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A classification of commutative parabolic Hecke algebras

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ABSTRACT

Let (W,S) be a Coxeter system with $I \subseteq S$ such that the parabolic subgroup W_I is finite. Associated to this data there is a *Hecke algebra* $\mathscr{H}$ and a *parabolic Hecke algebra* $\mathscr{H}^I = \mathbf{1}_I \mathscr{H} \mathbf{1}_I$ (over a ring $\mathbb{Z}[q_s]_{s\in S}$). We give a complete classification of the commutative parabolic Hecke algebras across all Coxeter types.

Introduction

Parabolic Hecke algebras $\mathscr{H}^I$ arise naturally as algebras of P_I bi-invariant functions on semisimple Lie (or Kac–Moody) groups G defined over finite fields, where P_I is a parabolic subgroup of type I. As such they play an important role in the representation of these groups, in particular in studying the representations which have a P_I-fixed vector. If $\mathscr{H}^I$ is commutative then (G, P_I) is a *Gelfand pair*. In this case the representation theory of $\mathscr{H}^I$ is considerably simplified, and this leads to powerful results about representations of the group G. See, for example, [3,19,20] for the affine case. Thus it is a natural question to ask when these algebras are commutative.

Hecke algebras can be defined more generally, without reference to Kac–Moody groups as follows. Let (W,S) be a Coxeter system, and let $(q_s)_{s\in S}$ be a family of commuting indeterminants with $q_s = q_t$ if and only if s and t are conjugate in W. The *Hecke algebra* is the associative $\mathbb{Z}[q_s]_{s\in S}$ algebra $\mathscr{H}$ with free basis $\{T_w \mid w \in W\}$ and relations given by Eqs. (1.1) in Section 1.2. Suppose that $I \subseteq S$ is such that the parabolic subgroup $W_I = \langle \{s \mid s \in I\}\rangle$ is finite. The *I-parabolic Hecke algebra* $\mathscr{H}^I$ is

$$\mathscr{H}^I = \mathbf{1}_I \mathscr{H} \mathbf{1}_I, \quad \text{where } \mathbf{1}_I = \sum_{w\in W_I} T_w.$$

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It is these algebras (and their specialisations with $q_s \geqslant 1$) that we study here. We give a complete classification of the pairs (W, I) with W irreducible such that $\mathscr{H}^I$ is commutative.

Let us put this result into perspective by surveying known results on the commutativity of parabolic Hecke algebras. Assume throughout that W is irreducible. Consider the *spherical case* (that is, $|W| < \infty$). The case $|S \backslash I| = 1$ (that is, W_I is a maximal parabolic subgroup of W) is classical, dating back to Iwahori [13] with proofs appearing in [8] (see also [6, Theorem 10.4.11]). It turns out that the statement is very neat in this case: $\mathscr{H}^I$ is commutative if and only if each minimal length W_I double coset representative is an involution. This statement does not hold in general (however we obtain a similar equivalence in Theorem 2.2). The proof in [8] uses elegant representation theory of the Coxeter group W, along with counting arguments, semisimplicity of the Hecke algebra, and Tits' Deformation Theorem. These techniques do not readily generalise to the infinite case, as we lose the counting arguments, semisimplicity, and the Deformation Theorem.

The spherical case with $|S \backslash I| = 1$ is also analysed in [17] via incidence structures and permutation representations. In particular [17, Section 4] gives a thorough analysis of the classical types, and in [17, Section 6] the question of studying the spherical case with $|S \backslash I| > 1$ is raised. It is shown in [14, Lemma III.3.5] that if W is of type A_n and $|S \backslash I| > 1$ then $\mathscr{H}^I$ is noncommutative. The main result in [2] extends this to show that if W is spherical and $|S \backslash I| > 1$ then $\mathscr{H}^I$ is noncommutative. We give a very short proof of this fact across *all* Coxeter types in Section 3 (it appears to have been previously known only for the spherical types via a case by case argument involving computer calculations for the exceptional types).

Now suppose that W is *affine* (see Section 1.1). If $I = S \backslash \{i\}$ with i a *special vertex* then it is well known that $\mathscr{H}^I$ is commutative. This result is important in the representation theory of semisimple Lie groups defined over local fields such as the p-adics (see [19,20]). The question of whether commutative parabolic Hecke algebras exist in the affine case with i not a special vertex is natural, yet to our knowledge has not been treated in the literature. It follows from our classification that there are in fact no such commutative parabolic Hecke algebras.

Now consider the case that W is non-affine and infinite. In [16, Theorem 3.5] it is shown that maximal parabolic Hecke algebras arising from group actions on locally finite thick buildings of type W are noncommutative. (However there is a mistake in the proof which needs to be fixed. Lécureux's Lemma 3.4 only holds for simple reflections, but is used for general reflections in the proof of his Theorem 3.5.) Such buildings can only exist if $m_{st} \in \{2, 3, 4, 6, 8, \infty\}$ for each $s, t \in S$ because the Feit–Higman Theorem restricts the possible rank 2 residues. If W is *crystallographic* (that is, $m_{st} \in \{2, 3, 4, 6, \infty\}$, cf. [15, p. 25]) then existence of such a building is guaranteed via Kac–Moody theory.

In summary, it appears that the following cases are not treated in the literature: (i) $|S \backslash I| > 1$ (for general Coxeter types), (ii) the affine case with $I = S \backslash \{i\}$ and i non-special, and (iii) the non-crystallographic non-affine infinite cases. It also appears that the existing techniques do not readily generalise to treat these cases. In this paper we give a systematic and complete classification of commutative parabolic Hecke algebras. Our proof uses a uniform technique to cover all cases (including the known cases). As a consequence it turns out that the three cases listed above give noncommutative parabolic Hecke algebras.

Let us briefly outline the structure of this paper. Section 1 gives standard definitions and background on Coxeter groups and Hecke algebras, and in Section 2 we state our classification theorem (Theorem 2.1). We also develop some elementary tests for commutativity and noncommutativity that will be used in Section 3, where we give the proof of the classification theorem. The proof has two parts. First we prove that those cases listed in Theorem 2.1 give rise to commutative parabolic Hecke algebras. This is achieved using Lemma 2.5, which is inspired by the statement of [8, Theorem 3.1]. Next we show that all remaining cases are noncommutative. This involves some Coxeter graph combinatorics to reduce the analysis to a finite number of cases. In each of these cases a word in the Coxeter group is exhibited, which when fed into our noncommutativity test (Proposition 2.8) proves that the parabolic Hecke algebra is noncommutative. We note that in order to apply our word arguments and diagram combinatorics to the general infinite cases, it is in fact necessary to give our elementary proof of the known noncommutative spherical cases. In Appendix A we make some comments on the structure of double cosets, and list the words we used to deduce noncommutativity.

1. Definitions

This section recalls some standard definitions and results on Coxeter groups, Hecke algebras, and specialisations of Hecke algebras. Standard references include [1,4,12,18].

1.1. Coxeter groups

A *Coxeter system* (W, S) is a group W generated by a set S with relations

$$(st)^{m_{st}} = 1 \quad \text{for all } s, t \in S,$$

where $m_{ss} = 1$ and $m_{st} \in \mathbb{Z}_{\geqslant 2} \cup \{\infty\}$ for all $s \neq t$. If $m_{st} = \infty$ then it is understood that there is no relation between s and t. We will always assume that $|S|$ is finite. The *Coxeter matrix* of (W, S) is $M = (m_{st})$. Let $M' = (c_{st})$ be the matrix with $c_{st} = -\cos(\pi/m_{st})$.

The *length* $\ell(w)$ of $w \in W$ is

$$\ell(w) = \min\{n \in \mathbb{N} \mid w = s_1 \cdots s_n \text{ with } s_1, \ldots, s_n \in S\}.$$

An expression $w = s_1 \cdots s_n$ with $n = \ell(w)$ is called a *reduced expression* for w.

The *Coxeter graph* (or *Coxeter diagram*) of (W, S) is the graph with vertex set S and with $s, t \in S$ joined by an edge if and only if $m_{st} \geqslant 3$. If $m_{st} \geqslant 4$ then the corresponding edge is labelled by m_{st}. A Coxeter system (W, S) is *irreducible* if its Coxeter graph is connected.

