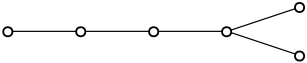
The Product Monomial Crystal

Joel Gibson
The University of Sydney
Supervisor: Dr. Oded Yacobi

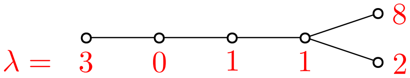
Presented at AustMS 2018, The University of Adelaide

December 5, 2018

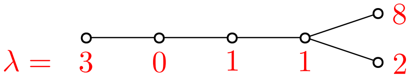
Motivation: Nakajima quiver varieties



MOTIVATION: NAKAJIMA QUIVER VARIETIES

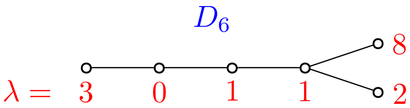


MOTIVATION: NAKAJIMA QUIVER VARIETIES



$\mathcal{M}(\lambda)$

Motivation: Nakajima quiver varieties

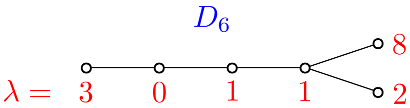


$\mathcal{M}(\lambda)$

$\mathfrak{g} = \mathfrak{so}(12)$

$$\mathfrak{g} \curvearrowright H_{\mathrm{top}}(\mathcal{M}(\lambda), \mathbb{C}) \cong V(\lambda)$$

Motivation: Nakajima quiver varieties



$\mathcal{M}(\lambda)$

$\mathfrak{g} = \mathfrak{so}(12)$

$\mathfrak{g} \curvearrowright H_{\text{top}}(\mathcal{M}(\lambda), \mathbb{C})$
$\cong V(\lambda)$

$\rho_{\mathbf{R}} : \mathbb{C}^\times \to \text{Aut}(\mathcal{M}(\lambda))$

$\mathfrak{g} \curvearrowright H_{\text{top}}(\mathcal{M}(\lambda)^{\rho_{\mathbf{R}}}, \mathbb{C})$
$\cong ???$

SETUP

Fix some Lie-theoretic data:

1. $\mathfrak{g}$ a semisimple simply-laced complex Lie algebra $\mathfrak{g}$.
2. $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ a choice of Cartan and Borel.

SETUP

Fix some Lie-theoretic data:

1. $\mathfrak{g}$ a semisimple simply-laced complex Lie algebra $\mathfrak{g}$.
2. $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ a choice of Cartan and Borel.

$\mathfrak{sl}_3$ example:

$$\underbrace{\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}}_{\mathfrak{h}} \subseteq \underbrace{\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}}_{\mathfrak{b}} \subseteq \underbrace{\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}}_{\mathfrak{g}=\mathfrak{sl}_3}$$

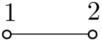
Setup

Fix some Lie-theoretic data:

1. $\mathfrak{g}$ a semisimple simply-laced complex Lie algebra $\mathfrak{g}$.
2. $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ a choice of Cartan and Borel.

$$\mathfrak{sl}_3 \text{ example:} \quad \underbrace{\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}}_{\mathfrak{h}} \subseteq \underbrace{\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}}_{\mathfrak{b}} \subseteq \underbrace{\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}}_{\mathfrak{g}=\mathfrak{sl}_3}$$

Then, for free, get

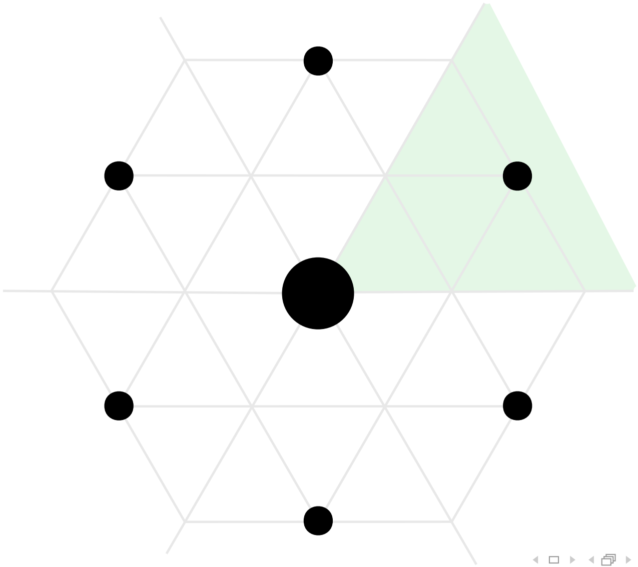
1. A Dynkin diagram I, a simple graph. $I =$ 

