THE EQUIVARIANT EULER CHARACTERISTIC OF REAL COXETER TORIC VARIETIES

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Abstract. Let W be a crystallographic Weyl group, and let $\mathcal{T}_W$ be the complex toric variety attached to the fan of cones corresponding to the reflecting hyperplanes of W, and its weight lattice. The real locus $\mathcal{T}_W(\mathbb{R})$ is a smooth, connected, compact manifold with a W-action. We give a formula for the Euler characteristic of $\mathcal{T}_W(\mathbb{R})$ as a generalised character of W. In type A_{n-1} for n odd, one obtains a generalised character of Sym_n whose degree is (up to sign) the n^{th} Euler number.

1. Introduction

Let Φ be a crystallographic root system in a Euclidean space V, and let Π be a simple subsystem of Φ. We assume that Φ spans V, and write $r = |\Pi| = \dim V$. Let W be the Weyl group of Φ. As described in [F] or [L], there is a complex toric variety $\mathcal{T}_W$ associated to this data, which is defined by the fan of rational convex polyhedral cones into which V is divided by the reflecting hyperplanes of W, and the weight lattice $M = \{\omega \in V \mid \langle \omega, \alpha \rangle \in \mathbb{Z}, \forall \alpha \in \phi\}$. As shown in [DPS, Section IV], $\mathcal{T}_W$ coincides with the Hessenberg variety in which the chosen subset of negative roots is $-\Pi$.

The nonsingular complex projective variety $\mathcal{T}_W$ has a real structure defined by complex conjugation σ on $\mathbb{C}$, and the real locus $\mathcal{T}_W(\mathbb{R})$ is the set of fixed points of σ on $\mathcal{T}_W$.

Proposition 1. (i) *With the above notation, $\mathcal{T}_W(\mathbb{R})$ is a smooth, connected, compact manifold of dimension r.*
(ii) *$\mathcal{T}_W(\mathbb{R})$ has a real cell decomposition in which the cells C_w are indexed by $w \in W$, and $C_w \cong \mathbb{R}^{d(w)}$ where*
$$d(w) = |\{\alpha \in \Pi \mid w(\alpha) \in \Phi^-\}|.$$

Proof. In part (i), everything but the connectedness follows from the fact that $\mathcal{T}_W$ is a nonsingular projective variety of dimension r (see, for example, [Sh, Chapter II, §2.3] – it is trivial that $\mathcal{T}_W(\mathbb{R})$ is nonempty). Connectedness follows from part (ii), since there is a unique 0-dimensional cell (more generally, the real locus of any nonsingular projective toric variety is connected, because one can use a filtration defined as in [F, §5.2] instead of the cell decomposition). The

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analogue of part (ii) for the complex variety may be found in [DPS], and it is clear that the complex cells are defined over $\mathbb{R}$, i.e. are fixed by σ. $\square$

Corollary 2. *(cf. Remark 7) The Euler characteristic of $\mathcal{T}_W(\mathbb{R})$ is given by*

$$\chi(\mathcal{T}_W(\mathbb{R})) = \sum_{w \in W} (-1)^{d(w)},$$

and is zero if r is odd.

Proof. The formula for Euler characteristic follows immediately from part (ii) of Proposition 1. Any odd-dimensional compact manifold has zero Euler characteristic. $\square$

Now the complex variety $\mathcal{T}_W$ has an obvious W-action, which is defined over $\mathbb{R}$, i.e. commutes with σ; hence the manifold $\mathcal{T}_W(\mathbb{R})$ has a W-action. The cohomology spaces $H^i(\mathcal{T}_W(\mathbb{R}), \mathbb{Q})$ are $\mathbb{Q}W$-modules, and so may be considered as elements of the (rational) Grothendieck ring $R(W)$.

Definition 3. *The equivariant Euler characteristic of $\mathcal{T}_W(\mathbb{R})$ is the following element of $R(W)$:*

$$\Lambda_W = \sum_i (-1)^i H^i(\mathcal{T}_W(\mathbb{R}), \mathbb{Q}).$$

Definition 4. *Let $\pi_W^{(2)}$ be the permutation character of W on the finite set $M/2M$, where M is the weight lattice as above. For any subset $J \subseteq \Pi$, let $\pi_{W_J}^{(2)}$ be the corresponding character of the parabolic subgroup W_J, regarded as the Weyl group of the parabolic subsystem Φ_J in the subspace $V_J = \mathbb{R}J$.*

In Section 2, we will prove the following formula.