Finite Coxeter groups are called *spherical Coxeter groups*. These are precisely the Coxeter groups whose matrix M' is positive definite. The irreducible spherical Coxeter groups are classified (see [7,4,12]).

Coxeter groups which are not finite but contain a normal abelian subgroup such that the corresponding quotient group is finite are called *affine Coxeter groups*. These are precisely the Coxeter groups whose matrix M' is positive semidefinite but not positive definite. The irreducible affine Coxeter groups are classified (see [4,12]). In each case the Coxeter graph of an irreducible affine Coxeter group is obtained from the Coxeter matrix of an irreducible spherical Coxeter graph by adding one extra vertex (usually labelled 0). The vertices of the affine Coxeter graph which are in the orbit of 0 under the action of the group of diagram automorphisms are called the *special vertices*.

When it is necessary to fix a labelling of the generators of a spherical or affine Coxeter group we will adopt the conventions from [4]. The *Bruhat partial order* $\leqslant$ on a Coxeter system (W, S) can be described as follows. If $v, w \in W$ then $v \leqslant w$ if and only if there is a reduced expression $w = s_1 \cdots s_n$ such that v is equal to a *subexpression* of $s_1 \cdots s_n$ (that is, an expression obtained by deleting factors). If $v \leqslant w$ then v is equal to a subexpression of *every* reduced expression of w. The *deletion condition* says that if $w = s_1 \cdots s_n$ with $n > \ell(w)$ then there exist indices $i < j$ such that $w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_n$, where $\hat{s}$ indicates that the factor s is omitted. The deletion condition holds for Coxeter groups (in fact it characterises them).

For $I \subseteq S$ let W_I be the subgroup of W generated by I. Each double coset $W_I w W_I$ has a unique minimal length representative [1, Proposition 2.23]. This representative is called *I-reduced*, and we let

$$R_I = \{w \in W \mid w \text{ is } I\text{-reduced}\}.$$

Thus R_I indexes the decomposition of W into $W_I w W_I$ double cosets. It is useful to note that a reduced expression for $w \in R_I$ cannot start or end with a letter in I. In particular, if $S \backslash I = \{s\}$ then every reduced expression for $w \in R_I$ must start and end with s.

A subset $I \subseteq S$ is *spherical* if the group W_I is finite. Coxeter systems (W, S) such that there exists a spherical subset $I = S \backslash \{i\}$ are called *nearly finite Coxeter groups* in [10]. This class includes the spherical and irreducible affine groups, but also many more Coxeter groups.

1.2. Hecke algebras

Let (W, S) be a Coxeter system, and let q_s, $s \in S$, be commuting indeterminants such that $q_s = q_t$ if and only if s and t are conjugate in W. Let $\mathcal{R} = \mathbb{Z}[q_s]_{s\in S}$ be the polynomial ring in q_s, $s \in S$, with integer coefficients. The condition on the parameters implies that the expression $q_w = q_{s_1} \cdots q_{s_\ell} \in \mathcal{R}$ does not depend on the particular choice of reduced expression $w = s_1 \cdots s_\ell$.

The Hecke algebra $\mathscr{H} = \mathscr{H}(W, S)$ is the associative $\mathcal{R}$-algebra with free basis $\{T_w \mid w \in W\}$ (as an $\mathcal{R}$-module) and multiplication laws

$$T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) = \ell(w) + 1, \\ q_s T_{ws} + (q_s - 1) T_w & \text{if } \ell(ws) = \ell(w) - 1. \end{cases} \tag{1.1}$$

If I is a spherical subset of S then the element

$$\mathbf{1}_I = \sum_{w\in W_I} T_w$$

is in $\mathscr{H}$ (since the sum is finite). This element has the following attractive properties, where for finite subsets $X \subseteq W$ the *Poincaré polynomial* of X is $X(q) = \sum_{w\in X} q_w$.

Lemma 1.1. *The element $\mathbf{1}_I$ satisfies $T_w \mathbf{1}_I = \mathbf{1}_I T_w = q_w \mathbf{1}_I$ for all $w \in W_I$, and $\mathbf{1}_I^2 = W_I(q)\mathbf{1}_I$.*

Proof. By induction it suffices to show that $T_s \mathbf{1}_I = \mathbf{1}_I T_s = q_s \mathbf{1}_I$ for each $s \in I$. We have

$$\mathbf{1}_I T_s = \sum_{w\in W_I} T_w T_s.$$

Split the sum into two parts, over the sets $W_I^{\pm} = \{w \in W_I \mid \ell(ws) = \ell(w) \pm 1\}$. Using the defining relations (1.1) and the fact that $W_I^{+} s = W_I^{-}$ shows that $\mathbf{1}_I T_s = q_s \mathbf{1}_I$. The $T_s \mathbf{1}_I$ case is similar, using the formula $T_s T_w = q_s T_{sw} + (q_s - 1) T_w$ if $\ell(sw) = \ell(w) - 1$ (which follows from (1.1)). The fact that $\mathbf{1}_I^2 = W_I(q)\mathbf{1}_I$ follows immediately. □

The *structure constants* $c_{u,v;w} \in \mathbb{Z}[q_s]_{s\in S}$ of $\mathscr{H}$ relative to the basis $\{T_w \mid w \in W\}$ are defined by the equations

$$T_u T_v = \sum_{w\in W} c_{u,v;w} T_w \quad \text{for all } u, v \in W. \tag{1.2}$$

Lemma 1.2. *The structure constants $c_{u,v;w}$ are polynomials in $\{q_s - 1 \mid s \in S\}$ with nonnegative integer coefficients.*

Proof. Induction on $\ell(v)$, with $\ell(v) = 0$ trivial. If $\ell(vs) = \ell(v) + 1$ then $T_u T_{vs} = (T_u T_v) T_s$. Expanding the left-hand side of this equation using (1.2) and the right-hand side using (1.2) and (1.1) gives

$$c_{u,vs;w} = \begin{cases} c_{u,v;ws} q_s & \text{if } \ell(ws) = \ell(w) + 1, \\ c_{u,v;ws} + c_{u,v;w}(q_s - 1) & \text{if } \ell(ws) = \ell(w) - 1. \end{cases}$$

By the induction hypothesis $c_{u,v;w}$ and $c_{u,v;ws}$ are polynomials in $\{q_s - 1 \mid s \in S\}$ with nonnegative integer coefficients, and so $c_{u,vs;w}$ is too (since $q_s = 1 + (q_s - 1)$). □

1.3. Parabolic Hecke algebras

Let $\mathscr{H}$ be the Hecke algebra with Coxeter system (W,S) and let $I \subseteq S$ be spherical. The *I-parabolic Hecke algebra* is

$$\mathscr{H}^I = \mathbf{1}_I \mathscr{H} \mathbf{1}_I.$$

We note that in general $\mathscr{H}^I$ is not unital (as $W_I(q)$ is not an invertible element of $\mathbb{Z}[q_s]_{s\in S}$).

Let I be spherical and let $w \in R_I$ be I-reduced. We define

$$T_w^I = \frac{W_I(q)}{W_{I\cap wIw^{-1}}(q)} \mathbf{1}_I T_w \mathbf{1}_I.$$

The Poincaré polynomial $W_I(q)$ is divisible by $W_{I\cap wIw^{-1}}(q)$ (this follows from Eq. (1.3) below and statement (a) immediately following (1.3)), and so the quotient is really an element of the coefficient ring $\mathcal{R} = \mathbb{Z}[q_s]_{s\in S}$.

The set $\{T_w^I \mid w \in R_I\}$ is a linear basis for $\mathscr{H}^I$ (Proposition 1.3). Let $c^I_{u,v;w}$, $u,v,w \in R_I$, be the structure constants of $\mathscr{H}^I$ relative to this basis, defined by the equations

$$T_u^I T_v^I = \sum_{w\in R_I} c^I_{u,v;w} T_w^I \quad \text{for } u,v \in R_I.$$

If $I = \emptyset$ then $\mathbf{1}_I = 1$ (the identity in $\mathscr{H}$), and so $T_w^I = T_w$ and $\mathscr{H}^I = \mathscr{H}$. Thus $c^{\emptyset}_{u,v;w} = c_{u,v;w}$ are the structure constants appearing in (1.2). Part (ii) of the following proposition relates the structure constants $c^I_{u,v;w}$ to the more elementary structure constants $c_{u,v;w}$.

