\) to \(\text{End}_{U_q} \left( V_{\mathfrak{g}(q)}^{\otimes n} \right)\) is surjective. Equivalently, it suffices to prove this is true with \(\mathcal{A}\) replaced by any field \(K\) which is an \(\mathcal{A}\)-algebra.

In this section we shall give a proof of Theorem 1.5 in the case where \(m \geq n\). Henceforth, we shall assume that \(K\) is field, and \(\zeta\) is the image of \(q\) in \(K\) and \(m \geq n\). Note that, in this case, by [29, Proposition 4.2],

\[
\dim \text{End}_{U_q} \left( V_{\mathfrak{g}(q)}^{\otimes n} \right) = \text{End}_{U_q} \left( V_{\mathfrak{g}(q)} \right) = \dim \mathfrak{S}_n(-q^{2m+1}, q) = \dim \mathfrak{S}_n(\zeta^{2m+1}, \zeta).
\]

Therefore, in order to prove Theorem 1.5 in the case \(m \geq n\), it suffices to show that \(\varphi_C\) is injective.

Our strategy to prove the injectivity of \(\varphi_C\) is similar to that used in [10, Section 3], but some extra technical difficulties do arise due to the complexity of the action on \(n\)-tensor space in this quantized case. First, we make some convention on the left and right place permutation actions. Throughout the rest of this paper, for any \(\sigma, \tau \in \mathfrak{S}_n, a \in \{1, 2, \ldots, n\}\), we set

\[
(a)(\sigma\tau) = ((a)\sigma)\tau, \quad (\sigma\tau)(a) = \sigma(\tau(a)).
\]

In particular, we have \(\sigma(a) = (a)\sigma^{-1}\). Therefore, for any \(i = (i_1, i_2, \ldots, i_n) \in I(2m, n), w \in \mathfrak{S}_n\),

\[
viw = (i_1, i_2, \ldots, i_n)w = (i_{w(1)}, i_{w(2)}, \ldots, i_{w(n)}),
\]

which gives the so-called right place permutation action:

\[
v_1 w = (v_{i_1} \otimes \cdots \otimes v_{i_n})w = v_{i_{w(1)}} \otimes \cdots \otimes v_{i_{w(n)}} = v_iw.
\]

For each \(w \in \mathfrak{S}_n\), the element \(T_w\) (resp., \(\hat{T}_w\)) is well defined in the BMW algebra \(\mathfrak{B}_n(-\zeta^{2m+1}, \zeta)\) (resp., in the Hecke algebra \(\mathcal{H}_K(\mathfrak{S}_n)\)) because of the braid relations. Precisely,

\[
T_w = T_{j_1}T_{j_2} \cdots T_{j_k} \in \mathfrak{B}_n(-\zeta^{2m+1}, \zeta), \quad \hat{T}_w = \hat{T}_{j_1}\hat{T}_{j_2} \cdots \hat{T}_{j_k} \in \mathcal{H}_K(\mathfrak{S}_n),
\]

for any reduced expression \(s_{j_1}s_{j_2} \cdots s_{j_k}\) of \(w\).

Set

\[
\hat{\beta} := \sum_{1 \leq i \leq 2m} \left( qE_{i,i} \otimes E_{i,i} \right) + \sum_{1 \leq i,j \leq 2m, i \neq j} E_{i,j} \otimes E_{j,i} + (q - q^{-1}) \sum_{1 \leq i,j \leq 2m} \left( E_{i,i} \otimes E_{j,j} \right).
\]

For \(i = 1, 2, \ldots, n - 1\), we set

\[
\hat{\beta}_i := id_{V_{\otimes i-1}} \otimes \hat{\beta} \otimes id_{V_{\otimes n-i-1}}.
\]

By [33], the map \(\hat{\varphi}\) which sends each \(\hat{T}_i\) to \(\hat{\beta}_i\) for \(i = 1, 2, \ldots, n - 1\) can be naturally extended to a representation of \(\mathcal{H}_q(\mathfrak{S}_n)\) on \(V_{\otimes n}^\vee\).

**Lemma 5.1.** Let \(i = (i_1, i_2, \ldots, i_n) \in I(2m, n)\). Suppose that \(i_j \neq i_k\) for any \(1 \leq j, k \leq n\). Then for any \(w \in \mathfrak{S}_n\),

\[
v_iT_w = v_i\hat{T}_w;
\]

if furthermore \(i_1 > i_2 > \cdots > i_n\), then \(v_iT_w = v_iw\).

**Proof.** This follows directly from the definition of action (see the formulae given above (3.2)).
Let \( q \) be an indeterminate over \( \mathbb{Z} \). Let \( \tilde{R} \) be the ring
\[
\tilde{R} := \mathbb{Z}[r, r^{-1}, q, q^{-1}, x]/\langle (1 - x)(q - q^{-1}) + (r - r^{-1}) \rangle.
\]
\( \tilde{R} \) naturally becomes an \( R \)-algebra (with \( z \) acting as \( q - q^{-1} \)). We regard \( \mathcal{A} \) as an \( \tilde{R} \)-algebra by sending \( r \) to \( -q^{2m+1} \) and \( x \) to \( 1 - \sum_{i=-m}^{m} q^{2i} \). The resulting \( \mathcal{A} \)-algebra is exactly \( \mathcal{B}_n(-q^{2m+1}, q)_{\mathcal{A}} \). We refer the reader to the beginning of Section 3 to understand how \( \mathcal{A} \) is an \( R \)-algebra. Let \( \mathcal{B}_n(r, q) := \mathcal{B}_n(r, x, z) \otimes_R \tilde{R} \).

In [22], a cellular basis for \( \mathcal{B}_n(r, q) \) indexed by certain bitableaux was constructed by Enyang. The advantage of that basis is that it is explicitly described in terms of generators and amenable to computation. In the remaining part of this section we shall use Enyang’s results in [22]. We first recall some notations and notions.

For each natural number \( n \) and each integer \( f \) with \( 0 \leq f \leq \lfloor n/2 \rfloor \), set
\[
\nu := \nu_f := ((2^f), (n - 2f)), \quad \text{where} \quad (2^f) := (2, 2, \ldots, 2) \quad \text{and} \quad (n - 2f)
\]
as partitions of \( 2f \) and \( n - 2f \) respectively. So \( \nu \) is a bipartition of \( n \). Let \( t' \) be the standard \( \nu \)-bitableau in which the numbers \( 1, 2, \ldots, n \) appear in order along successive rows of the first component tableau, and then in order along successive rows of the second component tableau. We define
\[
\mathcal{D}_\nu := \left\{ d \in \mathfrak{S}_n \mid t'^d \text{ is row standard and the first column of } t^{(1)} \text{ is an increasing sequence when read from top to bottom} \right\}.
\]
For each partition \( \lambda \) of \( n - 2f \), we denote by \( \text{Std}(\lambda) \) the set of all the standard \( \lambda \)-tableaux with entries in \( \{2f + 1, \ldots, n\} \). The initial tableau \( t^\lambda \) in this case has the numbers \( 2f + 1, \ldots, n \) in order along successive rows.

**Lemma 5.2.** Let \( d \in \mathcal{D}_\nu \). Assume that \( d = d's_j \) with \( \ell(d) = \ell(d') + 1 \), where \( 1 \leq j \leq n - 1 \). Then \( d' \in \mathcal{D}_{\nu_j} \).

**Proof.** Since \( d = d's_j \) and \( \ell(d) = \ell(d') + 1 \), we get \((j)(d')^{-1} < (j + 1)(d')^{-1}\). It follows that \( j, j + 1 \) can not both sit in the second component of \( t'^d \). If \( j, j + 1 \) sits in different components of \( t'^d \), then the lemma follows immediately. So it suffices to consider the case where both \( j, j + 1 \) sits in the first component of \( t'^d \). But \( d \in \mathcal{D}_f \), we deduce that \( j, j + 1 \) must be located in different rows and can not be both located in the first column of \( t'^{(2j)}d' \), which implies that \( d' \in \mathcal{D}_{\nu_j} \) (as \( t'^d \) and \( t'^d \) differ only in the positions of \( j, j + 1 \)). \( \square \)

Recall that (cf. [22]) the map \( T_i \mapsto T_i, E_i \mapsto E_i, \forall 1 \leq i \leq n - 1 \) extends naturally to an algebra anti-automorphism of \( \mathcal{B}_n(-q^{2m+1}, q)_{\mathcal{A}} \). We denote this anti-automorphism by “\( ^s \)”.

**Lemma 5.3.** ([22]) For each \( \lambda \vdash n - 2f \), \( s, t \in \text{Std}(\lambda) \), let \( m_{\lambda,s,t} \) denote the canonical image in \( \mathcal{B}_n(-q^{2m+1}, q)_{\mathcal{A}} \) of the corresponding Murphy basis element (cf. [45]) of the Hecke algebra \( H_{\mathcal{A}}(\mathfrak{S}_{\{2f+1, \ldots, n\}}) \). Then the set
\[
\left\{ T_{d_1}^sE_1E_3\cdots E_{2f-1}m_{\lambda,s,t}T_{d_2} \middle| 0 \leq f \leq \lfloor n/2 \rfloor, \lambda \vdash n - 2f, s, t \in \text{Std}(\lambda), d_1, d_2 \in \mathcal{D}_\nu, \text{ where } \nu := ((2^f), (n - 2f)) \right\}
\]
is a cellular basis of the BMW algebra \( \mathcal{B}_n(-q^{2m+1}, q)_{\mathcal{A}} \).

As a consequence, by combining Lemma 5.3 and [22, (3.3)], we have

**Corollary 5.4.** With the above notations, the set
\[
\left\{ T_{d_1}^sE_1E_3\cdots E_{2f-1}T_{\sigma}T_{d_2} \middle| 0 \leq f \leq \lfloor n/2 \rfloor, \sigma \in \mathfrak{S}_{\{2f+1, \ldots, n\}}, d_1, d_2 \in \mathcal{D}_\nu, \text{ where } \nu := ((2^f), (n - 2f)) \right\}
\]
is a basis of the BMW algebra \( \mathcal{B}_n(-q^{2m+1}, q)_{\mathcal{A}} \).
By base change, we can apply the previous results to the specialized algebra $B_n(-\zeta^{2m+1}, \zeta)$. The main result in this section is

**Theorem 5.5.** Suppose $m \geq n$. Then the natural homomorphism

$$\varphi_C : B_n(-\zeta^{2m+1}, \zeta) \rightarrow \text{End}_K(V^{\otimes n})$$

is injective. In particular, $\varphi_C$ maps $B_n(-\zeta^{2m+1}, \zeta)$ isomorphically onto $\text{End}_K(\gamma(\mathfrak{sp}_{2n}))$. Thus it is enough to calculate $\text{ann}_{B_n(-\zeta^{2m+1}, \zeta)}(V^\otimes n)$.

To prove the theorem, it suffices to show the annihilator $\text{ann}_{B_n(-\zeta^{2m+1}, \zeta)}(V^\otimes n)$ is $\{0\}$. Note that

$$\text{ann}_{B_n(-\zeta^{2m+1}, \zeta)}(V^\otimes n) = \bigcap_{v \in V^\otimes n} \text{ann}_{B_n(-\zeta^{2m+1}, \zeta)}(v).$$

For each integer $f$ with $0 \leq f \leq [n/2]$, we denote by $B^{(f)}$ the two-sided ideal of $B_n(-\zeta^{2m+1}, \zeta)$ generated by $E_1 E_2 \cdots E_{2f-1}$. Note that $B^{(f)}$ is spanned by all the basis elements whose indexing diagrams contain at least $2f$ horizontal edges (edges in each of the top and the bottom rows in the diagrams). We recall a notion introduced in [10]. For $t \in T(2m,n)$, an ordered pair $(s,t)$ $(1 \leq s \leq t \leq n)$ is called a symplectic pair in $i$ if $i_s = (i_t)'$. Two ordered pairs $(s,t)$ and $(u,v)$ are called disjoint if $\{s,t\} \cap \{u,v\} = \emptyset$. We define the symplectic length $\ell_s(v_t) = \ell_s(i_t)$ to be the maximal number of disjoint symplectic pairs $(s,t)$ in $i$. Note that if $f > \ell_s(v_t)$, then clearly $B^{(f)} \subseteq \text{ann}(v_t)$.

**Lemma 5.6.** $\text{ann}_{B_n(-\zeta^{2m+1}, \zeta)}(V^\otimes n) \subseteq B^{(1)}$.

*Proof.* Since $m \geq n$, the tensor $v := v_n \otimes v_{n-1} \otimes \cdots \otimes v_1$ is defined. Note that $i \neq j$ for any $i, j \in \{1, 2, \cdots, n\}$. Applying Lemma 5.1, we deduce that $v T_w = v \hat{T}_w$ for any $w \in \mathfrak{S}_n$. Now $B^{(1)}$ is contained in the annihilator of $v \hat{T}_w$, hence is contained in the intersection of all annihilators of $v \hat{T}_w$, as $w$ ranges over $\mathfrak{S}_n$. Hence $B^{(1)}$ annihilates the subspace $S$ spanned by the $v \hat{T}_w$, where $w$ runs through $\mathfrak{S}_n$.

On the other hand, since $m \geq n$, it is well known (cf. [21]) that the annihilator of $v$ in the Hecke algebra $\mathcal{H}_K(\mathfrak{S}_n)$ is $\{0\}$. Therefore, we conclude that $\text{ann}_{B_n(-\zeta^{2m+1}, \zeta)}(V^\otimes n) \subseteq B^{(1)}$.$\square$

Suppose that we have already shown $\text{ann}_{B_n(-\zeta^{2m+1}, \zeta)}(V^\otimes n) \subseteq B^{(f)}$ for some natural number $f \geq 1$. We want to show $\text{ann}_{B_n(-\zeta^{2m+1}, \zeta)}(V^\otimes n) \subseteq B^{(f+1)}$. If $f > [n/2]$ then $\text{ann}_{B_n(-\zeta^{2m+1}, \zeta)}(V^\otimes n) \subseteq B^{(f)} = 0$ implies that $\text{ann}_{B_n(-\zeta^{2m+1}, \zeta)}(V^\otimes n) = 0 \subseteq B^{(f+1)}$ and we are done. Thus we may assume $f \leq [n/2]$.

For $i := (i_1, \cdots, i_n) \in I(2m,n)$, we define $\text{WT}(i) = \lambda = (\lambda_1, \cdots, \lambda_{2m})$, where $\lambda_j$ is the number of times that $v_j$ occurs as tensor factor in $v_i$ for each $1 \leq j \leq 2m$. We call $\text{WT}(i)$ the $GL_{2m}$-weight of $v_i$. Note that for a given composition $\lambda$ of $n$, the simple tensors of $GL_{2m}$-weight $\lambda$ span a $\mathcal{H}_K(\mathfrak{S}_n)$-submodule $M^\lambda$ of $V^\otimes n$, thus

$$V^\otimes n = \bigoplus_{\lambda \in \Lambda(2m,n)} M^\lambda$$

as $\mathcal{H}_K(\mathfrak{S}_n)$-module, where $\Lambda(2m,n)$ denotes the set of compositions of $n$ into not more than $(2m)$ parts. It is well-known that $M^\lambda$ is isomorphic to the permutation representation of $\mathcal{H}_K(\mathfrak{S}_n)$ corresponding to $\lambda$. 
As a consequence, each element $v \in V^{\otimes n}$ can be written as a sum
$$v = \sum_{\lambda \in \Lambda(2m,n)} v_{\lambda}$$
for uniquely determined $v_{\lambda} \in M^{\lambda}$.