2. A weight lattice P, and dominant weights P^+.

Characters of representations

$\mathfrak{g} = \mathfrak{sl}_3$

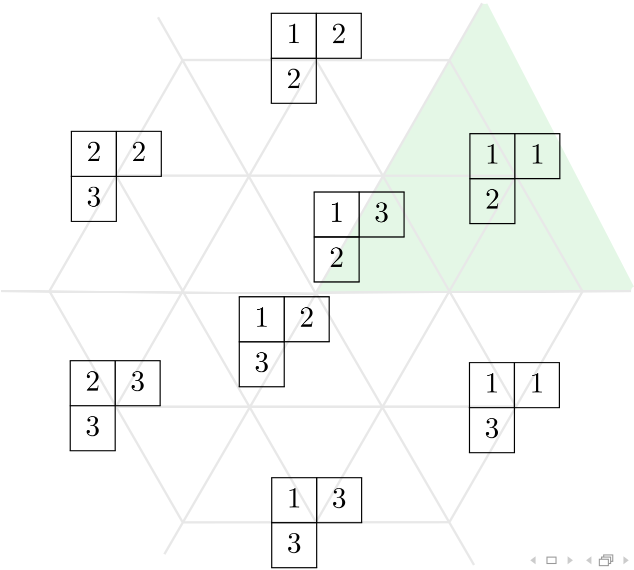
CHARACTERS OF REPRESENTATIONS



$\mathfrak{g} = \mathfrak{sl}_3$

$\mathfrak{sl}_3 \curvearrowright V$, with
$V = \mathfrak{sl}_3$

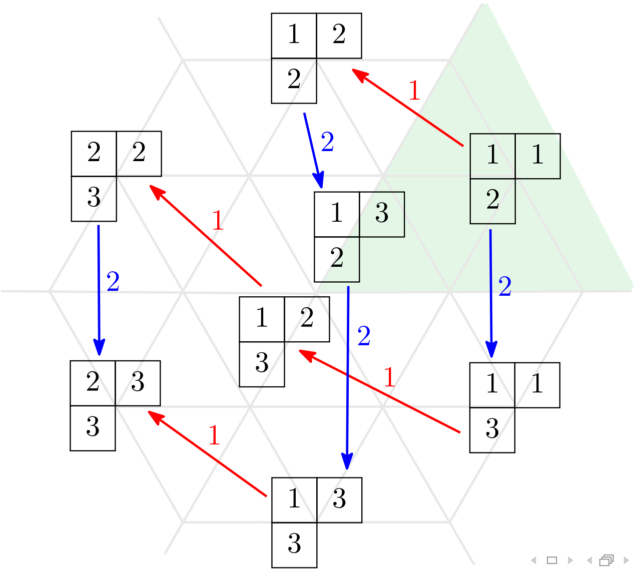
Characters of representations



$\mathfrak{g} = \mathfrak{sl}_3$

$\mathfrak{sl}_3 \curvearrowright V$, with
$V = \mathfrak{sl}_3$

Characters of representations



$\mathfrak{g} = \mathfrak{sl}_3$

$\mathfrak{sl}_3 \curvearrowright V$, with $V = \mathfrak{sl}_3$

A $\mathfrak{g}$-CRYSTAL IS...

A $\mathfrak{g}$-crystal is a *combinatorial shadow* of a $\mathfrak{g}$-representation.

A $\mathfrak{g}$-crystal is...

A $\mathfrak{g}$-crystal is a *combinatorial shadow* of a $\mathfrak{g}$-representation.