Proposition 1.3. *Let $I \subseteq S$ be spherical.*

(i) *For $w \in R_I$ we have*

$$T_w^I = W_I(q) \sum_{z\in W_I w W_I} T_z,$$

and $\{T_w^I \mid w \in R_I\}$ is a linear basis for $\mathscr{H}^I$.

(ii) *Let $u,v,w \in R_I$. For any $z \in W_I w W_I$ we have*

$$c^I_{u,v;w} = W_I(q) \sum_{\substack{x\in W_I u W_I \\ y \in W_I v W_I}} c_{x,y;z}.$$

Proof. Let $W_{I,w}$ be the subgroup of W_I stabilising wW_I under left multiplication, and let $M_{I,w}$ be a fixed set of minimal length representatives of cosets in $W_I/W_{I,w}$. Notice that $s \in S \cap W_{I,w}$ if and only if $s \in W_I$ and $s \in wW_I w^{-1}$, and hence (see [1, Lemma 2.25])

$$W_{I,w} = W_I \cap wW_I w^{-1} = W_{I\cap wIw^{-1}}. \tag{1.3}$$

If $w \in R_I$ then (see [1, §2.3.2]):

(a) Each $u \in W_I$ can be written in exactly one way as $u = xy$ with $x \in M_{I,w}$ and $y \in W_{I,w}$. Moreover $\ell(u) = \ell(x) + \ell(y)$ for any such expression.

(b) Each $v \in W_I w W_I$ can be written in exactly one way as $v = xwy$ with $x \in M_{I,w}$ and $y \in W_I$. Moreover $\ell(v) = \ell(x) + \ell(w) + \ell(y)$ for any such expression.

Using (a) we have

$$\mathbf{1}_I T_w \mathbf{1}_I = \sum_{u \in W_I} T_u T_w \mathbf{1}_I = \sum_{x \in M_{I,w}} \sum_{y \in W_{I,w}} T_x T_y T_w \mathbf{1}_I.$$

Since w is I-reduced we have $\ell(yw) = \ell(y) + \ell(w)$ for each $y \in W_{I,w}$, and $yw = wy'$ for some $y' \in W_I$ with $\ell(wy') = \ell(w) + \ell(y')$. This implies that $q_{y'} = q_y$, and (1.1) and Lemma 1.1 give

$$T_y T_w \mathbf{1}_I = T_{yw} \mathbf{1}_I = T_{wy'} \mathbf{1}_I = T_w T_{y'} \mathbf{1}_I = q_y T_w \mathbf{1}_I.$$

Thus by (1.3) we have $\sum_{y \in W_{I,w}} T_x T_y T_w \mathbf{1}_I = W_{I \cap wIw^{-1}}(q) T_x T_w \mathbf{1}_I$, and hence by (b) we compute

$$T_w^I = W_I(q) \sum_{x \in M_{I,w}} T_x T_w \mathbf{1}_I = W_I(q) \sum_{x \in M_{I,w}} \sum_{y \in W_I} T_x T_w T_y = W_I(q) \sum_{z \in W_I w W_I} T_z.$$

This formula shows that $\{T_w^I \mid w \in R_I\}$ is a linearly independent set (since double cosets are either equal or disjoint, and $\{T_w \mid w \in W\}$ is a basis for $\mathscr{H}$). It also spans $\mathscr{H}^I$, for if $z \in W$ then $z \in W_I w W_I$ for some $w \in R_I$, and since w is I-reduced we have $z = xwy$ with $x \in W_I$, $y \in W_I$, and $\ell(z) = \ell(x) + \ell(w) + \ell(y)$. Then using (1.1) and Lemma 1.1 we have $\mathbf{1}_I T_z \mathbf{1}_I = \mathbf{1}_I T_x T_w T_y \mathbf{1}_I = q_x q_y \mathbf{1}_I T_w \mathbf{1}_I$. This completes the proof of (i).

To prove (ii) we use (i) and the expansion $T_x T_y = \sum_z c_{x,y;z} T_z$ to write

$$T_u^I T_v^I = W_I(q)^2 \sum_{\substack{x \in W_I u W_I \\ y \in W_I v W_I}} T_x T_y = W_I(q)^2 \sum_{z \in W} \Bigg(\sum_{\substack{x \in W_I u W_I \\ y \in W_I v W_I}} c_{x,y;z} \Bigg) T_z.$$

On the other hand we have

$$T_u^I T_v^I = \sum_{w \in R_I} c_{u,v;w}^I T_w^I = W_I(q) \sum_{w \in R_I} \Bigg(c_{u,v;w}^I \sum_{z \in W_I w W_I} T_z \Bigg).$$

The result follows by comparing coefficients of T_z in these expressions. □

Remark 1.4. The structure constants $c_{u,v;w}^I$ in the spherical case are studied in [5] and [11]. In the affine case formulae are available using *positively folded alcove walks* (see [22]).

1.4. Specialisations of the Hecke algebra

One is often interested in *specialisations* of the Hecke algebra, where the parameters q_s, $s \in S$, are chosen to be specific complex numbers. Let us briefly describe this construction. Let $\boldsymbol{\tau} = (\tau_s)_{s \in S}$ be a sequence of complex numbers with $\tau_s = \tau_t$ whenever s and t are conjugate in W. Let $\psi : \mathcal{R} \to \mathbb{C}$ be the ring homomorphism given by $\psi(q_s) = \tau_s$ for each $s \in S$. Then $\mathbb{C}$ becomes a $(\mathbb{C}, \mathcal{R})$-bimodule via $(\lambda, \mu, x) \mapsto \lambda\mu\psi(x)$ for all $\lambda, \mu \in \mathbb{C}$ and $x \in \mathcal{R}$. The *specialised Hecke algebra* is $\mathscr{H}_{\boldsymbol{\tau}} = \mathbb{C} \otimes_{\mathcal{R}} \mathscr{H}$. This is an algebra over $\mathbb{C}$ with basis $\{1 \otimes T_w \mid w \in W\}$. Note that the specialisation of $\mathscr{H}$ with $\tau_s = 1$ for all $s \in S$ is equal to the group algebra of W.

Let $\mathscr{H}_{\boldsymbol{\tau}}^I$ be the specialisation of $\mathscr{H}^I$ with parameters $\boldsymbol{\tau} = (\tau_s)$. Our classification of commutative parabolic Hecke algebras applies to the 'generic' parabolic Hecke algebras $\mathscr{H}^I$ (defined over

$\mathbb{Z}[q_s]_{s\in S}$) *and* to the specialisations $\mathscr{H}_{\boldsymbol{\tau}}^I$ with $\tau_s \in \mathbb{R}$ and $\tau_s \geqslant 1$ for all $s \in S$. Potential problems arise for other values of τ_s, since our argument in Corollary 2.7, which relies on Corollary 1.5 below, breaks down.

The structure constants of the specialised algebra $\mathscr{H}_{\boldsymbol{\tau}}^I$ are obtained by applying the evaluation homomorphism $\psi : \mathbb{Z}[q_s]_{s\in S} \to \mathbb{C}$ with $\psi(q_s) = \tau_s$ to the structure constants of the generic algebra $\mathscr{H}^I$.

