NILPOTENT ORBITS OF LINEAR AND CYCLIC QUIVERS AND KAZHDAN-LUSZTIG POYNOMIALS OF TYPE A

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Abstract. The intersection cohomologies of closures of nilpotent orbits of linear (respectively, cyclic) quivers are known to be described by Kazhdan-Lusztig polynomials for the symmetric group (respectively, the affine symmetric group). We explain how to simplify this description using a combinatorial cancellation procedure, and derive some consequences for representation theory.

Contents

1. Introduction

This paper is concerned with formulas for the intersection cohomologies of closures of nilpotent orbits of linear and cyclic quivers. By fundamental results in geometric representation theory, these intersection cohomologies control certain features of the representations of affine Hecke algebras and quantum affine algebras. There is a well-known formula in the linear case due to Zelevinsky, using Kazhdan-Lusztig polynomials of the symmetric group; there is an analogous formula in the cyclic case due to Lusztig, using Kazhdan-Lusztig polynomials of the affine symmetric group. The main point of this paper is that both formulas can be rewritten in terms of Kazhdan-Lusztig polynomials for different (potentially smaller) symmetric or affine symmetric groups, by applying a combinatorial 'cancellation' procedure due to

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Billey and Warrington. The rewritten formulas in the linear quiver case have already appeared, in a representation-theoretic guise, in the work of Suzuki and others; in the cyclic quiver case they are new.

In the remainder of the introduction we will survey the main results and their representation-theoretic consequences; the other sections give the proofs, concentrating on the combinatorial side of the story. Sections 2 and 4 are purely combinatorial, explaining the concept of 'cancellation' for the symmetric group and affine symmetric group respectively. Most (perhaps all) of the results in Section 2 are known, but we will go over them in detail to provide a reference for the generalizations to the affine case in Section 4. Sections 3 and 5 connect these combinatorial results to the problem of computing intersection cohomology.

Throughout the paper, all vector spaces, algebras and varieties are over $\mathbb{C}$.

Consider the linear quiver of type A_∞, with vertex set $\mathbb{Z}$ and arrows $i \to i+1$ for all $i \in \mathbb{Z}$. Finite-dimensional representations of this quiver are parametrized by *multisegments*: a *segment* is a nonempty finite interval $[i,j]$ in $\mathbb{Z}$, and a multisegment is a finite formal sum of segments. Now fix a $\mathbb{Z}$-graded finite-dimensional vector space $V = \bigoplus_{i\in\mathbb{Z}} V_i$. Let $d_i = \dim V_i$, $d = \dim V$. A representation of the quiver on V is simply an element of

$$\mathcal{N}_V = \{\varphi \in \mathrm{End}(V) \,|\, \varphi(V_i) \subseteq V_{i+1},\ \forall i \in \mathbb{Z}\}.$$

Two such representations are isomorphic if they are in the same orbit for $G_V = \{g \in GL(V) \,|\, g(V_i) = V_i, \forall i\}$, acting on $\mathcal{N}_V$ by conjugation. Since all elements of $\mathcal{N}_V$ are nilpotent as endomorphisms of V, we call these *nilpotent orbits*. They are clearly in bijection with $M_{(d_i)}$, the set of multisegments such that each i occurs d_i times as an element of a segment. For $\mathbf{m} \in M_{(d_i)}$, let $\mathcal{O}_\mathbf{m}$ denote the corresponding orbit in $\mathcal{N}_V$. We put a partial order $\preceq$ on $M_{(d_i)}$ by setting $\mathbf{m} \preceq \mathbf{m}'$ if and only if $\mathcal{O}_\mathbf{m}$ is contained in the closure $\overline{\mathcal{O}_{\mathbf{m}'}}$ of $\mathcal{O}_{\mathbf{m}'}$.

The extent to which $\overline{\mathcal{O}_{\mathbf{m}'}}$ is singular at the points of $\mathcal{O}_\mathbf{m}$ is measured by an *intersection cohomology* polynomial $IC_{\mathbf{m},\mathbf{m}'} \in \mathbb{N}[q]$, defined by

$$IC_{\mathbf{m},\mathbf{m}'} = \sum_i \dim \mathcal{H}^{2i}_\mathbf{m} IC(\overline{\mathcal{O}_{\mathbf{m}'}})\, q^i,$$

where $IC(\overline{\mathcal{O}_{\mathbf{m}'}})$ is the intersection cohomology complex of $\overline{\mathcal{O}_{\mathbf{m}'}}$, and $\mathcal{H}^{2i}_\mathbf{m}$ denotes the stalk at a point of $\mathcal{O}_\mathbf{m}$ of the $(2i)$th cohomology sheaf (it turns out that all odd-degree cohomology sheaves of $IC(\overline{\mathcal{O}_{\mathbf{m}'}})$ vanish). Note that $IC_{\mathbf{m},\mathbf{m}'}$ is nonzero if and only if $\mathbf{m} \preceq \mathbf{m}'$, and is 1 if $\mathbf{m} = \mathbf{m}'$. Hence the inverse matrix $(IC^{\langle -1\rangle}_{\mathbf{m},\mathbf{m}'})_{\mathbf{m},\mathbf{m}'\in M_{(d_i)}}$ of $(IC_{\mathbf{m},\mathbf{m}'})_{\mathbf{m},\mathbf{m}'\in M_{(d_i)}}$ has entries in $\mathbb{Z}[q]$; moreover $IC^{\langle -1\rangle}_{\mathbf{m},\mathbf{m}'}$ is zero unless $\mathbf{m} \preceq \mathbf{m}'$, and is 1 if $\mathbf{m} = \mathbf{m}'$.

From the viewpoint of geometric representation theory, the poset $M_{(d_i)}$, together with these IC polynomials, is a model for certain "blocks" of representations of Lie-theoretic algebras of type A. More concretely, the algebras listed below each have a collection of finite-dimensional *standard* modules $\{M_{\mathbf{m}} \,|\, \mathbf{m} \in M_{(d_i)}\}$ and a collection of finite-dimensional simple modules $\{L_{\mathbf{m}} \,|\, \mathbf{m} \in M_{(d_i)}\}$, which are related by the following (equivalent) equations in the Grothendieck group of modules:

$$
\begin{aligned}
[M_{\mathbf{m}}] &= \sum_{\mathbf{m}' \in M_{(d_i)}} IC_{\mathbf{m},\mathbf{m}'}(1)\,[L_{\mathbf{m}'}],\ \forall \mathbf{m} \in M_{(d_i)},\\
[L_{\mathbf{m}}] &= \sum_{\mathbf{m}' \in M_{(d_i)}} IC^{\langle -1\rangle}_{\mathbf{m},\mathbf{m}'}(1)\,[M_{\mathbf{m}'}],\ \forall \mathbf{m} \in M_{(d_i)}.
\end{aligned}
\tag{1.1}
$$

So the sum of the coefficients of $IC_{\mathbf{m},\mathbf{m}'}$ is a composition multiplicity of a standard module; the individual coefficients record the composition multiplicities in a certain Jantzen-like filtration. For each algebra, the general definition of standard modules allows segments of arbitrary complex numbers, not just integers; but the problem of computing composition multiplicities can be reduced to the integer case. The algebras in question, and references to the definitions and results, are as follows.

(1) The affine Hecke algebra $\widehat{\mathcal{H}_d}$ attached to GL_d, specialized at a parameter which is not a root of unity (as in [5, Definition 12.3.1]). The standard and simple modules were defined by Zelevinksy in [24], and (1.1) was conjectured in [25] (see also [19]). Ginzburg proved (1.1) for standard modules defined in a geometric way (see [6, Theorem 8.6.23]). The fact that Ginzburg's standard modules coincide with Zelevinsky's in the Grothendieck group is usually deduced from the Induction Theorem of Kazhdan and Lusztig (see [2] – nowadays the best version of the Induction Theorem to use is [15, Theorem 7.11]), though it should be regarded as a comparatively easy case of that result.

(2) The corresponding degenerate affine Hecke algebra (as in [5, Definition 12.3.2]). The definitions of standard modules (as in [21, Section 2.2]) are analogous to case (1), and indeed (1.1) in this case can be deduced from case (1) by the results of Lusztig in [11] – he also gave a proof specific to this case in [13].

(3) The quantum affine algebra $U_\epsilon(\widehat{\mathfrak{sl}_r})$, specialized at a parameter ϵ which is not a root of unity (as in [5, Section 12.2A]). Here the standard modules are tensor products of fundamental evaluation modules corresponding to the segments, so we need r to be greater than or equal to the length of the longest segment involved (otherwise, we could just stipulate that any module indexed by a multisegment containing a segment of length $> r$ is zero). Equation (1.1) can be deduced from case (1) by Frobenius-Schur duality (see [5, Section 12.3D]). Alternatively, with a geometric definition of standard modules, (1.1) was proved by Ginzburg and Vasserot (see [23, Theorem 3]). It then follows that the two kinds of standard modules are the same in the Grothendieck group (see [23, Proposition 18]), which can presumably also be proved directly.

(4) The Yangian $Y(\mathfrak{sl}_r)$ (as in [5, Section 12.1A]). The standard modules as defined by Drinfeld in [7] are analogous to those in case (3), and (1.1) in this case can be deduced either from case (3) or case (2) using the results in [7].

This profusion of representation-theoretic meanings of the polynomials $IC_{\mathbf{m},\mathbf{m}'}$ and $IC^{\langle -1\rangle}_{\mathbf{m},\mathbf{m}'}$ is the main reason to be interested in computing them; but it is also why, in this paper, the clear-cut geometric definition is given greater prominence.

A classic result of Zelevinsky ([26, Corollary 1], see Theorem 3.2 below) identifies the polynomials $IC_{\mathbf{m},\mathbf{m}'}$ with Kazhdan-Lusztig polynomials of the symmetric group S_d. More precisely, it provides an isomorphism of posets between $M_{(d_i)}$ and a lower ideal $M'_{(d_i)}$ of the poset of maximal-length representatives of $S_{(d_i)}$–$S_{(d_i)}$ double cosets (under Bruhat order), where $S_{(d_i)}$ is the parabolic (i.e. Young) subgroup of S_d determined by the composition $d = \sum_i d_i$; and the polynomials attached to these posets (IC polynomials for $M_{(d_i)}$, and Kazhdan-Lusztig polynomials for $M'_{(d_i)}$) coincide under this isomorphism. Zelevinsky's proof is geometric, embedding the nilpotent orbits $\mathcal{O}_{\mathbf{m}}$ as open subvarieties of certain Schubert varieties, and using the fact that the intersection cohomologies of the latter are described by Kazhdan-Lusztig polynomials; but the point is that the poset $M'_{(d_i)}$ and its Kazhdan-Lusztig polynomials can be defined (and, in principle, computed) purely combinatorially.

More recent work of Suzuki ([21]) implicitly generalizes this result, providing a family of poset isomorphisms between various upper ideals of $M_{(d_i)}$ and combinatorially-defined posets. To explain this we adopt the notation of [17], which views multisegments as "generalized skew-shapes". For $\lambda, \mu \in \mathbb{Z}^k$, write $\lambda \supseteq \mu$ if $\lambda_i \geq \mu_i$ for all $1 \leq i \leq k$, and if this holds define a multisegment

$$\lambda/\mu = \sum_{i=1}^{k} [\mu_i - i + 1, \lambda_i - i], \tag{1.2}$$

where any "empty segments" of the form $[s+1, s]$ are ignored. The reason for the notation is that if λ and μ are *partitions*, i.e. $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ and similarly for μ, then the segments are exactly the rows of the skew-shape diagram usually called λ/μ, where each box is replaced by its *content.* Since the order of terms in (1.2) is unimportant, $\lambda/\mu = (w \cdot \lambda)/(w \cdot \mu)$ for all $w \in S_k$, where the "dot action" of S_k on $\mathbb{Z}^k$ is defined as usual by

$$(w \cdot \lambda)_i - i = \lambda_{w^{-1}(i)} - w^{-1}(i). \tag{1.3}$$

A fundamental domain for this dot action is

$$D_k := \{\lambda \in \mathbb{Z}^k \mid \lambda_1 - 1 \geq \lambda_2 - 2 \geq \cdots \geq \lambda_k - k\},$$

so we can write any multisegment in the *standard form*

$$\lambda/(w \cdot \mu) \text{ where } \begin{cases} \lambda, \mu \in D_k,\ w \in S_k,\ \lambda \supseteq w \cdot \mu, \text{ and} \\ w \text{ has maximal length in } W_\lambda w W_\mu. \end{cases} \tag{1.4}$$

Here W_λ and W_μ are the stabilizers of λ and μ for the dot action, which are clearly parabolic subgroups of S_k. Note that this expression in standard form is not uniquely determined by the multisegment, but rather by the multisegment together with a chosen multiset of empty segments.

Example. Let $\mathbf{m}$ be the multisegment $[1,2] + [2,2] + [3,3]$. The most economical way to express this in the form (1.4) is to take $k = 3$, $\lambda = (4,4,5)$, $\mu = (3,3,3)$, and w to be the transposition $(2,3)$. Another way is to take $k = 4$, $\lambda = (4,4,5,5)$, $\mu = (3,3,4,4)$, and w to be the transposition $(2,4)$; this effectively adds the empty segment $[2,1]$.

If $\lambda, \mu \in D_k$, we define

$$S_k[\lambda, \mu] = \{w \in S_k \mid \lambda \supseteq w \cdot \mu\} \text{ and}$$
$$S_k[\lambda, \mu]^\circ = \{w \in S_k[\lambda, \mu] \mid w \text{ has maximal length in } W_\lambda w W_\mu\}.$$

These are posets under Bruhat order; in fact we will see that $S_k[\lambda, \mu]$ is a lower ideal of S_k, so in particular $S_k[\lambda, \mu] \neq \emptyset \Leftrightarrow \lambda \supseteq \mu$. With this notation, the generalized form of Zelevinsky's result can be stated as follows.

Theorem 1.1. *Let $\lambda, \mu \in D_k$ be such that $\lambda \supseteq \mu$, $\lambda/\mu \in M_{(d_i)}$.*

(1) *The map* $w \mapsto \lambda/(w\cdot\mu)$ *is an isomorphism of posets between* $S_k[\lambda,\mu]^\circ$ *and* $\{\mathbf{m}' \in M_{(d_i)} \mid \lambda/\mu \preceq \mathbf{m}'\}$.
(2) *For* $w, w' \in S_k[\lambda,\mu]^\circ$, $IC_{\lambda/(w\cdot\mu),\lambda/(w'\cdot\mu)} = P_{w,w'}$, *a Kazhdan-Lusztig polynomial of* S_k.