Following [10], we consider the subgroup $\Pi$ of $\mathfrak{S}_{(1, \ldots, 2f)} \leq \mathfrak{S}_n$ permuting the rows of $t^{(1)}$ but keeping the entries in the rows fixed. The group $\Pi$ normalizes the stabilizer $\mathfrak{S}_{(2f)}$ of $t^{(1)}$ in $\mathfrak{S}_{2f}$. We set $\Psi := \mathfrak{S}_{(2f)} \rtimes \Pi$. By [10, Lemma 3.7], we have
$$\mathfrak{S}_{2f} = \bigsqcup_{d \in D_f} \Psi d,$$
where “$\bigsqcup$” means a disjoint union. We set $D_f := D_{v_f} \cap \mathfrak{S}_{2f}$.

Lemma 5.7. Let $d \in D_f$. Then for any $w \in \Psi$, $\ell(wd) \geq \ell(d)$.

Proof. Let $w \in \Psi$. By definition, we can write $w = w''w'$, where $w'' \in \mathfrak{S}_{(2f)}$, $w' \in \Pi$. Note that $\mathfrak{S}_{(2f)}$ is generated by $s_1, s_3, \ldots, s_{2f-1}$, and $\Pi$ is generated by $s_1, s_2, \ldots, s_{f-1}$, where $s_i := s_{2i-1}s_{2i-1}^{-1}s_{2i+1}s_{2i}$ for $i = 1, 2, \ldots, f - 1$.

We claim that $\ell(w) = \ell(w''w'd) \geq \ell(w'd)$ in this case, which follows directly from the counting of the number of inversions and the fact that for any $\sigma \in \mathfrak{S}_n$,
$$\ell(\sigma) = \left\{(i, j) \mid 1 \leq i < j \leq n, (i)\sigma > (j)\sigma \right\}.$$  
Therefore, it remains to show that $\ell(w'd) \geq \ell(d)$.

Note that the subgroup generated by $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_{f-1}$ is isomorphic to the symmetric group $\mathfrak{S}_f$. We use $\ell$ to denote the length function of $\mathfrak{S}_f$ with respect to the generators $\tilde{s}_i, i = 1, \ldots, f - 1$. We use induction on $\ell(w')$. If $\ell(w') = 1$, then $w' = \tilde{s}_i = s_{2i}s_{2i-1}s_{2i+1}s_{2i}$. In this case, our claim $\ell(w'd) \geq \ell(d)$ follows directly from the counting of the number of inversions. Suppose that for any $w' \in \Pi$ with $\ell(w') = k - 1$, we have $\ell(w'd) \geq \ell(d)$. Let $w' \in \Pi$ with $\ell(w') = k$. We can write $w' = \tilde{s}_j w'$, where $1 \leq j \leq f - 1$, such that $\ell(w') = \ell(w') + 1$. Now counting the number of inversions, it is easy to see that $\ell(w'd) = \ell(s_j w'd) \geq \ell(w'd)$. On the other hand, by induction hypothesis, $\ell(w'd) \geq \ell(d)$. Therefore, $\ell(w'd) \geq \ell(d)$, as required. This completes the proof of the lemma. \hfill $\square$

Let $P_f := \{(i_1, \ldots, i_{2f}) \mid 1 \leq i_1 < \cdots < i_{2f} \leq n\}$. For each $J \in P_f$, we use $d_J$ to denote the unique element in $D_{v_f}$ such that the first component of $t^{(1)}d_J$ is the tableau obtained by inserting the integers in $J$ in increasing order along successive rows of $t^{(1)}$. Let $D_{(2f,n-2f)}$ be the set of distinguished right coset representatives of $\mathfrak{S}_{(2f,n-2f)}$ in $\mathfrak{S}_n$. Clearly $d_J \in D_{(2f,n-2f)}$, and every element of $D_{(2f,n-2f)}$ is of the form $d_J$ for some $J \in P_f$. The following result is well known.

Lemma 5.8. Let $J = (i_1, i_2, \ldots, i_{2f}) \in P_f$. Then
$$(s_{2f}s_{2f+1} \cdots s_{i_{2f}-1})(s_{2f-1}s_{2f} \cdots s_{i_{2f}-1})(s_{2f} \cdots s_{i_{2f}-1})(s_{2f-1} \cdots s_{i_{2f}-1})$$
is a reduced expression of $d_J$.

By [10, Lemma 3.8], $D_{v_f} = \bigsqcup_{J \in P_f} D_J d_J$.

Definition 5.9. We define
$$c = (c_1, c_2, \ldots, c_{2f})$$
$$= \left((m - f + 1)', \ldots, (m - 1)', m', m, m - 1, \ldots, m - f + 1\right).$$
Therefore, there exists an element of the first column of $\ell$ of proof of the statements (1) and (3). Let

$$(a)d_0 = \begin{cases} (a+1)/2, & \text{if } a \in \{1, 3, \cdots, 2f - 1\}, \\ 2f + 1 - a/2, & \text{if } a \in \{2, 4, \cdots, 2f\}. \end{cases}$$

Then $d_0 \in D_f$. Counting the number of inversions, we deduce that $\ell(d_0) = f(f - 1)$. On the other hand, by direct verification, we know that

$$d_0 = (s_2f - 2s_2f - 1)(s_2f - 4s_2f - 3s_2f - 2s_2f - 1) \cdots (s_2s_3 \cdots s_2f - 2s_2f - 1).$$

It follows that

$$(5.10) \quad (s_2f - 2s_2f - 1)(s_2f - 4s_2f - 3s_2f - 2s_2f - 1) \cdots (s_2s_3 \cdots s_2f - 2s_2f - 1)$$

is a reduced expression of $d_0$.

**Definition 5.11.** We define

$$v_{c_0} = v_{c_0}d_0^{-1} = v_{c_0(m - f + 1)} \otimes v_{c_0(m - f + 1)'} \cdots \otimes v_{c_0(m - 1)' \otimes v_{c_0(m - 1)}' \otimes v_{c_0m} \otimes v_{c_0m'}}.$$

**Lemma 5.12.** Let $d \in D_f, J_0 := \{n - 2f + 1, n - 2f + 2, \cdots, n\}$.

1. There exists $w \in \mathfrak{S}_n$, such that $d_0 = dw$ and $\ell(d_0) = \ell(d) + \ell(w)$;
2. For any $J \in \mathcal{P}_f$, there exists $w' \in \mathfrak{S}_n$, such that $d_{J_0} = d_Jw'$ with $\ell(d_{J_0}) = \ell(d_J) + \ell(w')$;
3. For any $d \in D_f$ with $d \neq d_0d_{J_0}$, there exists integer $1 \leq j \leq n - 1$ such that $d_{s_j} \in D_f$ and $\ell(d_{s_j}) = \ell(d) + 1$.

**Proof.** The statement (2) is a well-known result, see e.g., [11]. We only give the proof of the statements (1) and (3).

First, we claim that there exists an element $w_1 \in \mathfrak{S}_2f$, such that $dw_1 \in D_f, \ell(dw_1) = \ell(d) + \ell(w_1)$ and the numbers $1, 2, \cdots, f$ are located in the first column of $t^{(2^f)}d$. In fact, let $1 \leq a \leq f$ be the smallest integer which is not located in the first column of $t^{(2^f)}d$, then for any integer $b$ which is located in the first column of $t^{(2^f)}d$, we must have $b \geq a + 1$. Furthermore, any integer between $a$ and $b - 1$ can only be located in the first $a - 1$ rows of the second column of $t^{(2^f)}$. Now let

$$w_1 := s_b - 2s_b - 2 \cdots s_a.$$  

It is easy to see that $dw_1 \in D_f, \ell(dw_1) = \ell(d) + \ell(w_1)$, and $1, 2, \cdots, a$ are located in the first column of $t^{(2^f)}$. Using induction on $a$, we can find an element $w' \in \mathfrak{S}_2f$, such that $dw' \in D_f, \ell(dw') = \ell(d) + \ell(w')$ and the numbers $1, 2, \cdots, f$ are located in the first column of $t^{(2^f)}$. Let $w_{0,f}$ be the unique element in $\mathfrak{S}_2f$ such that

$$(a)w_{0,f} = \begin{cases} (a + 1)/2, & \text{if } a \in \{1, 3, \cdots, 2f - 1\}, \\ f + a/2, & \text{if } a \in \{2, 4, \cdots, 2f\}. \end{cases}$$

Then, $dw' = w_{0,f}w_1'$ for some $w_1' \in \mathfrak{S}_{(f + 1, f + 2, \cdots, 2f)}$.

Let $w_0' \in \mathfrak{S}_{(f + 1, f + 2, \cdots, 2f)}$ be defined by

$$(f + 1, f + 2, \cdots, 2f)w_0' = (2f, 2f - 1, \cdots, f + 1).$$

Then $w_0'$ is the unique longest element in $\mathfrak{S}_{(f + 1, f + 2, \cdots, 2f)}$. It is well known that there exists $w'' \in \mathfrak{S}_{(f + 1, f + 2, \cdots, 2f)}$ such that $w_0'' = w_0''w''$ and $\ell(w_0'') = \ell(w_0') + \ell(w'')$.

It is clear that

$$\ell(dw''w'') = \ell(w_0'f w_0'w_0') = \ell(w_0'f) + \ell(w_0') + \ell(w_0'f) + \ell(w_0') + \ell(w'') = \ell(w_0'f w_0') + \ell(w'') = \ell(dw'') + \ell(w'') = \ell(d) + \ell(w') + \ell(w''),$$

Therefore,

$$\ell(d) + \ell(w') = \ell(dw''w'') \leq \ell(d) + \ell(w'') \leq \ell(d) + \ell(w') + \ell(w'').$$
which forces $\ell(w''w') = \ell(w') + \ell(w'')$. Since $d_0 = dw_0,\ell w_0' = dw_0,\ell w_1'' = d(w''w')$. The statement (1) is proved.

Let $d \in \mathcal{D}_{cf}$ with $d \neq d_0J_0$. We can write $d = d_1d_2J_1$, where $d_1 \in \mathcal{D}_f, J_1 \in \mathcal{P}_f$. For any $d' \in \mathcal{D}_{(2f,n-2f)}$ and any integer $1 \leq j \leq n-1$ satisfying $d''s_j$ and $\ell(d') = \ell(d') + 1$, it is well known that $d'' \in \mathcal{D}_{(2f,n-2f)}$. If $J_1 \neq J_0$, then the statement (3) follows directly from this well-known fact and the statement (2).

Now we assume $J_1 = J_0$. Then $d_1 \neq d_0$. By statement (1), we can find $s_1 \in \mathcal{G}_{2f}$ such that $d_1s_1 \in \mathcal{D}_f$ and $\ell(d_1s_1) = \ell(d_1) + 1$. Then $d^{-1}_{J_0}s_1d_{J_0} = s_j$ for some $s_j \in \mathcal{G}_{(n-2f,2f)}$. Note that

$$\ell(ds_j) = \ell(d_1ds_1s_j) = \ell(d_1s_1) + \ell(ds_1) = \ell(d_1) + 1 + \ell(ds_1)$$

as required. \hfill \Box

We define

$$I_f := \left\{ v \in \mathcal{V} \mid 1 \leq b_{n-2f} < \ldots < b_1 \leq m - f \right\}.$$ 

It is clear that $\ell_s(v \otimes v_\delta) = f$ for all $v \in I_f$.

For an arbitrary element $v \in \mathcal{V}^{\otimes n}$, we say the simple tensor $v = v_{i_1} \otimes \ldots \otimes v_{i_a}$ is involved in $v$, if $v_j$ has nonzero coefficient in writing $v$ as a linear combination of the basis $\{ v \}_{j \in I(2m, n)}$ of $\mathcal{V}^{\otimes n}$. For later use, we note the following very useful fact: for any $(i_1, i_2), (j_1, j_2) \in I(2m, 2)$,

$$v_{j_1} \otimes v_{j_2} \text{ is involved in } (v_{i_1} \otimes v_{i_2})^{J'} \text{ only if } j_1 \leq i_2 \text{ and } j_2 \geq i_1.$$ 

**Lemma 5.14.** Let $s, i_1, \ldots, i_a$ be integers such that

1. $1 \leq s \leq f$;
2. $\ell_s(i_1, \ldots, i_a) = 0$;
3. for each integer $1 \leq t \leq \alpha$, either $1 \leq i_t < m - f + 1$ or $m' \leq i_t \leq (m - f + s + 1)$.

Let $d$ be a distinguished right coset representative of

$$\mathcal{G}_{(1,2,\ldots,2s)} \times \mathcal{G}_{(2s+1,\ldots,2s+a)}$$

in $\mathcal{G}_{2s+a}$. Let $J := \{ a+1, a+2, \ldots, a+2s \}$. Let

$$\tilde{v} = v_{i_1} \otimes \ldots \otimes v_{i_a},$$

$$\tilde{w} = v_{(m-f+1')} \otimes v_{(m-f+2')} \otimes \ldots \otimes v_{(m-f+s)} \otimes \ldots \otimes v_{m-f+1}.$$ 

Then

$$(\tilde{v} \otimes \tilde{w})I_{j-1} = \zeta^s \delta_{d,d_j} \tilde{v} \otimes \tilde{w} + \sum_{u \in I(2m,2s+a)} A_u v_{u_1} \otimes \ldots \otimes v_{u_{2s+a}},$$

for some $z \in \mathbb{Z}$, and $A_u \neq 0$ only if

4. $\ell_s(u_1, \ldots, u_{2s}) < s$; and
5. any integer $x$ with $(m - f + 1)' < x \leq 2m$ or $m - f + s + 1 \leq x \leq m$ does not appear in $(u_1, \ldots, u_{2s})$.

**Proof.** We write

$$j_1 = (1)d, \quad j_2 = (2)d, \quad \ldots j_{2s} = (2s)d.$$ 

Then $1 \leq j_1 < j_2 < \cdots < j_{2s} \leq 2s + a$, and $d = d_j$ if and only if $j_t = a + t$ for each integer $1 \leq t \leq 2s$. Note that

$$\frac{1}{(s_{j_1-1}s_{j_2-1}s_{j_3} \cdots s_{j_{2s}+1})}$$

is a reduced expression of $d^{-1}$. 