Corollary 1.5. *If $\tau_s \geqslant 1$ for all $s \in S$ then $\psi(c^I_{u,v;w}) \geqslant 0$, and if the constant term of $c_{u,v;w}$ when written as a polynomial in the variables $q_s - 1$ is nonzero then $\psi(c^I_{u,v;w}) > 0$.*

Proof. By Lemma 1.2 the claim is true for $I = \emptyset$ (where $c^I_{u,v;w} = c_{u,v;w}$), and by Proposition 1.3 we see that the claim holds for general (spherical) I, since $W_I(\boldsymbol{\tau}) > 0$ if $\tau_s \geqslant 1$ for all $s \in S$. $\square$

Remark 1.6. If $\tau_s = p^n$ for all $s \in S$ with p a prime then $\mathscr{H}_{\boldsymbol{\tau}} \cong \mathcal{C}_c(B\backslash G/B)$. Here G is a Kac–Moody group of type W over the finite field $\mathbb{F}_{p^n}$ (see [23]), B is the standard Borel subgroup of G, and $\mathcal{C}_c(B\backslash G/B)$ is the convolution algebra of B bi-invariant functions $f : G \to \mathbb{C}$ supported on finitely many B double cosets. For such a Kac–Moody group to exist it is necessary and sufficient that $m_{st} \in \{2, 3, 4, 6, \infty\}$ for each $s, t \in S$ (see [15, Proposition 1.3.21]). Similarly $\mathscr{H}_{\boldsymbol{\tau}}^I \cong \mathcal{C}_c(P_I\backslash G/P_I)$ where P_I is the standard I-parabolic subgroup $P_I = \bigsqcup_{w\in W_I} BwB$.

Remark 1.7. Suppose that $\tau_s = \tau$ for all $s \in S$. If W is spherical then $\mathscr{H}_{\boldsymbol{\tau}}$ is isomorphic to the group algebra of W for all values of $\tau \in \mathbb{C}^\times$ except for roots of the Poincaré polynomial $W(\tau)$ [9, §68A]. This statement is usually *not* true for infinite Coxeter groups W (see [24, §11.7]).

2. Commutativity of $\mathscr{H}^I$

2.1. Statement of results

The following classification theorem is the main result of this paper. The proof is given in the next section after giving some preliminary observations in this section. We use Bourbaki [4] conventions for the labelling of the nodes of spherical and affine Coxeter systems. In the H_3 and H_4 cases (where there is no explicit labelling given in [4]) we take $m_{12} = 3$ and $m_{23} = 5$ in the H_3 case, and $m_{12} = m_{23} = 3$ and $m_{34} = 5$ in the H_4 case.

If X_n is a spherical Coxeter diagram and if i is a vertex of X_n then we write $X_{n,i}$ to denote the case where (W, S) has type X_n and $I = S\backslash\{i\}$. Similarly if $\tilde{X}_n$ is an affine diagram then the notation $\tilde{X}_{n,i}$ means that (W, S) has type $\tilde{X}_n$ and $I = S\backslash\{i\}$.

Theorem 2.1. *Let (W, S) be irreducible, let $I \subseteq S$ be spherical, and let $\boldsymbol{\tau} = (\tau_s)$ with $\tau_s \geqslant 1$ for each $s \in S$. The I-parabolic Hecke algebras $\mathscr{H}^I$ and $\mathscr{H}_{\boldsymbol{\tau}}^I$ are noncommutative if $|S\backslash I| > 1$. If $I = S\backslash\{i\}$ then $\mathscr{H}^I$ and $\mathscr{H}_{\boldsymbol{\tau}}^I$ are commutative in the cases*

- *$A_{n,i}$ $(1 \leqslant i \leqslant n)$, $B_{n,i}$ $(1 \leqslant i \leqslant n)$, $D_{n,i}$ $(1 \leqslant i \leqslant n/2$ or $i = n-1, n)$, $E_{6,1}$, $E_{6,2}$, $E_{6,6}$, $E_{7,1}$, $E_{7,2}$, $E_{7,7}$, $E_{8,1}$, $E_{8,8}$, $F_{4,1}$, $F_{4,4}$, $H_{3,1}$, $H_{3,3}$, $H_{4,1}$, $I_2(p)_i$ $(i = 1, 2)$, and*
- *all affine cases $\tilde{X}_{n,i}$ with i a special type,*

and noncommutative otherwise.

As a consequence of this classification it turns out that we have the following uniform statement which has the same flavour as [8, Theorem 3.1]. The proof of Theorem 2.2 is given at the end of Section 3.

Theorem 2.2. *With the hypothesis of Theorem 2.1, the algebras $\mathscr{H}^I$ and $\mathscr{H}_{\boldsymbol{\tau}}^I$ are commutative if and only if there is an automorphism π of the Coxeter diagram such that*

(a) $\pi(I) = I$,
(b) $\pi(w) = w^{-1}$ *for all* $w \in R_I$, *and*
(c) $q_{\pi(s)} = q_s$ *for all* $s \in S$.

Remark 2.3. Suppose that the Coxeter system (W, S) is not irreducible. Let $S = S_1 \cup \cdots \cup S_n$ be the decomposition of the nodes of the Coxeter graph into connected components, and let $W_j = \langle S_j \rangle$ for each $j = 1, \ldots, n$. It is elementary that

$$\mathscr{H}(W, S) \cong \mathscr{H}(W_1, S_1) \oplus \cdots \oplus \mathscr{H}(W_n, S_n).$$

Let $I \subseteq S$ be spherical, and let $I_j = I \cap S_j$. Then $\mathbf{1}_I = \mathbf{1}_{I_1} \cdots \mathbf{1}_{I_n}$, and it follows that

$$\mathscr{H}^I(W, S) \cong \mathscr{H}^{I_1}(W_1, S_1) \oplus \cdots \oplus \mathscr{H}^{I_n}(W_n, S_n).$$

Thus $\mathscr{H}^I(W, S)$ is commutative if and only if each $\mathscr{H}^{I_j}(W_j, S_j)$ is commutative. Thus we will henceforth assume that the (W, S) is irreducible.

Remark 2.4. In the spherical case (except for H_3 and H_4) commutativity of $X_{n,i}$ is dealt with in [8, Theorem 3.1] (see also [6, Theorem 10.4.11]). We give a different elementary proof here. In fact our proof technique for the general case makes it crucial for us to give our proof of the spherical case.

2.2. Initial observations

By induction on $\ell(y)$ we see that $c_{x,y;z} = c_{y^{-1},x^{-1};z^{-1}}$, and so by Proposition 1.3 we see that

$$c^I_{v^{-1},u^{-1};w^{-1}} = W_I(q) \sum_{\substack{x \in W_I v W_I \\ y \in W_I u W_I}} c_{x^{-1},y^{-1};z^{-1}} = W_I(q) \sum_{\substack{y \in W_I u W_I \\ x \in W_I v W_I}} c_{y,x;z} = c^I_{u,v;w}, \tag{2.1}$$

where z is any element of the double coset $W_I w^{-1} W_I$. Thus if each $w \in R_I$ is an involution then $c^I_{u,v;w} = c^I_{v,u;w}$, and so the algebra $\mathscr{H}^I$ is commutative. It turns out that in the spherical case this is an equivalence: $\mathscr{H}^I$ is commutative if and only if each element of R_I is an involution (see [8, Theorem 3.1] and Claim 1 in Section 3 below). However it is not an equivalence in arbitrary type (as the affine cases with special vertices show).

The following lemma is modelled on [21, Theorem 5.21 and Theorem 5.24].

Lemma 2.5. *Suppose that there is an automorphism π of the Coxeter graph satisfying conditions* (a), (b) *and* (c) *of Theorem* 2.2. *Then the algebras $\mathscr{H}^I$ and $\mathscr{H}^I_{\boldsymbol{\tau}}$ (for any specialisation $\tau_s \in \mathbb{C}$) are commutative.*

Proof. We claim that the property $q_{\pi(s)} = q_s$ implies that

$$c_{x,y;z} = c_{\pi(x),\pi(y);\pi(z)} \quad \text{for all } x, y, z \in W. \tag{2.2}$$

We argue by induction on $\ell(y)$, with $\ell(y) = 0$ trivial. If $\ell(sy) > \ell(y)$, then expanding $T_x T_{sy} = (T_x T_s) T_y$ in two ways using (1.1) gives

$$c_{x,sy;z} = \begin{cases} c_{xs,y;z} & \text{if } \ell(xs) > \ell(x), \\ q_s c_{xs,y;z} + (q_s - 1) c_{x,y;z} & \text{if } \ell(xs) < \ell(x). \end{cases}$$

By the induction hypothesis and property (c) we have $c_{x,sy;z} = c_{\pi(x),\pi(sy);\pi(z)}$, hence (2.2).