Zelevinsky's original result is the special case where $k = d$, λ is such that each integer i occurs d_i times in $(\lambda_1 - 1, \cdots, \lambda_d - d)$, and $\mu = \lambda - (1, 1, \cdots, 1)$. In this case λ/μ is the trivial multisegment $\sum_i d_i[i,i]$ (corresponding to the zero orbit), so the image of the isomorphism in part (1) is all of $M_{(d_i)}$; the parabolic subgroups W_λ and W_μ both equal $S_{(d_i)}$, and $S_d[\lambda,\mu]^\circ$ is the poset $M'_{(d_i)}$ mentioned above.

The fact that Theorem 1.1 is true in general means that in expressing $\mathbf{m}$ and $\mathbf{m}'$ as $\lambda/(w\cdot\mu)$ and $\lambda/(w'\cdot\mu)$, any of the empty segments which occur in the Zelevinsky case can be "cancelled" without changing the Kazhdan-Lusztig polynomial (or, indeed, new ones can be added). For example, at the extreme, Theorem 1.1 shows that $IC_{\mathbf{m},\mathbf{m}'}$ can be identified with a Kazhdan-Lusztig polynomial of $S_{k(\mathbf{m})}$, where $k(\mathbf{m})$ is the number of segments of $\mathbf{m}$ (smaller than d, unless $\mathbf{m}$ is trivial). In Section 3 we will use a result of Billey and Warrington, which provides for just such cancellations in Kazhdan-Lusztig polynomials of symmetric groups, to deduce Theorem 1.1 from Zelevinsky's theorem. The essence of the result is stated in Theorem 3.3, using some matrix notation which will be introduced in §2.

As already mentioned, Theorem 1.1 cannot be considered new, since in the context of the representation theory of the degenerate affine Hecke algebra (case (2) above) it follows from Suzuki's results in [21]. With notation as in the Theorem, he defines an exact functor F_λ from the category $\mathcal{O}$ of representations of $\mathfrak{gl}_k$ to the category of finite-dimensional modules for the degenerate affine Hecke algebra associated to GL_d, and shows that it takes the Verma module $M(w\cdot\mu)$ to the standard module $M_{\lambda/(w\cdot\mu)}$ and the simple module $L(w\cdot\mu)$ to the simple module $L_{\lambda/(w\cdot\mu)}$ for all $w \in S_k[\lambda,\mu]^\circ$. Part (1) and the $q = 1$ specialization of part (2) of Theorem 1.1 then follow from the known Kazhdan-Lusztig conjecture for $\mathfrak{gl}_k$ (combine [21, (5.2.1) and (5.2.2)] – the historical remarks following [21, (5.2.3)] properly apply only to the Zelevinsky case). Moreover, by [21, Theorem 5.3.5] the Kazhdan-Lusztig polynomials $P_{w,w'}$ for $w' \in S_k[\lambda,\mu]^\circ$ record multiplicities in a Jantzen-type filtration of $M_{\lambda/(w\cdot\mu)}$, whose definition clearly depends only on the multisegment (i.e. not on the empty segments); since (2) of Theorem 1.1 is true in the Zelevinsky case, it must be true in general. (As well as [21], see [17] and [1] for the analogous results in the case of the affine Hecke algebra and Yangian respectively).

One corollary concerns those multisegments λ/μ where $\lambda, \mu \in D_k$ satisfy $W_\lambda = W_\mu = \{1\}$, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$ (these are the "placed skew-shapes" of [18]): for such λ/μ, it follows from Theorem 1.1 that

$$IC^{\langle -1\rangle}_{\lambda/\mu,\lambda/(w\cdot\mu)} = \varepsilon(w), \text{ for all } w \in S_k[\lambda,\mu], \tag{1.5}$$

where ε denotes the sign character. Thus the corresponding simple modules (called *calibrated* for the affine Hecke algebra in [18] and *tame* for the Yangian in [16]) can be written as an alternating sum of standard modules in the Grothendieck group. Representation-theoretically, this reflects the existence of a BGG-like resolution of these simple modules (transferred by the appropriate functor from the BGG resolution of the $\mathfrak{gl}_k$-module $L(\mu)$) – see [21, Theorem 5.1.1] and [17, (4.13)].

The justification for re-proving Theorem 1.1 in §3 below is that the combinatorics involved generalizes immediately to the case of cyclic quivers, as we will now explain.

Fix a positive integer n, and consider the cyclic quiver of type $\widetilde{A_{n-1}}$, with vertex set $\mathbb{Z}/n\mathbb{Z}$ and arrows $\bar{i} \to \overline{i+1}$ for all $\bar{i} \in \mathbb{Z}/n\mathbb{Z}$. Finite-dimensional *nilpotent* representations of this quiver are parametrized by multisegments as before, except that there is no difference between segments $[i,j]$ and $[i',j']$ when $i'-i = j'-j$ is a multiple of n. Fix a $(\mathbb{Z}/n\mathbb{Z})$-graded finite-dimensional vector space $V = \bigoplus_{\bar{i}\in\mathbb{Z}/n\mathbb{Z}} V_{\bar{i}}$, and set $d_i = \dim V_{\bar{i}}$, $d = \dim V$. We define

$$\mathcal{N}_V = \{\varphi \in \mathrm{End}(V) \,|\, \varphi(V_{\bar{i}}) \subseteq V_{\overline{i+1}},\ \forall \bar{i} \in \mathbb{Z}/n\mathbb{Z},\ \varphi \text{ nilpotent}\},$$

and consider G_V-orbits in $\mathcal{N}_V$. These are in bijection with $M_{(d_i),n}$, the set of multisegments (in this modulo n sense) such that each congruence class $\bar{i}$ occurs d_i times among the elements of the segments. For $\mathbf{m} \in M_{(d_i),n}$, let $\mathcal{O}_{\mathbf{m}}$ denote the corresponding nilpotent orbit, and define a partial order $\preceq$ and polynomials $IC_{\mathbf{m},\mathbf{m}'} \in \mathbb{N}[q]$ and $IC^{\langle -1\rangle}_{\mathbf{m},\mathbf{m}'} \in \mathbb{Z}[q]$ in the same way as before.

These polynomials too have representation-theoretic significance. The specialized quantum affine algebra $U_\zeta(\widehat{\mathfrak{sl}_r})$, where ζ^2 is a primitive nth root of 1, has a collection of standard modules $\{M_{\mathbf{m}} \,|\, \mathbf{m} \in M_{(d_i),n}\}$ and a collection of simple modules $\{L_{\mathbf{m}} \,|\, \mathbf{m} \in M_{(d_i),n}\}$, satisfying the equivalent equations

$$\begin{aligned} [M_{\mathbf{m}}] &= \sum_{\mathbf{m}'\in M_{(d_i),n}} IC_{\mathbf{m},\mathbf{m}'}(1)\,[L_{\mathbf{m}'}],\ \forall \mathbf{m} \in M_{(d_i),n}, \\ [L_{\mathbf{m}}] &= \sum_{\mathbf{m}'\in M_{(d_i),n}} IC^{\langle -1\rangle}_{\mathbf{m},\mathbf{m}'}(1)\,[M_{\mathbf{m}'}],\ \forall \mathbf{m} \in M_{(d_i),n}. \end{aligned} \tag{1.6}$$

(See [23, Theorem 3] – again, for small r we have to disregard multisegments containing a segment of length $> r$.) The same is true for the affine Hecke algebra $\widehat{\mathcal{H}_d}$ specialized at a primitive nth root of unity, except that there the simple modules are parametrized by the smaller set of *aperiodic* multisegments (see [10, Section 2]), so we have to set $[L_{\mathbf{m}}] = 0$ if $\mathbf{m}$ is not aperiodic.

The analogue of Zelevinsky's result for cyclic quivers was proved by Lusztig in [12, §11] (it is stated below as Theorem 5.2). This identifies $IC_{\mathbf{m},\mathbf{m}'}$ with a Kazhdan-Lusztig polynomial of the affine symmetric group $\widetilde{S_d}$ (the Coxeter group of type $\widetilde{A_{d-1}}$). In Section 4, we will show that a version of Billey and Warrington's cancellation works for the affine symmetric group. As a consequence, we get an analogue of Theorem 1.1 in this setting, for which a representation-theoretic proof does not yet exist.

To state it requires extending the dot action of S_k on $\mathbb{Z}^k$ to $\widetilde{S_k}$, so that the extra Coxeter generator s_0 acts by

$$(s_0 \cdot \lambda)_i = \begin{cases} \lambda_k - k + 1 + n, & \text{if } i = 1, \\ \lambda_i, & \text{if } 2 \le i \le k-1, \\ \lambda_1 - 1 + k - n, & \text{if } i = k. \end{cases}$$

It is then clear that $(w \cdot \lambda)/(w \cdot \mu) = \lambda/\mu$ for all $w \in \widetilde{S_k}$, where the multisegments are now interpreted in the modulo n sense. A fundamental domain for the action of $\widetilde{S_k}$ on $\mathbb{Z}^k$ is

$$\widetilde{D_k} := \{\lambda \in \mathbb{Z}^k \mid \lambda_1 - 1 \ge \lambda_2 - 2 \ge \cdots \ge \lambda_k - k \ge \lambda_1 - n - 1\},$$

and the corresponding standard form for multisegments is

$$\lambda/(w \cdot \mu) \text{ where } \begin{cases} \lambda, \mu \in \widetilde{D_k},\ w \in \widetilde{S_k},\ \lambda \supseteq w \cdot \mu, \text{ and} \\ w \text{ has maximal length in } \widetilde{W_\lambda} w \widetilde{W_\mu}, \end{cases} \tag{1.7}$$

where $\widetilde{W_\lambda}$ and $\widetilde{W_\mu}$ denote the stabilizers of λ and μ in $\widetilde{S_k}$ (proper parabolic subgroups, hence finite). For $\lambda, \mu \in \widetilde{D_k}$, we define

$$\widetilde{S_k}[\lambda, \mu] = \{w \in \widetilde{S_k} \mid \lambda \supseteq w \cdot \mu\} \text{ and}$$
$$\widetilde{S_k}[\lambda, \mu]^\circ = \{w \in \widetilde{S_k}[\lambda, \mu] \mid w \text{ has maximal length in } \widetilde{W_\lambda} w \widetilde{W_\mu}\}.$$

We will see in §5 that, as in the symmetric group case, $\widetilde{S_k}[\lambda, \mu]$ is a (finite) lower ideal of $\widetilde{S_k}$ for Bruhat order. We can now state a generalization of Lusztig's result.

Theorem 1.2. *Let $\lambda, \mu \in \widetilde{D_k}$ be such that $\lambda \supseteq \mu$, $\lambda/\mu \in M_{(d_i),n}$.*

(1) *The map* $w \mapsto \lambda/(w \cdot \mu)$ *is an isomorphism of posets between* $\widetilde{S_k}[\lambda,\mu]^\circ$ *and* $\{\mathbf{m}' \in M_{(d_i),n} \,|\, \lambda/\mu \preceq \mathbf{m}'\}$.
(2) *For* $w, w' \in \widetilde{S_k}[\lambda,\mu]^\circ$, $IC_{\lambda/(w\cdot\mu),\lambda/(w'\cdot\mu)} = P_{w,w'}$, *a Kazhdan-Lusztig polynomial of* $\widetilde{S_k}$.

(An alternative statement using matrix notation is given in Theorem 5.3.) It is natural to wonder whether there is a representation-theoretic functor which "explains" this Theorem too.

As in the linear quiver case, Theorem 1.2 implies that $IC_{\mathbf{m},\mathbf{m}'}$ can be identified with a Kazhdan-Lusztig polynomial of $\widetilde{S_{k(\mathbf{m})}}$, where $k(\mathbf{m})$ is the number of segments of $\mathbf{m}$; this immediately implies the main result of [8], that $IC_{\mathbf{m},\mathbf{m}'} = 1$ when $\mathbf{m} \preceq \mathbf{m}'$, $k(\mathbf{m}) = 2$.

Another consequence of Theorem 1.2 is an analogue of (1.5), concerning those multisegments λ/μ where $\lambda,\mu \in \widetilde{D_k}$ satisfy $\widetilde{W_\lambda} = \widetilde{W_\mu} = \{1\}$; this means that $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k \geq \lambda_1 - n + k$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq \mu_1 - n + k$. For such λ/μ, it follows from Theorem 1.2 that

$$IC^{\langle -1\rangle}_{\lambda/\mu,\lambda/(w.\mu)} = \varepsilon(w), \text{ for all } w \in \widetilde{S_k}[\lambda,\mu]. \tag{1.8}$$

So once more the corresponding simple modules can be written as an alternating sum of standard modules in the Grothendieck group; probably this indicates a BGG-like resolution.

Theorems 1.1 and 1.2 combine well with the method used by Varagnolo and Vasserot in [22] to determine the decomposition numbers of $U_\zeta(\mathfrak{gl}_r)$ where ζ^2 is a primitive nth root of 1. Suppose we want to compute the multiplicity of the simple module $L_\zeta(\mu')$ in the Weyl module $V_\zeta(\lambda')$, where λ and μ are partitions with at most k parts all of size $\leq r$, and λ' and μ' are the transpose partitions (regarded as dominant integral weights for $\mathfrak{gl}_r$). By definition, $V_\zeta(\lambda')$ is the specialization at ζ of the simple module $V_q(\lambda')$ for the generic $U_q(\mathfrak{gl}_r)$. Now using a suitable normalization of the evaluation map $U_q(\widehat{\mathfrak{sl}_r}) \to U_q(\mathfrak{gl}_r)$, we can regard $V_q(\lambda')$ as the simple $U_q(\widehat{\mathfrak{sl}_r})$-module $L_{\lambda/0}$ (see [22, Section 12.2]). By (1.5), we have the equation

$$[L_{\lambda/0}] = \sum_{w \in S_k[\lambda,0]} \varepsilon(w)\,[M_{\lambda/(w\cdot 0)}]. \tag{1.9}$$

Now let $w_\lambda, w_\mu, w_0 \in \widetilde{S_k}$ be such that $w_\lambda \cdot \lambda, w_\mu \cdot \mu, w_0 \cdot 0 \in \widetilde{D_k}$. As noted in [22, Section 12.3], the specialization at ζ of the standard module $M_{\lambda/(w\cdot 0)}$ is merely the $U_\zeta(\widehat{\mathfrak{sl}_r})$-standard module of the same name, which in standard form is $M_{(w_\lambda\cdot\lambda)/(w_\lambda w w_0^{-1})^\circ\cdot(w_0\cdot 0)}$, where $(w_\lambda w w_0^{-1})^\circ$ is the longest element of $\widetilde{W_{w_\lambda\cdot\lambda}} w_\lambda w w_0^{-1} \widetilde{W_{w_0\cdot 0}}$. So in the Grothendieck group of $U_\zeta(\widehat{\mathfrak{sl}_r})$-modules,

$$[V_\zeta(\lambda')] = \sum_{w\in S_k[\lambda,0]} \varepsilon(w)\,[M_{(w_\lambda\cdot\lambda)/(w_\lambda w w_0^{-1})^\circ\cdot(w_0\cdot 0)}]. \tag{1.10}$$

Now as noted in [22, Section 12.2], $L_\zeta(\mu')$ regarded as a simple $U_\zeta(\widehat{\mathfrak{sl}_r})$-module is $L_{\mu/0} = L_{(w_\mu\cdot\mu)/(w_\mu w_0^{-1})^\circ\cdot(w_0\cdot 0)}$, where $(w_\mu w_0^{-1})^\circ$ is the longest element of $\widetilde{W_{w_\mu\cdot\mu}} w_\mu w_0^{-1} \widetilde{W_{w_0\cdot 0}}$. Using Theorem 1.2, we obtain

(1.11)

$$[V_\zeta(\lambda') : L_\zeta(\mu')] = \begin{cases} \sum_{w\in S_k} \varepsilon(w)\, P_{(w_\lambda w w_0^{-1})^\circ,(w_\mu w_0^{-1})^\circ}(1), & \text{if } \mu \in \widetilde{S_k}\cdot\lambda, \\ 0, & \text{otherwise.} \end{cases}$$

In the first case, summing over all of S_k rather than just $S_k[\lambda,0]$ introduces no new terms, since $(w_\mu w_0^{-1})^\circ \in \widetilde{S_k}[w_\mu\cdot\mu, w_0\cdot 0] = \widetilde{S_k}[w_\lambda\cdot\lambda, w_0\cdot 0]$, so the Kazhdan-Lusztig polynomial can only be nonzero when $w_\lambda w w_0^{-1} \in \widetilde{S_k}[w_\lambda\cdot\lambda, w_0\cdot 0]$, i.e. $\lambda \supseteq w\cdot 0$.