$\mathfrak{g}$ crystals form a semisimple category, with simples indexed by dominant weights.

A $\mathfrak{g}$-CRYSTAL IS...

A $\mathfrak{g}$-crystal is a *combinatorial shadow* of a $\mathfrak{g}$-representation.

$\mathfrak{g}$ crystals form a semisimple category, with simples indexed by dominant weights.

The category of crystals is *monoidal*: the underlying set of $C_1 \otimes C_2$ is $C_1 \times C_2$.

The decomposition numbers match those in $\mathfrak{g}$-mod:

$$[B(\nu) : B(\lambda) \otimes B(\mu)] = [V(\nu) : V(\lambda) \otimes V(\mu)]$$

A $\mathfrak{g}$-crystal is...

A $\mathfrak{g}$-crystal is a *combinatorial shadow* of a $\mathfrak{g}$-representation.

$\mathfrak{g}$ crystals form a semisimple category, with simples indexed by dominant weights.

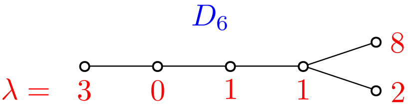
The category of crystals is *monoidal*: the underlying set of $C_1 \otimes C_2$ is $C_1 \times C_2$.

The decomposition numbers match those in $\mathfrak{g}$-mod:

$$[B(\nu) : B(\lambda) \otimes B(\mu)] = [V(\nu) : V(\lambda) \otimes V(\mu)]$$

... but there is no functor $\mathfrak{g}$-mod $\rightarrow$ $\mathfrak{g}$-crystals.

Reminder: Nakajima quiver varieties



$$\mathcal{M}(\lambda)$$

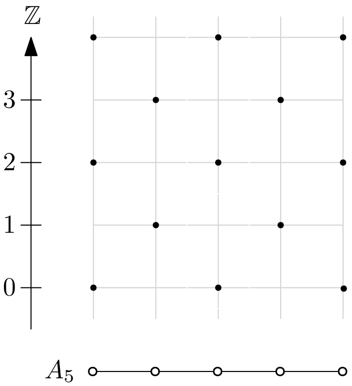
$$\mathfrak{g} = \mathfrak{so}(12)$$

$$\mathfrak{g} \curvearrowright H_{\mathrm{top}}(\mathcal{M}(\lambda), \mathbb{C}) \cong V(\lambda)$$

$$\rho_{\mathbf{R}} : \mathbb{C}^{\times} \to \mathrm{Aut}(\mathcal{M}(\lambda))$$

$$\mathfrak{g} \curvearrowright H_{\mathrm{top}}(\mathcal{M}(\lambda)^{\rho_{\mathbf{R}}}, \mathbb{C}) \cong ???$$

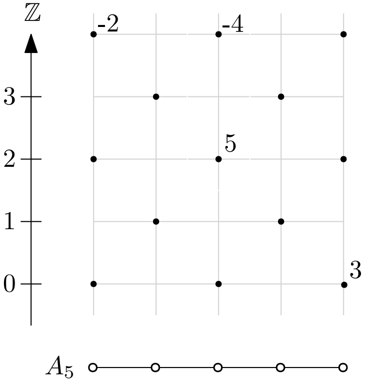
MONOMIAL CRYSTAL



Partition $I = I_0 \sqcup I_1$ into a bipartite graph.

$$L := \{(i,h) \in I \times \mathbb{Z} \mid \mathsf{parity}(i) = \mathsf{parity}(h)\}$$

Monomial crystal

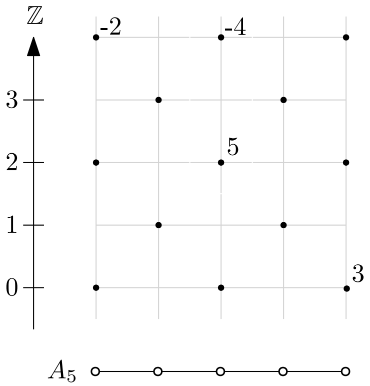


Partition $I = I_0 \sqcup I_1$ into a bipartite graph.

$$L := \{(i, h) \in I \times \mathbb{Z} \mid \mathsf{parity}(i) = \mathsf{parity}(h)\}$$

The *monomial crystal* is the set $\mathbb{Z}L$.