By properties (a) and (b) if w is I-reduced then $\pi(W_I w W_I) = W_I w^{-1} W_I = (W_I w W_I)^{-1}$. Using this observation, by Proposition 1.3 and (2.2) we have $c^I_{\pi(u),\pi(v);\pi(w)} = c^I_{u,v;w}$.

On the other hand, by (b) and (2.1) we have $c^I_{\pi(u),\pi(v);\pi(w)} = c^I_{u^{-1},v^{-1};w^{-1}} = c^I_{v,u;w}$. Thus $c^I_{u,v;w} = c^I_{v,u;w}$. So $\mathscr{H}^I$ is commutative, and hence $\mathscr{H}^I_{\boldsymbol{\tau}}$ is commutative for each specialisation. □

Lemma 2.6. *Let $u, v, w \in R_I$. If $c^I_{u,v;w} \neq 0$ then there exist $u' \leqslant u$, $v' \leqslant v$, and $y \in W_I$ such that $w = u'yv'$ and $\ell(w) = \ell(u') + \ell(y) + \ell(v')$.*

Proof. Recall that T^I_u is a scalar times $\mathbf{1}_I T_u \mathbf{1}_I$. Thus $T^I_u T^I_v = \sum c^I_{u,v;w} T^I_w$ is a scalar times

$$\mathbf{1}_I T_u \mathbf{1}_I \cdot \mathbf{1}_I T_v \mathbf{1}_I = W_I(q) \mathbf{1}_I T_u \mathbf{1}_I T_v \mathbf{1}_I = W_I(q) \sum_{z \in W_I} \mathbf{1}_I T_u T_z T_v \mathbf{1}_I. \tag{2.3}$$

Since $v \in R_I$ we have $T_z T_v = T_{zv}$ for each $z \in W_I$. An induction on $\ell(u)$ using (1.1) shows that $T_u T_{zv}$ is a linear combination of terms $T_{u'zv}$ with $u' \leqslant u$. Therefore the right-hand side of (2.3) is a linear combination of terms $\{\mathbf{1}_I T_x \mathbf{1}_I \mid x \in u' W_I v,\ u' \leqslant u\}$. It follows from Lemma 1.1 that for each $x \in W$, $\mathbf{1}_I T_x \mathbf{1}_I$ is a nonzero scalar multiple of $\mathbf{1}_I T_{x'} \mathbf{1}_I$, where x' is the unique I-reduced element of $W_I x W_I$ (see the proof of Proposition 1.3). Therefore the right-hand side of (2.3) is a linear combination of terms $\mathbf{1}_I T_{x'} \mathbf{1}_I$ with x' being the I-reduced element of a double coset of the form $W_I u' W_I v W_I$ with $u' \leqslant u$.

Hence if $c^I_{u,v;w} \neq 0$ then $w \in W_I u' W_I v W_I$ for some $u' \leqslant u$, and so $w = w_1 u' w_2 v w_3$ with $w_1, w_2, w_3 \in W_I$. By repeated applications of the deletion condition we obtain a reduced word $w = w'_1 u'' w'_2 v' w'_3$ with $w'_1, w'_2, w'_3 \in W_I$ and $u'' \leqslant u$ and $v' \leqslant v$. But every reduced expression for an I-reduced word starts and ends with elements from $S \backslash I$. Thus $w'_1 = w'_3 = 1$, and so $w = u'' w'_2 v'$ with $\ell(w) = \ell(u'') + \ell(w'_2) + \ell(v')$, completing the proof. □

Thus we obtain the following general test for noncommutativity.

Corollary 2.7. *Let $u, v, w \in R_I$. Suppose that $w = uzv$ with $\ell(w) = \ell(u) + \ell(z) + \ell(v)$ and $z \in W_I$. If there do not exist u', v', z' with $u' \leqslant u$, $v' \leqslant v$, and $z' \in W_I$ such that $w = v'z'u'$ and $\ell(w) = \ell(v') + \ell(z') + \ell(u')$, then $\mathscr{H}^I$ and $\mathscr{H}^I_{\boldsymbol{\tau}}$ (with $\tau_s \geqslant 1$) are noncommutative.*

Proof. Let $\psi : \mathbb{Z}[q_s]_{s \in S} \to \mathbb{C}$ be the evaluation homomorphism with $\psi(q_s) = \tau_s \geqslant 1$ for each $s \in S$. We claim that if $w = uzv$ with $z \in W_I$ and $\ell(w) = \ell(u) + \ell(z) + \ell(v)$ then $c^I_{u,v;w} \neq 0$ and $\psi(c^I_{u,v;w}) > 0$. To see this, note that by Proposition 1.3 and the defining relations (1.1) we have

$$\begin{aligned} c^I_{u,v;uzv} &= W_I(q)(c_{uz,v;uzv} + \text{positive linear combination of other } c_{x,x';x''} \text{ terms}) \\ &= W_I(q)(1 + \text{positive linear combination of other } c_{x,x';x''} \text{ terms}), \end{aligned}$$

from which the result follows (see Lemma 1.2 and Corollary 1.5).

On the other hand, by Lemma 2.6 and the assumptions of the corollary we have $c^I_{v,u;w} = 0$ (and hence $\psi(c^I_{u,v;w}) = 0$ too), and so the algebras $\mathscr{H}^I$ and $\mathscr{H}^I_{\boldsymbol{\tau}}$ are noncommutative. □

The following more specific test for noncommutativity will be used frequently.

Proposition 2.8. *Let $I = S \backslash \{i\}$. Suppose that there is an element $w \in R_I$ such that $w = u w_I i$ with $u \in R_I$, $w_I \in W_I$, and $\ell(w) = \ell(u) + \ell(w_I) + 1$. Fix reduced expressions for u and w_I, and suppose that:*

(1) *the induced decomposition $w = u w_I i$ has the minimal number of i factors amongst all possible reduced expressions for w, and*

(2) *there is a generator $k \in I$ that appears in w_I but not in u, and that in every reduced expression for w with the minimal number of i factors no occurrence of this k generator appears between the first two i generators of the expression.*

Then $\mathscr{H}^I$ and $\mathscr{H}^I_{\boldsymbol{\tau}}$ (with $\tau_s \geqslant 1$) are noncommutative.

Proof. By Corollary 2.7 it is sufficient to show that w cannot be written as $w = i'z'u'$ with $i' \in \{\mathrm{id}, i\}$, $u' \leqslant u$, $z' \in W_I$, and $\ell(w) = \ell(i') + \ell(z') + \ell(u')$. Suppose we have such an expression. By (1) we see that $i' = i$, and that u' has the same number of i factors as u does. In particular, u' starts and ends with an i. Since u' contains no k factors we see that z' must contain some k factors. Then these factors are between the first two i generators, contradicting (2). □

3. Proof of Theorem 2.1

We use the following notation. If X_n is a spherical Coxeter type with nodes $1, 2, \dots, n$ then X_n^i is the Coxeter graph obtained by attaching a new node (labelled 0) to the i node of X_n by a single bond. Similarly, X_n^{ij} with $i \neq j$ indicates that this new node is connected to i and j by single bonds, and X_n^{ii} indicates that 0 is joined to i by a double bond. This notation naturally extends, and, for example, $F_4^{1,1} \times E_7^{2,5,6}$ indicates that a new node 0 is connected to the 1 node of an F_4 diagram by a double bond, and to the 2, 5 and 6 nodes of an E_7 diagram by single bonds. Also, recall the notation $X_{n,i}$ and $\tilde{X}_{n,i}$ from the beginning of Section 2.1.

Recall that we assume throughout that (W, S) is irreducible. The proof of Theorem 2.1 is achieved via the following 6 claims. The first claim shows that if $|S \backslash I| > 1$ then $\mathscr{H}^I$ is noncommutative, allowing us to focus on the maximal parabolic case $I = S \backslash \{i\}$. The second and third claims deal with the commutative spherical and affine cases. In Claim 4 we produce a list of noncommutative cases. This library of noncommutative cases is used in Claims 5 and 6 to show that all cases other than those listed in Theorem 2.1 are noncommutative.