Theorem 5. *The equivariant Euler characteristic Λ_W of $\mathcal{T}_W(\mathbb{R})$ is given as an element of the Grothendieck ring $R(W)$ by*

$$\Lambda_W = \varepsilon_W \sum_{J \subseteq \Pi} (-1)^{|J|} \operatorname{Ind}_{W_J}^W(\pi_{W_J}^{(2)}),$$

where ε_W denotes the sign character of W.

In Section 3, we will deduce the following more explicit formula in type A.

Theorem 6. *Suppose that Φ is of type A_{n-1}, so that $W \cong \mathrm{Sym}_n$. Then*

$$\Lambda_{\mathrm{Sym}_n} = \sum_{m \geq 0} (-1)^m \sum_{\substack{n_1, n_2, \cdots, n_m \geq 2 \\ n_1, n_2, \cdots, n_m \text{ even} \\ n_1 + n_2 + \cdots + n_m \leq n}} \operatorname{Ind}_{\mathrm{Sym}_{n-n_1-\cdots-n_m} \times \mathrm{Sym}_{n_1} \times \cdots \times \mathrm{Sym}_{n_m}}^{\mathrm{Sym}_n} (\varepsilon_{n_1, \cdots, n_m}),$$

where $\varepsilon_{n_1, \cdots, n_m}$ is the linear character whose restriction to the $\mathrm{Sym}_{n-n_1-\cdots-n_m}$ factor is trivial and whose restriction to the Sym_{n_i} factors is the sign character.

An intriguing question suggested by the form of Theorem 6 is whether the inner sum equals the individual cohomology space $H^m(\mathcal{T}_{\mathrm{Sym}_n}(\mathbb{R}), \mathbb{Q})$ as Sym_n-module.

Remark 7. Theorem 5 implies the following formula for the nonequivariant Euler characteristic:

$$\chi(\mathcal{T}_W(\mathbb{R})) = \sum_{J \subseteq \Pi} [W : W_J]\, (-2)^{|J|}. \tag{8}$$

The fact that this equals the formula given in Corollary 2 is the $q = -1$ case of the following identity, alluded to in [L, Remark 3.6]:

$$\sum_{J \subseteq \Pi} [W : W_J]\, (q-1)^{|J|} = \sum_{w \in W} q^{d(w)}. \tag{9}$$

The two sides of (9) are the expressions for the Poincaré polynomial of $\mathcal{T}_W$ resulting from the viewpoints of [L] and [DPS] respectively. There is an elementary proof of (9): rewrite $[W : W_J]$ as $|\{w \in W \mid w(J) \subseteq \Phi^-\}|$, expand $(q-1)^{|J|}$ using the Binomial Theorem, and apply the Inclusion-Exclusion Principle.

In Section 4, we will compare the equivariant Euler characteristic of $\mathcal{T}_W(\mathbb{R})$ with the $q = -1$ specialization of the equivariant Poincaré polynomial of $\mathcal{T}_W$.

2. Proof of Theorem 5

Since the cell decomposition of $\mathcal{T}_W$ defined in [DPS] is not preserved under the action of W, it is not convenient for the computation of the equivariant Euler characteristic. Instead we use the decomposition into torus orbits, as in [L]. Although we will not need it, we remark that it is possible to give a W-stable CW-complex construction of $\mathcal{T}_W(\mathbb{R})$, using [D, Theorem 2.4].

Let G be a semisimple complex algebraic group corresponding to the root datum $(\mathbb{Z}\Phi, \Phi, M, \Phi^\vee)$ (thus, G is of adjoint type), and let T be a maximal torus of G with cocharacter group $Y(T) \cong M$. By construction, $\mathcal{T}_W$ carries an action of T.

Since T is defined over $\mathbb{Z}$, it has a canonical real structure. The Weyl group elements give automorphisms of T which preserve the real locus $T(\mathbb{R})$.

Definition 10. *Define the generalised character $\phi_W \in R(W)$ by*

$$\phi_W(w) = \chi_c(T(\mathbb{R})^w),$$

where χ_c denotes the compact-supports Euler characteristic. For any $J \subseteq \Pi$, let ϕ_{W_J} be the corresponding element of $R(W_J)$.