In the special case that λ and μ have trivial stabilizers in $\widetilde{S_k}$ (i.e. $\lambda_1 - 1, \cdots, \lambda_k - k$ have different residues modulo n, and similarly for μ – this requires $k \le n$, which automatically implies $w_0 = 1$), (1.11) becomes

$$[V_\zeta(\lambda') : L_\zeta(\mu')] = \begin{cases} \sum_{w\in S_k} \varepsilon(w)\, P_{w_\lambda w, w_\mu}(1), & \text{if } \mu \in \widetilde{S_k}\cdot\lambda, \\ 0, & \text{otherwise.} \end{cases} \tag{1.12}$$

This is the form of the answer given by Soergel in [20, Conjecture 7.1] for the equivalent problem of computing tilting module multiplicities for $U_\zeta(\mathfrak{gl}_k)$.

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2. Cancellation for the symmetric group

In this section we explain the combinatorial result of Billey and Warrington on which our approach depends. Fix a positive integer d, and let S_d be the group of permutations of $[1,d] = \{1,\cdots,d\}$. For $i \in [1,d-1]$, we define $s_i \in S_d$ to be the transposition interchanging i and $i+1$; as everyone knows, $s_1,\cdots,s_{d-1}$ form a set of Coxeter generators for S_d of type A_{d-1}. We thus have a length function $\ell : S_d \to \mathbb{N}$, a Bruhat order $\le$, and Kazhdan-Lusztig polynomials $P_{y,w} \in \mathbb{N}[q]$ for

$y, w \in S_d$ (which are nonzero iff $y \leq w$). Good references for Kazhdan-Lusztig polynomials are [9, Chapter 7] and [20] (where the notation is somewhat different).

The length function and the Bruhat order have well-known combinatorial descriptions. Define the *inversion statistics*

$$\mathrm{inv}_i(w) = |\{i' < i \,|\, w(i') > w(i)\}|, \ \mathrm{Inv}_i(w) = |\{i' > i \,|\, w(i') < w(i)\}|,$$

for any $w \in S_d$ and $i \in [1, d]$. These are related by $\mathrm{Inv}_i(w) = \mathrm{inv}_i(w) + w(i) - i$. Then

$$\ell(w) = \sum_{i \in [1,d]} \mathrm{inv}_i(w) = \sum_{i \in [1,d]} \mathrm{Inv}_i(w). \tag{2.1}$$

A special case of Bruhat order is that for all $i \in [1, d-1]$,

$$ws_i < w \text{ if and only if } w(i) > w(i+1). \tag{2.2}$$

The general description, due to Deodhar, is as follows:

Proposition 2.1. *If $y, w \in S_d$, $y \leq w$ if and only if for all $i, j \in [1, d]$,*

$$|\{i' \leq i \,|\, y(i') \geq j\}| \leq |\{i' \leq i \,|\, w(i') \geq j\}|.$$

In other words, for all $i \in [1, d]$ and $m \in [1, i]$, the mth largest element in $y[1, i]$ is less than or equal to the mth largest element in $w[1, i]$. If $y \leq w$, we write $[y, w]$ for the Bruhat interval $\{x \in S_d \,|\, y \leq x \leq w\}$.

We now come to the key definition.

Definition. If $y \leq w$ in S_d, we say that $i \in [1, d]$ is *cancellable* for the interval $[y, w]$ if $y(i) = w(i)$, $\mathrm{inv}_i(y) = \mathrm{inv}_i(w)$, and $\mathrm{Inv}_i(y) = \mathrm{Inv}_i(w)$. (Clearly any two of these conditions imply the third.)

The reason for the name 'cancellable' is that Bruhat order and Kazhdan-Lusztig polynomials are preserved under the operation of 'cancelling the common action on i' from the permutations in question, in the following sense. For all $i \in [1, d]$, let $\sigma_i : [1, d] \setminus \{i\} \to [1, d-1]$ be the unique order-preserving bijection. For $w \in S_d$, we define $w^{\hat{i}} \in S_{d-1}$ by

$$w^{\hat{i}} = \sigma_{w(i)} \circ w \circ \sigma_i^{-1}.$$

It is clear from either formula in (2.1) that

$$\ell(w^{\hat{i}}) = \ell(w) - \mathrm{inv}_i(w) - \mathrm{Inv}_i(w). \tag{2.3}$$

The following result combines Lemmas 17 and 39 of [3], but we will spell out the proof for later reference.

Proposition 2.2. *Suppose that i is cancellable for $[y, w]$.*

(1) *For any $x \in [y, w]$, $x(i) = y(i)$ and $\mathrm{inv}_i(x) = \mathrm{inv}_i(y)$. Hence i is cancellable for any sub-interval of $[y, w]$.*

(2) *$x \mapsto x^{\hat{i}}$ is an isomorphism of posets between $[y,w]$ and $[y^{\hat{i}}, w^{\hat{i}}]$, which reduces all lengths by the same amount.*

(3) *For any $u, v \in [y,w]$, $P_{u,v} = P_{u^{\hat{i}},v^{\hat{i}}}$.*

Proof. Set $j = y(i) = w(i)$, $m = \mathrm{inv}_i(y)+1 = \mathrm{inv}_i(w)+1$, and suppose $y \leq x \leq w$. Now $y[1,i]$ and $w[1,i]$ each have exactly m elements $\geq j$ and $m-1$ elements $> j$. By Proposition 2.1, the same is true of $x[1,i]$. Similarly, $y[1,i-1]$ and $w[1,i-1]$ each have exactly $m-1$ elements $\geq j$ and $m-1$ elements $> j$, so the same is true of $x[1,i-1]$. Thus $x(i) = j$ and $\mathrm{inv}_i(x) = m-1$, proving (1). Moreover, it is clear from Proposition 2.1 that $y^{\hat{i}} \leq x^{\hat{i}} \leq w^{\hat{i}}$. The construction of the map $[y^{\hat{i}}, w^{\hat{i}}] \to [y,w] : x \mapsto \tilde{x}$ inverse to $x \mapsto x^{\hat{i}}$ is easy:

$$\tilde{x}(i') = \begin{cases} j, & \text{if } i' = i, \\ \sigma_j^{-1}(x(\sigma_i(i'))), & \text{if } i' \neq i. \end{cases}$$

This proves the isomorphism part of (2), and the statement about lengths follows from (1). In light of parts (1) and (2), it clearly suffices to prove (3) in the case $u = y$, $v = w$. We prove this by induction on $\ell(w)$, it being trivial if $w = 1$. Choose one of the Coxeter generators, say s, such that $ws < w$. We now have three cases.

Case 1: $s = s_{i-1}$. This means that $w(i-1) > j$, so $w[1,i-2]$ has only $m-2$ elements $> j$. Therefore the same is true of $y[1,i-2]$, so $y(i-1) > j$, i.e. $ys < y$. Moreover, $ws[1,i-1]$ has only $m-2$ elements $> j$, so $y \not\leq ws$. Under these circumstances we have (see (2.5) below)

$$P_{y,w} = P_{ys,ws}. \tag{2.4}$$

Obviously $i-1$ is cancellable for $[ys, ws]$, so by the induction hypothesis, $P_{ys,ws} = P_{(ys)^{\widehat{i-1}},(ws)^{\widehat{i-1}}}$. But $(ys)^{\widehat{i-1}} = y^{\hat{i}}$ and $(ws_{i-1})^{\widehat{i-1}} = w^{\hat{i}}$, so we have the result.

Case 2: $s = s_i$. This means that $w(i+1) < j$, so $w[1,i+1]$ has only $m-1$ elements $> j$. Therefore the same is true of $y[1,i+1]$, so $y(i+1) < j$, i.e. $ys < y$. Moreover, $ws[1,i]$ has only $m-1$ elements $\geq j$, so $y \not\leq ws$. The proof proceeds as in Case 1, with $i+1$ in place of $i-1$.

Case 3: $s \neq s_{i-1}, s_i$. The fundamental recursive property of Kazhdan-Lusztig polynomials ([9, Section 7.11, (23)]) tells us that

$$P_{y,w} = P_{y',ws} + qP_{y's,ws} - \sum_{\substack{y \leq z < ws \\ zs < z}} \mu(z,ws)\, q^{(\ell(w)-\ell(z))/2} P_{y,z}, \tag{2.5}$$

where $\mu(z,ws)$ is the coefficient of $q^{(\ell(ws)-\ell(z)-1)/2}$ in $P_{z,ws}$, and y' is the minimum of y and ys in Bruhat order. All the nonzero Kazhdan-Lusztig polynomials involved in the right-hand side are indexed by elements of the interval $[y', ws]$, for which i is cancellable. By the induction hypothesis, they can all be replaced by the analogous polynomials for the interval $[(y')^{\hat{i}}, (ws)^{\hat{i}}]$, and the result follows. $\square$

We now recall (and extend slightly) the matrix notation used in [26]. Let $(b_i)_{i\in[1,n]}$ be an n-tuple of nonnegative integers whose sum is d, and let $(c_j)_{j\in[1,n']}$ be an n'-tuple of nonnegative integers whose sum is also d. To avoid notational clutter, we make the convention for the rest of this section that the range of the variables i and i' will be $[1, n]$ unless otherwise specified, and that of the variables j and j' will be $[1, n']$. We will use boldface letters such as $\mathbf{m}$ and $\mathbf{m}'$ for the $(n \times n')$-matrices whose entries are written with the corresponding ordinary letters $m_{i,j}$ and $m'_{i,j}$. Let $M_{(b_i);(c_j)}$ be the set of all $(n \times n')$-matrices $\mathbf{m}$ satisfying:

(1) $m_{i,j} \in \mathbb{N}$, for all i, j,
(2) $\sum_j m_{i,j} = b_i$, for all i, and
(3) $\sum_i m_{i,j} = c_j$, for all j.

If any b_i or c_j is 0, the corresponding row or column must always be zero and is therefore irrelevant, but it will be convenient to allow this possibility. We will use an obvious notation for the sums of various sectors of a matrix:

$$m_{\leq i,\geq j} = \sum_{\substack{i'\leq i\\ j'\geq j}} m_{i',j'},\ m_{\leq i,j} = \sum_{i'\leq i} m_{i',j},\ m_{i,\geq j} = \sum_{j'\geq j} m_{i,j'},$$

and similarly $m_{\geq i,\leq j}$, etc. Note that for $\mathbf{m} \in M_{(b_i);(c_j)}$,

$$\begin{aligned} m_{\geq i,\leq j} &= c_1 + c_2 + \cdots + c_j - m_{\leq i-1,\leq j} \\ &= c_1 + \cdots + c_j - b_1 - \cdots - b_{i-1} + m_{\leq i-1,\geq j+1}. \end{aligned} \tag{2.6}$$

The matrices in $M_{(b_i);(c_j)}$ parametrize double cosets of S_d with respect to certain parabolic subgroups. Namely, write $[1, d]$ as the disjoint union of blocks $B_1, \cdots, B_n$ such that all elements of B_i are less than all elements of B_{i+1}, and $|B_i| = b_i$. (Because we are allowing some b_i to be zero, some of these blocks could be empty.) Let $S_{(b_i)}$ be the subgroup of S_d which preserves each B_i separately; this is a parabolic subgroup isomorphic to $S_{b_1} \times \cdots \times S_{b_n}$. Similarly define blocks C_j of sizes c_j, and the parabolic subgroup $S_{(c_j)}$. We define a surjective map $\psi : S_d \to M_{(b_i);(c_j)}$ by

$$\psi(w)_{i,j} = |w(B_i) \cap C_j|.$$

The fibres of ψ are exactly the double cosets $S_{(c_j)}wS_{(b_i)}$, so ψ induces a bijection $S_{(c_j)}\backslash S_d / S_{(b_i)} \leftrightarrow M_{(b_i);(c_j)}$. For $\mathbf{m} \in M_{(b_i);(c_j)}$, let $w_{\mathbf{m}} \in S_d$ be the longest element in the corresponding double coset.