If \( \ell(d) = 1 \), i.e., \( d = 1 \), then there is nothing to prove. In general, let
\[
d'^{-1} = (s_{j_2-1} \cdots s_3 s_2)(s_{j_3-1} \cdots s_4 s_3) \cdots (s_{j_2-s-1} \cdots s_2 s_{2s-1}).
\]
Then
\[
d'^{-1} = (s_{j_2-1} \cdots s_3 s_2)d'^{-1}(s_{j_2-s-1} \cdots s_{2s} s_{2s}), \quad \ell(d') = \ell(d') + j_1 + j_{2s} - 2s - 1,
\]
and \( d' \) is a distinguished right coset representative of
\[
\mathcal{G}(2s, \cdots, 2s - 1) \times \mathcal{G}(2s, \cdots, 2s + a - 1)
\]
in \( \mathcal{G}(2s, \cdots, 2s + a - 1) \). Note that since \( j_1 \leq a + 1 \), any simple tensor involved in
\[
(\tilde{v} \otimes \tilde{w}) T_{j_1-1} \cdots T_2 T_1
\]
is of the form
\[
v_{i_2} \otimes v_{i_3} \otimes \cdots \otimes v_{i_a} \otimes v_{t_{j+1}} \otimes \tilde{w}' \otimes v_{(m-f+s)'},
\]
where
\[
\tilde{w}' = v_{(m-f+s)'}, \quad \otimes \cdots \otimes v_{(m-f+s)'} \otimes v_{m-f+s+} \cdots \otimes v_{m-f+1}.
\]
It suffices to consider the following three cases:

**Case 1.** \( 1 \leq j_1 \leq a \). Then \( \hat{i}_{a+1} = m - f + s \). Since \( \ell_s(i_1, \cdots, i_a) = 0 \), by the definition of \( \beta' \), it is easy to see that
\[
\mathrm{WT}(i_1, \cdots, i_a) = \mathrm{WT}(\hat{i}_1, \cdots, \hat{i}_a).
\]
In particular, either \( 1 \leq \hat{i}_1 < m - f + 1 \) or \( m' < \hat{i}_1 \leq (m - f + s + 1)' \). We define
\[
\tilde{v}'' = v_{i_2} \otimes v_{i_3} \otimes \cdots \otimes v_{i_a} \otimes \tilde{v}' \otimes v_{(m-f+s)} \otimes \cdots \otimes v_{m-f+1}.
\]

Then our conclusion follows easily from induction on \( a \).

**Case 2.** \( j_1 = a + 1 \) and \( \hat{i}_1 \neq (m - f + 1)' \). Then we must have \( j_2 = a + 1 \) for each integer \( 1 \leq t \leq 2s \). By the definition of \( \beta' \), it is easy to see that \( \hat{i}_1 \neq (m - f + 1)' \)
implies that either \( 1 \leq \hat{i}_1 < m - f + 1 \) or \( m' < \hat{i}_1 \leq (m - f + s + 1)' \). We define
\[
\tilde{v}''_2 = v_{i_2} \otimes v_{i_3} \otimes \cdots \otimes v_{i_a} \otimes v_{i_{a+1}}, \quad \tilde{w}'' = \tilde{w}.
\]

By induction on \( \ell(d) \), we deduce that
\[
(\tilde{v}''_2 \otimes \tilde{w}'') T_{d'-1} = \zeta^{z} \tilde{w}'' \otimes \tilde{v}'' + \sum_{u \in I(2m, 2s + a - 2)} A_u v_{u_1} \otimes \cdots \otimes v_{u_{2s + a - 2}}.
\]
for some \( z \in \mathbb{Z} \) and \( A_u \neq 0 \) only if

(a1) \( \ell_s(u_1, \cdots, u_{2s-2}) < s - 1 \); and

(a2) any integer \( x \) with \( 1 \leq x < m - f + 1 \) or \( m' \leq x \leq (m - f + s)' \) does not appear in \( (u_1, \cdots, u_{2s-2}) \).

It remains to consider
\[
\left( \tilde{v}_{i_2} \otimes \tilde{w}'' \otimes \tilde{v}'' \otimes v_{m-f+1} \right) T_{a + 2s - 1} \cdots T_{2s+1} T_{2s},
\]
\[
\left( \tilde{v}_{i_2} \otimes v_{u_1} \otimes \cdots \otimes v_{u_{2s-1}} \otimes v_{m-f+1} \right) T_{a + 2s - 1} \cdots T_{2s+1} T_{2s},
\]
where \( u_1, \cdots, u_{2s-2} \) satisfy the conditions (a1), (a2) above. Note that under the action of \( T_{a + 2s - 1} \cdots T_{2s+1} T_{2s} \), the first \( (2s-1) \) parts do not change, while by (5.13) the \( 2s \) position will be replaced by a vector of the form \( v_p \) with \( p \leq m - f + 1 \). Now using the condition (a2), our conclusion follows immediately.

**Case 3.** \( j_1 = a + 1 \) and \( \hat{i}_1 = (m - f + 1)' \). Then we also must have \( \hat{i}_{2s+1} = i_{2s} \), \( j_t = a + t \) and \( \hat{i}_t = i_{t-1} \) for each integer \( 2 \leq t \leq 2s \). In this case, our conclusion follows from the same argument used in the proof of Case 2. \( \square \)
Lemma 5.15. Let $b \in I_f$, $v = v_b \otimes v_c \in V^\otimes n$. Let $w \in D_{\nu f}$. If $w \neq d_0d_0^\sigma$, then

$T_w^*e_1e_3 \cdots e_{2f-1} \in \text{ann}(v)$.

Proof. Let $w \in D_{\nu f}$. We write $w = d_1d_f$, where $d_1 \in D_f, J \in P_f$. Then

$vT_w^* = (v_b \otimes v_c)t_{d_1}T_{d_f}^{-1}$.

Let $J = (i_1, i_2, \cdots, i_{2f})$. By Lemma 5.8,

$(s_{i_1-1} \cdots s_{2s_1})(s_{i_2-1} \cdots s_{3s_2}) \cdots (s_{i_{2f-1}-1} \cdots s_{2f} s_{2f-1})(s_{i_{2f}-1} \cdots s_{2f+1} s_{2f})$

is a reduced expression of $d_f^{-1}$. Using Lemma 5.14, we get that

$$(v_b \otimes v_c)t_{d_f}^{-1} = \zeta \delta_{d, \nu} (v_c \otimes v_b) + \sum_{\nu'} A_{\nu'} \nu'$,

for some integer $z$ and some $A_{\nu'} \in K$, where the subscript $\nu'$ runs over all the simple $n$-tensor such that its first (2f)-parts have symplectic length less than $f$. It follows that if $J \neq J_0$, then we are done. Henceforth, we assume that $J = J_0$, then $d_1 \neq d_0$ and

$vT_w^*e_1e_3 \cdots e_{2f-1} = \zeta \delta_{d, \nu} (v_c \otimes v_b)T_{d_f}^{-1}e_1e_3 \cdots e_{2f-1},$

for some integer $a$.

By (5.10),

$$\sigma = (s_{2f-1} s_{2f-2} \cdots s_{3s_2})(s_{2f-1} s_{2f-2} \cdots s_{5s_4}) \cdots (s_{2f-1} s_{2f-2} s_{3f-4} s_{2f-2}^{-1})$$

is a reduced expression of $d_f^{-1}$. By Lemma 5.12, for any $d \in D_f$, $d$ is less or equal than $d_f$ in the Bruhat order. It follows that there is a subexpression of $\sigma$ which is equal to a reduced expression of $d_f^{-1}$. Combining this with the definitions of the operator $/\ell$ and the indices $b, c$, it is easy to see that

$$(v_c \otimes v_b)t_{d_f}^{-1} = v_cT_{d_f}^{-1} \otimes v_b = \zeta \delta_{d, \nu} v_c d_0^{-1} \otimes v_b + \sum_{\nu'} B_{\nu'} \nu',$$

for some integer $z \in Z$ and some $B_{\nu'} \in K$, where the subscript $\nu'$ runs over all the simple $n$-tensor $v_j \otimes \cdots \otimes v_j$ such that there exists $1 \leq s \leq f$ satisfying $j_{2s-1} \neq (j_{2s})'$. Now using the fact that $d_1 \neq d_0$, it is easy to see that $(v_c \otimes v_b)t_{d_1}e_1e_3 \cdots e_{2f-1} = 0$. Hence, $vT_w^*e_1e_3 \cdots e_{2f-1} = 0$, as required. \hfill $\square$

We are now ready to prove the key lemma from which our main result in this section will follow easily.

Lemma 5.18. Let $S$ be the subset

$$\left\{ T_{d_1}^* e_1e_3 \cdots e_{2f-1} t_{d_2} \right\} = \left\{ d_1, d_2 \in D_{\nu f}, d_1 \neq d_0d_0^\sigma, \sigma \in \mathfrak{S}_{2f+1, \cdots, n} \right\}$$

of the basis (5.4) of $\mathfrak{B}_n(-\zeta^{2m+1}, \zeta)$, and let $U$ be the subspace spanned by $S$. Then

$$B^{(f)} \bigcap \left( \bigcap_{b \in I_f} \text{ann}(v_b \otimes v_c) \right) = B^{(f+1)} \oplus U.$$  

Proof. By definition of $I_f$, $\ell_v(v_b) = 0$. Hence $\ell_v(v_b \otimes v_c) = f$. It follows that $B^{(f+1)} \subseteq \text{ann}(v_b \otimes v_c)$. This, together with the Lemma 5.15, shows that the right-hand side is contained in the left-hand side.
Now let \( x \in B^{(f)} \cap (\bigcap_{b \in I_f} \text{ann}(v_b \otimes v_c)) \). We want to show that \( x \in B^{(f+1)} \oplus U \).

Using the basis (5.4) of the BMW algebra \( \mathfrak{B}_n(-\zeta^{2m+1}, \zeta) \), we may assume that

\[
x = T_{d_0d_0} E_1 E_3 \cdots E_{2f-1} \left( \sum_{d \in D_{\nu_f}} z_d T_d \right),
\]

where \( \nu = \nu_f = (2f, (n-2f)) \) and the elements \( z_d \), \( d \in D_{\nu_f} \) are taken from the \( K \)-linear subspace spanned by \( \{ T_w \mid w \in \mathfrak{S}_{(2f+1, \ldots, n)} \} \). We then have to show \( x = 0 \), or equivalently, to show that \( z_d = 0 \) for each \( d \in D_{\nu_f} \).

Let \( d \in D_{\nu_f} \). We write \( z_d = \sum_{\sigma \in \mathfrak{S}_{(2f+1, \ldots, n)}} B_{\sigma} T_{\sigma} \), where \( B_{\sigma} \in K \) for each \( \sigma \).

Suppose that \( v_b z_d = 0 \) for any \( b \in I_f \). By the definition of \( I_f \), it is easy to see that

\[
0 = v_b z_d = \sum_{\sigma \in \mathfrak{S}_{(2f+1, \ldots, n)}} B_{\sigma} v_b T_{\sigma}.
\]

However, since \( m-2f > n-2f \), the Hecke algebra \( \mathcal{H}_K(\mathfrak{S}_{(2f+1, \ldots, n)}) \) acts faithfully on \( v_b \). This implies \( B_{\sigma} = 0 \) for each \( \sigma \in \mathfrak{S}_{(2f+1, \ldots, n)} \). Thus \( z_d = 0 \), as required.

Therefore, to show that \( z_d = 0 \), it suffices to show that \( v_b z_d = 0 \) for any \( b \in I_f \).

We divide the proof into two steps:

**Step 1.** We first prove that \( z_{d_0d_0} = 0 \), equivalently, \( v_b z_{d_0d_0} = 0 \) for any \( b \in I_f \).

Let \( b \in I_f \).

\[
0 = (v_b \otimes v_c) x = \sum_{d \in D_{\nu_f}} (v_b \otimes v_c) T_{d_0} T_{d_0}^{-1} E_1 E_3 \cdots E_{2f-1} z_d T_d
\]

\[
= \zeta^2 \sum_{d \in D_{\nu_f}} (v_b E_1 E_3 \cdots E_{2f-1} \otimes v_b) z_d T_d
\]

\[
= \zeta^2 \sum_{d \in D_{\nu_f}} (v_b E_1 E_3 \cdots E_{2f-1} \otimes v_b z_d) T_d,
\]

for some integer \( z \in \mathbb{Z} \), where the third equality follows from (5.16) and (5.17).

By [10, Lemma 3.8], for each \( d \in D_{\nu_f} \), we can write \( d = d_1 d_f \), where \( d_1 \in D_{\nu_f} \), \( J \in P_f \), and \( \ell(d) = \ell(d_1) + \ell(d_f) \). Hence \( T_d = T_{d_1} T_{d_f} \). Therefore

\[
0 = \sum_{d \in D_{\nu_f}} \left( v_b E_1 E_3 \cdots E_{2f-1} \otimes v_b z_d \right) T_d
\]

\[
= \sum_{J \in P_f} \sum_{d_1 \in D_{\nu_f}} \left( v_b E_1 E_3 \cdots E_{2f-1} \otimes v_b z_{d_1 d_f} \right) T_d T_{d_f}
\]

\[
= \sum_{J \in P_f} \sum_{d_1 \in D_{\nu_f}} \left( v_b E_1 E_3 \cdots E_{2f-1} T_{d_1} \otimes v_b z_{d_1 d_f} \right) T_{d_f}.
\]

We want to show that \( v_b z_{d_0d_0} = 0 \). Note that

\[
v_b E_1 E_3 \cdots E_{2f-1} = \sum_{1 \leq i_1, \ldots, i_f \leq 2m} \pm \zeta^{a_i} v_{i_1} \otimes v_{i_1} \otimes v_{i_2} \otimes v_{i_2} \otimes \cdots \otimes v_{i_f} \otimes v_{i_f},
\]

for some \( a_i \in \mathbb{Z} \). Note also that each simple tensor \( v_b \) involved \( v_b z_{d_0d_0} \) has the same \( GL_{2m} \)-weight as \( v_b \). Let \( (i_1, \cdots, i_f) \in I(2m, f) \). We claim that
(a) for any $\tilde{b}, \bar{b} \in I(2m, n - 2f), \tilde{j} \in I(2m, 2f)$ with $WT(\tilde{b}) = WT(\bar{b})$ and $\ell_s(j) = f$, the simple $n$-tensor $(v_{\mathfrak{c}_j} \otimes v_{\bar{b}})T_{d_j}$ is involved in

$$(v_{\tilde{j}_1} \otimes v_{\tilde{j}_2} \otimes \cdots \otimes v_{\tilde{j}_f} \otimes v_{\bar{b}})T_{d_j}$$

if and only if $(\tilde{j}_1, \tilde{j}_2, \ldots, \tilde{j}_f) = c_{d_0}J_0$ and $\bar{b} = \bar{b}$.

(b) for any $d_1 \in \mathfrak{S}_f$ with $d_1 \leq d_0$ (in the Bruhat order), the simple $(2f)$-tensor $v_{\mathfrak{c}_j} \otimes v_{\bar{b}}$ is involved in

$$(v_{\tilde{j}_1} \otimes v_{\tilde{j}_2} \otimes \cdots \otimes v_{\tilde{j}_f} \otimes v_{\bar{b}})T_{d_j}$$

if and only if $(\tilde{j}_1, \tilde{j}_2, \ldots, \tilde{j}_f) = ((m - f + 1)', \ldots, (m - 1)', m')$ and $d_1 = d_0$.

Once these two claims are proved to be true, it is easy to see that the identity $v_{\mathfrak{c}_j} \otimes v_{\bar{b}} = 0$ follows at once. Therefore, it remains to prove the claims (a) and (b).