Proposition 11. *In $R(W)$, the following equation holds:*

$$\Lambda_W = \sum_{J \subseteq \Pi} \mathrm{Ind}_{W_J}^{W} \phi_{W_J}.$$

Proof. By the Lefschetz fixed-point formula, the trace of an element $w \in W$ on Λ_W is given by $\chi(\mathcal{T}_W(\mathbb{R})^w)$. Since $\mathcal{T}_W(\mathbb{R})$ is compact, we may replace χ with the compact-supports Euler characteristic χ_c, which is an "Eulerian function" in the sense of [DL] (i.e. additive over decompositions of spaces into locally closed subspaces).

As noted in [L, proof of Theorem 1.1], $\mathcal{T}_W$ is the disjoint union of the orbits of T on $\mathcal{T}$; these orbits are tori, so that $\mathcal{T}_W = \coprod_{xW_J} T(xW_J)$, where the union is over all cosets xW_J of all parabolic subgroups W_J ($J \subseteq \Pi$). We have $\dim(T(xW_J)) = |J|$, and $Y(T(xW_J)) = xM_J$, where M_J is the weight lattice of W_J as in §1. This decomposition is stable under complex conjugation σ, and the pieces are permuted by W: $w \cdot T(xW_J) = T(wxW_J)$, whence it follows that

$$\mathcal{T}_W(\mathbb{R})^w = \coprod_{\substack{xW_J \\ wxW_J = xW_J}} T(xW_J)(\mathbb{R})^w,$$

and by the additivity of the compact-supports Euler characteristic,

$$\chi_c(\mathcal{T}_W(\mathbb{R})^w) = \sum_{\substack{xW_J \\ wxW_J = xW_J}} \chi_c(T(xW_J)(\mathbb{R})^w).$$

But if $wxW_J = xW_J$, then $T(xW_J)^w = x \cdot T(W_J)^{x^{-1}wx}$. So we obtain

$$\chi_c(\mathcal{T}_W(\mathbb{R})^w) = \sum_{\substack{xW_J \\ wxW_J = xW_J}} \chi_c(T(W_J)(\mathbb{R})^{x^{-1}wx}) = \sum_{\substack{xW_J \\ wxW_J = xW_J}} \phi_{W_J}(x^{-1}wx).$$

The Proposition now follows from Frobenius' formula for induced characters. □

In view of Proposition 11, Theorem 5 follows from the following result, applied to every parabolic subgroup W_J of W.

Proposition 12. *We have* $\phi_W = (-1)^r \, \epsilon_W \, \pi_W^{(2)}$.

Proof. We have an isomorphism of topological groups $T(\mathbb{R}) \cong Y(T) \otimes_{\mathbb{Z}} \mathbb{R}^\times$, where $\mathbb{R}^\times$ is viewed as a $\mathbb{Z}$-module via its abelian group structure, and the topology on $Y(T) \otimes_{\mathbb{Z}} \mathbb{R}^\times$ is such that $\{0\} \otimes \mathbb{R}^\times$ is an open and closed subgroup homeomorphic to $\mathbb{R}^\times$. For any $w \in W$, the action of w on $T(\mathbb{R})$ corresponds to the action of $w \otimes \mathrm{id}$ on $Y(T) \otimes_{\mathbb{Z}} \mathbb{R}^\times$. Moreover, we have another isomorphism of topological groups $\mathbb{R} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} \mathbb{R}^\times$, defined by $(x, a) \mapsto (-1)^a e^x$. Hence

$$T(\mathbb{R}) \cong Y(T) \otimes_{\mathbb{Z}} (\mathbb{R} \oplus \mathbb{Z}/2\mathbb{Z}) \cong (Y(T) \otimes_{\mathbb{Z}} \mathbb{R}) \oplus (Y(T) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \cong V \oplus M/2M,$$

where V has its usual topology, and $M/2M$ is a finite discrete set. This isomorphism respects the action of W on both sides, W acting diagonally on the right side.