Claim 1. *If $|S \backslash I| > 1$ then $\mathscr{H}^I$ and $\mathscr{H}^I_{\boldsymbol{\tau}}$ (with $\tau_s \geqslant 1$) are noncommutative.*

Proof. Choose vertices $s, t \in S \backslash I$ with $s \neq t$ at minimal length in the (connected) Coxeter graph of W. Then $s, t \in R_I$, and if $s, s_1, \dots, s_n, t$ is a minimal length path in the Coxeter diagram then $s_1, \dots, s_n \in W_I$. The I-reduced element $w = ss_1 \cdots s_n t$ satisfies $\ell(w) = \ell(s) + \ell(s_1 \cdots s_n) + \ell(t)$. But w cannot be written as $w = t'z's'$ with $t' \leqslant t$, $s' \leqslant s$, $z' \in W_I$, and $\ell(w) = \ell(t') + \ell(z') + \ell(s')$, for there is exactly one reduced expression for w, and this reduced expression has one s, and one t, and the t is to the right of the s. Thus by Corollary 2.7 the algebra $\mathscr{H}^I$ (and its specialisations with $\tau_s \geqslant 1$) is noncommutative. (Compare with [2].) □

Claim 2. *The spherical cases listed in Theorem 2.1 are commutative.*

Proof. It is well known that in each case listed the minimal length double coset representatives are involutions (see Proposition A.1 for the $E_{8,1}$ example). Thus Lemma 2.5 applies (with π being trivial), and so the algebras are commutative. □

Claim 3. *If $I = S \backslash \{i\}$ with i a special node of an affine diagram then $\mathscr{H}^I$ is commutative.*

Proof. Let (W, S) be an irreducible Coxeter system of affine type, and let $I = S \backslash \{i\}$, where i is a special type. Then $\mathscr{H}^I$ (and hence $\mathscr{H}_{\boldsymbol{\tau}}$ for all specialisations) is commutative by Lemma 2.5 with the diagram automorphism π from that lemma being opposition in the spherical residue. In more detail: We may assume that $i = 0$. Let Q be the coroot lattice of the associated root system, and let P be the coweight lattice, with dominant cone P^+. Let $W_0 = W_{S \backslash \{0\}}$. Then $W \cong Q \rtimes W_0$, and $\{t_\lambda \mid \lambda \in Q \cap P^+\}$ is a set of $W_0 \backslash W / W_0$ representatives, where t_λ is the translation by λ. So the double cosets satisfy

$(W_0 t_\lambda W_0)^{-1} = W_0 t_\lambda^{-1} W_0 = W_0 t_{-\lambda} W_0 = W_0 t_{\lambda^*} W_0$, where $\lambda^* = -w_0\lambda$, with w_0 being the longest element of W_0. It follows that the minimal length element m_λ of $W_0 t_\lambda W_0$ satisfies $m_\lambda^{-1} = m_{\lambda^*}$. Hence the automorphism π of the Coxeter diagram given by $\pi(0) = 0$ and $\alpha_{\pi(j)} = -w_0\alpha_j$ for $j = 1, \ldots, n$ satisfies $\pi(m_\lambda) = m_\lambda^{-1}$ for all $\lambda \in Q \cap P^+$. By construction we have $\pi(I) = I$, and considering the connected affine diagrams we have $q_{\pi(s)} = q_s$ for all $s \in S$. Thus by Lemma 2.5 $\mathscr{H}^I$ is commutative (and hence $\mathscr{H}^I_{\boldsymbol{\tau}}$ is too). $\square$

Claim 4. *All of the cases listed in Tables* 1, 2 *and* 3 *in Appendix* A.2 *are noncommutative.*

Proof. We say that an element $w \in W$ has an *essentially unique expression* if every reduced expression for w is obtained from a given reduced expression of w by a sequence of 'commutations' (that is, Coxeter moves of the form $st = ts$). It is routine to check that all of the words in Tables 1, 2 and 3 in Appendix A.2 have essentially unique expressions, except for the $H_{4,4}$, $\tilde{F}_{4,4}$, $\tilde{E}_{8,1}$ and H_4^1 words. These words will be dealt with below. For those words with essentially unique expressions it is easy to check that the triple (u, w_I, k) provided in the table satisfies the hypothesis of Proposition 2.8, except for the $B_2^{1,2}$, B_4^3, E_8^1, $H_3^{1,1}$, $I_2(5)^{1,1}$ and $I_2(7)^1$ words, and so the associated algebras are noncommutative. For example, consider the D_5^3 word $w = uw_I0$ with $u = 03243120$, $w_I = 3543$ and $k = 5$. To see that there are no $131 \mapsto 313$ Coxeter moves available one considers each triple $(1, 3, 1)$ in the given reduced decomposition for w and verifies that there is no sequence of commutations that make these three generators adjacent. One such triple is $w = 03\underline{1}24\underline{3}\,\underline{1}2035430$, and it is clear that it is impossible to make the first $\underline{1}$ adjacent to the $\underline{3}$ using commutations. Continuing in this fashion one verifies that this word has an essentially unique expression. It is now clear that the word is reduced and I-reduced, and that *every* reduced expression for w has the property that the $k = 5$ generator does not appear between the first two 0 generators. Thus Proposition 2.8 applies, and so the algebra is noncommutative.

It remains to deal with the $H_{4,4}$, $\tilde{F}_{4,4}$, $\tilde{E}_{8,1}$, H_4^1, $B_2^{1,2}$, B_4^3, $H_3^{1,1}$, $I_2(5)^{1,1}$ and $I_2(7)^1$ words (these are marked with a $*$ in Appendix A.2). The $H_{4,4}$ word $w = uw_I4$ with $u = 434323434$ and $w_I = 123$ has only one possible Coxeter move ($323 \mapsto 232$). The only Coxeter move available in the resulting expression $w = 4342324341234$ is the move $232 \mapsto 323$ taking us back to the original expression. Therefore *every* reduced expression for w is obtained from one of

$$4343234341234,$$

$$4342324341234$$

by using only commutations. Hence it is clear that the $k = 1$ generator can never appear in between the first two 4 generators of a reduced expression for w, and so Proposition 2.8 applies.

The $\tilde{F}_{4,4}$ word $w = uw_I4$ with $u = 43231234$ and $w_I = 3231230123$ has exactly one possible Coxeter move ($343 \mapsto 434$). The only Coxeter move in the resulting expression is the one returning us to the original expression. Thus, as in the $H_{4,4}$ case, we readily see that Proposition 2.8 (with $k = 0$) applies.

Consider the $\tilde{E}_{8,1}$ word $w = 134562453413245\underline{676}805432456781345672456345243$1. The only Coxeter move possible initially is the $676 \mapsto 767$ move. After making this move we get $w = 13456245341324\underline{5}7\underline{6}780\underline{5}43245678134567245634524 31$. The only new Coxeter move available is the $565 \mapsto 656$ move, giving $w = 134562453413247656780432456781345672456345243$1. There are now no new Coxeter moves, and so *every* reduced expression for w is obtained from one of

$$1345624534132456768054324567813456724563452431,$$

$$1345624534132457678054324567813456724563452431,$$

$$134562453413247656780432456781345672456345243 1$$

using commutations alone. Thus it is clear that the 0 generator can never be between the first two 1 generators, and so Proposition 2.8 applies.

The details for the H_4^1 word $w = uw_I0$ with $u = 012343210$ and $w_I = 434234123243412343 21$ are as follows. Arguing as above one sees that every reduced expression for w is obtained from one of the following three expressions by commuting generators:

$$01234321043423412324341234321 0,$$

$$01234321043423413234341234321 0,$$

$$01234321043423432123434234321 0.$$

It follows that every reduced expression for w has at least three 4s between the last two 0 generators. Thus there is no reduced expression $w = 0zu'$ with $u' \leqslant u$ and $z \in W_I$ because such an expression has at *most* one 4 between the last two 0s. Thus Corollary 2.7 proves noncommutativity.

Consider the $B_2^{1,2}$ word $w = uw_I0$ with $u = 01210$ and $w_I = 212$. This word has *exactly one* reduced expression, and this expression has exactly two 2s in between the last two 0 generators. Hence there is no reduced expression of the form $w = 0zu'$ with $z \in W_I$ and $u' \leqslant u$, for each such expression has at most one 2 between the last two 0s. Thus Corollary 2.7 proves noncommutativity.