Note that in the case when $n = n' = d$ and all $b_i = c_j = 1$, the parabolic subgroups are trivial, and we have merely passed from elements of S_d to the corresponding permutation matrices (or their transposes, depending on your convention). In general, the permutation $w_{\mathbf{m}}$ can be constructed from the matrix $\mathbf{m}$ as follows: assuming that the images of $B_{i'}$ for $i' < i$ have been determined, we send successive various-sized sub-blocks of B_i to the various C_js, according to the entries of the ith row of $\mathbf{m}$ read from right to left. Within each sub-block, we successively take the largest element of C_j still unused. More formally, if a is the sth element of B_i, then $w_{\mathbf{m}}(a) \in C_j$ where j is maximal such that $m_{i,\geq j} \geq s$. Specifically, $w_{\mathbf{m}}(a)$ is the tth largest element of C_j where

$$t = m_{\leq i-1,j} + s - m_{i,\geq j+1}. \tag{2.7}$$

Example. Take $d = 9$, $n = n' = 4$, and define b_i, c_j so that

$$B_1 = \{1\},\ B_2 = \{2,3,4,5\},\ B_3 = \{6,7,8\},\ B_4 = \{9\},$$
$$C_1 = \{1,2\},\ C_2 = \{3,4,5\},\ C_3 = \{6,7,8\},\ C_4 = \{9\}.$$

Let us construct $w_{\mathbf{m}}$ where

$$\mathbf{m} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first row tells us that $w_{\mathbf{m}}(1)$ is an element of C_1; we take the largest element, namely 2. The second row tells us that $w_{\mathbf{m}}(B_2)$ consists of one element of C_3, two elements of C_2, and one of C_1, in that order. Taking the largest elements not yet used, we set $w_{\mathbf{m}}(2) = 8$, $w_{\mathbf{m}}(3) = 5$, $w_{\mathbf{m}}(4) = 4$, and $w_{\mathbf{m}}(5) = 1$. Continuing in this way, we see that $w_{\mathbf{m}}$ is the permutation 285417639 (in 'one-line' notation).

We define a length function $\ell : M_{(b_i);(c_j)} \to \mathbb{N}$ by $\ell(\mathbf{m}) = \ell(w_{\mathbf{m}})$, and a partial order on $M_{(b_i);(c_j)}$ by

$$\mathbf{m} \leq \mathbf{m}' \Leftrightarrow w_{\mathbf{m}} \leq w_{\mathbf{m}'}.$$

These can be described as follows.

Proposition 2.3. *Let* $\mathbf{m}, \mathbf{m}' \in M_{(b_i);(c_j)}$.

(1) $\ell(\mathbf{m}) = \sum_{i,j} m_{i,j} m_{\leq i,\geq j} - \sum_{i,j} \binom{m_{i,j}+1}{2}$.

(2) $\mathbf{m} \leq \mathbf{m}'$ *if and only if, for all* i, j,
$$m_{\leq i,\geq j} \leq m'_{\leq i,\geq j}.$$

(3) $\mathbf{m} \leq \mathbf{m}'$ *if and only if, for all* i, j,
$$m_{\geq i,\leq j} \leq m'_{\geq i,\leq j}.$$

Proof. Let a be the largest element of $B_i \cap w_{\mathbf{m}}^{-1}(C_j)$. Then for $1 \le k \le m_{i,j}$, $a-k+1$ is the kth largest element of $B_i \cap w_{\mathbf{m}}^{-1}(C_j)$. Clearly

$$\mathrm{inv}_{a-k+1}(w_{\mathbf{m}}) = m_{\le i, \ge j} - k. \tag{2.8}$$

Summing this over all i, j, and $1 \le k \le m_{i,j}$ gives (1). To prove (2), fix i and j, and let b be the largest element of $\cup_{i' \le i} B_{i'}$ and c the smallest element of $\cup_{j' \ge j} C_{j'}$. If $\mathbf{m} \le \mathbf{m}'$, then by Proposition 2.1, we have

$$|\{a \le b \,|\, w_{\mathbf{m}}(a) \ge c\}| \le |\{a \le b \,|\, w_{\mathbf{m}'}(a) \ge c\}|,$$

which exactly says that $m_{\le i, \ge j} \le m'_{\le i, \ge j}$. Conversely, suppose we know that $m_{\le i, \ge j} \le m'_{\le i, \ge j}$ and $m_{\le i-1, \ge j} \le m'_{\le i-1, \ge j}$. For all $1 \le k \le b_i$, we have

$$|\{a \le b-k+1 \,|\, w_{\mathbf{m}}(a) \ge c\}| = \max\{m_{\le i, \ge j} - k + 1, m_{\le i-1, \ge j}\} \tag{2.9}$$

and similarly for $\mathbf{m}'$, so our assumption implies

$$|\{a \le b-k+1 \,|\, w_{\mathbf{m}}(a) \ge c\}| \le |\{a \le b-k+1 \,|\, w_{\mathbf{m}'}(a) \ge c\}|.$$

Combining these statements for all j tells us that for all m, the mth largest element of $w_{\mathbf{m}}[1, b-k+1]$ lies in a block C_j prior or equal to that containing the mth largest element of $w_{\mathbf{m}'}[1, b-k+1]$. Remembering how $w_{\mathbf{m}}$ and $w_{\mathbf{m}'}$ are constructed from $\mathbf{m}$ and $\mathbf{m}'$, this implies that for all m, the mth largest element of $w_{\mathbf{m}}[1, b-k+1]$ is less than or equal to the mth largest element of $w_{\mathbf{m}'}[1, b-k+1]$. Letting i and k vary, we get $w_{\mathbf{m}} \le w_{\mathbf{m}'}$ by Proposition 2.1, so (2) is proved. One way to deduce (3) from (2) is to use (2.6). Another way is to recall that $w_{\mathbf{m}} \le w_{\mathbf{m}'}$ if and only if $w_{\mathbf{m}}^{-1} \le w_{\mathbf{m}'}^{-1}$; clearly the inverse of $w_{\mathbf{m}}$ is the permutation $w_{\mathbf{m}^t}$ associated to the transpose matrix $\mathbf{m}^t \in M_{(c_j);(b_i)}$, and the condition in (3) is the transpose of the condition in (2). $\square$

We can also define Kazhdan-Lusztig polynomials indexed by pairs of elements of $M_{(b_i);(c_j)}$: $P_{\mathbf{m},\mathbf{m}'} = P_{w_{\mathbf{m}}, w_{\mathbf{m}'}}$. By definition we have

$$P_{\mathbf{m},\mathbf{m}'} \ne 0 \Rightarrow \mathbf{m} \le \mathbf{m}', \text{ and } P_{\mathbf{m},\mathbf{m}} = 1. \tag{2.10}$$

So the inverse matrix $(P^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'})_{\mathbf{m},\mathbf{m}' \in M_{(b_i);(c_j)}}$ of $(P_{\mathbf{m},\mathbf{m}'})_{\mathbf{m},\mathbf{m}' \in M_{(b_i);(c_j)}}$ has entries in $\mathbb{Z}[q]$ which also satisfy (2.10). In fact, we can express these entries in terms of those of the original matrix, as follows. Recall the Kazhdan-Lusztig inversion formula ([9, Section 7.14, (24)]):

$$\sum_{x \in S_d} \varepsilon(xy)\, P_{xw_0^{(d)}, yw_0^{(d)}}\, P_{x,w} = \delta_{y,w}, \tag{2.11}$$

where $\varepsilon(z) = (-1)^{\ell(z)}$ and $w_0^{(d)}$ is the longest element of S_d. Using the fact that $P_{x, w_{\mathbf{m}''}} = P_{x', w_{\mathbf{m}''}}$ for all $x' \in S_{(c_j)} x S_{(b_i)}$ ([9, Section 7.14, Corollary]), we get

$$P^{\langle -1\rangle}_{\mathbf{m},\mathbf{m}'} = \sum_{x\in S_{(c_j)}w_{\mathbf{m}'}S_{(b_i)}} \varepsilon(xw_{\mathbf{m}})\, P_{xw_0^{(d)},w_{\mathbf{m}}w_0^{(d)}}. \tag{2.12}$$

A general Kazhdan-Lusztig polynomial $P_{y,w}$, $y,w\in S_d$, can be expressed in the form $P_{\mathbf{m},\mathbf{m}'}$ in various ways. The most trivial takes $n=n'=d$ and all $b_i=c_j=1$, so that there is no difference between permutations and matrices. At the other extreme of usefulness, we can take (B_i) to be the collection consisting of the maximal intervals on which w is decreasing, and (C_j) the same thing for w^{-1}. With these choices, w is clearly the longest element in its double coset $S_{(c_j)}wS_{(b_i)}$, so $P_{y,w}$ depends only on the double coset of y; in other words, $P_{y,w}=P_{\psi(y),\psi(w)}$ where $\psi: S_d\to M_{(b_i);(c_j)}$ is as above.

Example. Let $y=128456379$ and $w=587429316$ in S_9. The blocks B_i and C_j determined by w are exactly those used in the previous example. Indeed, $w=w_{\mathbf{m}'}$ where

$$\mathbf{m}'=\begin{pmatrix}0&1&0&0\\1&1&2&0\\1&1&0&1\\0&0&1&0\end{pmatrix}.$$

Now $\psi(y)$ is the matrix $\mathbf{m}$ from the previous example, so the permutation $w_{\mathbf{m}}=285417639$ found there is the longest element in the double coset $S_{(c_j)}yS_{(b_i)}$. Using the criteria in Proposition 2.3, it is easy to check that $\mathbf{m}\le\mathbf{m}'$. The above principle means in this case that

$$P_{y,w}=P_{\mathbf{m},\mathbf{m}'}=P_{w_{\mathbf{m}},w}.$$

The advantage of the latter form is that 2 is cancellable for $[w_{\mathbf{m}},w]$. Since $w_{\mathbf{m}}^{\hat{2}}=25417638$ and $w^{\hat{2}}=57428316$, we get $P_{y,w}=P_{25417638,57428316}$.

In order to be able to perform such a cancellation directly on matrices, we note the following.

Proposition 2.4. *Let* $\mathbf{m}\in M_{(b_i);(c_j)}$, $a\in B_i\cap w_{\mathbf{m}}^{-1}(C_j)$. *Let* $\mathbf{e}$ *be the matrix with* $e_{i,j}=1$, *all other entries zero.*

(1) $w_{\mathbf{m}}^{\hat{a}}=w_{\mathbf{m}-\mathbf{e}}$.
(2) $\ell(\mathbf{m})-\ell(\mathbf{m}-\mathbf{e})$ *equals each of the following:*

$$\begin{aligned}m_{\le i,\ge j}+m_{\ge i,\le j}-m_{i,j}-1 &= m_{\le i-1,\ge j}+m_{\ge i+1,\le j}+b_i-1\\ &= m_{\le i,\ge j+1}+m_{\ge i,\le j-1}+c_j-1.\end{aligned}$$

Proof. Part (1) is clear from the explicit construction of $w_{\mathbf{m}}$ given above, and (2) follows easily from (1) of Proposition 2.3. ☐

Definition. If $\mathbf{m} \le \mathbf{m}'$ in $M_{(b_i);(c_j)}$, we say that $(i,j) \in [1,n] \times [1,n']$ is *cancellable* for the interval $[\mathbf{m}, \mathbf{m}']$ if

(1) $m_{i,j} \ge 1$.
(2) $m_{\le i-1, \ge j} = m'_{\le i-1,\ge j}$, or equivalently $m_{\ge i, \le j-1} = m'_{\ge i,\le j-1}$.
(3) $m_{\le i, \ge j+1} = m'_{\le i,\ge j+1}$, or equivalently $m_{\ge i+1, \le j} = m'_{\ge i+1,\le j}$.

These equivalences follow from (2.6).

Proposition 2.5. *Suppose that* $\mathbf{m} \le \mathbf{m}'$ *in* $M_{(b_i);(c_j)}$ *and* (i,j) *is cancellable for* $[\mathbf{m}, \mathbf{m}']$. *Let* $\mathbf{e}$ *be the matrix with* $e_{i,j} = 1$, *all other entries zero.*

(1) *For any* $\mathbf{m}^1 \in [\mathbf{m}, \mathbf{m}']$,
  (a) $m^1_{i,j} \ge m_{i,j}$,
  (b) $m^1_{\le i-1,\ge j} = m_{\le i-1,\ge j}$, *and*
  (c) $m^1_{\le i,\ge j+1} = m_{\le i,\ge j+1}$.
  Hence (i,j) *is cancellable for any sub-interval of* $[\mathbf{m}, \mathbf{m}']$.
(2) *The map* $\mathbf{m}^1 \mapsto \mathbf{m}^1 - \mathbf{e}$ *is an isomorphism of posets between* $[\mathbf{m}, \mathbf{m}']$ *and* $[\mathbf{m} - \mathbf{e}, \mathbf{m}' - \mathbf{e}]$, *which reduces all lengths by the same amount.*
(3) *For any* $\mathbf{m}^1, \mathbf{m}^2 \in [\mathbf{m}, \mathbf{m}']$, $P_{\mathbf{m}^1,\mathbf{m}^2} = P_{\mathbf{m}^1-\mathbf{e},\mathbf{m}^2-\mathbf{e}}$.
(4) *For any* $\mathbf{m}^1, \mathbf{m}^2 \in [\mathbf{m}, \mathbf{m}']$, $P^{\langle -1\rangle}_{\mathbf{m}^1,\mathbf{m}^2} = P^{\langle -1\rangle}_{\mathbf{m}^1-\mathbf{e},\mathbf{m}^2-\mathbf{e}}$.