It follows that for any $w \in W$, we have a homeomorphism $T(\mathbb{R})^w \cong V^w \times (M/2M)^w$, and hence

$$\chi_c(T(\mathbb{R})^w) = \chi_c(V^w) \, |(M/2M)^w| = (-1)^{\dim V^w} \pi_W^{(2)}(w),$$

because V^w is a real vector space. The result now follows from the fact that $\varepsilon_W(w) = (-1)^{r-\dim V^w}$. $\square$

3. Type A

In this section we restrict attention to the case where Φ is of type A_{n-1}, so that $W \cong \mathrm{Sym}_n$, the symmetric group of degree $n \geq 1$. In this case, the quantity $d(w)$ defined in Proposition 1 is the number of descents of the permutation w, and it follows from Corollary 2 that

$$\chi(\mathcal{T}_{\mathrm{Sym}_n}(\mathbb{R})) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} E_n, & \text{if } n \text{ is odd,} \end{cases} \tag{13}$$

where E_n is the Euler number, i.e. the coefficient of $\frac{x^n}{n!}$ in the Taylor series of $\tan(x)$ (see [Sta, Section 3.16]).

We have a simple expression for the permutation character $\pi^{(2)}_{\mathrm{Sym}_n}$.

Proposition 14. *For any $n \geq 1$, the following equation holds in $R(\mathrm{Sym}_n)$:*

$$\pi^{(2)}_{\mathrm{Sym}_n} = \sum_{\substack{0 \leq s \leq n \\ s \text{ even}}} \mathrm{Ind}^{\mathrm{Sym}_n}_{\mathrm{Sym}_{n-s} \times \mathrm{Sym}_s}(1).$$

Proof. If n is odd, this reflects an isomorphism of sets with an action of Sym_n: the subgroups $\mathrm{Sym}_{n-s} \times \mathrm{Sym}_s$ referred to in the statement are precisely the stabilizers of representatives of the Sym_n-orbits in $M/2M$. However, this is not the case for n even, so we shall prove the equality on the level of characters. Since $\mathrm{Ind}^{\mathrm{Sym}_n}_{\mathrm{Sym}_{n-s} \times \mathrm{Sym}_s}(1)(w)$ is the number of w-stable subsets of $\{1, 2, \cdots, n\}$ which have s elements, this amounts to proving that $\pi^{(2)}_{\mathrm{Sym}_n}(w)$ is the number of w-stable subsets of $\{1, 2, \cdots, n\}$ which have an even number of elements, for any $w \in \mathrm{Sym}_n$.

Now the weight lattice M referred to in Definition 4 may be identified with $\mathbb{Z}^n/\mathbb{Z}(1, 1, \cdots, 1)$, where Sym_n acts by permuting the coordinates. Hence we may identify $M/2M$ with $(\mathbb{Z}/2\mathbb{Z})^n/(\mathbb{Z}/2\mathbb{Z})(1, 1, \cdots, 1)$. For any $w \in \mathrm{Sym}_n$, $\pi^{(2)}_{\mathrm{Sym}_n}(w)$ is, by definition, the number of w-fixed elements of this set. Hence $2\pi^{(2)}_{\mathrm{Sym}_n}(w)$ is the number of elements in the set

$$\{(a_1, a_2, \cdots, a_n) \in (\mathbb{Z}/2\mathbb{Z})^n \,|\, \text{either } a_{w(i)} = a_i, \forall i, \text{ or } a_{w(i)} = a_i + 1, \forall i\}.$$

Note that if w has a cycle of odd length, the case that $a_{w(i)} = a_i + 1$ for all i cannot occur. It is easy to deduce the following formula, in which $c(w)$ denotes the number of cycles of w:

$$\pi^{(2)}_{\mathrm{Sym}_n}(w) = \begin{cases} 2^{c(w)-1} & \text{if } w \text{ has a cycle of odd length,} \\ 2^{c(w)} & \text{if all the cycles of } w \text{ have even length.} \end{cases} \tag{15}$$

Clearly the right-hand side equals the number of w-stable subsets of $\{1, 2, \cdots, n\}$ which have an even number of elements, as required. $\square$

We shall now deduce Theorem 6 from Proposition 14 and Theorem 5. It is convenient to consider $\bigoplus_{n\geq 0} R(\mathrm{Sym}_n)$, which as usual we regard as an $\mathbb{N}$-graded ring via the induction product:

$$\chi_1.\chi_2 = \mathrm{Ind}_{\mathrm{Sym}_{n_1}\times\mathrm{Sym}_{n_2}}^{\mathrm{Sym}_{n_1+n_2}}(\chi_1 \boxtimes \chi_2),\ \text{for } \chi_1 \in R(\mathrm{Sym}_{n_1}),\ \chi_2 \in R(\mathrm{Sym}_{n_2}).$$

(The identity element of this ring is the trivial character of Sym_0, which we will write simply as 1.) In fact, to work with all n simultaneously, we need the completion $\widehat{\bigoplus}_{n\geq 0} R(\mathrm{Sym}_n)$. In this ring, every element with degree-0 term equal to 1 has a multiplicative inverse.