MONOMIAL CRYSTAL



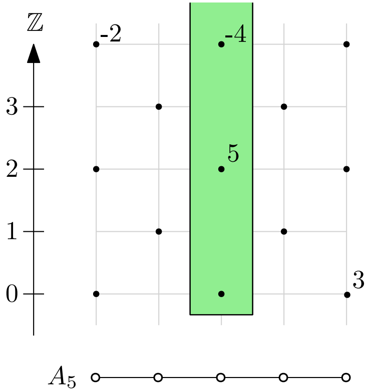
Partition $I = I_0 \sqcup I_1$ into a bipartite graph.

$$L := \{(i,h) \in I \times \mathbb{Z} \mid \mathsf{parity}(i) = \mathsf{parity}(h)\}$$

The *monomial crystal* is the set $\mathbb{Z}L$.

$$\langle \mathrm{wt}(b), \alpha_i^\vee \rangle = \text{sum in column } i$$
$$\mathrm{wt}(b) = -2\varpi_1 + \varpi_3 + 3\varpi_5$$

MONOMIAL CRYSTAL



Partition $I = I_0 \sqcup I_1$ into a bipartite graph.

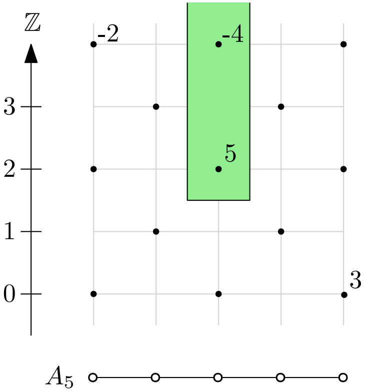
$$L := \{(i,h) \in I \times \mathbb{Z} \mid \text{parity}(i) = \text{parity}(h)\}$$

The *monomial crystal* is the set $\mathbb{Z}L$.

$$\langle \mathrm{wt}(b), \alpha_i^\vee \rangle = \text{sum in column } i$$
$$\mathrm{wt}(b) = -2\varpi_1 + \varpi_3 + 3\varpi_5$$

Computing arrow $i = 3$...

Monomial crystal



Partition $I = I_0 \sqcup I_1$ into a bipartite graph.

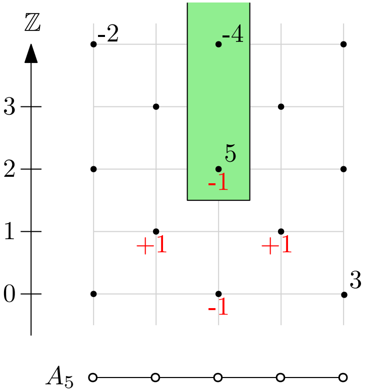
$$L := \{(i,h) \in I \times \mathbb{Z} \mid \mathsf{parity}(i) = \mathsf{parity}(h)\}$$

The *monomial crystal* is the set $\mathbb{Z}L$.

$$\langle \mathrm{wt}(b), \alpha_i^\vee \rangle = \text{sum in column } i$$
$$\mathrm{wt}(b) = -2\varpi_1 + \varpi_3 + 3\varpi_5$$

Computing arrow $i = 3$...

Monomial crystal



Partition $I = I_0 \sqcup I_1$ into a bipartite graph.

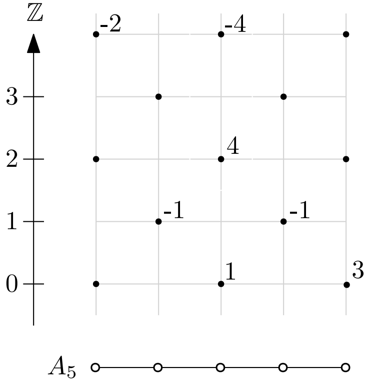
$$L := \{(i,h) \in I \times \mathbb{Z} \mid \mathsf{parity}(i) = \mathsf{parity}(h)\}$$

The *monomial crystal* is the set $\mathbb{Z}L$.

$$\langle \mathrm{wt}(b), \alpha_i^\vee \rangle = \text{sum in column } i$$
$$\mathrm{wt}(b) = -2\varpi_1 + \varpi_3 + 3\varpi_5$$

Computing arrow $i = 3$...