Proof. Let $\mathbf{m}^1 \in [\mathbf{m}, \mathbf{m}']$. By (2) of Proposition 2.3, we have

$$m_{\le i-1,\ge j} \le m^1_{\le i-1,\ge j} \le m'_{\le i-1,\ge j} = m_{\le i-1,\ge j},$$

which proves (1b), and (1c) is similar. It follows that

$$\begin{aligned} m^1_{i,j} - m_{i,j} &= (m^1_{i,j} + m^1_{\le i-1,\ge j} + m^1_{\le i,\ge j+1}) - (m_{i,j} + m_{\le i-1,\ge j} + m_{\le i,\ge j+1}) \\ &= (m^1_{\le i,\ge j} + m^1_{\le i-1,\ge j+1}) - (m_{\le i,\ge j} + m_{\le i-1,\ge j+1}) \\ &= (m^1_{\le i,\ge j} - m_{\le i,\ge j}) + (m^1_{\le i-1,\ge j+1} - m_{\le i-1,\ge j+1}) \\ &\ge 0, \end{aligned}$$

by Proposition 2.3 again. So (1a) is proved. Thus $\mathbf{m}^1 - \mathbf{e} \in M_{(\widetilde{b_i});(\widetilde{c_j})}$, where

$$\widetilde{b_{i'}} = b_{i'} - \delta_{i,i'},\ \widetilde{c_{j'}} = c_{j'} - \delta_{j,j'}.$$

Given this, the first part of (2) is obvious from either description of the partial order given in Proposition 2.3, and the second part from (2) of Proposition 2.4. To prove (3), let a be the largest element of $B_i \cap w_{\mathbf{m}}^{-1}(C_j)$, i.e. the $(m_{i,\ge j})$th element of B_i. We want to show that a is cancellable for $[w_{\mathbf{m}}, w_{\mathbf{m}'}]$. Let

$$\delta = m'_{i,\ge j} - m_{i,\ge j} = m'_{\le i,j} - m_{\le i,j} = m'_{\le i,\ge j} - m_{\le i,\ge j} \ge 0.$$

(These are equal because (i,j) is cancellable, and nonnegative because $\mathbf{m} \leq \mathbf{m}'$.) By the above chain of equalities applied to $\mathbf{m}^1 = \mathbf{m}'$, we have

$$m'_{i,j} - m_{i,j} = \delta + (m'_{\leq i-1,\geq j+1} - m_{\leq i-1,\geq j+1}) \geq \delta,$$

so

$$m'_{i,\geq j+1} = m'_{i,\geq j} - m'_{i,j} = m_{i,\geq j} + \delta - m'_{i,j} < m_{i,\geq j} \leq m'_{i,\geq j},$$

which means that $w_{\mathbf{m}'}(a) \in C_j$. From (2.7) we see that $w_{\mathbf{m}'}(a) = w_{\mathbf{m}}(a)$. Moreover,

$$\mathrm{inv}_a(w_{\mathbf{m}}) = m_{\leq i,\geq j} - 1 = m'_{\leq i,\geq j} - \delta - 1 = \mathrm{inv}_a(w_{\mathbf{m}'}),$$

so a is cancellable for $[w_{\mathbf{m}}, w_{\mathbf{m}'}]$. In particular, for any $\mathbf{m}^1 \in [\mathbf{m}, \mathbf{m}']$, $w_{\mathbf{m}^1}(a) \in C_j$, which by (1) of Proposition 2.4 implies $w^{\hat{a}}_{\mathbf{m}^1} = w_{\mathbf{m}^1-\mathbf{e}}$. Then part (3) follows from (3) of Proposition 2.2, and part (4) follows formally from parts (2) and (3). $\square$

Example. With matrices $\mathbf{m}$, $\mathbf{m}'$ defined as in previous examples, $(2,3)$ is cancellable for $[\mathbf{m}, \mathbf{m}']$, corresponding to the fact that 2 is cancellable for $[w_{\mathbf{m}}, w_{\mathbf{m}'}]$. Performing the cancellation directly on the matrices, we get

$$\mathbf{m} - \mathbf{e} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \mathbf{m}' - \mathbf{e} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The reader can check that these matrices correspond to the permutations 25417638 and 57428316 found earlier.

3. Nilpotent orbits of the linear quiver

We now return to the set-up of the first part of the introduction, so V is a d-dimensional $\mathbb{Z}$-graded vector space, with $d_i = \dim V_i$. For convenience, we adjust the grading so that $d_i \neq 0 \Rightarrow i \in [1,n]$, for some positive integer n (so we are effectively considering the linear quiver of type A_n). Throughout this section, the variables i,j range over $[1,n]$ unless otherwise specified.

We saw in §1 that the G_V-orbits in $\mathcal{N}_V$ are in bijection with the set $M_{(d_i)}$ of multisegments in which i occurs d_i times as an element of a segment. Following [26], we change this parametrization by multisegments to a parametrization by matrices. We identify each $\mathbf{m} \in M_{(d_i)}$

with the $(n \times n)$-matrix $(m_{i,j})$, where
(3.1)

$$m_{i,j} = \begin{cases} \text{multiplicity of the segment } [i,j], & \text{if } i \le j, \\ \text{number of segments } [k,l] \text{ where } k \le j,\ l \ge i, & \text{if } j = i-1, \\ 0, & \text{if } j < i-1. \end{cases}$$

It is clear that this matrix lies in the set $M_{(d_i);(d_j)}$, as defined in the previous section. So we have identified $M_{(d_i)}$ with a subset of $M_{(d_i);(d_j)}$, which can be described as follows.

Proposition 3.1. *Let* $M'_{(d_i)} = \{\mathbf{m} \in M_{(d_i);(d_j)} \,|\, m_{i,j} = 0,\ \forall j < i-1\}$.

(1) $M'_{(d_i)}$ *is a lower ideal of the poset* $M_{(d_i);(d_j)}$.
(2) *If* $\mathbf{m} \in M'_{(d_i)}$, *then for all* $i \ge j$, $m_{\le i, \ge j} = d_j + d_{j+1} + \cdots + d_i$.
(3) *For* $\mathbf{m}, \mathbf{m}' \in M'_{(d_i)}$, $\mathbf{m} \le \mathbf{m}'$ *if and only if* $m_{\le i, \ge j} \le m'_{\le i, \ge j}$ *for all* $i < j$.
(4) *If* $\mathbf{m} \in M'_{(d_i)}$, *then for all* $i \in [2, n]$, $m_{i,i-1} = m_{\le i-1, \ge i}$.
(5) $M'_{(d_i)} = M_{(d_i)}$.

Proof. An element $\mathbf{m} \in M_{(d_i);(d_j)}$ lies in $M'_{(d_i)}$ if and only if $m_{\ge i, \le j} = 0$ for all i, j such that $j < i-1$, so (1) follows from (3) of Proposition 2.3. For (2), since $m_{\ge i+1, \le j-1} = 0$, (2.6) gives

$$m_{\le i, \ge j} = d_1 + \cdots + d_i - d_1 - \cdots - d_{j-1} = d_j + \cdots + d_i,$$

as required. Part (3) then follows from (2) of Proposition 2.3. For (4), we have

$$m_{i,i-1} = d_i - m_{i,\ge i} = m_{\le i, \ge i} - m_{i, \ge i} = m_{\le i-1, \ge i}.$$

From (4) and the $i = j$ case of (2) it follows that every matrix in $M'_{(d_i)}$ arises from a multisegment in $M_{(d_i)}$ by the rule (3.1), whence (5). $\square$

As mentioned in the introduction, the identification of $M_{(d_i)}$ with $M'_{(d_i)}$ is a poset isomorphism: the geometrically-defined partial order $\preceq$ on $M_{(d_i)}$ is the restriction of the partial order $\le$ on $M_{(d_i);(d_j)}$. This is part of Zelevinsky's result [26, Corollary 1], which we can state (with some supplementary detail) as follows.

Theorem 3.2. *Let* $\mathbf{m}, \mathbf{m}' \in M_{(d_i)}$.

(1) $\dim \mathcal{O}_{\mathbf{m}} = \ell(\mathbf{m}) - \sum_i \binom{d_i}{2}$.
(2) $\mathcal{O}_{\mathbf{m}} \subseteq \overline{\mathcal{O}_{\mathbf{m}'}} \Leftrightarrow \mathbf{m} \le \mathbf{m}'$.
(3) $\mathcal{H}^i IC(\overline{\mathcal{O}_{\mathbf{m}'}}) = 0$ *for* i *odd*.
(4) $IC_{\mathbf{m},\mathbf{m}'} = P_{\mathbf{m},\mathbf{m}'}$.
(5) $IC^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'} = P^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'}$.

Proof. For reference in §5, we recall Zelevinsky's proof. Define the partial flag variety $\mathcal{B}_{(d_i)}$ to be the set of collections of subspaces $(W_i)_{i\in[0,n]}$ of V such that $W_0 = 0$, and for all $i \in [1,n]$, $W_{i-1} \subset W_i$ and $\dim W_i/W_{i-1} = d_i$; this is naturally a nonsingular projective variety of dimension $\binom{d}{2} - \sum_i \binom{d_i}{2}$. We define a 'base-point' (U_i) in $\mathcal{B}_{(d_i)}$ by $U_i = V_1 \oplus \cdots \oplus V_i$. Relative to this base-point, $\mathcal{B}_{(d_i)}$ decomposes into Schubert cells $\mathcal{B}_{\mathbf{m}}$ for $\mathbf{m} \in M_{(d_i);(d_j)}$. Explicitly, $\mathcal{B}_{\mathbf{m}}$ consists of those (W_i) such that for all $i, j \in [1,n]$,

$$\dim \frac{W_i \cap U_j}{W_i \cap U_{j-1} + W_{i-1} \cap U_j} = m_{i,j}.$$

The analogues of (1)–(4) for these Schubert cells (for all of $M_{(d_i);(d_j)}$) are well known. Let $\mathcal{B}'_{(d_i)}$ be the closed subvariety of $\mathcal{B}_{(d_i)}$ defined by requiring $W_i \supset U_{i-1}$; from the description of $M_{(d_i)}$ as $M'_{(d_i)}$, it is easy to see that $\mathcal{B}'_{(d_i)} = \bigcup_{\mathbf{m}\in M_{(d_i)}} \mathcal{B}_{\mathbf{m}}$.

Now we define a morphism $\mathcal{N}_V \to \mathcal{B}'_{(d_i)} : \varphi \mapsto (W_i(\varphi))$ by the rule

$$W_i(\varphi) = U_{i-1} \oplus \{v + \varphi(v) + \varphi^2(v) + \cdots + \varphi^{n-i}(v) \mid v \in V_i\}.$$

An easy check shows that this morphism maps $\mathcal{O}_{\mathbf{m}}$ into $\mathcal{B}_{\mathbf{m}}$ for all $\mathbf{m} \in M_{(d_i)}$. Moreover, it gives an isomorphism between $\mathcal{N}_V$ and the open subvariety of $\mathcal{B}'_{(d_i)}$ defined by requiring

$$W_i \cap \bigoplus_{i'>i} V_{i'} = 0, \ \forall i \in [1,n].$$

Hence each $\mathcal{O}_{\mathbf{m}}$ is embedded as an open subvariety of the Schubert cell $\mathcal{B}_{\mathbf{m}}$, and (1)–(4) follow. Since $M_{(d_i)}$ is a lower ideal of $M_{(d_i);(d_j)}$, (5) is an automatic consequence of (4). $\square$

In view of (3) of Proposition 3.1, part (2) of Theorem 3.2 says that $\mathcal{O}_{\mathbf{m}} \subseteq \overline{\mathcal{O}_{\mathbf{m}'}}$ if and only if for all $i < j$, $m_{\leq i,\geq j} \leq m'_{\leq i,\geq j}$. Now if $\varphi \in \mathcal{O}_{\mathbf{m}}$, then for $i \leq j$, $m_{\leq i,\geq j} = \mathrm{rk}\, \varphi^{j-i}|_{V_i}$. So we recover the well-known fact that $\mathcal{O}_{\mathbf{m}} \subseteq \overline{\mathcal{O}_{\mathbf{m}'}}$ if and only if for all $i < j$, $\mathrm{rk}\, \varphi^{j-i}|_{V_i} \leq \mathrm{rk}\, (\varphi')^{j-i}|_{V_i}$ for any $\varphi \in \mathcal{O}_{\mathbf{m}}$ and $\varphi' \in \mathcal{O}_{\mathbf{m}'}$. (Of course the "only if" direction is obvious.) We can define an element $\mathbf{m}^{\max} \in M_{(d_i)}$ uniquely by the requirement that for $i \leq j$, $m^{\max}_{\leq i,\geq j}$ equals the maximum possible rank, namely $\min\{d_i, d_{i+1}, \cdots, d_j\}$. It follows that $\mathbf{m} \leq \mathbf{m}^{\max}$ for all $\mathbf{m} \in M_{(d_i)}$, and the orbit $\mathcal{O}_{\mathbf{m}^{\max}}$ is dense in $\mathcal{N}_V$.

As foreshadowed in the introduction, Theorem 3.2 is only one of many possible ways to express a particular $IC_{\mathbf{m},\mathbf{m}'}$ as a Kazhdan-Lusztig polynomial: the below-diagonal entries prescribed by (3.1) correspond to one particular choice of "empty segments". A more general statement is the following.

Theorem 3.3. *Let* $b_1, \cdots, b_n, c_1, \cdots, c_n \in \mathbb{N}$ *be such that*

$$b_1 = d_1, \ c_n = d_n, \ \text{and } d_i - b_i = d_{i-1} - c_{i-1}, \, \forall i \in [2, n].$$

Define an $(n \times n)$*-matrix* $\mathbf{a}$ *by*

$$a_{i,j} = \begin{cases} d_i - b_i, & \text{if } j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $M_{(d_i)}^{(b_i);(c_j)} = \{\mathbf{m} \in M_{(d_i)} \,|\, m_{i,i-1} \geq d_i - b_i, \, \forall i \in [2, n]\}$.

(1) $M_{(d_i)}^{(b_i);(c_j)}$ *is an upper ideal of* $M_{(d_i)}$.
(2) *The map* $\mathbf{m} \mapsto \mathbf{m} - \mathbf{a}$ *is an isomorphism of posets between* $M_{(d_i)}^{(b_i);(c_j)}$ *and* $\{\widetilde{\mathbf{m}} \in M_{(b_i);(c_j)} \,|\, \widetilde{m}_{i,j} = 0, \, \forall j < i-1\}$.
(3) *For any* $\mathbf{m}', \mathbf{m}'' \in M_{(d_i)}^{(b_i);(c_j)}$, $IC_{\mathbf{m}',\mathbf{m}''} = P_{\mathbf{m}'-\mathbf{a},\mathbf{m}''-\mathbf{a}}$.
(4) *For any* $\mathbf{m}', \mathbf{m}'' \in M_{(d_i)}^{(b_i);(c_j)}$, $IC_{\mathbf{m}',\mathbf{m}''}^{\langle -1 \rangle} = P_{\mathbf{m}'-\mathbf{a},\mathbf{m}''-\mathbf{a}}^{\langle -1 \rangle}$.

Proof. For $\mathbf{m} \in M_{(d_i)}$, $m_{i,i-1}$ equals $m_{\geq i, \leq i-1}$, so (1) follows from (3) of Proposition 2.3. For (2), the fact that the given map is a bijection is obvious, and by Proposition 2.3 it preserves partial orders. Now this map is the composition of maps of the form $\mathbf{m} \mapsto \mathbf{m} - \mathbf{e}$ as in Proposition 2.5 and their inverses $\mathbf{m} \mapsto \mathbf{m} + \mathbf{e}$, where the positions which are being altered are all of the form $(i, i-1)$. Since all matrices involved have zero entries in positions (i, j) where $j < i-1$, conditions (2) and (3) of the definition of cancellability always hold. So Proposition 2.5 implies (3) and (4) with $P_{\mathbf{m}',\mathbf{m}''}$ and $P_{\mathbf{m}',\mathbf{m}''}^{\langle -1 \rangle}$ in place of $IC_{\mathbf{m}',\mathbf{m}''}$ and $IC_{\mathbf{m}',\mathbf{m}''}^{\langle -1 \rangle}$, and Theorem 3.2 gives the result. $\square$

Corollary 3.4. *For* $\mathbf{m} \in M_{(d_i)}$, *define an* $(n \times n)$*-matrix* $\mathbf{m}^-$ *by*

$$m_{i,j}^- = \begin{cases} m_{i,j}, & \text{if } j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\langle \mathbf{m} \rangle = \{\mathbf{m}' \in M_{(d_i)} \,|\, \mathbf{m} \leq \mathbf{m}'\} = [\mathbf{m}, \mathbf{m}^{\max}]$.