The type A case of Theorem 5 says that

$$\varepsilon_{\mathrm{Sym}_n}\Lambda_{\mathrm{Sym}_n} = \sum_{m\geq 0}(-1)^{n-m}\sum_{\substack{n_1,n_2,\cdots,n_m\geq 1\\ n_1+n_2+\cdots+n_m=n}}\mathrm{Ind}_{\mathrm{Sym}_{n_1}\times\cdots\times\mathrm{Sym}_{n_m}}^{\mathrm{Sym}_n}(\pi^{(2)}_{\mathrm{Sym}_{n_1}}\boxtimes\cdots\boxtimes\pi^{(2)}_{\mathrm{Sym}_{n_m}}).$$

This can be translated into the following equality in $\widehat{\bigoplus}_{n\geq 0} R(\mathrm{Sym}_n)$:

$$1 + \sum_{n\geq 1}(-1)^n\,\varepsilon_{\mathrm{Sym}_n}\Lambda_{\mathrm{Sym}_n} = \Big(1 + \sum_{n\geq 1}\pi^{(2)}_{\mathrm{Sym}_n}\Big)^{-1}. \tag{16}$$

But Proposition 14 amounts to the following equality in $\widehat{\bigoplus}_{n\geq 0} R(\mathrm{Sym}_n)$:

$$1 + \sum_{n\geq 1}\pi^{(2)}_{\mathrm{Sym}_n} = \Big(\sum_{n\geq 0}1_{\mathrm{Sym}_n}\Big).\Big(\sum_{\substack{n\geq 0\\ n \text{ even}}}1_{\mathrm{Sym}_n}\Big). \tag{17}$$

We now substitute (17) into (16) and use the well-known fact that the multiplicative inverse of $\sum_{n\geq 0}1_{\mathrm{Sym}_n}$ is $\sum_{n\geq 0}(-1)^n\,\varepsilon_{\mathrm{Sym}_n}$, to obtain:

$$1 + \sum_{n\geq 1}(-1)^n\,\varepsilon_{\mathrm{Sym}_n}\Lambda_{\mathrm{Sym}_n} = \Big(\sum_{n\geq 0}(-1)^n\,\varepsilon_{\mathrm{Sym}_n}\Big).\Big(1 + \sum_{\substack{n\geq 2\\ n \text{ even}}}1_{\mathrm{Sym}_n}\Big)^{-1}. \tag{18}$$

Applying the ring involution of $\widehat{\bigoplus}_{n\geq 0} R(\mathrm{Sym}_n)$ which maps $\chi \in R(\mathrm{Sym}_n)$ to $(-1)^n\varepsilon_{\mathrm{Sym}_n}\chi \in R(\mathrm{Sym}_n)$ gives:

$$1 + \sum_{n\geq 1}\Lambda_{\mathrm{Sym}_n} = \Big(\sum_{n\geq 0}1_{\mathrm{Sym}_n}\Big).\Big(1 + \sum_{\substack{n\geq 2\\ n \text{ even}}}\varepsilon_{\mathrm{Sym}_n}\Big)^{-1}. \tag{19}$$

Extracting the degree-n term on both sides, we obtain Theorem 6.

4. Comparison of $\mathcal{T}_W(\mathbb{R})$ and $\mathcal{T}_W$

There is a large class of complex varieties X with real structures for which the compact-supports Euler characteristic of the real locus $X(\mathbb{R})$ can be obtained from the rational point counting function $P_X(q) = |X(\mathbb{F}_q)|$, if $P_X(q)$ is a polynomial, by setting $q = -1$. Such varieties X are called computable in [KL, §5]. The condition that $P_X(q)$ be polynomial relates to the structure of the cohomology of X (cf. [KL]). It applies to toric varieties as follows.