Monomial crystal



Partition $I = I_0 \sqcup I_1$ into a bipartite graph.

$L := \{(i,h) \in I \times \mathbb{Z} \mid \text{parity}(i) = \text{parity}(h)\}$

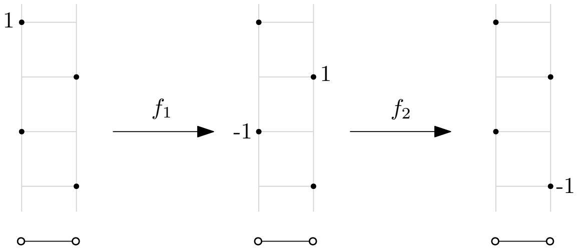
The *monomial crystal* is the set $\mathbb{Z}L$.

$$\langle \mathrm{wt}(b), \alpha_i^\vee \rangle = \text{sum in column } i$$
$$\mathrm{wt}(b) = -2\varpi_1 + \varpi_3 + 3\varpi_5$$

Computing arrow $i = 3$...

Fundamental monomial crystals

The crystal generated by $(i, c) \in L$ is a *fundamental* crystal, written $B(i, c)$.



The basic crystal $B(1, c)$ in type A_2.

> *Theorem (Kashiwara)*
>
> The crystal $B(i, c)$ is isomorphic to $B(\varpi_i)$, the irreducible crystal of highest weight ϖ_i.

The product monomial crystal

Let $\mathbf{R} = \{(i_1, c_1), \ldots, (i_r, c_r)\}$ be a multiset.

- Each $B(i_k, c_k) \subseteq \mathbb{Z}L$ is a finite crystal isomorphic to $B(\varpi_{i_k})$.
- Let $B(\mathbf{R}) \subseteq \mathbb{Z}L$ be their sum:

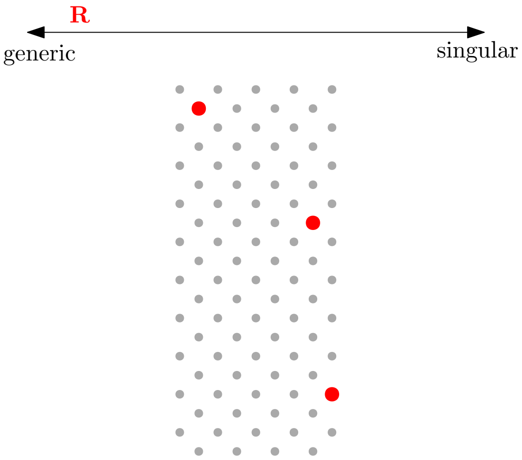
$$B(\mathbf{R}) = \{b_1 + \cdots + b_r \mid b_k \in B(i_k, c_k)\}$$

- Redundancies may occur: $|B(\mathbf{R})| \leq |B(i_1, c_1)| \cdots |B(i_r, c_r)|$

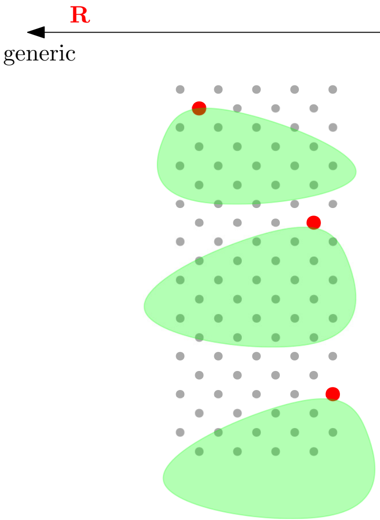
> *Theorem (Kamnitzer, Tingley, Webster, Weekes, Yacobi)*
>
> $B(\mathbf{R})$ is a subcrystal of $\mathbb{Z}L$.