Suppose that $(v_{\mathfrak{c}_j} \otimes v_{\bar{b}})d_{j_0}$ is involved in

$$(v_{\tilde{j}_1} \otimes v_{\tilde{j}_2} \otimes \cdots \otimes v_{\tilde{j}_f} \otimes v_{\bar{b}})T_{d_j}.$$ 

By definition,

$$(v_{\mathfrak{c}_j} \otimes v_{\bar{b}})d_{j_0} = v_{\bar{b}} \otimes v_{\mathfrak{c}_j}.$$

Let $J = (j_1, j_2, \ldots, j_{2f})$. Then

$$(s_{2f}\ldots s_{2f+1}\ldots s_{2f-1})(s_{2f-1}s_{2f-2}\ldots s_{2f-1})\cdots (s_1s_2\ldots s_{j_1-1})$$

is a reduced expression of $d_J$. Note that $1 \leq j_1 < \cdots < j_{2f} \leq 2f$. If $j_{2f} \neq n$, then the rightmost vector of any simple tensor involved in $(v_{\tilde{j}_1} \otimes v_{\tilde{j}_2} \otimes \cdots \otimes v_{\tilde{j}_f} \otimes v_{\bar{b}})T_{d_j}$ must be $v_{\bar{b}_{j_{2f}}}$, which is impossible (because $\bar{b}_{j_{2f}} \leq m - f$). Therefore, we deduce that $j_{2f} = n$. Let $\Sigma$ be the set of all the simple $n$-tensor $v$ which is involved in

$$(v_{\tilde{j}_1} \otimes v_{\tilde{j}_2} \otimes \cdots \otimes v_{\tilde{j}_f} \otimes v_{\bar{b}})T_{2f}T_{2f+1}\cdots T_{n-1}.$$ 

Note that $\tilde{b}_t \leq m - f$ for each $t$. We claim that for each integer $t$ with $1 \leq t \leq n - 2f$, $\tilde{b}_t \neq (\tilde{j}_{2f})'$.

In fact, if $1 \leq t \leq n - 2f$ is the smallest integer such that $\tilde{b}_t = (\tilde{j}_{2f})'$, then the $(2f + t)$th position of any simple tensor involved in

$$(v_{\tilde{j}_1} \otimes v_{\tilde{j}_2} \otimes \cdots \otimes v_{\tilde{j}_f} \otimes v_{\bar{b}})(T_{2f}T_{2f+1}\cdots T_{2f+t-1})$$

is a vector $v_a$ with either $a > (m - f + 1)'$ or $a < (m - f + 1)$. It follows (from the definition of $\beta'$ and (5.13)) that the $n$th position of any simple tensor involved in

$$(v_{\tilde{j}_1} \otimes v_{\tilde{j}_2} \otimes \cdots \otimes v_{\tilde{j}_f} \otimes v_{\bar{b}})(T_{2f}T_{2f+1}\cdots T_{n-1})$$

is a vector $v_a$ with either $a \leq m - f$ or $a \geq (m - f)'$. Since the action of $(T_{2f}T_{2f-1}\cdots T_{2f-1})\cdots (T_1T_2\cdots T_{j_1-1})$ on any simple $n$-tensor does not change its rightmost vector, we deduce that $v_{\bar{b}} \otimes v_{\mathfrak{c}_j}$ can not be involved in

$$(v_{\tilde{j}_1} \otimes v_{\tilde{j}_2} \otimes \cdots \otimes v_{\tilde{j}_f} \otimes v_{\bar{b}})(T_{2f}T_{2f+1}\cdots T_{n-1})(T_{2f-1}T_{2f}\cdots T_{j_{2f-1}-1})$$

$$\cdots (T_1T_2\cdots T_{j_1-1}),$$

a contradiction.

Therefore, $\tilde{b}_t \neq (\tilde{j}_{2f})'$ for any $1 \leq t \leq n - 2f$. It follows that $v = v_{\tilde{j}_1} \otimes v_{\tilde{j}_2} \otimes \cdots \otimes v_{\tilde{j}_{2f-1}} \otimes v_{\bar{b}} \otimes v_{\tilde{j}_{2f}}$ is the unique simple $n$-tensor in $\Sigma$ such that $v_{\bar{b}} \otimes v_{\mathfrak{c}_j}$ is involved in

$v(T_{2f-1}T_{2f}\cdots T_{j_{2f-1}-1})\cdots (T_1T_2\cdots T_{j_1-1}).$
In particular, we deduce that \( \tilde{j}_{2f} = (c_0d_0)_{2f} = m - f + 1 \). Now we are in a position to use induction on \( n \). It follows easily that

\[
(j_1, j_2, \cdots, j_{2f-1}) = (n - 2f + 1, n - 2f + 2, \cdots, n - 1),
\]

\[
(\tilde{j}_1, \cdots, \tilde{j}_{2f-1}) = ((c_0d_0)_{1}, \cdots, (c_0d_0)_{2f-1}), \quad \tilde{b} = \tilde{b}.
\]

Conversely, by the definition of \( \beta' \), \((v_{c_0d_0} \otimes v_{\tilde{b}})T_{d_0} = v_{\tilde{b}} \otimes v_{c_0d_0} \). This proves the claim (a).

We now turn to the claim (b). By Lemma 5.1 and direct verification, it is easy to see that the simple \((2f)\)-tensor \( v_{c_0d_0} \) is involved in

\[
(v_{m-f+1} \otimes v_{m-f+2} \otimes \cdots \otimes v_{m})T_{d_0}.
\]

This proves one direction of the claim (b). Now suppose that the simple \((2f)\)-tensor \( v_{c_0d_0} \) is involved in

\[
(v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \cdots \otimes v_{i_{2f}})(T_{d_1},
\]

where \((i_1, \cdots, i_{2f}) \in \mathcal{I}(2m, f) \) with \( d_1 \leq d_0 \) (in the Bruhat order). Then \( d_1 \) has a reduced expression which is a subexpression of (5.10). Hence we can write \( d_1 = d'_1d''_1 \), where \( d'_1 \) is a subexpression of

\[
(s_2\cdots s_{2f-1})(s_{2f-3}f_2\cdots s_{2f-1}),
\]

\( d''_1 \) is a subexpression of \( s_2s_3\cdots s_{2f-2} \), such that \( \ell(d'_1) = \ell(d'_1') + \ell(d''_1) \). Then \( T_{d_1} = T_{d'_1}T_{d''_1} \).

By definition of \( d''_1 \), any simple tensor involved in

\[
(v_{i_1} \otimes v_{i_1'} \otimes v_{i_2} \otimes v_{i_2'} \cdots \otimes v_{i_{2f-1}})(T_{d'_1},
\]

is of the form

\[
v_{i_1} \otimes v_{i_1'} \otimes v_{i_2} \otimes v_{i_2'} \cdots \otimes v_{i_{2f-2}},
\]

where \( l = (l_1, l_2, \cdots, l_{2f-2}) \in \mathcal{I}(2m, 2f - 2) \) with \( \ell_5(l) = f - 1 \). By assumption, we can choose one such simple \( n \)-tensor, say

\[
v_{c_0d_0} := v_{i_1} \otimes v_{i_1'} \otimes v_{i_2} \otimes v_{i_2'},
\]

such that \( v_{c_0d_0} \) is involved in \( vT_{d'_1} \). By definition of \( d''_1' \), it is easy to see that

\[
i_1 = (m - f + 1)' \). We claim that

\[
(b1) \quad (l_1, l_2, \cdots, l_{2f-2}) = ((m - f + 2)', (m - f + 3)', \cdots, m', m, \cdots, m - f + 3, m - f + 2);
\]

\[
(b2) \quad d''_1' = s_2s_3\cdots s_{2f-2}.
\]

If both (b1) and (b2) are true, then the claim (b) follows easily from induction on \( f \). Therefore, it suffices to prove the two claims (b1) and (b2).

Recall that \( c_0d_0 = ((m - f + 1)', (m - f + 2)', (m - f + 3)', \cdots, m', m - 1', \cdots, m - f + 1) \). If \( l_1 < (m - f + 2)' \), then (by (5.13)) the second position of any simple \((2f)\)-tensor involved in \( \nu[1]T_{d''_1} \) is always occupied by a vector \( v_a \) with \( a < (m - f + 2)' \), which is impossible (because \( v_{c_0d_0} \) is involved in \( vT_{d''_1} \)). Hence \( l_1 \geq (m - f + 2)' \). By similar reason, we can deduce that \( l_1 \) can not be strictly bigger than \((m - f + 1)\). Therefore, \( l_1 \in \{(m - f + 1)', (m - f + 2)' \} \). Assume that \( l_1 = (m - f + 1)' \). Then by (5.13) and the fact that \( \ell(l) = f - 1 \), it is easy to see for any simple \((2f)\)-tensor \( v_{k_1} \otimes \cdots \otimes v_{k_{2f}} \) involved in \( \nu[1]T_{d''_1} \), we have \( k_t \leq m - f + 1 \) for some \( t \geq 3 \), a contradiction. This forces \( l_1 = (m - f + 2)' \). Repeating the same argument, we deduce that for any integer \( 1 \leq t \leq f - 1, l_t = (m - f + t + 1)' \). Now since \( \ell_5(l) = f - 1 \), it follows that \( \{l_f, l_{f+1}, \cdots, l_{2f-2}\} = \{m - f + 1', (m - f + 2)', \cdots, (m - 1)\} \). In particular, \( l_t \neq m' \) for any \( t \). Using the same arguments as before, we easily deduce that for each integer \( f \leq t \leq 2f - 2, l_t = m - t + f \). This proves (b1). Now (b2) follows
We divide the discussion into cases: $K$ where $0 \in \sigma$ such that $x$ is involved in $x$. Let $S'$ be the subset of the basis (5.4) of $\mathfrak{B}_n(-\zeta^{2m+1}, \zeta)$, and let $U'$ be the subspace spanned by $S'$. By the main results we obtained in Step 1, we know that

$$B^{(f)} \left( \bigcap_{b \in I_f} \mathrm{ann}(v_b \otimes v_c) \right) \subseteq B^{(f+1)} \oplus U \oplus U'.$$

We want to prove that

$$B^{(f)} \left( \bigcap_{b \in I_f} \mathrm{ann}(v_b \otimes v_c) \right) \subseteq B^{(f+1)} \oplus U.$$

If this is not the case, then by Lemma 5.15,

$$U' \cap B^{(f)} \left( \bigcap_{b \in I_f} \mathrm{ann}(v_b \otimes v_c) \right) \neq 0.$$

Let $\tilde{\Sigma}$ be the set of those $d_2 \in D_{\nu_j}$, such that there exist some $x' \in U' \cap B^{(f)} \left( \bigcap_{b \in I_f} \mathrm{ann}(v_b \otimes v_c) \right)$ satisfying

$$T_{d_0d_{j_0}}^* E_1 E_3 \cdots E_{2f-1} T_{\sigma_2} T_{d_2}$$

is involved in $x'$. We choose $d_2 \in \tilde{\Sigma}$ such that $\ell(d_2)$ is as big as possible.

By the definition of $U'$, $d_2 \neq d_0d_{j_0}$. It follows from Lemma 5.12 that we can find an integer $j$ with $1 \leq j \leq n - 1$, such that $d_2s_j \in D_{\nu_j}$ and $\ell(d_2s_j) = \ell(d_2) + 1$. Let

$$x' \in U' \cap B^{(f)} \left( \bigcap_{b \in I_f} \mathrm{ann}(v_b \otimes v_c) \right)$$

such that $T_{d_0d_{j_0}}^* E_1 E_3 \cdots E_{2f-1} T_{d_2} T_{d_3}$ is involved in $x'$. We claim that there exist $\tau \in \mathfrak{S}_{\{2f+1, \ldots, n\}}$, $d_3 \in D_{\nu_j}$ with $\ell(d_3) > \ell(d_2)$, such that

$$T_{d_0d_{j_0}}^* E_1 E_3 \cdots E_{2f-1} T_{\sigma_2} T_{d_3}$$

is involved in $x'T_{d_3}$.

We write

$$x' = A_0 T_{d_0d_{j_0}}^* E_1 E_3 \cdots E_{2f-1} z_{d_2} T_{d_2} + \sum_{\sigma \in \mathfrak{S}_{\{2f+1, \ldots, n\}} \setminus \{d_0d_{j_0}\} \setminus \{d_3\}} A_{d_0d_{j_0}d_3} E_1 E_3 \cdots E_{2f-1} T_{\sigma_2} T_{d_3},$$

where $0 \neq z_{d_2} \in K \cdot \text{Span} \{T_w | w \in \mathfrak{S}_{\{2f+1, \ldots, n\}} \}$, $T_{\sigma_2}$ is involved in $z_{d_2}$, $0 \neq A_0 \in K$, $A_{d_0d_{j_0}d_3} \in K$ for each $d_0d_{j_0}, \sigma$.

Note that $T_{d_0d_{j_0}}^* E_1 E_3 \cdots E_{2f-1} T_{d_2} T_{d_3} = T_{d_0d_{j_0}}^* E_1 E_3 \cdots E_{2f-1} T_{d_2} T_{d_3}$. It remains to analyze how each $T_{d_0d_{j_0}}^* E_1 E_3 \cdots E_{2f-1} T_{\sigma_2} T_{d_3}$ is expressed as a linear combination of the basis elements given in Corollary 5.4. Our purpose is to show that $T_{d_0d_{j_0}}^* E_1 E_3 \cdots E_{2f-1} T_{\sigma_2} T_{d_3}$ is not involved in each

$$T_{d_0d_{j_0}}^* E_1 E_3 \cdots E_{2f-1} T_{\sigma_2} T_{d_3} T_j.$$
Case 1. \( \ell(d'_s s_j) = \ell(d'_2) + 1 \). In this case, \( T_{d'_s} T_j = T_{d'_2} \). We write \( d'_s s_j = w d_4 \), where \( w \in \mathcal{S}_{\nu_j} \) and \( \nu' d_4 \) is row standard. Then \( \ell(w d_4) = \ell(w) + \ell(d_4) \). Furthermore, we can write \( T_{w_4} = x_4 T_{w_4'} \), where

\[
x_4 \in (T_1, T_3, \ldots, T_{2f-1}), \ w'_4 \in \mathcal{S}_{\nu_j^{(z)}}.
\]

We have

\[
T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{d_0}^* T_{d'_2}^* T_{d'_2} = T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{w_4} T_{d_4}
\]

for some \( A_{\bar{k}} \in K \), where the second equality follows from the fact that \( E_{2i-1} T_{2i-1} = r^{-1} E_{2i-1} \) for each \( i \) and \([22, (3.2)]\). Now we use \([22, Proposition\ 3.7]\) to express \( T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{d_4} \) as a linear combination of the basis elements given in Corollary 5.4. Note that in the notation of \([22, Proposition\ 3.7]\), it is easy to check (in all the three cases listed in \([22, Proposition\ 3.7]\)) that \( u = s_{2j} s_{2j+1} s_{2j-1} s_{2j} u \), \( \ell(u) \leq \ell(w) \), \( \ell(v) = \ell(w) - 1 \), \( \ell(v') \leq \ell(w) - 1 \). It follows that each \( T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{d_4} \) can be expressed as a linear combination of the basis elements of the form

\[
T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_{s'} T_{d_2'},
\]

where \( s'' \in \mathcal{S}_{\{2f+1, \ldots, n\}} \), \( d_2' \in \mathcal{D}_{\nu_j} \), with either

1. \( \ell(d_2') < \ell(d_4) \); or
2. \( \ell(d_2') = \ell(d_4) \) and \( d_4 = zd_2' \) for some \( z \in \Pi \).