Consider the B_4^3 word $w = uw_I0$ with $u = 03430$ and $w_I = 234123$. It is clear that every reduced expression for w has at least one 2 in between the last two 0 generators. Thus there is no reduced expression of the form $w = 0zu'$ with $z \in W_I$ and $u' \leqslant u$ (since such expressions have no 2s in between the last two 0 generators) and so Corollary 2.7 proves noncommutativity.

Consider the E_8^1 word $w = uw_I0$ with $u = 0134254310$, $w_I = 65423456781342543654276543 1$. This word has an essentially unique expression, and so it is clear that every reduced expression for w has at least two 2s in between the last two 0 generators. Hence there is no reduced expression of the form $w = 0zu'$ with $z \in W_I$ and $u' \leqslant u$, for each such expression has either zero or one 2s between the last two 0s.

Consider the $H_3^{1,1}$ word $w = uw_I0$ with $u = 010$ and $w_I = 232132321$. Every reduced expression for w has at least two 2s in between the last two 0 generators. Thus there is no reduced expression of the form $w = 0zu'$ with $z \in W_I$ and $u' \leqslant u$, and so Corollary 2.7 proves noncommutativity. Similarly, for the $I_2(5)^{1,1}$ word $w = uw_I0$ with $u = 010$ and $w_I = 2121$ every reduced expression for this word has at least one 2 in between the last two 0 generators. So Corollary 2.7 proves noncommutativity. Finally, every reduced expression for the $I_2(7)^1$ word $w = uw_I0$ with $u = 012120$ and $w_I = 12121$ has exactly three 1s in between the last two 0 generators, and as above, Corollary 2.7 proves noncommutativity. □

Claim 5. *All spherical and affine cases other than those listed in Theorem 2.1 are noncommutative.*

Proof. Claim 4 above has provided us with a library of noncommutative examples. We use this library to deal with the remaining cases via the following obvious fact: If $I \subseteq S$ is spherical, and if S' is such that $I \subseteq S' \subseteq S$, and if the parabolic Hecke algebra $\mathscr{H}^I(W_{S'}, S')$ is noncommutative, then $\mathscr{H}^I(W, S)$ is noncommutative too (and the same holds for specialisations with $\tau_s \geqslant 1$). This is clear, since the former algebra is a subalgebra of the latter.

It is now straightforward to show that all remaining spherical and affine cases are noncommutative. For example $E_{7,5}$ is noncommutative since the 5 node of E_7 plays the role of the 5 node in an E_6 residue, and $E_{6,5}$ is noncommutative by our library. Similarly $\tilde{E}_{8,2}$ is noncommutative since the 2 node of $\tilde{E}_8$ plays the role of the 2 node in an E_8 residue, and $E_{8,2}$ is noncommutative. □

Claim 6. *All infinite non-affine cases are noncommutative.*

Proof. The reduction arguments in this proof rely on the following fact. If Proposition 2.8 (or Corollary 2.7) has been used to prove noncommutativity for an I-parabolic Hecke algebra with Coxeter data m_{st}, then the I-parabolic Hecke algebras with Coxeter data $\overline{m}_{st} \geqslant m_{st}$ for all $s, t \in S$ are also noncommutative. This fact is proved formally in the following lemma.

Lemma 3.1. *Let (W, S) be a Coxeter system with Coxeter matrix $M = (m_{st})$. Let $I \subseteq S$. Suppose there exist $w, u, v, z \in W$ such that $w = uzv$, $u, v, w \in R_I$, $z \in W_I$, and $\ell(w) = \ell(u) + \ell(z) + \ell(v)$, and that there exist no $u', v', z' \in W$ with $w = v'z'u'$, $u' \leqslant u$, $v' \leqslant v$, $z' \in W_I$, and $\ell(w) = \ell(v') + \ell(z') + \ell(u')$.*

Let $(\overline{W}, S)$ be a Coxeter system with Coxeter matrix $\overline{M} = (\overline{m}_{st})$. Suppose that $I \subseteq S$ is spherical (for $\overline{W}$), and let $\overline{\mathscr{H}}^I = \mathscr{H}^I(\overline{W}, S)$ be the associated I-parabolic Hecke algebra. If $\overline{m}_{st} \geqslant m_{st}$ for all $s, t \in S$ then the algebras $\overline{\mathscr{H}}^I$ and $\overline{\mathscr{H}}^I_{\boldsymbol{\tau}}$ (with $\tau_s \geqslant 1$) are noncommutative.

Proof. Let $w = uzv$ be a reduced expression in W with $u, v, w \in R_I$, $z \in W_I$, and $\ell(w) = \ell(u) + \ell(z) + \ell(v)$. We claim that the corresponding conditions hold when the expression for w is read in $\overline{W}$. Since uzv is reduced in W, it cannot contain a subword in two letters i, j of length larger than m_{ij}. Hence any elementary transformation of uzv in $\overline{W}$ involves a subword in i, j of length $m_{ij} = \overline{m}_{ij}$, and thus can also be carried out in W. Since we cannot produce a subword of the form ss by carrying out elementary transformations in W, uzv must also be reduced in $\overline{W}$, and so the expression for w is reduced when read in $\overline{W}$. Since z is a word with letters in I, it is in $\overline{W}_I$ when read in $\overline{W}$. Next we claim that the words u, v, w, when read in $\overline{W}$, are still I-reduced: Suppose for instance that w is not I-reduced when read in $\overline{W}$. Then ws or sw is not reduced in $\overline{W}$ for some $s \in I$. By the exchange condition, w can be rewritten (in $\overline{W}$) as a reduced word starting or ending in s. But as before, all the elementary transformations which transform w into some $w's$ (or sw') in $\overline{W}$ (with the word w' of smaller length than w) can also be carried out in W, contradicting that w is I-reduced in W.

Now assume, by way of contradiction, that in $\overline{W}$ the word $w = uzv$ can also be written as $v'z'u'$ with $v' \leqslant v$, $u' \leqslant u$, $z' \in \overline{W}_I$, and $\ell(w) = \ell(v') + \ell(z') + \ell(u')$. Note first that, with the same argument as before, the transformation $uzv \mapsto v'z'u'$ can be carried out in W as well (and the result $v'z'u'$ is of course still reduced in W). The word z' has all letters in I, and so represents an element of W_I when read in W. Finally we claim that if $u' \leqslant u$ in $\overline{W}$ then also $u' \leqslant u$ in W (and similarly for v and v'). Since $u' \leqslant u$ in $\overline{W}$ there exists a subword u'' of u which, when read in $\overline{W}$, is equal to u'. Applying the deletion condition if necessary, we may assume that u'' is reduced in $\overline{W}$. Hence there exist elementary transformations $u' \mapsto u''$ in $\overline{W}$. But u' is reduced in W, and so all these elementary transformations can be carried out in W as well, proving that $u' \leqslant u$ in W also. This completes the proof that our assumptions on u, v, w, z in W are violated. So u', v', z' as described cannot exist in $\overline{W}$, which implies by Corollary 2.7 that the Hecke algebras $\overline{\mathscr{H}}^I$ and $\overline{\mathscr{H}}^I_{\boldsymbol{\tau}}$ (with $\tau_s \geqslant 1$) are noncommutative. $\square$

Suppose that W is neither spherical nor affine. Let $I \subseteq S$ be spherical, and suppose that $\mathscr{H}^I$ is commutative and not in Tables 1, 2 and 3 in Appendix A.2. By Claim 1 we see that $|S \backslash I| = 1$, and so by relabelling nodes if necessary we may assume that $I = S \backslash \{0\}$.

We will prove the following reductions based on the neighbourhood of 0 in the Coxeter graph:

- The valency of 0 is at most 2, and so the diagram $I = S \backslash \{0\}$ has 1 or 2 connected components.
- If 0 has valency 1 then the bond number p is either 3 or 4.
- If 0 has valency 2 then the bond numbers $p \leqslant q$ are $(p, q) = (3, 3)$ or $(3, 4)$.