(1) *For all* $\mathbf{m}' \in \langle \mathbf{m} \rangle$, $\mathbf{m}' - \mathbf{m}^- \in M_{(d_i - m_{i,i-1});(d_j - m_{j+1,j})}$.
(2) *The map* $\mathbf{m}' \mapsto \mathbf{m}' - \mathbf{m}^-$ *is an isomorphism of posets between* $\langle \mathbf{m} \rangle$ *and* $[\mathbf{m} - \mathbf{m}^-, \mathbf{m}^{\max} - \mathbf{m}^-]$.
(3) *For any* $\mathbf{m}', \mathbf{m}'' \in \langle \mathbf{m} \rangle$, $IC_{\mathbf{m}',\mathbf{m}''} = P_{\mathbf{m}'-\mathbf{m}^-,\mathbf{m}''-\mathbf{m}^-}$.
(4) *For any* $\mathbf{m}', \mathbf{m}'' \in \langle \mathbf{m} \rangle$, $IC_{\mathbf{m}',\mathbf{m}''}^{\langle -1 \rangle} = P_{\mathbf{m}'-\mathbf{m}^-,\mathbf{m}''-\mathbf{m}^-}^{\langle -1 \rangle}$.

Proof. Apply Theorem 3.3 with

$$b_i = d_i - m_{i,i-1}, \ c_{i-1} = d_{i-1} - m_{i,i-1}$$

for all $i \in [2, n]$, and restrict to the upper ideal $\langle \mathbf{m} \rangle$ of $M_{(d_i)}^{(b_i);(c_j)}$. $\square$

Note that the polynomials $P_{\mathbf{m}'-\mathbf{m}^-,\mathbf{m}''-\mathbf{m}^-}$ in (3) are Kazhdan-Lusztig polynomials for $S_{k(\mathbf{m})}$, where $k(\mathbf{m})$ is the number of segments of $\mathbf{m}$, which is also the sum of the entries of $\mathbf{m}-\mathbf{m}^-$.

Finally, we have to connect Theorem 3.3 to the notation of the introduction, in order to prove Theorem 1.1. We have elements $\lambda, \mu \in D_k$; we can clearly assume that all $\lambda_s - s, \mu_s - s + 1$ for $1 \leq s \leq k$ lie in $[1, n]$. Define

$$b_i = |\{s \,|\, \mu_s - s + 1 = i\}|, \;\; c_j = |\{s \,|\, \lambda_s - s = j\}|.$$

Then the subgroups $S_{(b_i)}$ and $S_{(c_j)}$ of S_k are exactly the conjugates under $w_0^{(k)}$ of the dot stabilizers W_μ and W_λ (this reversal comes about because the sequences $(\lambda_s - s)$ and $(\mu_s - s)$ are decreasing). The map $\psi : S_k \to M_{(b_i);(c_j)}$ as defined in the previous section satisfies

$$\psi(w_0^{(k)} w w_0^{(k)})_{i,j} = |\{s \,|\, \mu_s - s + 1 = i, \lambda_{w(s)} - w(s) = j\}|. \tag{3.2}$$

So $S_k[\lambda, \mu] = \{w \in S_k \,|\, \psi(w_0^{(k)} w w_0^{(k)})_{i,j} = 0, \, \forall j < i - 1\}$, which shows that it is indeed a lower ideal of S_k. Moreover, $w \mapsto \psi(w_0^{(k)} w w_0^{(k)})$ gives an isomorphism of posets between $S_k[\lambda, \mu]^\circ$ and $\{\widetilde{\mathbf{m}} \in M_{(b_i);(c_j)} \,|\, \widetilde{m}_{i,j} = 0, \, \forall j < i-1\}$, and the polynomials attached to these posets correspond, since

$$P_{w,w'} = P_{w_0^{(k)} w w_0^{(k)}, w_0^{(k)} w' w_0^{(k)}} = P_{\psi(w_0^{(k)} w w_0^{(k)}), \psi(w_0^{(k)} w' w_0^{(k)})} \tag{3.3}$$

for all $w, w' \in S_k[\lambda, \mu]^\circ$. Now as in Theorem 1.1, assume that $\lambda/\mu \in M_{(d_i)}$; it follows immediately that (b_i) and (c_j) satisfy the conditions of Theorem 3.3. By (3.2), for all $w \in S_k[\lambda, \mu]^\circ$, the multisegment $\lambda/(w \cdot \mu)$ when viewed as a matrix has the same diagonal and above-diagonal entries as $\psi(w_0^{(k)} w w_0^{(k)})$; hence $\lambda/(w \cdot \mu) = \psi(w_0^{(k)} w w_0^{(k)}) + \mathbf{a}$ where $\mathbf{a}$ is as in Theorem 3.3. Thus Theorem 1.1 follows from Theorem 3.3.

4. Cancellation for the affine symmetric group

We now want to extend the results of §2 to the affine symmetric group. Again fix a positive integer d. Let $\widehat{S_d}$ be the group of permutations w of the set $\mathbb{Z}$ such that $w(i + d) = w(i) + d$, for all $i \in \mathbb{Z}$. An element $w \in \widehat{S_d}$ is determined by its *window* $(w(1), w(2), \cdots, w(d))$, which can be any collection of representatives of the congruence classes mod d, in any order. The subgroup of $\widehat{S_d}$ which preserves $[1, d]$ is clearly isomorphic to S_d.

The group $\widehat{S_d}$ is the 'extended' affine symmetric group: it can be written as a semi-direct product $\langle\tau\rangle \ltimes \widetilde{S_d}$, where

$$\widetilde{S_d} = \{w \in \widehat{S_d} \,|\, \sum_{i=1}^{d} w(i) = \sum_{i=1}^{d} i\}$$

is the actual affine symmetric group, and τ is the element of infinite order sending i to $i+1$ for all $i \in \mathbb{Z}$. In general, $w \in \tau^{a(w)}\widetilde{S_d}$ where

$$a(w) = \frac{1}{d}(\sum_{i=1}^{d} w(i) - \sum_{i=1}^{d} i).$$

Note that for any $i \in \mathbb{Z}$, the set $w(-\infty, i]$ can be obtained from $(-\infty, i + a(w)]$ by changing finitely many elements (keeping distinctness). In other words, for m sufficiently large, the mth largest element in $w(-\infty, i]$ is $i + a(w) - m + 1$.

If $d = 1$, $\widetilde{S_d}$ is the trivial group and $\widehat{S_d} = \langle\tau\rangle$. If $d \geq 2$, we define $s_i \in \widetilde{S_d}$ for all $i \in \mathbb{Z}$ by

$$s_i(j) = \begin{cases} j+1, & \text{if } j \equiv i \text{ mod } d, \\ j-1, & \text{if } j \equiv i+1 \text{ mod } d, \\ j, & \text{otherwise.} \end{cases}$$

Thus $s_i = s_{i'}$ iff $i \equiv i'$ mod d. It is well known that $s_0, s_1, \cdots, s_{d-1}$ form a set of Coxeter generators for $\widetilde{S_d}$ of type $\widetilde{A_{d-1}}$. They thus determine a length function $\ell : \widetilde{S_d} \to \mathbb{N}$, a Bruhat order $\leq$ on $\widetilde{S_d}$, and Kazhdan-Lusztig polynomials $P_{y,w} \in \mathbb{N}[q]$ for $y, w \in \widetilde{S_d}$, all of which are invariant under conjugation by τ. We extend these to $\widehat{S_d}$ in the standard way:

$$\begin{aligned} \ell(w) &= \ell(\tau^{-a(w)}w), \\ y \leq w &\Leftrightarrow a(y) = a(w), \tau^{-a(y)}y \leq \tau^{-a(w)}w, \\ P_{y,w} &= \begin{cases} P_{\tau^{-a(y)}y, \tau^{-a(w)}w}, & \text{if } a(y) = a(w), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We define inversion statistics as in the finite case:

$$\mathrm{inv}_i(w) = |\{i' < i \,|\, w(i') > w(i)\}|, \ \mathrm{Inv}_i(w) = |\{i' > i \,|\, w(i') < w(i)\}|,$$

for any $w \in \widehat{S_d}$ and $i \in \mathbb{Z}$ (these sets are finite, even though i' runs over $\mathbb{Z}$.) Clearly $\mathrm{inv}_{i+d}(w) = \mathrm{inv}_i(w)$, $\mathrm{Inv}_{i+d}(w) = \mathrm{Inv}_i(w)$, and

$$\begin{aligned}
&\mathrm{Inv}_i(w) - \mathrm{inv}_i(w)\\
&= |(-\infty, w(i)] \setminus w(-\infty, i]| - |w(-\infty, i] \setminus (-\infty, w(i)]|\\
&= |(-\infty, w(i)] \setminus (-\infty, i + a(w)]| - |(-\infty, i + a(w)] \setminus (-\infty, w(i)]|\\
&= w(i) - i - a(w).
\end{aligned}$$

The formula for ℓ on $\widehat{S_d}$ is analogous to that for S_d (see [4, Proposition 4.1(ii)]):

$$\ell(w) = \sum_{i\in[1,d]} \mathrm{inv}_i(w) = \sum_{i\in[1,d]} \mathrm{Inv}_i(w). \tag{4.1}$$

We also have

$$ws_i < w \text{ if and only if } w(i) > w(i + 1), \tag{4.2}$$

and a general description of Bruhat order along the lines of Proposition 2.1 (this is a rephrasing of [4, Theorem 6.5], trivially extended from $\widetilde{S_d}$ to $\widehat{S_d}$):

Proposition 4.1. *If $y, w \in \widehat{S_d}$, $y \leq w$ if and only if for all $i \in \mathbb{Z}$,*

$$|\{i' \leq i \,|\, y(i') \geq j\}| \leq |\{i' \leq i \,|\, w(i') \geq j\}|, \ \forall j, \text{ with equality for } j \ll 0.$$

In other words, for all positive integers m, the mth largest element in $y(-\infty, i]$ is less than or equal to the mth largest element in $w(-\infty, i]$, with equality for $m \gg 0$. It suffices to check this for $i \in [1, d]$.

The definition of cancellability is identical to the finite case:

Definition. If $y \leq w$ in $\widehat{S_d}$, we say that $i \in \mathbb{Z}$ is *cancellable* for the interval $[y, w]$ if $y(i) = w(i)$, $\mathrm{inv}_i(y) = \mathrm{inv}_i(w)$, and $\mathrm{Inv}_i(y) = \mathrm{Inv}_i(w)$. (Clearly any two of these conditions imply the third, and i is cancellable for $[y, w]$ iff $i + d$ is.)

However, the process of cancellation is not as uniquely defined as in the finite case: we need to choose order-preserving bijections $\sigma_{\bar{i}} : \mathbb{Z} \setminus \bar{i} \to \mathbb{Z}$ for all congruence classes $\bar{i}$ mod d. Then for any $w \in \widehat{S_d}$, we define $w^{\hat{i}} \in \widehat{S_{d-1}}$ by

$$w^{\hat{i}} = \sigma_{\overline{w(i)}} \circ w \circ \sigma_{\bar{i}}^{-1}.$$

Note that using different σ's would have the effect of multiplying $w^{\hat{i}}$ on left and right by powers of τ. Independently of the choice, we have

$$\ell(w^{\hat{i}}) = \ell(w) - \mathrm{inv}_i(w) - \mathrm{Inv}_i(w). \tag{4.3}$$

Example. Take $d = 3$, $y = \tau s_1 s_2$ and $w = \tau s_2 s_1 s_0 s_2$. Then y has window $(3, 4, 2)$ and w has window $(0, 7, 2)$. Since $\mathrm{inv}_3(y) = \mathrm{inv}_3(w) = 2$, 3 is cancellable for $[y, w]$. If we normalize $\sigma_{\bar{2}}$ and $\sigma_{\bar{3}}$ by requiring that they preserve 1, then $y^{\hat{3}}$ and $w^{\hat{3}}$ are the elements of $\widehat{S_2}$ with windows $(2, 3)$ and $(0, 5)$, namely τ and $\tau s_1 s_0$.

We can now extend Proposition 2.2 to the affine case.

Proposition 4.2. *Suppose that i is cancellable for $[y, w]$.*

(1) *For any $x \in [y, w]$, $x(i) = y(i)$ and $\mathrm{inv}_i(x) = \mathrm{inv}_i(y)$. Hence i is cancellable for any sub-interval of $[y, w]$.*
(2) *$x \mapsto x^{\hat{i}}$ is an isomorphism of posets between $[y, w]$ and $[y^{\hat{i}}, w^{\hat{i}}]$, which reduces all lengths by the same amount.*
(3) *For any $u, v \in [y, w]$, $P_{u,v} = P_{u^{\hat{i}},v^{\hat{i}}}$.*

Proof. The proof of part (1) is identical to that of part (1) of Proposition 2.2, with $[1, i]$ replaced by $(-\infty, i]$ and so on, and of course using Proposition 4.1 instead of Proposition 2.1. Similarly with part (2), where the inverse map $[y^{\hat{i}}, w^{\hat{i}}] \to [y, w] : x \mapsto \tilde{x}$ is now defined by

$$\tilde{x}(i') = \begin{cases} y(i) + kd, & \text{if } i' = i + kd, \\ \sigma_{\overline{y(i)}}^{-1}(x(\sigma_{\bar{i}}(i'))), & \text{if } i' \notin \bar{i}. \end{cases}$$

The proof of (3) is also mostly unchanged. Apart from replacing $[1, i]$ by $(-\infty, i]$ and so on, the only change is that in Case 1, we need not have $(ys)^{\widehat{i-1}} = y^{\hat{i}}$ and $(ws)^{\widehat{i-1}} = w^{\hat{i}}$, but rather have

$$(ys)^{\widehat{i-1}} = \tau^a y^{\hat{i}} \tau^b, \ (ws)^{\widehat{i-1}} = \tau^a w^{\hat{i}} \tau^b \text{ for some } a, b \in \mathbb{Z},$$

which still implies $P_{(ys)^{\widehat{i-1}},(ws)^{\widehat{i-1}}} = P_{y^{\hat{i}},w^{\hat{i}}}$ as required. □

We now introduce some affine matrix notation very similar to that in [14]. Let $(b_i)_{i\in\mathbb{Z}}$ be a $\mathbb{Z}$-tuple of nonnegative integers, periodic with period $n \geq 1$, such that $\sum_{i=1}^{n} b_i = d$; and let $(c_j)_{j\in\mathbb{Z}}$ be another such, with period $n' \geq 1$, such that $\sum_{j=1}^{n'} c_j = d$. Our notational convention now is that the range of the variables i, i', j, j' is all of $\mathbb{Z}$ unless otherwise specified. Let $M_{(b_i),n;(c_j),n'}$ be the set of all $(\mathbb{Z} \times \mathbb{Z})$-matrices $\mathbf{m}$ satisfying:

(1) $m_{i,j} \in \mathbb{N}$, for all i, j,
(2) $m_{i+n,j+n'} = m_{i,j}$, for all i, j,
(3) $\sum_j m_{i,j} = b_i$, for all i, and
(4) $\sum_i m_{i,j} = c_j$, for all j.