Note that \( \ell(d_2') \leq \ell(d_4) + 1 \) with equality holds only if \( d'_2 s_j = d_4 \). As a result, \( \ell(d_2') \leq \ell(d_2) + 1 \). If \( \ell(d_2') = \ell(d_2) + 1 \), then \( d'_2 \neq d_2 s_j \) and we are done. If \( \ell(d_2') = \ell(d_2) + 1 \), then \( \ell(d_2') = \ell(d_2) \), \( \ell(d_4') = \ell(d_4) = \ell(d_2') + 1 \) and \( d_2 s_j = d_4 = zd_2' \) for some \( z \in \Pi \). In this case we claim that \( d'_2 \neq d_2 s_j \). This is true because otherwise we would deduce that \( d'_2 = zd_2 \), which is impossible (since \( d'_2, d_2 \) are different elements in \( \mathcal{D}_{\nu_j} \)). This completes the proof in Case 1.

Case 2. \( \ell(d'_2 s_j) = \ell(d'_2) - 1 \). Then by Lemma 5.2, \( d'_2 s_j \in \mathcal{D}_{\nu_j} \). In this case (note that our \( T_j \) is \( \zeta^{-1} T_j \) in \([22\text{'s notation}]\), by \([22, Lemma\ 2.1]\),

\[
T_{d'_2} T_j = T_{d'_2 s_j} + (\zeta - \zeta^{-1})(T_{d'_2} + \zeta^{-2m-1} T_{d_2 s_j} E_j).
\]

Therefore,

\[
T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{d'_2} T_{d'_2} = T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{d'_2} s_j + (\zeta - \zeta^{-1}) T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{d'_2} s_j
\]

\[
+ (\zeta - \zeta^{-1}) \zeta^{-2m-1} T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{d_2} s_j E_j.
\]

By comparing their length, we see that \( d_2 s_j \notin \{ d'_2 s_j, d_2' \} \). Hence

\[
T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{d_2} s_j
\]

is not involved in

\[
T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{d_2} s_j + (\zeta - \zeta^{-1}) T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{d_2} s_j.
\]

Note that \( d_2 s_j, d_3 \in \mathcal{D}_{\nu_j} \) imply that \( j, j+1 \) are not in the same row of \( \nu' d_2 s_j \). Combining this with \([22, Lemma\ 3.4, 3.5, Proposition\ 3.3, 3.4] \), we deduce that \( T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{d_2} s_j \) is not involved in

\[
(\zeta - \zeta^{-1}) \zeta^{-2m-1} T_{d_0 d_0}^* E_1 E_3 \cdots E_{2f-1} T_s T_{d_2} s_j E_j,
\]

as required. This completes the proof in Case 2.
As a consequence, we can deduce that $T_{d_0,d_2}^*E_1E_3\cdots E_{2f-1}T_{\sigma_2}T_{d_2}s_j$ is always involved in $x'T_j$. Note that $x'T_j \in B^{(f)} \cap \left( \bigcap_{b \in I_f} \operatorname{ann}(v_b \otimes v_{T_j}) \right)$ and $\ell(d_2s_j) > \ell(d_2)$. We get a contradiction to our choice of $d_2$. This completes the proof of the lemma.

**Theorem 5.19.** With the above notations, we have that

$$B^{(f)} \cap \operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}(V^{\otimes n}) \subseteq B^{(f+1)}.$$ 

**Proof.** Suppose that

$$B^{(f)} \cap \operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}(V^{\otimes n}) \nsubseteq B^{(f+1)}.$$

By Lemma 5.18, we can find an element $x$ in $B^{(f)} \cap \operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}(V^{\otimes n})$ of the following form:

$$x = z + y,$$

where $z \in B^{(f+1)}$, $0 \neq y \in U$.

We write

$$y = \sum_{d_1,d_2,\sigma} A_{d_1,d_2,\sigma} T_{d_1}^*E_1E_3\cdots E_{2f-1}T_{\sigma}T_{d_2},$$

where the subscripts run over all elements $d_1,d_2,\sigma$ such that $d_1,d_2 \in D_\nu$ and $d_1 \neq d_0d_2$. Let $\Sigma$ be the set of those $d_1 \in D_\nu$, such that there exist some $x \in B^{(f)} \cap \operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}(V^{\otimes n})$, $d_2 \in D_\nu$, $\sigma \in \mathfrak{S}_{2f+1,\ldots,n}$ satisfying

$$T_{d_1}^*E_1E_3\cdots E_{2f-1}T_{\sigma}T_{d_2} \text{ is involved in } y.$$ 

We choose $d_1' \in \Sigma$ such that $\ell(d_1')$ is as big as possible.

By definition of $U$, $d_1' \neq d_0d_2$. It follows from Lemma 5.12 that we can find an integer $j$ with $1 \leq j \leq n - 1$, such that $d_1's_j \in D_\nu$ and $\ell(d_1's_j) = \ell(d_1') + 1$. Let $x \in B^{(f)} \cap \operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}(V^{\otimes n})$, $d_2' \in D_\nu$, $\sigma' \in \mathfrak{S}_{2f+1,\ldots,n}$ be such that $T_{d_1}^*E_1E_3\cdots E_{2f-1}T_{\sigma}T_{d_2}$ is involved in $y$. Note that

$$T_j^*z + T_j^*y = T_j^*x \in B^{(f)} \cap \operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}(V^{\otimes n}), T_j^*z \in B^{(f+1)}.$$

Our purpose is to show that there exist some $d_2' \in D_\nu$, $\sigma' \in \mathfrak{S}_{2f+1,\ldots,n}$, such that $T_{d_1's_j}^*E_1E_3\cdots E_{2f-1}T_{\sigma}T_{d_2}$ is involved in $T_j^*y$. If this true, then we get a contradiction to our choice of $d_1'$, and we are done.

We write

$$y = A_0T_{d_1'}E_1E_3\cdots E_{2f-1}T_{\sigma}T_{d_2'} + \sum_{d_1,d_2,\sigma} A_{d_1,d_2,\tau} T_{d_1}^*E_1E_3\cdots E_{2f-1}T_{\sigma}T_{d_2},$$

where $0 \neq A_0 \in K$, $A_{d_1,d_2,\sigma} \in K$ for each $d_1,d_2 \in D_\nu$, $\sigma \in \mathfrak{S}_{2f+1,\ldots,n}$.

Note that $T_j^*T_{d_1}^*E_1E_3\cdots E_{2f-1}T_{\sigma}T_{d_2'} = T_{d_1's_j}^*E_1E_3\cdots E_{2f-1}T_{\sigma}T_{d_2'}$. Using the same argument as in the proof of Step 2 of Lemma 5.18, we can show that

$$T_{d_1's_j}^*E_1E_3\cdots E_{2f-1}T_{\sigma}T_{d_2'}$$

is not involved in $T_j^*T_{d_1}^*E_1E_3\cdots E_{2f-1}T_{\sigma}T_{d_2}$, as required. This completes the proof of the theorem.

**Proof of Theorem 1.5 in the case where** $m \geq n$: It follows easily from Lemma 5.6, Theorem 5.19 and induction on $f$ that

$$\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}(V^{\otimes n}) = 0.$$
This proves the injectivity of \( \varphi_\mathcal{C} \), and hence \( \varphi_\mathcal{C} \) must map \( \mathfrak{B}_n(-\zeta^{2m+1}, \zeta) \) isomorphically onto \( \text{End}_{U_K}(\text{sp}_{2m}(V_K^{\otimes n})) \) in this case.

6. Proof of Theorem 1.5 in the case where \( m < n \)

The purpose of this section is to give a proof of Theorem 1.5 in the case where \( m < n \). Our strategy is similar to but technically more difficult than [10, Section 4].

Let \( m_0 \) be a natural number with \( m_0 \geq n \). Let \( \tilde{\mathcal{V}}_\mathcal{a} \) be a free \( \mathcal{A} \)-module of rank \( 2m_0 \). Assume that \( \tilde{\mathcal{V}}_\mathcal{a} \) is equipped with a skew bilinear form \( (, ) \) as well as an ordered basis \( \{ v_1, v_2, \ldots, v_{2m_0} \} \) satisfying

\[
(v_i, v_j) = \begin{cases} 1, & \text{if } i + j = 2m_0 + 1 \text{ and } i < j; \\ -1, & \text{if } i + j = 2m_0 + 1 \text{ and } i > j; \\ 0 & \text{otherwise.} \end{cases}
\]

For any \( \mathcal{A} \)-algebra \( K \), we set \( \tilde{V}_K := \tilde{\mathcal{V}}_\mathcal{a} \otimes_\mathcal{A} K \). Let \( \zeta \) be the image of \( q \) in \( K \). Let \( \iota \) be the \( K \)-linear injection from \( V_K \) into \( \tilde{V}_K \) defined by

\[
\sum_{i=1}^{2m_0} k_i v_i \mapsto \sum_{i=1}^{2m_0} k_i v_{i+m_0-m}, \quad \forall k_1, \ldots, k_{2m_0} \in K.
\]

Let \( \pi \) be the \( K \)-linear surjection from \( \tilde{V}_K \) into \( V_K \) defined by

\[
\sum_{i=1}^{2m_0} k_i v_i \mapsto \sum_{i=1}^{2m_0} k_i v_i, \quad \forall k_1, \ldots, k_{2m_0} \in K.
\]

We set \( \tilde{\iota} := \zeta^{m_0} \iota, \tilde{\pi} := \zeta^{m_0} \pi \). We regard \( \mathbb{C} \) as an \( \mathcal{A} \)-algebra by specializing \( q \) to 1. As before, we identify \( \text{sp}(V_\mathbb{C}) \) with \( \text{sp}_{2m}(\mathbb{C}) \) and \( \text{sp}(V_\mathbb{C}) \) with \( \text{sp}_{2m_0}(\mathbb{C}) \). Then, \( \iota \) induces an identification of \( \text{sp}_{2m}(\mathbb{C}) \) with the Lie subalgebra of \( \text{sp}_{2m_0}(\mathbb{C}) \) consisting of the following block diagonal matrices:

\[
\left\{ \begin{array}{c} \text{diag}(0, \ldots, 0, A, 0, \ldots, 0) \\ \text{(m_0 - m) copies} \\ \text{(m_0 - m) copies} \end{array} \right| A \in \text{sp}_{2m_0}(\mathbb{C}) \}.
\]

Henceforth let \( K \) be a field which is an \( \mathcal{A} \)-algebra, we set

\[
\tilde{\mathfrak{g}} := \text{sp}_{2m_0}(\mathbb{C}), \quad \mathfrak{g} := \text{sp}_{2m}(\mathbb{C}), \quad V = V_K, \quad \tilde{V} = \tilde{V}_K.
\]

The inclusion \( \mathfrak{g} \subseteq \tilde{\mathfrak{g}} \) naturally induces an injection

\[
\mathcal{U}_Q(\mathfrak{g}) \hookrightarrow \mathcal{U}_Q(\tilde{\mathfrak{g}})
\]

\[
e_i \mapsto e_{i+m_0-m}, \quad e_i \mapsto e_{i+m_0-m}, \quad k_i \mapsto k_{i+m_0-m}, \quad i = 1, 2, \ldots, m.
\]

By restriction, we get an injection \( \mathcal{U}_\mathcal{A}(\mathfrak{g}) \hookrightarrow \mathcal{U}_\mathcal{A}(\tilde{\mathfrak{g}}) \). By base change, we get a natural map \( \mathcal{U}_K(\mathfrak{g}) \rightarrow \mathcal{U}_K(\tilde{\mathfrak{g}}) \). It is easy to see that

\[
\mathbb{Z}^{\otimes 2n} \left( \left( V^{\otimes 2n} \right)^{\mathcal{U}_K(\mathfrak{g})} \right) \subseteq \left( V^{\otimes 2n} \right)^{\mathcal{U}_K(\mathfrak{g})}.
\]

For each integer \( i \) with \( 1 \leq i \leq 2m \), we define

\[
w_i = \begin{cases} v_i, & \text{if } 1 \leq i \leq m; \\ (-1)^{i-m-1} v_i, & \text{if } m + 1 \leq i \leq 2m. \end{cases}
\]
Then the natural representation of $U_{Q(q)}(g)$ on $V_{Q(q)}$ is given by

\[ e_i w_j = \begin{cases} w_i, & \text{if } j = i + 1, \\ w_{2m+1-(i+1)}, & \text{if } j = 2m + 1 - i, \\ 0, & \text{otherwise;} \end{cases} \]

\[ f_i w_j = \begin{cases} w_{i+1}, & \text{if } j = i, \\ w_{2m+1-i}, & \text{if } j = 2m + 1 - (i + 1), \\ 0, & \text{otherwise;} \end{cases} \]

\[ k_i w_j = \begin{cases} q w_j, & \text{if } j = i \text{ or } j = 2m + 1 - (i + 1), \\ q^{-1} w_j, & \text{if } j = i + 1 \text{ or } j = 2m + 1 - i, \\ w_j, & \text{otherwise}, \end{cases} \]

\[ k_m w_j = \begin{cases} q w_j, & \text{if } j = m, \\ q^{-1} w_j, & \text{if } j = m + 1, \\ w_j, & \text{otherwise}, \end{cases} \]

where $1 \leq i \leq m - 1$, $1 \leq j \leq 2m$. By definition (cf. [30, (8.18)]), $\{w_i\}_{1 \leq i \leq 2m}$ is a canonical basis of the $U_{Q(q)}(g)$-module $V_{Q(q)}$ in the sense of [41]. Similarly, the natural $U_{Q(q)}(g)$-module $V_{Q(q)}$ has a canonical basis $\{\tilde{w}_i\}_{1 \leq i \leq 2m_0}$ such that

\[ e_i \tilde{w}_j = \begin{cases} \tilde{w}_j, & \text{if } j = i + 1, \\ \tilde{w}_{2m_0+1-(i+1)}, & \text{if } j = 2m_0 + 1 - i, \\ 0, & \text{otherwise;} \end{cases} \]

\[ f_i \tilde{w}_j = \begin{cases} \tilde{w}_{i+1}, & \text{if } j = i, \\ \tilde{w}_{2m_0+1-i}, & \text{if } j = 2m_0 + 1 - (i + 1), \\ 0, & \text{otherwise;} \end{cases} \]

\[ e_{m_0} \tilde{w}_j = \begin{cases} \tilde{w}_{m_0}, & \text{if } j = m_0 + 1, \\ 0, & \text{otherwise}, \end{cases} \]

\[ f_{m_0} \tilde{w}_j = \begin{cases} \tilde{w}_{m_0+1}, & \text{if } j = m_0, \\ 0, & \text{otherwise}, \end{cases} \]

\[ k_i \tilde{w}_j = \begin{cases} q \tilde{w}_j, & \text{if } j = i \text{ or } j = 2m_0 + 1 - (i + 1), \\ q^{-1} \tilde{w}_j, & \text{if } j = i + 1 \text{ or } j = 2m_0 + 1 - i, \\ \tilde{w}_j, & \text{otherwise}, \end{cases} \]

\[ k_{m_0} \tilde{w}_j = \begin{cases} q \tilde{w}_j, & \text{if } j = m_0, \\ q^{-1} \tilde{w}_j, & \text{if } j = m_0 + 1, \\ \tilde{w}_j, & \text{otherwise}, \end{cases} \]

where $1 \leq i \leq m_0 - 1$, $1 \leq j \leq 2m_0$. Note that the subspace $\tilde{V}_{Q(q)}$ spanned by $\{\tilde{w}_j\}_{m_0-1 \leq i \leq m_0+m}$ is stable under the action of the subalgebra $U_{Q(q)}(g)$, and it is canonically isomorphic to $U_{Q(q)}(g)$-module $V_{Q(q)}$.