Write $Gr^{\ell}_{\mathbf{F}} Gr^{m}_{\bar{\mathbf{F}}} H^j_c(X)$ for the (ℓ, m)-graded part of the Hodge filtration of the cohomology of the complex variety X. It is clear from the results of [KL, §2] that X is computable if

$$P_X(q) \in \mathbb{Z}[q], \text{ and } \sigma \text{ acts as } (-1)^{\ell} \text{ on } Gr^{\ell}_{\mathbf{F}} Gr^{\ell}_{\bar{\mathbf{F}}} H^j_c(X) \tag{20}$$

for each j and ℓ.

The results of [F] and [KL, Proposition 5.2] make it evident that any toric variety X satisfies (20). This is because X is a union of locally closed subvarieties isomorphic to tori, which all satisfy (20).

In the case of a nonsingular projective toric variety X, we have that $H^i(X, \mathbb{Q}) = H^i_c(X, \mathbb{Q}) = 0$ for i odd, and $H^{2i}_c(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = Gr^i_{\mathbf{F}} Gr^i_{\bar{\mathbf{F}}} H^{2i}_c(X)$, whence $P_X(q) = \sum_i \dim H^{2i}(X)\, q^i$. It follows that

$$\chi(X(\mathbb{R})) = \sum_i (-1)^i \dim H^{2i}(X, \mathbb{Q}). \tag{21}$$

For nonsingular projective toric varieties, there is also a degree-by-degree statement involving cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Proposition 22. *Let X be a nonsingular projective complex toric variety. There is an isomorphism*

$$H^{2i}(X, \mathbb{Z}/2\mathbb{Z}) \cong H^i(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$$

for any i, which is equivariant for any automorphism of X defined over $\mathbb{R}$.

Proof. This follows from [K, Theorem 0.1]. □

We deduce an equivariant version of (21).

Corollary 23. *Let X be a nonsingular projective complex toric variety, and α an automorphism of X of finite odd order, defined over $\mathbb{R}$. Then*

$$\sum_i (-1)^i \operatorname{tr}(\alpha, H^i(X(\mathbb{R}), \mathbb{Q})) = \sum_i (-1)^i \operatorname{tr}(\alpha, H^{2i}(X, \mathbb{Q})).$$

Proof. Let A be the cyclic group of odd order generated by α. It suffices to prove the following equation in the rational Grothendieck ring $R(A)$:

$$\sum_i (-1)^i H^i(X(\mathbb{R}), \mathbb{Q}) = \sum_i (-1)^i H^{2i}(X, \mathbb{Q}). \tag{24}$$

Since $|A|$ is odd, there is an isomorphism $\tau : R(A) \to R_{\mathbb{Z}/2\mathbb{Z}}(A)$, where $R_{\mathbb{Z}/2\mathbb{Z}}(A)$ denotes the Grothendieck ring of $(\mathbb{Z}/2\mathbb{Z})A$-modules: for any $\mathbb{Q}A$-module M, $\tau(M)$ is defined by choosing an integral form of M and reducing modulo 2. So it suffices to prove the equation in $R_{\mathbb{Z}/2\mathbb{Z}}(A)$ obtained from (24) by applying τ to both sides. On the left-hand side, the universal coefficient theorem implies that

$$\tau\left(\sum_i (-1)^i H^i(X(\mathbb{R}), \mathbb{Q})\right) = \sum_i (-1)^i H^i(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}). \tag{25}$$

On the right-hand side, the fact that the integral homology of X vanishes in odd degrees [F, §5.2] implies that $\tau(H^{2i}(X, \mathbb{Q})) = H^{2i}(X, \mathbb{Z}/2\mathbb{Z})$. So the result follows from Proposition 22. □

Examples abound to show that the restriction to odd-order elements is necessary (consider $X = \mathbb{P}^1$, $\alpha : z \mapsto z^{-1}$). Thus, for a group of even order acting on a toric variety, the equivariant Euler characteristic of the real locus cannot be simply deduced from knowledge of the action on the cohomology groups of the complex variety.

In the case where $X = \mathcal{T}_W$ with $\alpha = w$ an element of W, the left-hand side of Corollary 23 is computed by Theorem 5, and a formula for the right-hand side may be deduced from [L, Theorem 1.1]. The comparison gives nothing new, and one would not expect it to, because the proofs of both theorems use the same reduction to the case of a torus, and the analogue of Corollary 23 where X is a torus (for cohomology with compact supports) is easy.