The crystal $B(\mathbf{R})$ is called the *product monomial crystal* associated to the data $\mathbf{R}$.

BETWEEN GENERIC AND SINGULAR

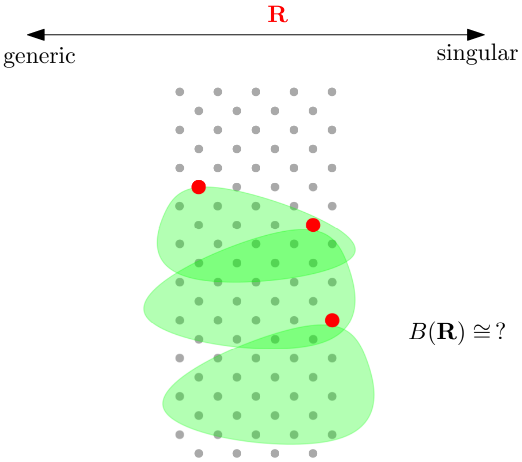


BETWEEN GENERIC AND SINGULAR



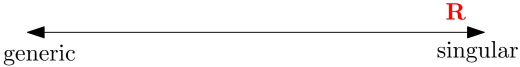
$$B(\mathbf{R}) \cong B(\varpi_2) \otimes B(\varpi_8) \otimes B(\varpi_9)$$

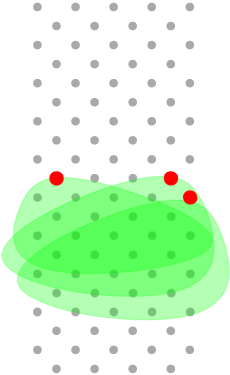
BETWEEN GENERIC AND SINGULAR



$B(\mathbf{R}) \cong ?$

BETWEEN GENERIC AND SINGULAR





$$B(\mathbf{R}) \cong B(\varpi_2 + \varpi_8 + \varpi_9)$$

My contributions

Natural question: can we describe $B(\mathbf{R})$ for arbitrary $\mathbf{R}$?

> *Theorem (G, 2018)*
>
> In any simply-laced type, there is a Demazure-type formula giving the character of $B(\mathbf{R})$. This formula consists of Demazure operators π_i, and multiplications by the fundamental weights ϖ_i.

The character formula is proved using a novel method for analysing $B(\mathbf{R})$ through *Demazure truncations*.

SCHUR FUNCTORS

λ a partition, $\mathbb{S}_\lambda : \mathsf{Vect}_\mathbb{C} \to \mathsf{Vect}_\mathbb{C}$ a "Schur functor".

$\mathbb{S}_\lambda(V)$ is the image of d_λ:

$$d_\lambda : \mathrm{Alt}^{\mathrm{cols}\,\lambda}(V) \xrightarrow{\text{comult}} V^{\otimes\lambda} \xrightarrow{\text{mult}} \mathrm{Sym}^{\mathrm{rows}\,\lambda}(V)$$

Schur functors

λ a partition, $\mathbb{S}_\lambda : \mathsf{Vect}_\mathbb{C} \to \mathsf{Vect}_\mathbb{C}$ a "Schur functor".

$\mathbb{S}_\lambda(V)$ is the image of d_λ:

$$d_\lambda : \mathrm{Alt}^{\mathrm{cols}\,\lambda}(V) \xrightarrow{\mathsf{comult}} V^{\otimes\lambda} \xrightarrow{\mathsf{mult}} \mathrm{Sym}^{\mathrm{rows}\,\lambda}(V)$$

For $\lambda = (3, 1)$,

$$d_\lambda : \bigwedge^2(V) \otimes \bigwedge^1(V) \otimes \bigwedge^1(V) \to S^3(V) \otimes S^1(V)$$

$$(v_1 \wedge v_2) \otimes v_3 \otimes v_4 \mapsto \begin{array}{|c|c|c|}\hline v_1 & v_3 & v_4 \\ \hline v_2 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|c|}\hline v_2 & v_3 & v_4 \\ \hline v_1 \\ \cline{1-1}\end{array} \mapsto v_1v_3v_4 \otimes v_2 - v_2v_3v_4 \otimes v_1$$