For each integer $i$ with $1 \leq i \leq 2m$, we define $w^*_i \in V^*_\mathcal{A} := \text{Hom}_\mathcal{A}(V_{\mathcal{A}}, \mathcal{A})$ by $w^*_i(v) = \langle w_i, v \rangle$, $\forall v \in V_{\mathcal{A}}$. Then $\{w^*_i\}_{i=1}^{2m}$ is an $\mathcal{A}$-basis of $V^*_\mathcal{A}$, and $w^*_i$ is a highest weight vector of weight $\varepsilon_i$. Furthermore, the map $w^*_1 \mapsto w_1$ can be naturally
extended to an $U_{sf}(g)$-module isomorphism $\tau: V_{sf}^* \cong V_{sf}$ such that

$$\tau(w_i^*) = \begin{cases} q^{1-i}w_i, & \text{if } 1 \leq i \leq m; \\ q^{-i}w_i, & \text{if } m+1 \leq i \leq 2m, \end{cases}$$

where the $U_{sf}(g)$-structure on $V_{sf}^*$ is defined via the antipode $S$. In a similar way, we can define an $A$-basis $\{\tilde{w}_i\}_{i=1}^{2m}$ of $\tilde{V}_{sf}$, and an $U_{sf}(g)$-module isomorphism $\tilde{\tau}: \tilde{V}_{sf}^* \cong \tilde{V}_{sf}$. Let $\tilde{\tau}_{sf}$ be the free $A$-submodule generated by

$$\{\tilde{w}_{m_0-m+1}, \tilde{w}_{m_0-m+2}, \ldots, \tilde{w}_{m_0+m}\}.$$

Set $\tilde{V} := \tilde{V}_{sf} \otimes A K$. Note that the algebra $U_K(g)$ acts on $\tilde{V}$ via the natural map $U_K(g) \to U_K(\hat{g})$. The resulting $U_K(g)$-module $\tilde{V}$ is naturally isomorphic to the natural $U_K(g)$-module $V$ via the correspondence

$$\tilde{w}_i \mapsto w_{i-m_0+m}, \quad \text{for } i = m_0 - m + 1, m_0 - m + 2, \ldots, m_0 + m.$$

Recall our definitions of $\tilde{\tau}, \tilde{\tau}$ at the beginning of this section. We define a linear map $\Theta_0$ as follows:

$$\Theta_0 : \text{End}(\tilde{V}^n) \to \text{End}(V^\otimes n),$$

$$f \mapsto (\tilde{\tau}^\otimes n) \circ f \circ (\tilde{\tau}^\otimes n).$$

One can verify directly that

$$\Theta_0(\text{End}_{U_K(\hat{g})}(\tilde{V}^\otimes n)) \subseteq \text{End}_{U_K(g)}(\tilde{V}^\otimes n) \cong \text{End}_{U_K(g)}(V^\otimes n),$$

where the last isomorphism comes from the natural $U_K(g)$-module isomorphism $\tilde{V} \cong V$.

By Corollary 5.4, the BMW algebra $B_n(-q^{2m+1}, q)$ has a basis

$$\{T_{d_1}^\nu E_1 E_3 \cdots E_{2f-1} T_{d_2}^\sigma | 0 \leq f \leq [n/2], \lambda \vdash n-2f, \sigma \in \mathfrak{S}_{(2f+1, \cdots, n)}, d_1, d_2 \in D_{\nu}, \nu := ((2f), (n-2f))\}.$$

The same is true for the BMW algebra $B_n(-q^{2m+1}, q)$. To distinguish its basis elements with those of $B_n(-q^{2m+1}, q)$, we denote them by

$$\tilde{T}_{d_1}^\nu \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_{d_2}^\sigma,$$

where $\tilde{T}_i, \tilde{E}_i$ are standard generators of $B_n(-q^{2m+1}, q)$. We define an $A$-linear isomorphism $\Theta_1$ from the BMW algebra $B_n(-q^{2m+1}, q)_{sf}$ to the BMW algebra $B_n(-q^{2m+1}, q)$ as follows:

$$\Theta_1(\tilde{T}_{d_1}^\nu \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_{d_2}^\sigma) = q^{m(m+m)} n^d_1 E_1 E_3 \cdots E_{2f-1} T_{d_2},$$

for each $0 \leq f \leq [n/2], \lambda \vdash n-2f, s, t \in \text{Std}(\lambda)$ and $d_1, d_2 \in D_{\nu}$. By base change, we get a $K$-linear isomorphism $B_n(-q^{2m+1}, \zeta) \cong B_n(-q^{2m+1}, \zeta)$, which will be still denoted by $\Theta_1$.

By the main result in last section, we know that the natural homomorphism $\varphi_C$ from $B_n(-q^{2m+1}, \zeta)$ to $\text{End}_{U_K(\hat{g})}(\tilde{V}^\otimes n)$ is always an isomorphism. Therefore, in order to prove Theorem 1.5 (in the case where $m < n$), it suffices to prove the following lemma.

**Lemma 6.1.** With the notations as above,

(1) the following diagram of maps

$$\begin{array}{ccc}
B_n(-q^{2m+1}, \zeta) & \xrightarrow{\varphi_C} & \text{End}_{U_K(\hat{g})}((\tilde{V}^\otimes n) \\
\Theta_1 \downarrow & & \Theta_0 \downarrow \\
B_n(-q^{2m+1}, \zeta) & \xrightarrow{\varphi_C} & \text{End}_{U_K(g)}(V^\otimes n) 
\end{array}$$
is commutative;
(2) the map \( \Theta_0 : \text{End}_{U_K(\tilde{g})}(\tilde{V} \otimes^n) \rightarrow \text{End}_{U_K(\tilde{g})}(V \otimes^n) \)
is surjective.

The remaining part of this section is devoted to the proof of Lemma 6.1. The proof of Lemma 6.1 (2) is almost the same as [10, Section 4], which we just sketch here. First, we note that the following diagram of maps
\[
\begin{array}{ccc}
\text{End}_{U_K(\tilde{g})}(\tilde{V} \otimes^n) & \xrightarrow{\sim} & (\tilde{V} \otimes^n \otimes (\tilde{V}^*) \otimes^n U_K(\tilde{g})) \\
\downarrow_{\Theta_0} & & \downarrow_{\Theta_0} \\
\text{End}_{U_K(\tilde{g})}(V \otimes^n) & \xrightarrow{\sim} & (V \otimes^n \otimes (V^*) \otimes^n U_K(\tilde{g}))
\end{array}
\]
is commutative.\(^3\) Note that (by the theory of tilting modules)
\[
(V \otimes^n) U_K(\tilde{g}) \cong (V \otimes^n (V^*) \otimes^n) U_K(\tilde{g}) \cong \text{End}_{U_K(\tilde{g})}(V \otimes^n)
\]
\[
\cong \text{End}_{U_{\mathcal{A}}(\tilde{g})}(V \otimes^\mathcal{A} K) \cong (V \otimes^\mathcal{A} K) \otimes^\mathcal{A} K.
\]
Therefore, to prove Lemma 6.1(2), it suffices to show that
\[
\tilde{V} \otimes^2 (\tilde{V} \otimes^n U_K(\tilde{g})) = (V \otimes^n) U_K(\tilde{g}),
\]
equivalently, to show that
\[
\tilde{V} \otimes^2 (\tilde{V} \otimes^n U_K(\tilde{g})) = (V \otimes^n) U_{\mathcal{A}}(\tilde{g}).
\]

Let \( M := (V_{Q(\gamma)}) \otimes^n \). By [41, (27.3)], the \( U_{Q(\gamma)}(\tilde{g}) \)-module \( M \) is a based module. There is a canonical basis \( B \) of \( M \), in Lusztig’s notation ([41, (27.3.2)]), where each element in \( B \) is of the form \( w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_{2n}} \), and \( w_{i_1} \circ \cdots \circ w_{i_{2n}} \) is equal to \( w_{i_1} \otimes \cdots \otimes w_{i_{2n}} \) plus a linear combination of elements \( w_{i_1} \otimes \cdots \otimes w_{j_{2n}} \) with \( (w_{j_1}, \cdots, w_{j_{2n}}) < (w_{i_1}, \cdots, w_{i_{2n}}) \) and with coefficients in \( v^{-1}Z[v^{-1}] \), where \( n < n' \) is a partial order defined in [41, (27.3.1)]. In particular, \( B \) is an \( \mathcal{A} \)-basis of \( V_{\mathcal{A}} \otimes^n \). By [41, (27.2.1)], there is a partition
\[
B = \bigsqcup_{\lambda \in X^+} B[\lambda].
\]

Let
\[
B[\neq 0] := \bigsqcup_{\lambda \in C_+} B[\lambda], \quad M[\neq 0] := \sum_{b \in B[\neq 0]} \mathcal{A}b.
\]

By [41, (27.1),(27.2)] and the discussion in [10, Section 4], we know that the isomorphism \( (\tau^{-1}) \otimes^n : V_{\mathcal{A}} \otimes^n \rightarrow (V^*) \otimes^n \cong (V_{\mathcal{A}} \otimes^n)^* \) induces an isomorphism
\[
(V_{\mathcal{A}} \otimes^n) U_{\mathcal{A}}(\tilde{g}) \cong (V_{\mathcal{A}} \otimes^n / M[\neq 0] \otimes^\mathcal{A}).
\]

All the above have a counterpart with respect to \( \tilde{V} \), which we will just put the symbol \( \sim \). Therefore, we have the notations \( \tilde{M} := (\tilde{V}_{Q(\gamma)}) \otimes^n \tilde{B}, \tilde{w}_{i_1} \circ \tilde{w}_{i_2} \circ \cdots \circ \tilde{w}_{i_{2n}}, \tilde{M}[\neq 0] \otimes^\mathcal{A}, \) and we also have an isomorphism
\[
(\tilde{V} \otimes^n) U_{\mathcal{A}}(\tilde{g}) \cong (\tilde{V} \otimes^n \tilde{M}[\neq 0]) \otimes^\mathcal{A}.
\]

---

\(^3\)This is the point where we have to use the isomorphisms \( \sim, \tilde{\sim} \) instead of \( \sim, \tilde{\sim} \).
Lemma 6.3. With the notations as above, the following diagram of maps
\[
\begin{array}{ccc}
(V_{\mathfrak{a}} \otimes_{\mathfrak{a}} \mathbb{R})^* & \longrightarrow & (V_{\mathfrak{a}} \otimes_{\mathfrak{a}} \mathbb{R})^* \\
\sim & \downarrow & \sim \\
(V_{\mathfrak{a}} \otimes_{\mathfrak{a}} \mathbb{R})^* & \longrightarrow & (V_{\mathfrak{a}} \otimes_{\mathfrak{a}} \mathbb{R})^*
\end{array}
\]
is commutative. In particular, we have
\[
\tilde{\theta}^{\otimes 2n} (M[\neq 0]_{\mathfrak{a}}) \subseteq \tilde{M}[\neq 0]_{\mathfrak{a}}.
\]
Proof. This follows from direct verification.

Therefore, to prove (6.2), it suffices to show that
\[
(\tilde{\theta}^{\otimes 2n})^* \left( (V_{\mathfrak{a}} \otimes_{\mathfrak{a}} \mathbb{R})^* \right) = (V_{\mathfrak{a}} \otimes_{\mathfrak{a}} \mathbb{R})^*.
\]

Let
\[
J_0 := \left\{(i_1, \cdots, i_{2n}) \in I(2m, 2n) \mid w_{i_1} \cdots w_{i_{2n}} \in B[0]\right\},
\]
\[
\tilde{J}_0 := \left\{(i_1, \cdots, i_{2n}) \in I(2m_0, 2n) \mid \tilde{w}_{i_1} \cdots \tilde{w}_{i_{2n}} \in \tilde{B}[0]\right\}.
\]

Lemma 6.5. ([10, Corollary 4.5]) With the above notation, the set
\[
\left\{w_{i_1} \otimes \cdots \otimes w_{i_{2n}} + M[\neq 0]_{\mathfrak{a}} \mid (i_1, \cdots, i_{2n}) \in J_0\right\}
\]
forms an $\mathfrak{a}$-basis of $V_{\mathfrak{a}} \otimes_{\mathfrak{a}} \mathbb{R}$, and the set
\[
\left\{\tilde{w}_{i_1} \otimes \cdots \otimes \tilde{w}_{i_{2n}} + \tilde{M}[\neq 0]_{\mathfrak{a}} \mid (i_1, \cdots, i_{2n}) \in \tilde{J}_0\right\}
\]
forms an $\mathfrak{a}$-basis of $\tilde{V}_{\mathfrak{a}} \otimes_{\mathfrak{a}} \mathbb{R}$.

We set
\[
J_0[m_0 - m] := \left\{(m_0 - m + i_1, \cdots, m_0 - m + i_{2n}) \mid (i_1, \cdots, i_{2n}) \in J_0\right\}.
\]

Theorem 6.6. With the above notation, $J_0[m_0 - m] \subseteq \tilde{J}_0$.

Proof. This is proved by using the same argument as in the proof of [10, Theorem 4.7]).

Now Lemma 6.5 and Theorem 6.6 imply that $\tilde{\theta}^{\otimes 2n}$ maps $V_{\mathfrak{a}} \otimes_{\mathfrak{a}} \mathbb{R}$ onto an $\mathfrak{a}$-direct summand of $\tilde{V}_{\mathfrak{a}} \otimes_{\mathfrak{a}} \mathbb{R}$. It follows that
\[
(\tilde{\theta}^{\otimes 2n})^* \left( (V_{\mathfrak{a}} \otimes_{\mathfrak{a}} \mathbb{R})^* \right) = (V_{\mathfrak{a}} \otimes_{\mathfrak{a}} \mathbb{R})^*.
\]

which proves (6.4). This completes the proof of Lemma 6.1 (2).