It is easy to see that for $\mathbf{m} \in M_{(b_i),n;(c_j),n'}$, $m_{i,j} = 0$ for $|j - i| \gg 0$; so sums of the form $m_{i,\geq j}$, $m_{\leq i,j}$, $m_{\leq i,\geq j}$ and $m_{\geq i,\leq j}$ are finite. We have the following substitute for (2.6). For fixed i, $m_{\geq i,\leq j_0} = 0$ for all j_0 sufficiently negative, and for j greater than such j_0,

$$m_{\geq i,\leq j} = c_{j_0+1} + \cdots + c_j - m_{\leq i-1,\geq j_0+1} + m_{\leq i-1,\geq j+1}. \tag{4.4}$$

The matrices in $M_{(b_i),n;(c_j),n'}$ parametrize double cosets of $\widehat{S_d}$ with respect to proper parabolic subgroups of $\widetilde{S_d}$. Namely, write $\mathbb{Z}$ as the disjoint union of (possibly empty) blocks B_i such that all elements of B_i are less than all elements of B_{i+1}, and $|B_i| = b_i$. It follows that $B_{i+n} = B_i + d$. Note that the collection (B_i) is determined by (b_i) up to translation (i.e. a power of τ). Let $S_{(B_i)}$ be the subgroup of $\widehat{S_d}$ which preserves each B_i separately; this is a parabolic subgroup of $\widetilde{S_d}$ isomorphic to $S_{b_1} \times \cdots \times S_{b_n}$. (It is determined by (b_i) up to conjugation by a power of τ.) Similarly define blocks C_j of sizes c_j, and the parabolic subgroup $S_{(C_j)}$. We define a surjective map $\psi : \widehat{S_d} \to M_{(b_i),n;(c_j),n'}$ by

$$\psi(w)_{i,j} = |w(B_i) \cap C_j|.$$

The fibres of ψ are exactly the double cosets $S_{(C_j)} w S_{(B_i)}$, so ψ induces a bijection $S_{(C_j)} \backslash \widehat{S_d} \,/\, S_{(B_i)} \leftrightarrow M_{(b_i),n;(c_j),n'}$. For $\mathbf{m} \in M_{(b_i),n;(c_j),n'}$, let $w_{\mathbf{m}} \in \widehat{S_d}$ be the longest element in the corresponding double coset.

The permutation $w_{\mathbf{m}}$ can be read off the matrix $\mathbf{m}$ by exactly the same prescription as in the finite case (remembering that i, i', j, j' now range over all of $\mathbb{Z}$).

Example. Let $d = 7$, $n = 2$, $n' = 3$, and define

$$b_i = \begin{cases} 3, & \text{if } i \equiv 1 \bmod 2, \\ 4, & \text{if } i \equiv 0 \bmod 2, \end{cases} \qquad c_j = \begin{cases} 3, & \text{if } j \equiv 1 \bmod 3, \\ 2, & \text{if } j \equiv 2 \bmod 3, \\ 2, & \text{if } j \equiv 0 \bmod 3. \end{cases}$$

Let $\mathbf{m} \in M_{(b_i),2;(c_j),3}$ be the following matrix:

$$\begin{matrix}
\ddots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
& 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\cdots & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
& 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
\cdots & 0 & 0 & 0 & \mathbf{0} & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
& 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & \cdots \\
& 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & \\
& & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \ddots
\end{matrix}$$

where the $\mathbf{0}$ is the $(1,1)$ entry. We choose (B_i) and (C_j) so that $B_1 = C_1 = \{1, 2, 3\}$. The row containing $\mathbf{0}$ tells us that $w_{\mathbf{m}}(B_1)$ consists of one element of $C_6 = \{13, 14\}$ and two of $C_2 = \{4, 5\}$, in that order. Since the $(0, 6)$ entry is 1, the largest element of C_6 'has already been used' in $w_{\mathbf{m}}(B_0)$, so we set $w_{\mathbf{m}}(1) = 13$, $w_{\mathbf{m}}(2) = 5$, $w_{\mathbf{m}}(3) = 4$. Treating the next row similarly, we find that $w_{\mathbf{m}}$ is the element of $\widehat{S}_7$ with window $(13, 5, 4, 21, 10, 9, 1)$.

We define a length function $\ell : M_{(b_i),n;(c_j),n'} \to \mathbb{N}$ by $\ell(\mathbf{m}) = \ell(w_{\mathbf{m}})$, and a partial order on $M_{(b_i),n;(c_j),n'}$ by

$$\mathbf{m} \leq \mathbf{m}' \Leftrightarrow w_{\mathbf{m}} \leq w_{\mathbf{m}'}.$$

Since the map $\mathbf{m} \mapsto w_{\mathbf{m}}$ depends on the choice of (B_i) and (C_j) only modulo left and right multiplication by fixed powers of τ, these definitions are independent of this choice. Indeed, they can be described in an analogous way to Proposition 2.3:

Proposition 4.3. *Let* $\mathbf{m}, \mathbf{m}' \in M_{(b_i),n;(c_j),n'}$.

(1) $\ell(\mathbf{m}) = \sum_{i\in[1,n],j} m_{i,j} m_{\leq i,\geq j} - \sum_{i\in[1,n],j} \binom{m_{i,j}+1}{2}$.
(2) $\mathbf{m} \leq \mathbf{m}'$ *if and only if, for all* $i \in \mathbb{Z}$,

$$m_{\leq i,\geq j} \leq m'_{\leq i,\geq j},\ \forall j,\ \textit{with equality for } j \ll 0.$$

(3) $\mathbf{m} \leq \mathbf{m}'$ *if and only if, for all* $j \in \mathbb{Z}$,

$$m_{\geq i,\leq j} \leq m'_{\geq i,\leq j},\ \forall i,\ \textit{with equality for } i \ll 0.$$

Proof. The proof is mostly identical to that of Proposition 2.3, using (4.1) and Proposition 4.1 instead of (2.1) and Proposition 2.1. In the proof of (3), the argument using (2.6) no longer makes sense, but the argument using transposes does. ☐

As in §2, we define $P_{\mathbf{m},\mathbf{m}'} = P_{w_{\mathbf{m}},w_{\mathbf{m}'}}$ for $\mathbf{m}, \mathbf{m}' \in M_{(b_i),n;(c_j),n'}$. From (2) of Proposition 4.3 it is clear that each interval $[\mathbf{m}, \mathbf{m}']$ in the poset $M_{(b_i),n;(c_j),n'}$ is finite, so the inverse matrix $(P^{\langle -1\rangle}_{\mathbf{m},\mathbf{m}'})_{\mathbf{m},\mathbf{m}'\in M_{(b_i),n;(c_j),n'}}$ of $(P_{\mathbf{m},\mathbf{m}'})_{\mathbf{m},\mathbf{m}'\in M_{(b_i),n;(c_j),n'}}$ is well defined.

The matrix definition of cancellability is identical to the finite case.

Definition. If $\mathbf{m} \leq \mathbf{m}'$ in $M_{(b_i),n;(c_j),n'}$, we say that $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ is *cancellable* for $[\mathbf{m}, \mathbf{m}']$ if

(1) $m_{i,j} \geq 1$.
(2) $m_{\leq i-1,\geq j} = m'_{\leq i-1,\geq j}$, or equivalently $m_{\geq i,\leq j+1} = m'_{\geq i,\leq j+1}$.
(3) $m_{\leq i,\geq j+1} = m'_{\leq i,\geq j+1}$, or equivalently $m_{\geq i+1,\leq j} = m'_{\geq i+1,\leq j}$.

These equivalences follow from (4.4), bearing in mind (2) of Proposition 4.3. Clearly (i, j) is cancellable iff $(i + n, j + n')$ is.

Proposition 4.4. *Suppose that* $\mathbf{m} \leq \mathbf{m}'$ *in* $M_{(b_i),n;(c_j),n'}$ *and* (i,j) *is cancellable for* $[\mathbf{m},\mathbf{m}']$. *Let* $\mathbf{e}$ *be the matrix with* $e_{i+kn,j+kn'} = 1$ *for all* k, *all other entries zero.*

(1) *For any* $\mathbf{m}^1 \in [\mathbf{m},\mathbf{m}']$,
   (a) $m^1_{i,j} \geq m_{i,j}$,
   (b) $m^1_{\leq i-1,\geq j} = m_{\leq i-1,\geq j}$, *and*
   (c) $m^1_{\leq i,\geq j+1} = m_{\leq i,\geq j+1}$.
   Hence (i,j) *is cancellable for any sub-interval of* $[\mathbf{m},\mathbf{m}']$.
(2) *The map* $\mathbf{m}^1 \mapsto \mathbf{m}^1 - \mathbf{e}$ *is an isomorphism of posets between* $[\mathbf{m},\mathbf{m}']$ *and* $[\mathbf{m}-\mathbf{e},\mathbf{m}'-\mathbf{e}]$, *which reduces all lengths by the same amount.*
(3) *For any* $\mathbf{m}^1,\mathbf{m}^2 \in [\mathbf{m},\mathbf{m}']$, $P_{\mathbf{m}^1,\mathbf{m}^2} = P_{\mathbf{m}^1-\mathbf{e},\mathbf{m}^2-\mathbf{e}}$.
(4) *For any* $\mathbf{m}^1,\mathbf{m}^2 \in [\mathbf{m},\mathbf{m}']$, $P^{\langle -1\rangle}_{\mathbf{m}^1,\mathbf{m}^2} = P^{\langle -1\rangle}_{\mathbf{m}^1-\mathbf{e},\mathbf{m}^2-\mathbf{e}}$.

Proof. Completely analogous to the proof of Proposition 2.5, using the analogue of Proposition 2.4. □

5. Nilpotent orbits of the cyclic quiver

We now return to the set-up of the latter part of §1, so V is a d-dimensional $\mathbb{Z}/n\mathbb{Z}$-graded vector space, and $d_i = \dim V_{\bar{i}}$ for all $i \in \mathbb{Z}$. We saw in §1 that the G_V-orbits in $\mathcal{N}_V$ are in bijection with the set $M_{(d_i),n}$ of multisegments (in the modulo n sense) such that each congruence class $\bar{i}$ occurs d_i times among the elements of the segments. As in §3, we will identify each $\mathbf{m} \in M_{(d_i),n}$ with a matrix $(m_{i,j})$, this time in $M_{(d_i),n;(d_j),n}$; the definition of $m_{i,j}$ is exactly the same as (3.1). The resulting subset of $M_{(d_i),n;(d_j),n}$ is described as follows.

Proposition 5.1. *Let* $M''_{(d_i),n} = \{\mathbf{m} \in M_{(d_i),n;(d_j),n} \,|\, m_{i,j} = 0, \forall j < i-1\}$.

(1) $M''_{(d_i),n}$ *is a lower ideal of the poset* $M_{(d_i),n;(d_j),n}$.
(2) *For all* $\mathbf{m} \in M''_{(d_i),n}$, *there is some* $f(\mathbf{m}) \in \mathbb{Z}$ *such that*
$$m_{\leq i,\geq j} + f(\mathbf{m}) = d_j + \cdots + d_i, \ \forall i \geq j.$$
(3) *For* $\mathbf{m},\mathbf{m}'$ *in* $M''_{(d_i),n}$, $\mathbf{m} \leq \mathbf{m}'$ *if and only if* $f(\mathbf{m}) = f(\mathbf{m}')$ *and* $m_{\leq i,\geq j} \leq m'_{\leq i,\geq j}$ *for all* $i < j$.
(4) $M'_{(d_i),n} = \{\mathbf{m} \in M''_{(d_i),n} \,|\, f(\mathbf{m}) = 0\}$ *is a lower ideal of the poset* $M_{(d_i),n;(d_j),n}$.
(5) *If* $\mathbf{m} \in M'_{(d_i),n}$, *then* $m_{i,i-1} = m_{\leq i-1,\geq i}$ *for all* i.
(6) $M'_{(d_i),n} = M_{(d_i),n}$.

Proof. As in the finite case, (1) is immediate from (3) of Proposition 4.3. (2) comes from the fact that for $i \geq j$,

$$m_{\leq i, \geq j} = d_j + \cdots + d_{i-1} + m_{\leq i, \geq i} = d_{j+1} + \cdots + d_i + m_{\leq j, \geq j}.$$

Using this, (3) comes from (2) of Proposition 4.3, and (4) is an immediate consequence of (1) and (3). (5) is proved in the same way as (4) of Proposition 3.1. From (5) and the $i = j$ case of (2) it follows that every matrix in $M'_{(d_i),n}$ arises from a multisegment in $M_{(d_i),n}$, whence (6). $\square$

We can now state Lusztig's affine analogue of Theorem 3.2:

Theorem 5.2. *Let* $\mathbf{m}, \mathbf{m}' \in M_{(d_i),n}$.

(1) $\dim \mathcal{O}_{\mathbf{m}} = \ell(\mathbf{m}) - \sum_{i \in [1,n]} \binom{d_i}{2}$.
(2) $\mathcal{O}_{\mathbf{m}} \subseteq \overline{\mathcal{O}_{\mathbf{m}'}} \Leftrightarrow \mathbf{m} \leq \mathbf{m}'$.
(3) $\mathcal{H}^i IC(\overline{\mathcal{O}_{\mathbf{m}'}}) = 0$ *for i odd.*
(4) $IC_{\mathbf{m},\mathbf{m}'} = P_{\mathbf{m},\mathbf{m}'}$.
(5) $IC^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'} = P^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'}$.

Proof. As with Theorem 3.2, (5) follows from (4) because $M_{(d_i),n}$ is a lower ideal of $M_{(d_i),n;(d_j),n}$. Parts (1)–(4) were proved by Lusztig in [12, §11], but since the conventions there are slightly different, a sketch of a proof along the lines of the above proof of Theorem 3.2 may be helpful.

Form $\mathcal{V} = \mathbb{C}((t)) \otimes_{\mathbb{C}} V$, and consider lattices (free $\mathbb{C}[\![t]\!]$-submodules of rank d) in $\mathcal{V}$. Define $\widehat{\mathcal{B}}_{(d_i),n}$ to be the set of collections of lattices $(\mathcal{M}_i)_{i \in \mathbb{Z}}$ such that for all $i \in \mathbb{Z}$:

(1) $\mathcal{M}_{i-1} \subset \mathcal{M}_i$,
(2) $\dim_{\mathbb{C}} \mathcal{M}_i / \mathcal{M}_{i-1} = d_i$, and
(3) $\mathcal{M}_{i-n} = t\mathcal{M}_i$.