(GENERALISED) SCHUR MODULES

By functorality, $G \curvearrowright V \implies G \curvearrowright \mathbb{S}_\lambda(V)$

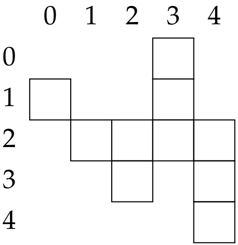
When $G = \mathrm{GL}_n(\mathbb{C})$, the $\mathbb{S}_\lambda(\mathbb{C}^n)$ is called the *Schur module* for λ.

(Generalised) Schur modules

By functorality, $G \curvearrowright V \implies G \curvearrowright \mathbb{S}_\lambda(V)$

When $G = \mathrm{GL}_n(\mathbb{C})$, the $\mathbb{S}_\lambda(\mathbb{C}^n)$ is called the *Schur module* for λ.

Let $D \subseteq \mathbb{N} \times \mathbb{N}$ be a subset of cardinality d, for example



The functor $\mathbb{S}_D$ still makes sense. $\mathbb{S}_D(\mathbb{C}^n)$ is the *generalised Schur module* associated to the diagram D for GL_n.

CRYSTAL OF GENERALISED SCHUR MODULES

$\mathbb{S}_D(\mathbb{C}^n)$ is an $\mathfrak{sl}_n$-module: what is its crystal?

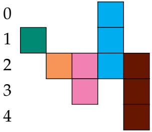
- GL_n-character of $\mathbb{S}_D(\mathbb{C}^n)$: Magyar, Reiner, Shimozono (1990s).

Theorem (G, 2018)

In type A, the crystal $B(\mathbf{R})$ is the crystal of a generalised Schur module, for a diagram D depending on $\mathbf{R}$. Conversely, this gives the crystal of every generalised Schur module for a column-convex diagram.

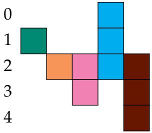
Correspondence of diagrams and multisets

1. Diagram D

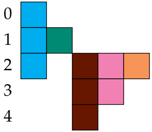


CORRESPONDENCE OF DIAGRAMS AND MULTISETS

1. Diagram D

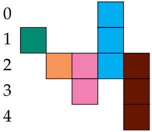


2. Reorder columns:

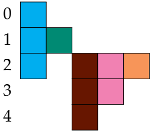


Correspondence of diagrams and multisets

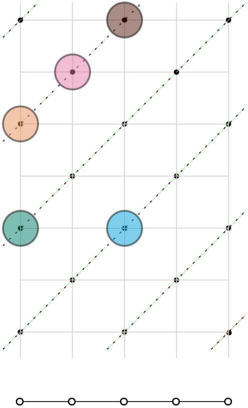
1. Diagram D



2. Reorder columns:

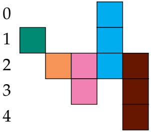


4. Place groups along diagonals:

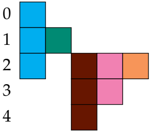


Correspondence of diagrams and multisets

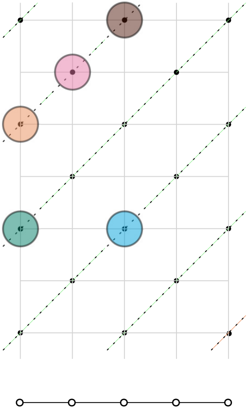
1. Diagram D



2. Reorder columns:



4. Place groups along diagonals:



5. $\mathbf{R} = \{(3,0), (1,0), (3,4), (2,3), (1,2)\}$

FUTURE DIRECTIONS

1. Truncations could apply to other monomial crystals.
2. Similar results should hold for simply-laced bipartite Kac-Moody types.
3. Do the truncations have a deeper meaning?