It remains to prove Lemma 6.1 (1). From now on and until the end of this paper we shall set, for any integer $i$ with $1 \leq i \leq 2m_0$,
\[
i' := 2m_0 + 1 - i.
\]

Note that both $\Theta_0$ and $\Theta_1$ are in general not algebra maps. Let $f$ be an integer with $0 \leq f \leq [n/2]$. By definition,
\[
\Theta_1(\tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1}) = \zeta^{(m_0+m)n} E_1 E_3 \cdots E_{2f-1}.
\]
Therefore,
\[
\varphi_C \Theta_1 \left( \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \right) = \zeta^{(m_0+m)n} \varphi_C \left( E_1 E_3 \cdots E_{2f-1} \right) \\
= \zeta^{(m_0+m)n} \varphi_C (E_1) \varphi_C (E_3) \cdots \varphi_C (E_{2f-1}) \\
= \Theta_0 \left( \varphi_C (E_1) \varphi_C (E_3) \cdots \varphi_C (E_{2f-1}) \right) \\
= \Theta_0 \varphi_C \left( E_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \right),
\]
where the third equality follows from the fact that different \( \varphi_C (\tilde{E}_{2i-1}) \) acts on pairwise non-intersected positions.

Let \( \sigma \in \mathfrak{S}_{\{2f+1, \ldots , n\}}, \nu := ((2^f), (n-2f)) \), \( d_1, d_2 \in \mathcal{D}_w \). Our purpose is to show that
\[
\varphi_C \Theta_1 \left( \tilde{T}_d \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_a \tilde{T}_b \right) = \Theta_0 \varphi_C \left( \tilde{T}_d \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_a \tilde{T}_b \right).
\]
Equivalently,
\[
\varphi_C (T_d E_1 E_3 \cdots E_{2f-1} T_a T_b) = \zeta^{-(m_0+m)n} \Theta_0 \varphi_C (\tilde{T}_d \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_a \tilde{T}_b).
\]
We divide the proof into four steps:

**Step 1.** We want to prove that \( \varphi_C \Theta_1 (\tilde{T}_w) = \Theta_0 \varphi_C (\tilde{T}_w) \) for any \( w \in \mathfrak{S}_n \).

We set \( \tilde{I}(2m, n) = \{ i \in I(2m, n) \mid m_0 - m + 1 \leq i_t \leq m_0 + m \text{ for each } t \} \).

Note that, by definition, \( \Theta_1 (\tilde{T}_w) = \zeta^{(m_0+m)n} T_w \) for each \( w \in \mathfrak{S}_n \). Hence it suffices to show that for any \( w \in \mathfrak{S}_n \),
\[
\varphi_C (T_w) = \zeta^{-(m_0+m)n} \Theta_0 \varphi_C (\tilde{T}_w).
\]

Let \( w \in \mathfrak{S}_n \), \( 1 \leq k \leq n-1 \), such that \( \ell (w_{sk}) = \ell (w) + 1 \). Let \( \tilde{I} \in \tilde{I}(2m, n) \), \( j \in I(2m_0, n) \), such that \( v_1 \) is involved in \( v_1 \tilde{T}_{w_{sk}} \). We claim that
\begin{itemize}
  \item [(A)] for any \( i \in I(2m_0, n) \) such that \( v_1 \) is involved in \( v_1 \tilde{T}_w \) and \( v_1 \) is involved in \( v_1 \tilde{T}_b \), if \( k = (l_{k+1})^j \), and \( m_0 - m + 1 \leq b_k \leq m_0 + m \text{ whenever } b \neq k \), then \( b_k \geq m_0 - m + 1 \).
\end{itemize}

If \( \ell (w) = 0 \), there is nothing to prove. Assume \( \ell (w) = a \geq 1 \). We fix a reduced expression of \( w \) as follows:
\[
w = s_i s_{i-1} \cdots s_{i-a}.
\]
We set \( q_{a+1} = k \). By assumption, \( v_1 \) is involved in \( v_1 \tilde{T}_{q_1} \cdots \tilde{T}_{q_a} \). It follows that there exist \( j^{[0]}, j^{[1]}, \ldots , j^{[a+1]} \in I(2m_0, n) \)
\[
\text{such that}
\begin{enumerate}
  \item[(1)] \( j^{[0]} = 1, j^{[a]} = 1, j^{[a+1]} = j \);
  \item[(2)] for each integer \( 1 \leq t \leq a + 1 \), \( v_{j^{[t]}} \) is involved in \( v_{j^{[t]}} \tilde{T}_{q_t} \).
\end{enumerate}

For each integer \( 2 \leq t \leq a + 1 \), we set \( w_t := s_{i_t} s_{i_{t-1}} \cdots s_{i_{t-1}} \).

For any integer \( t \) with \( 2 \leq t \leq a + 1 \), we define
\[
A_t := \sum_{1 \leq b \leq n} (b) w_t^{-1}, \quad B_t := \sum_{1 \leq b \leq n} \sum_{m_0 + m < j_{[t]}^{(t-1)} \leq 2 m_0} (b) w_t^{-1},
\]
\[
C_t := A_t - B_t.
\]
We claim that
\[(6.9) \quad 0 \leq C_2 \leq C_3 \leq \cdots \leq C_{n+1}.
\]
Recall by definition that $\bar{T}_{s_q}$ acts only on the $(q_t, q_t + 1)$ positions of the simple tensor $v_{j_{q_t}^{[t-1]}}$ in the same way as the operator $\beta'$ acts on $v_{j_{q_t}^{[t-1]}} \otimes v_{j_{q_t}^{[t-1]}}$. To compare $C_t$ with $C_{t+1}$, it suffices to check the simple tensors involved in
\[\left(v_{j_{q_t}^{[t-1]}} \otimes v_{j_{q_t}^{[t-1]}}\right) \beta'.\]
By the explicit definition of the operator $\beta'$, we need only consider the following six possibilities:

Case 1. \(\{j_{q}^{[t-1]}, j_{q_t+1}^{[t]}\} \not\subseteq \{m_0 - m + 1, m_0 - m + 2, \ldots, m_0 + m\}\). In this case, it is clear that $C_t = C_{t+1}$.

Case 2. \(j_{q_t}^{[t-1]} \not= (j_{q_t}^{[t-1]})'\). Then we have either \((j_{q_t}^{[t-1]}, j_{q_t+1}^{[t]}) = (j_{q_t+1}^{[t]}, j_{q_t}^{[t]})\) or \((j_{q_t}^{[t-1]}, j_{q_t+1}^{[t]}) = j_{q_t+1}^{[t]}, j_{q_t+1}^{[t]}\). In both cases, we still get that $C_t = C_{t+1}$.

Case 3. \(j_{q_t}^{[t-1]} = (j_{q_t}^{[t-1]})' > m_0 + m\). Then we must have
\[j_{q_t}^{[t]} = (j_{q_t}^{[t]})' < j_{q_t+1}^{[t-1]} < m_0 - m + 1.
\]
In this case, since
\[
\begin{align*}
(q_t)s_q^{[t-1]} &= q_t + 1, \quad (q_t + 1)s_q^{[t-1]} = q_t, \\
j_{q_t+1}^{[t-1]}, j_{q_t}^{[t]} &\leq m_0 - m + 1, \quad j_{q_t}^{[t-1]}, j_{q_t+1}^{[t]} > m_0 + m,
\end{align*}
\]
and \(b)s_q^{[t-1]} = b\) for any \(b \not\in \{q_t, q_t + 1\}\), it follows easily that $C_t = C_{t+1}$.

Case 4. \(m_0 - m + 1 < j_{q_t}^{[t-1]} = (j_{q_t}^{[t-1]})' \leq m_0 + m, \quad j_{q_t}^{[t]} = (j_{q_t}^{[t]})' < m_0 - m + 1\).

In this case, since \(\ell(w_{1}s_q) = \ell(w_t) + 1\), it follows that
\[
\begin{align*}
(q_t)w_t^{[t-1]} &= (q_t + 1)w_t^{[t-1]} > (q_t)w_t^{[t-1]} = (q_t + 1)w_{t+1}^{[t-1]}, \\
\end{align*}
\]
and \(b)s_q^{[t-1]} = b\) for any \(b \not\in \{q_t, q_t + 1\}\), it follows that
\[C_{t+1} = C_t + \left((q_t)w_t^{[t-1]} - (q_t + 1)w_{t+1}^{[t-1]}\right) > C_t.
\]

Case 5. \(j_{q_t}^{[t-1]} = (j_{q_t}^{[t-1]})' < m_0 - m + 1, \quad j_{q_t}^{[t-1]}, j_{q_t+1}^{[t]} \not= (j_{q_t}^{[t]}, j_{q_t+1}^{[t]}).\) In this case, we must have \(j_{q_t}^{[t]} = (j_{q_t}^{[t]})'\). Since \(\ell(w_{1}s_q) = \ell(w_t) + 1\), it follows that
\[
\begin{align*}
(q_t)w_t^{[t-1]} &= (q_t + 1)w_t^{[t-1]} > (q_t)w_t^{[t-1]} = (q_t + 1)w_{t+1}^{[t-1]}, \\
\end{align*}
\]
and \(b)s_q^{[t-1]} = b\) for any \(b \not\in \{q_t, q_t + 1\}\). If \(j_{q_t}^{[t]} > m_0 + m\), then it is clear that $C_t = C_{t+1}$; if \(j_{q_t}^{[t]} < m_0 - m + 1\), then
\[C_{t+1} = C_t - \left((q_t)w_t^{[t-1]} - (q_t + 1)w_{t+1}^{[t-1]}\right) + \left((q_t)w_{t+1}^{[t-1]} - (q_t + 1)w_{t+1}^{[t-1]}\right) > C_t,
\]
if \(m_0 - m + 1 \leq j_{q_t}^{[t]} \leq m_0 + m\), then
\[C_{t+1} = C_t - \left((q_t)w_t^{[t-1]} - (q_t + 1)w_{t+1}^{[t-1]}\right) > C_t.
\]

Case 6. \(j_{q_t}^{[t-1]} = (j_{q_t}^{[t-1]})' < m_0 - m + 1, \quad j_{q_t}^{[t-1]}, j_{q_t+1}^{[t]} = (j_{q_t}^{[t]}, j_{q_t+1}^{[t]}).\) Since
\[\ell(w_{1}s_q) = \ell(w_t) + 1\), it follows that
\[
\begin{align*}
(q_t)w_t^{[t-1]} &= (q_t + 1)w_t^{[t-1]} > (q_t)w_t^{[t-1]} = (q_t + 1)w_{t+1}^{[t-1]}, \\
\end{align*}
\]
and \((b) s_{q_1}^{-1} = b\) for any \(b \not\in \{q_k, q_t + 1\}\). Therefore
\[
C_{t+1} = C_t - \left( (q_t) w_{t}^{-1} - (q_t + 1) w_{t}^{-1} \right) + \left( (q_t) w_{t+1}^{-1} - (q_t + 1) w_{t+1}^{-1} \right) > C_t,
\]
as required. This proves our claim (6.8).

Since \(\ell(w_{a+1} s_k) = \ell(w_{a+1}) + 1\), it follows that \((k) w_{a+1}^{-1} < (k + 1) w_{a+1}^{-1}\). Now suppose that \(l_k < m_0 - m + 1\). Then by our assumption on \(l\), it is easy to see that
\[
C_{a+1} = (k) w_{a+1}^{-1} - (k + 1) w_{a+1}^{-1} < 0,
\]
which contradicts to (6.9). It follows that \(l_k \geq m_0 - m + 1\). This completes the proof of our claim (A).

Now we use induction on \(\ell(w)\) and the results (A) to prove our claim (6.8). If \(\ell(w) = 0\), there is nothing to prove. Let \(w = us_k\) with \(\ell(w) = \ell(u) + 1\). Suppose \(\varphi_C(T_u) = \zeta^{-(m_0 + m)n} \Theta_0 \varphi_C\left(\tilde{T}_u\right)\). Then for each \(i \in \tilde{I}(2m, n)\), we can write
\[
v_i \tilde{T}_u = \ell^{\otimes n}(v_{i-m_0+m} T_u) + \sum_{I \in \tilde{I}(2m, n) \setminus \tilde{I}(2m, n)} A_{i, I} v_j,
\]
where \(A_{i, I} \in K\) for each \(I\), and
\[i - m_0 + m := (i_1 - m_0 + m, \ldots, i_n - m_0 + m).\]
Therefore,
\[
v_i \tilde{T}_w = \left( \ell^{\otimes n}(v_{i-m_0+m} T_u) \right) \tilde{T}_k + \sum_{I \in \tilde{I}(2m, n) \setminus \tilde{I}(2m, n)} A_{i, I} v_k.
\]
Note that \(A_{i, I} \neq 0\) implies that \(v_i\) is involved in \(v_i \tilde{T}_w\).

We claim that \(\pi^{\otimes n}(v_i \tilde{T}_w) = 0\) whenever \(A_{i, I} \neq 0\). In fact, by the definition of \(\beta'\) and the fact that \(I \in \tilde{I}(2m_0, n) \setminus \tilde{I}(2m, n)\), it is easy to see that \(\pi^{\otimes n}(v_i \tilde{T}_w) \neq 0\) only if \(l_k = (l_{k+1})' < m_0 - m + 1\) and \(m_0 - m + 1 \leq l_k \leq m_0 + m\) whenever \(b \neq k, k + 1\). Applying our result (A), we know that this is impossible. This proves our claim.

Note also that
\[
\left( \ell^{\otimes n}(v_{i-m_0+m} T_u) \right) \tilde{T}_k = \ell^{\otimes n}(v_{i-m_0+m} T_u \tilde{T}_k) + \sum_{j \in \tilde{I}(2m, n) \setminus \tilde{I}(2m, n)} A_{i, j} v_j,
\]
where \(A_{i, j} \in K\) for each \(j\). As a consequence, we get that
\[
\pi^{\otimes n}(v_i \tilde{T}_w) = v_{i-m_0+m} T_w.
\]
Equivalently, \(\varphi_C\left(T_w\right) = \zeta^{-(m_0 + m)n} \Theta_0 \varphi_C\left(\tilde{T}_w\right)\), as required. This completes the proof of our claim (6.8). As a direct consequence, it is easy to see that for any \(\sigma \in \mathfrak{S}_{2f+1, \ldots, n}\),
\[
(6.10) \quad \varphi_C \Theta_1(\tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_\sigma) = \Theta_0 \varphi_C\left(E_1 E_3 \cdots E_{2f-1} T_\sigma\right).
\]

**Step 2.** We claim that for any \(d_i \in \mathcal{D}_{ns}\),
\[
\varphi_C \Theta_1\left(\tilde{T}_{d_i} \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_\sigma\right) = \Theta_0 \varphi_C\left(\tilde{T}_{d_i} \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_\sigma\right).
\]
By definition, \(\Theta_1\left(\tilde{T}_{d_i} \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_\sigma\right) = \zeta^{-(m_0 + m)n} \Theta_0 \varphi_C\left(\tilde{T}_{d_i} \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_\sigma\right)\). Therefore, our claim is equivalent to
\[
(6.11) \quad \varphi_C\left(\tilde{T}_{d_i} E_1 E_3 \cdots E_{2f-1} T_\sigma\right) = \zeta^{-(m_0 + m)n} \Theta_0 \varphi_C\left(\tilde{T}_{d_i} \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_\sigma\right).
\]
By (6.10), for any $i \in \tilde{I}(2m,n)$,
\[ v_i \tilde{T}_{d_{1i}} = i^{\otimes n} (v_{i-m_0+m} T_{d_{1i}^*}) + \sum_{l \in I(2m_0,n) \setminus \tilde{I}(2m,n)} \hat{A}_{l1} v_l, \]
where $\hat{A}_{l1} \in K$ for each $l$.