It is well known that $\widehat{\mathcal{B}}_{(d_i),n}$ has the structure of an increasing union of projective varieties. We define a base-point $(\mathcal{L}_i)$ in $\widehat{\mathcal{B}}_{(d_i),n}$ as follows. For any $i \in \mathbb{Z}$, let V_i denote $t^k V_{\bar{i}}$ where k is defined by $i + kn \in \{1, \cdots, n\}$. Define

$$\mathcal{L}_i = \widehat{\bigoplus_{j \leq i}} V_j, \ \forall i \in \mathbb{Z},$$

where $\widehat{\bigoplus}$ denotes completed direct sum. Relative to this base-point, $\widehat{\mathcal{B}}_{(d_i),n}$ decomposes into affine Schubert cells $\widehat{\mathcal{B}}_{\mathbf{m}}$ for $\mathbf{m} \in M_{(d_i),n;(d_j),n}$. Explicitly, $\widehat{\mathcal{B}}_{\mathbf{m}}$ consists of those $(\mathcal{M}_i)$ such that for all $i, j \in \mathbb{Z}$,

$$\dim_{\mathbb{C}} \frac{\mathcal{M}_i \cap \mathcal{L}_j}{\mathcal{M}_i \cap \mathcal{L}_{j-1} + \mathcal{M}_{i-1} \cap \mathcal{L}_j} = m_{i,j}.$$

The analogues of (1)–(4) for these affine Schubert cells (for all of $M_{(d_i),n;(d_j),n}$) are well known. Let $\widehat{\mathcal{B}}'_{(d_i),n}$ be the closed subvariety of $\widehat{\mathcal{B}}_{(d_i),n}$ defined by requiring $\mathcal{M}_i \supseteq \mathcal{L}_{i-1}$, $\dim_{\mathbb{C}} \mathcal{M}_i/\mathcal{L}_{i-1} = d_i$; from the description of $M_{(d_i),n}$ as $M'_{(d_i),n}$, it is easy to see that $\widehat{\mathcal{B}}'_{(d_i),n} = \bigcup_{\mathbf{m}\in M_{(d_i),n}} \widehat{\mathcal{B}}_{\mathbf{m}}$.

Now we define a morphism $\mathcal{N}_V \to \widehat{\mathcal{B}}'_{(d_i),n} : \varphi \mapsto (\mathcal{M}_i(\varphi))$ by the rule

$$\mathcal{M}_i(\varphi) = \mathcal{L}_{i-1} \oplus \{v + \varphi(v) + \varphi^2(v) + \cdots \mid v \in V_i\}.$$

(Since φ is nilpotent, this sum is actually finite.) An easy check shows that this morphism maps $\mathcal{O}_{\mathbf{m}}$ into $\widehat{\mathcal{B}}_{\mathbf{m}}$ for all $\mathbf{m} \in M_{(d_i),n}$. All that remains is to verify that it gives an isomorphism between $\mathcal{N}_V$ and the open subvariety of $\widehat{\mathcal{B}}'_{(d_i),n}$ defined by requiring

$$\mathcal{M}_i \cap \bigoplus_{i'>i} V_{i'} = 0,\ \forall i \in \mathbb{Z}.$$

The "dual" statement to this is what is proved in [12, §11]. □

Note that in contrast to the situation in §3, the poset $M_{(d_i),n}$ may have more than one maximal element.

We now come to the affine analogue of Theorem 3.3, a generalization of Theorem 5.2.

Theorem 5.3. *Let $b_i, c_j \in \mathbb{N}$ be such that*

$$b_{i+n} = b_i,\ c_{j+n} = c_j,\ \text{and } d_i - b_i = d_{i-1} - c_{i-1},\ \forall i,j \in \mathbb{Z}.$$

Define a $(\mathbb{Z}\times\mathbb{Z})$-matrix $\mathbf{a}$ by

$$a_{i,j} = \begin{cases} d_i - b_i, & \text{if } j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $M^{(b_i);(c_j)}_{(d_i),n} = \{\mathbf{m} \in M_{(d_i),n} \mid m_{i,i-1} \geq d_i - b_i,\ \forall i \in \mathbb{Z}\}$.

(1) *$M^{(b_i);(c_j)}_{(d_i),n}$ is an upper ideal of $M_{(d_i),n}$.*
(2) *The map $\mathbf{m} \mapsto \mathbf{m} - \mathbf{a}$ is an isomorphism of posets between $M^{(b_i);(c_j)}_{(d_i),n}$ and $\{\widetilde{\mathbf{m}} \in M_{(b_i),n;(c_j),n} \mid \widetilde{m}_{i,j} = 0,\ \forall j < i-1\}$.*
(3) *For any $\mathbf{m}', \mathbf{m}'' \in M^{(b_i);(c_j)}_{(d_i),n}$, $IC_{\mathbf{m}',\mathbf{m}''} = P_{\mathbf{m}'-\mathbf{a},\mathbf{m}''-\mathbf{a}}$.*
(4) *For any $\mathbf{m}', \mathbf{m}'' \in M^{(b_i);(c_j)}_{(d_i),n}$, $IC^{\langle -1\rangle}_{\mathbf{m}',\mathbf{m}''} = P^{\langle -1\rangle}_{\mathbf{m}'-\mathbf{a},\mathbf{m}''-\mathbf{a}}$.*

Proof. Completely analogous to the proof of Theorem 3.3, using Proposition 4.3, Proposition 4.4, and Theorem 5.2 in place of Proposition 2.3, Proposition 2.5, and Theorem 3.2. □

Corollary 5.4. *For* $\mathbf{m} \in M_{(d_i),n}$*, define a* $(\mathbb{Z} \times \mathbb{Z})$*-matrix* $\mathbf{m}^-$ *by*

$$m^-_{i,j} = \begin{cases} m_{i,j}, & \text{if } j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\langle \mathbf{m} \rangle = \{\mathbf{m}' \in M'_{(d_i),n} \,|\, \mathbf{m} \leq \mathbf{m}'\}$*, and let* $\mathbf{m}_1^{\max}, \cdots, \mathbf{m}_t^{\max}$ *be the maximal elements of* $\langle \mathbf{m} \rangle$*.*

(1) *For all* $\mathbf{m}' \in \langle \mathbf{m} \rangle$*,* $\mathbf{m}' - \mathbf{m}^- \in M_{(d_i - m_{i,i-1}),n;(d_j - m_{j+1,j}),n}$*.*
(2) *The map* $\mathbf{m}' \mapsto \mathbf{m}' - \mathbf{m}^-$ *is an isomorphism of posets between* $\langle \mathbf{m} \rangle$ *and* $\bigcup_{s=1}^t [\mathbf{m} - \mathbf{m}^-, \mathbf{m}_s^{\max} - \mathbf{m}^-]$*.*
(3) *For any* $\mathbf{m}', \mathbf{m}'' \in \langle \mathbf{m} \rangle$*,* $IC_{\mathbf{m}',\mathbf{m}''} = P_{\mathbf{m}'-\mathbf{m}^-,\mathbf{m}''-\mathbf{m}^-}$*.*
(4) *For any* $\mathbf{m}', \mathbf{m}'' \in \langle \mathbf{m} \rangle$*,* $IC^{\langle -1 \rangle}_{\mathbf{m}',\mathbf{m}''} = P^{\langle -1 \rangle}_{\mathbf{m}'-\mathbf{m}^-,\mathbf{m}''-\mathbf{m}^-}$*.*

Proof. Apply Theorem 5.3 with

$$b_i = d_i - m_{i,i-1}, \; c_{i-1} = d_{i-1} - m_{i,i-1}$$

for all $i \in \mathbb{Z}$, and restrict to the upper ideal $\langle \mathbf{m} \rangle$ of $M^{(b_i);(c_j)}_{(d_i),n}$. □

Note that the polynomials $P_{\mathbf{m}'-\mathbf{m}^-,\mathbf{m}''-\mathbf{m}^-}$ in (3) are Kazhdan-Lusztig polynomials for $\widetilde{S_{k(\mathbf{m})}}$, where $k(\mathbf{m})$ is the number of segments in $\mathbf{m}$, which is also the sum of the entries in rows 1 to n of $\mathbf{m} - \mathbf{m}^-$. As a corollary, we recover the main result of [8]:

Corollary 5.5. *If* $\mathbf{m} \leq \mathbf{m}'$ *in* $M_{(d_i),n}$*, and* $k(\mathbf{m}) = 2$*, then* $IC_{\mathbf{m},\mathbf{m}'} = 1$*.*

Proof. In $\widetilde{S}_2$ all nonzero Kazhdan-Lusztig polynomials are 1. □

Example. Let $d = 6$, $n = 3$, $d_1 = d_2 = d_3 = 2$. Let $\mathbf{m} \in M_{(d_i),3}$ be the multisegment $[1,2] + [2,3] + [3,4]$. Then $\langle \mathbf{m} \rangle$ has three maximal elements,

$$\mathbf{m}_1^{\max} = [1,6], \; \mathbf{m}_2^{\max} = [2,7], \; \text{and } \mathbf{m}_3^{\max} = [3,8].$$

Displaying only the rows indexed by $1, 2, 3$, we have

$$\mathbf{m} = \begin{pmatrix} \cdots & 1 & \mathbf{0} & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix},$$

$$\mathbf{m}_1^{\max} = \begin{pmatrix} \cdots & 1 & \mathbf{0} & 0 & 0 & 0 & 0 & 1 & \cdots \\ \cdots & 0 & 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 2 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$\mathbf{m} - \mathbf{m}^- = \begin{pmatrix} \cdots & 0 & \mathbf{0} & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix},$$

$$\mathbf{m}_1^{\max} - \mathbf{m}^- = \begin{pmatrix} \cdots & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 1 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

where the $\mathbf{0}$ is the $(1,1)$-entry. Setting $B_i = \{i\}$ and $C_j = \{j-1\}$, so that $w_{\mathbf{m}-\mathbf{m}^-}$ is the identity of $\widetilde{S_3}$, we find that $w_{\mathbf{m}_1^{\max}-\mathbf{m}^-}$ has window $(5, 0, 1)$, and is therefore $s_1 s_0 s_2 s_1$. Similarly $w_{\mathbf{m}_2^{\max}-\mathbf{m}^-} = s_2 s_1 s_0 s_2$ and $w_{\mathbf{m}_3^{\max}-\mathbf{m}^-} = s_0 s_2 s_1 s_0$. So $\mathbf{m}' \mapsto w_{\mathbf{m}'-\mathbf{m}^-}$ is an isomorphism between $\langle \mathbf{m} \rangle$ and $[1, s_1 s_0 s_2 s_1] \cup [1, s_2 s_1 s_0 s_2] \cup [1, s_0 s_2 s_1 s_0]$. Moreover,

$$IC_{\mathbf{m},\mathbf{m}_1^{\max}} = P_{1, s_1 s_0 s_2 s_1} = q + 1,$$

and similarly $IC_{\mathbf{m},\mathbf{m}_2^{\max}} = IC_{\mathbf{m},\mathbf{m}_3^{\max}} = q + 1$, while

$$IC^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'} = P^{\langle -1 \rangle}_{1, w_{\mathbf{m}'-\mathbf{m}^-}} = \varepsilon(w_{\mathbf{m}'-\mathbf{m}^-})$$

for all $\mathbf{m}' \in \langle \mathbf{m} \rangle$ (an example of (1.8)).

Finally, we must prove Theorem 1.2. We have elements $\lambda, \mu \in \widetilde{D_k}$; define λ_s and μ_s for all $s \in \mathbb{Z}$ by the rule

$$\lambda_{s+k} = \lambda_s + k - n, \ \mu_{s+k} = \mu_s + k - n.$$

Then $\lambda_s - s \geq \lambda_{s+1} - (s+1)$ for all $s \in \mathbb{Z}$, and similarly for μ; also,

$$(w \cdot \mu)_s - s = \mu_{w^{-1}(s)} - w^{-1}(s), \ \forall w \in \widetilde{S_k}, \ s \in [1, k].$$

Define

$$B_i = \{-s \,|\, \mu_s - s + 1 = i\}, \ C_j = \{-s \,|\, \lambda_s - s = j\}.$$

Then the subgroups $S_{(B_i)}$ and $S_{(C_j)}$ of $\widetilde{S_k}$ are exactly the images of $\widetilde{W_\mu}$ and $\widetilde{W_\lambda}$ under the automorphism $\tau : \widehat{S_k} \to \widehat{S_k}$ defined by $\tau(w)(i) = -w(-i)$. The map $\psi : \widehat{S_k} \to M_{(b_i),n;(c_j),n}$ as defined in §4 satisfies

$$\psi(\tau(w))_{i,j} = |\{s \in \mathbb{Z} \,|\, \mu_s - s + 1 = i, \lambda_{w(s)} - w(s) = j\}|. \tag{5.1}$$

So $\widetilde{S_k}[\lambda,\mu] = \{w \in \widetilde{S_k} \mid \psi(\tau(w))_{i,j} = 0, \forall j < i-1\}$, which shows that it is indeed a lower ideal of $\widetilde{S_k}$. Moreover, $w \mapsto \psi(\tau(w))$ gives an isomorphism of posets between $\widetilde{S_k}[\lambda,\mu]^\circ$ and

$$\{\widetilde{\mathbf{m}} \in M_{(b_i),n;(c_j),n} \mid \widetilde{\mathbf{m}} \geq \psi(1) \text{ and } \widetilde{m}_{i,j} = 0, \forall j < i-1\},$$

and the polynomials attached to these posets correspond, since

$$P_{w,w'} = P_{\tau(w),\tau(w')} = P_{\psi(\tau(w)),\psi(\tau(w'))} \tag{5.2}$$

for all $w, w' \in \widetilde{S_k}[\lambda,\mu]^\circ$. Now make the assumption of Theorem 1.2, that $\lambda/\mu \in M_{(d_i),n}$; it follows immediately that (b_i) and (c_j) satisfy the conditions of Theorem 5.3. By (5.1), for all $w \in \widetilde{S_k}[\lambda,\mu]^\circ$, the multisegment $\lambda/(w\cdot\mu)$ when viewed as a matrix has the same diagonal and above-diagonal entries as $\psi(\tau(w))$; hence $\lambda/(w\cdot\mu) = \psi(\tau(w)) + \mathbf{a}$ where $\mathbf{a}$ is as in Theorem 5.3. Thus Theorem 1.2 follows from Theorem 5.3.

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