However, we wish to point out a curious complement to Corollary 23, which appears to hold only in type A (except for odd-rank cases, where it follows from Poincaré duality).

Proposition 26. *If $w \in \mathrm{Sym}_n$ has even order, then*

$$\sum_i (-1)^i \operatorname{tr}(w, H^{2i}(\mathcal{T}_{\mathrm{Sym}_n}, \mathbb{Q})) = 0.$$

Proof. Recall the ring $\widehat{\bigoplus}_{n\geq 0} R(\mathrm{Sym}_n)$ from Section 3. The following generating-function formula holds in $\mathbb{Q}[q] \otimes_{\mathbb{Q}} \widehat{\bigoplus}_{n\geq 0} R(\mathrm{Sym}_n)$ (see [Ste, Theorem 6.2]):

$$1 + \sum_{n\geq 1} \sum_i q^i\, H^{2i}(\mathcal{T}_{\mathrm{Sym}_n}, \mathbb{Q}) = \frac{\sum_{n\geq 0} 1_{\mathrm{Sym}_n}}{1 - \sum_{n\geq 2}(q + q^2 + \cdots + q^{n-1}) 1_{\mathrm{Sym}_n}}. \tag{27}$$

Equation (27) can be deduced easily from [L, Theorem 1.1], which implies that the left-hand side is the multiplicative inverse of $1 - \sum_{n\geq 1} \gamma_{\mathrm{Sym}_n}$, where $\gamma_{\mathrm{Sym}_n}(w) = \det_V(q - w)$. Setting $q = -1$ in (27), we deduce the following equation in $\widehat{\bigoplus}_{n\geq 0}R(\mathrm{Sym}_n)$:

$$1+\sum_{n\geq 1}\sum_i(-1)^i\,H^{2i}(\mathcal{T}_{\mathrm{Sym}_n},\mathbb{Q})=\Big(\sum_{n\geq 0}1_{\mathrm{Sym}_n}\Big).\Big(1+\sum_{\substack{n\geq 2\\ n\text{ even}}}1_{\mathrm{Sym}_n}\Big)^{-1}. \tag{28}$$

Extracting the degree-n terms, we have an equality in $R(\mathrm{Sym}_n)$:

$$\begin{aligned}&\sum_i(-1)^i\,H^{2i}(\mathcal{T}_{\mathrm{Sym}_n},\mathbb{Q})\\ &\qquad=\sum_{m\geq 0}(-1)^m\sum_{\substack{n_1,n_2,\cdots,n_m\geq 2\\ n_1,n_2,\cdots,n_m\text{ even}\\ n_1+n_2+\cdots+n_m\leq n}}\mathrm{Ind}_{\mathrm{Sym}_{n-n_1-\cdots-n_m}\times\mathrm{Sym}_{n_1}\times\cdots\times\mathrm{Sym}_{n_m}}^{\mathrm{Sym}_n}(1).\end{aligned} \tag{29}$$

Note that the right-hand side of (29) coincides with the right-hand side of Theorem 6 when evaluated at any w satisfying $\varepsilon(w)=1$ (in particular, since this holds for all elements of odd order, Corollary 23 is visibly true.)

To conclude the proof, we must show that the right-hand side of (29) takes the value 0 on any w which has even order (i.e. contains an even cycle). But the value in question is $\sum_{m\geq 0}(-1)^m f_m(w)$, where $f_m(w)$ is the number of m-tuples $(A_1,A_2,\cdots,A_m)$ of disjoint w-stable nonempty subsets of $\{1,\cdots,n\}$ such that each $|A_i|$ is even. Let $C\subseteq\{1,2,\cdots,n\}$ be a cycle of w such that $|C|$ is even, and let y be the induced permutation of $\{1,2,\cdots,n\}\setminus C$. Since any such $(A_1,A_2,\cdots,A_m)$ must have either $C=A_i$ for some i, $C\subset A_i$ for some i, or $C\cap A_i=\emptyset$ for all i, we have

$$f_m(w)=mf_{m-1}(y)+(m+1)f_m(y), \tag{30}$$

where $f_{-1}(y)=0$. It follows immediately that $\sum_{m\geq 0}(-1)^m f_m(w)=0$. □

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