To prove (6.11), it suffices to show that
\[ (6.12) \quad \ell_s (l_1, \ldots, l_{2f}) < f \text{ whenever } \hat{A}_{l1} \neq 0. \]

It remains to prove (6.12). By [10, Lemma 3.8], we can write $d_1 = d_{11} d_{12}$, where $d_{11} \in D_f, J \in T_f$. Then,
\[ \tilde{T}_{d_{11}} = \tilde{T}_{d_{12}} \tilde{T}_{d_{11}} = \tilde{T}_{d_{12}}^{-1} \tilde{T}_{d_{11}}. \]

Since $d_{11}^{-1} \in \mathfrak{S}_{2f}$, the action of $\tilde{T}_{d_{11}^{-1}}$ does not change the symplectic length of the first $(2f)$ parts of any simple $n$-tensor. Therefore, using (6.8), we can assume without loss of generality that $d_{11} = 1$, and hence $d_1 = d_{11} \in D_{(2f,m-2f)}$. With this assumption, we claim that
\[ (6.13) \quad v_i \tilde{T}_{d_{12}} = i^{\otimes n} (v_{i-m_0+m} T_{d_{12}^*}) + \sum_{l \in I(2m_0,n) \setminus \tilde{I}(2m,n)} \hat{A}_{l1} v_l, \]
where $\hat{A}_{l1} \in K$ for each $l$, and $\hat{A}_{l1} \neq 0$ only if
\[ \{l_1, \ldots, l_{2f}\} \cap \{m_0 + m + 1, m_0 + m + 2, \ldots, 2m_0\} = \emptyset, \]
and either $l_1$ or $l_2$ belongs to $\{1, 2, \ldots, m_0 - m\}$. If this is true, then it is clear that $\hat{A}_{l1} \neq 0$ only if $\ell_s (l) < f$ and hence (6.12) follows.

Let $J = (j_1, j_2, \ldots, j_{2f})$. Then $1 \leq j_1 < j_2 < \cdots < j_{2f} \leq n$. By Lemma 5.8,
\[ (s_{j_1-1} \cdots s_{2} s_1) (s_{j_2-1} \cdots s_{2} s_2) \cdots (s_{j_{2f}-1} \cdots s_{2} s_{2f-1}) (s_{j_{2f}} - 1 \cdots s_{2} s_{2f+1}) \]
is a reduced expression of $d_{12}^{-1}$. If $f = 1$, then
\[ \tilde{T}_{d_{12}} = (\tilde{T}_{j_1-1} \cdots \tilde{T}_{2} \tilde{T}_1) (\tilde{T}_{j_2-1} \cdots \tilde{T}_{3} \tilde{T}_2). \]
In this case, suppose that $v_i$ is involved in $v_i \tilde{T}_{d_{12}}$, where $i \in \tilde{I}(2m,n)$. Then, there must exist
\[ l^{[0], l^{[1]}, \ldots, l^{[j_1+j_2-3]}]} \in I(2m_0, n) \]
such that
\begin{enumerate}
\item $l^{[0]} = i, l^{[j_1+j_2-3]} = 1$;
\item for each $1 \leq t \leq j_1 - 1$, $v_{l^{[t]}}$ is involved in $v_i (\tilde{T}_{j_1-t} \cdots \tilde{T}_{2} \tilde{T}_1)$;
\item for each $j_1 \leq t \leq j_1 + j_2 - 3$, $v_{l^{[t]}}$ is involved in $v_i (\tilde{T}_{j_1-t} \cdots \tilde{T}_{2} \tilde{T}_1) (\tilde{T}_{j_1-t} \cdots \tilde{T}_{2} \tilde{T}_1)$;
\item for each $1 \leq t \leq j_1 - 1$, $v_{l^{[t]}}$ is involved in $v_i (\tilde{T}_{j_1-t} \cdots \tilde{T}_{2} \tilde{T}_1)$;
\item for each $j_1 \leq t \leq j_1 + j_2 - 3$, $v_{l^{[t]}}$ is involved in $v_i (\tilde{T}_{j_1-t} \cdots \tilde{T}_{2} \tilde{T}_1)$.
\end{enumerate}

Now suppose $l \in I(2m_0,n) \setminus \tilde{I}(2m,n)$. If there exists an integer $1 \leq b \leq j_1$ such that
\[ m_0 - m_1 \leq l^{[b-1]}_{j_1 - b} = (l^{[b-1]}_{j_1 - b+1})' - m_0 + m, \]
then we choose such a $b$ which is maximal. By (5.13), we have $l_1 = l^{[j_1-1]}_1 < m_0 - m + 1$ and $l_2 = l^{[j_1-1]}_{j_2} \leq m_0 + m$. If there does not exist such a $b$, then there must exist an integer $j_1 \leq c \leq j_1 + j_2 - 3$ such that
\[ m_0 - m_1 \leq l^{[c-1]}_{j_1+j_2-c-1} = (l^{[c-1]}_{j_1+j_2-c})' - m_0 + m, \]
and $l^{[c]}_{j_1+j_2-c} < m_0 + m - 1$. 

We choose such a $c$ which is maximal. By (5.13), $l_2 < m_0 - m + 1$. The non-existence of $b$ implies that $m_0 - m + 1 \leq l_1 \leq m_0 + m$. This proves our claim in the case $f = 1$.

Now we assume that $f \geq 2$. We use induction on $n - 2f$. If $n - 2f = 0$, then $d_f = 1$, there is nothing to prove. We set

$$\hat{d}_f = (s_{2f-1}s_{2f} \cdots s_{j_{2f-1}})(s_{2f-2}s_{2f-1} \cdots s_{j_{2f-2}}) \cdots (s_1s_2 \cdots s_{j_{1}}).$$

Then $d_{f-1} = \hat{d}_f^{1/2}(s_{j_{2f-1}} \cdots s_{j_1})$ and

$$\ell(d_{f-1}^{-1}) = \ell(\hat{d}_f^{-1}) + j_{2f} - 2f.$$

If $j_{2f} \leq n - 1$, then $d_f \in \mathcal{D}(2f,n-1-2f)$, and we are done by induction hypothesis. If $j_{2f} = n$, then by induction hypothesis, we have

$$v_1T^*_{d_f} = \iota^{\otimes n}(v_{1-m_0+m}T^*_{d_f}) + \sum_{u \in I(2m_0,n)} \tilde{B}_{i,u}v_1,$$

where $\tilde{B}_{i,u} \in K$ for each $u$, and $\tilde{B}_{i,u} \neq 0$ only if

1. $m_0 - m + 1 \leq l_u \leq m_0 + m$; and
2. $\{i_1, \cdots, i_{2f}\} \cap \{m_0 + m + 1, m_0 + m + 2, \cdots, 2m_0\} = \emptyset$; and
3. either $l_1$ or $l_2$ belongs to $\{1, 2, \cdots, m_0 - m\}$.

It remains to check the simple tensors involved in $v_1\tilde{T}_{n-1} \tilde{T}_{n-2} \cdots \tilde{T}_{2f}$ as well as in

$$\iota^{\otimes n}(v_{1-m_0+m}T^*_{d_f})\tilde{T}_{n-1} \tilde{T}_{n-2} \cdots \tilde{T}_{2f} = \iota^{\otimes n}(v_{1-m_0+m}T^*_{d_f}) + \sum_{u \in I(2m_0,n)} \tilde{B}_{i,u}v_1,$$

where $\tilde{B}_{i,u} \in K$ for each $u$, and $\tilde{B}_{i,u} \neq 0$ only if

$$\{u_1, \cdots, u_{2f}\} \cap \{m_0 + m + 1, m_0 + m + 2, \cdots, 2m_0\} = \emptyset,$$

and either $u_1$ or $u_2$ belongs to $\{1, 2, \cdots, m_0 - m\}$. By the same reason, we can deduce that

$$v_1\tilde{T}_{n-1} \tilde{T}_{n-2} \cdots \tilde{T}_{2f-1} = \sum_{u \in I(2m_0,n)} \tilde{B}_{i,u}v_1,$$

where $\tilde{B}_{i,u} \in K$ for each $u$, and $\tilde{B}_{i,u} \neq 0$ only if

$$\{u_1, \cdots, u_{2f}\} \cap \{m_0 + m + 1, m_0 + m + 2, \cdots, 2m_0\} = \emptyset,$$

and either $u_1$ or $u_2$ belongs to $\{1, 2, \cdots, m_0 - m\}$. This completes the proof of (6.13), and hence the proof of (6.11).

**Step 3.** We want to show that for any $d_2 \in D_{v_f}$,

$$\varphi_C \Theta_1 \left( \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1}\tilde{T}_\sigma \tilde{T}_{d_2} \right) = \Theta_0 \varphi_C \left( \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1}\tilde{T}_\sigma \tilde{T}_{d_2} \right).$$

As before, it is equivalent to show that

$$\varphi_C \left( E_1 E_3 \cdots E_{2f-1} T_\sigma T_{d_2} \right) = \zeta^{- (m_0 + m)n} \Theta_0 \varphi_C \left( \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1}\tilde{T}_\sigma \tilde{T}_{d_2} \right).$$

Recall that $V^{\otimes n}$ has a basis consisting of all the simple $n$-tensors $v_i$, where $i \in I(2m_0,n)$. We ordered the elements of this basis as $X_1, X_2, \cdots, X_{(2m_0)^n}$ such that
the subset \( \{X_1, X_2, \cdots, X_{(2m)^n}\} \) is a basis of \( \iota_{(n)}(V_{(n)}) \). With this ordered basis \( \{X_1, \cdots, X_{(2m)^n}\} \) in mind, we identify \( \text{End}_K(\tilde{V}_{(n)}) \) with full matrix algebra \( M_{(2m)^n \times (2m)^n}(K) \), and identify the homomorphism

\[
\varphi_C : (\mathfrak{B}_n(-\zeta^{2m_0+1}, \zeta))^{\text{op}} \rightarrow \text{End}_K(\tilde{V}_{(n)})
\]

with a homomorphism

\[
\varphi_C : (\mathfrak{B}_n(-\zeta^{2m_0+1}, \zeta))^{\text{op}} \rightarrow M_{(2m_0)^n \times (2m_0)^n}(K).
\]

We claim that for any \( x \in \mathfrak{B}_n(-\zeta^{2m_0+1}, \zeta) \),

\[
(6.15) \quad \varphi_C(x^*) = (\varphi_C(x))^t,
\]

where \((\varphi_C(x))^t\) means the transpose of the matrix \(\varphi_C(x)\), and “\(^t\)” denotes the algebra anti-automorphism of \(\mathfrak{B}_n(-\zeta^{2m_0+1}, \zeta)\) defined in Section 5.

In fact, by direct verification, it is easy to see that for each \(1 \leq i \leq n-1\), both \(\varphi_C(T_i)\) and \(\varphi_C(E_i)\) are symmetric matrices (with respect to the above ordered basis). Since \(\mathfrak{B}_n(-\zeta^{2m_0+1}, \zeta)\) is generated by \(T_i, E_i, i = 1, 2, \cdots, n-1\) and \(\varphi_C\) is an algebra homomorphism, the claim (6.15) follows immediately.

The above argument applies equally well to \(V_{(n)}\). For each integer \(i\) with \(1 \leq i \leq (2m)^n\), let \(Y_i := \pi_{(n)}(X_i)\). Then, \(\{Y_1, \cdots, Y_{(2m)^n}\}\) is a basis of \(V_{(n)}\). With the ordered basis \(\{Y_1, \cdots, Y_{(2m)^n}\}\) in mind, we identify \(\text{End}_K(V_{(n)})\) with full matrix algebra \(M_{(2m)^n \times (2m)^n}(K)\), and identify the homomorphism

\[
\varphi_C : (\mathfrak{B}_n(-\zeta^{2m+1}, \zeta))^{\text{op}} \rightarrow \text{End}_K(V_{(n)})
\]

with a homomorphism

\[
(\varphi_C(x))^t = (\varphi_C(x))^t.
\]

As before, for any \(x \in \mathfrak{B}_n(-\zeta^{2m_0+1}, \zeta)\), we have \(\varphi_C(x^*) = (\varphi_C(x))^t\).

We define a linear map \(\Theta'_0\) from \(M_{(2m)^n \times (2m)^n}(K)\) to \(M_{(2m)^n \times (2m)^n}(K)\) as follows: for any \(M \in M_{(2m)^n \times (2m)^n}(K)\), \(\Theta'_0(M)\) is the submatrix of \(M\) in the upper-left corner, obtained from \(M\) by deleting the last \((2m_0)^n - (2m)^n\) rows and the last \((2m_0)^n - (2m)^n\) columns. Then, it is clear that

\[
(6.16) \quad \Theta'_0(M^t) = (\Theta'_0(M))^t.
\]

With the ordered bases \(\{X_1, X_2, \cdots, X_{(2m)^n}\}\) and \(\{Y_1, \cdots, Y_{(2m)^n}\}\) in mind, it is easy to check that we can identify the linear map \(\Theta_0\) as the restriction of \(\zeta^{(m_0+m)n} \Theta'_0\).
Applying (6.15), (6.16) and (6.11), we get
\[
\zeta^{-(m_0+m)n} \Theta_0 \varphi C \left( \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_d \right) \\
= \Theta_0 \varphi C \left( \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_d \right) \\
= \Theta_0 \varphi C \left( \left( \tilde{T}_d^* \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_{\sigma^{-1}} \right)^t \right) \\
= \Theta_0 \varphi C \left( \left( \tilde{T}_d^* \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_{\sigma^{-1}} \right)^t \right) \\
= \varphi C \left( T_d^* E_1 E_3 \cdots E_{2f-1} T_{\sigma^{-1}} \right) \\
= \varphi C \left( E_1 E_3 \cdots E_{2f-1} T_{\sigma^{-1}} \right),
\]
where \( \hat{A}_{ij} \in K \) for each \( j \).

By (6.12),
\[
v_1 \tilde{T}_d^* = \zeta^{\otimes n} \left( v_1 - m_0 + m T_{d_1}^* \right) + \sum_{l \in I(2m_0,n) \setminus I(2m,n)} \hat{A}_{ij} v_1,
\]
where \( \hat{A}_{ij} \in K \) for each \( l \). Therefore,
\[
v_1 \tilde{T}_d^* \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_d \tilde{T}_{\sigma} \tilde{T}_d = \zeta^{\otimes n} \left( v_1 - m_0 + m T_{d_1}^* \right) \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_d \tilde{T}_{\sigma} \tilde{T}_d.
\]
Now we use (6.14), it follows that
\[
\zeta^{\otimes n} \left( v_1 - m_0 + m T_{d_1}^* \right) \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_d \tilde{T}_{\sigma} \tilde{T}_d \\
= \zeta^{\otimes n} \left( v_1 - m_0 + m T_{d_1}^* \right) \tilde{E}_1 \tilde{E}_3 \cdots \tilde{E}_{2f-1} \tilde{T}_d \tilde{T}_{\sigma} \tilde{T}_d + \sum_{j \in I(2m_0,n) \setminus I(2m,n)} \hat{A}_{ij} v_j,
\]
where \( \hat{A}_{ij} \in K \) for each \( j \), as required. This completes the proof of (6.7). Hence we complete the proof of Lemma 6.1 (1).

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