For the first claim, suppose that 0 has valency 4 with bond numbers $3 \leqslant p \leqslant q \leqslant r \leqslant s$. If $(p, q, r, s) = (3, 3, 3, 3)$ then 0 is noncommutative in a $\tilde{D}_4$ residue, and if the bond numbers are different from $(3, 3, 3, 3)$ then we can use Lemma 3.1 to deduce noncommutativity. Thus 0 has valency at most 3. Suppose that 0 has valency 3 with bond numbers $3 \leqslant p \leqslant q \leqslant r$. If there is at least one vertex not connected to 0 then 0 is noncommutative in either a D_5 residue, or is noncommutative by Lemma 3.1 and comparison to a D_5 residue. Thus if 0 has valency 3 then S has exactly 4 vertices. Suppose that there are nodes $i, j \neq 0$ which are connected. The 'minimal' case is $A_1^1 \times A_2^{1,2}$ (which is in Tables 1, 2 and 3 in Appendix A.2), and all other bond number possibilities are noncommutative by Lemma 3.1. So suppose that S has exactly 4 nodes, and that there are no other bonds other than those which involve the 0 node. If $(p,q,r)=(3,3,3)$ then we have a D_4 diagram (contradicting the assumption that W is neither spherical nor affine). If $(p,q,r)=(3,3,4)$ then the 0 node is noncommutative in $\tilde{B}_3$, and Lemma 3.1 shows that all higher bond numbers also lead to noncommutative algebras. This completes the proof of the first statement.

To prove the second statement, if 0 has valency 1 with bond number at least 5, then we can compare a suitable residue with either $\tilde{B}_{3,0}$ (if 0 is not connected to an end vertex), or with $B_2^{1,1,1}$ (if 0 is connected with an end vertex and there are only three vertices), or with $H_{4,4}$ (if 0 is connected with an end vertex and there are at least four vertices) to deduce noncommutativity (applying Lemma 3.1).

To prove the third statement, suppose that the valency of 0 is 2 with bond numbers $(3,n)$, $n \geqslant 5$, or $(m,4)$, $m \geqslant 4$. Then we can compare an appropriate residue with $H_{3,2}$ or $\tilde{C}_{2,1}$ to deduce noncommutativity (applying Lemma 3.1).

The three bullet points above place severe restrictions on the Coxeter diagram $S = I \cup \{0\}$. We now eliminate each possibility using our noncommutative examples from the library in Appendix A. We will give examples of the arguments used.

Case 1: *The valency of* 0 *is* 1 *with* $p=3$. We consider each possible connected spherical diagram $I = S\backslash\{0\}$ and each possible way of connecting 0 with a single bond to make S. For example, suppose that $I = B_n$ with $n \geqslant 2$. The possible diagrams are B_n^i with $i = 1,\dots,n$. If $n=2$ then B_2^1 and B_2^2 both give B_3 diagrams, a contradiction, so assume that $n \geqslant 3$. We have $B_n^1 = B_{n+1}$ and $B_n^2 = \tilde{B}_n$ (a contradiction). Each diagram B_n^i with $2 < i < n < i+4$ has 0 as a noncommutative node in a B_{n-i+3}^3 (and these are all in our table). If $n \geqslant i+4$, then we have a B_7^3 residue, which is noncommutative by Lemma 3.1 and comparison with $E_{8,2}$. In B_n^n, $n \geqslant 4$, the node 0 is noncommutative in an $\tilde{F}_4$ residue. Thus $I = B_n$ is excluded.

Case 2: *The valency of* 0 *is* 1 *with* $p=4$. Again we consider each diagram. For example, suppose that $I = H_3$. The diagram $H_3^{1,1}$ is in our table, and $H_3^{2,2}$ and $H_3^{3,3}$ both have 0 as a noncommutative node in an $I_2(5)^{1,1}$ residue.

Case 3: *The valency of* 0 *is* 2 *with* $(p,q)=(3,3)$, *and* I *has one connected component*. For example suppose that $I = A_n$ with $n \geqslant 2$. The possibilities are $A_n^{i,j}$ with $1 \leqslant i < j \leqslant n$. The case $i=1$ and $j=n$ is excluded, for it gives an $\tilde{A}_n$ diagram. By looking in a residue it suffices to show that the 0 node is noncommutative in $A_k^{1,k-1}$ for each $k \geqslant 3$. The diagrams $A_3^{1,2}$ and $A_4^{1,3}$ are in Table 3 in Appendix A.2. The diagram $A_5^{1,4}$ is excluded by comparing it to an E_6 diagram and using Lemma 3.1. Specifically, if we decrease the bond $m_{12}=3$ in $A_5^{1,4}$ to $m_{12}=2$ then we get an E_6 diagram with 0 playing the role of the (noncommutative) 3 node. The diagram $A_6^{1,5}$ is excluded since it has 0 as a noncommutative node in an E_6 residue, and for $k \geqslant 7$ the $A_k^{1,k-1}$ diagram is excluded since it has 0 as the noncommutative $k-3$ node in a D_k residue.

Case 4: *The valency of* 0 *is* 2 *with* $(p,q)=(3,4)$, *and* I *has one connected component*. Suppose that 0 is connected to $i \in I$ by a single bond, and to $j \in I$ by a double bond (with $i \neq j$). The case where i and j are connected is excluded by Lemma 3.1 and the fact that 0 is noncommutative in $A_2^{1,1,2}$. So suppose that i and j are not connected. Since I is connected, j is connected to some $k \in I$ with $k \neq i$. Then 0 is noncommutative in an F_4 residue (incorporating $i, 0, j$ and k) or by comparison to an F_4 diagram (using Lemma 3.1).

Case 5: *The valency of* 0 *is* 2 *with* $(p,q)=(3,3)$, *and* I *has two connected components*. Let the connected components be I_1 and I_2. Suppose that 0 is connected to $i_1 \in I_1$ and $i_2 \in I_2$. If there are nodes $j_1, k_1 \in I_1$ connected to i_1 and $j_2, k_2 \in I_2$ connected to i_2 then 0 is a noncommutative node in a $\tilde{D}_6$ residue, or can be compared to such a vertex by Lemma 3.1. Therefore either i_1 or i_2 is an end node.

Suppose that i_1 is an end node of I_1, and that i_1 is connected to $j_1 \in I_1$. Assume that there exist neighbours $j_2, k_2 \in I_2$ of i_2. Then the 0 node is noncommutative by comparison with a $D_{7,4}$ diagram, using Lemma 3.1.

Suppose that there exist $j_2, k_2 \in I_2$ distinct neighbours of i_2, and that j_2 has a neighbour $m_2 \neq k_2$. Then the 0 node is noncommutative by comparison with an $E_{6,3}$ diagram.

There are now 2 possibilities remaining: (i) $I_1 = \{i_1\} = A_1$ and I_2 is a 'star' with 0 connected to the central node, or (ii) i_1 is an end node of I_1 and i_2 is an end node of I_2. (By a 'star' we mean a central node with other nodes hanging off it. None of these outer nodes are connected to other outer nodes, because the diagram I_2 cannot have a triangle since it is spherical.) Consider case (i). If any bond number of I_2 is $\geqslant 4$ then we compare with an $A_1^1 \times B_3^2$ diagram. So suppose that all bonds in I_2 are 3-bonds. If I_2 has at least four vertices then 0 is the noncommutative in an $A_1^1 \times D_4^2$ diagram. If I_2 has exactly three vertices then we have D_5, and if it has exactly 2 vertices then we have A_4. Thus case (i) is excluded.

We are left to consider the case when i_1 is an end node of I_1 and i_2 is an end node of I_2. We consider these case by case. For example, suppose that $I_1 = A_n$ and $I_2 = E_m$ for $m = 6, 7, 8$. By symmetry we can suppose that 0 is connected to the node 1 of A_n, and 0 is not connected to the node 6 of E_6. So the possibilities are $A_n^1 \times E_m^k$, with $k = 1, 2, m$. In $A_n^1 \times E_m^1$, the 0 node is noncommutative in an $E_{8,2}$ residue. In $A_n^1 \times E_m^2$, the 0 node is noncommutative in an $E_{7,6}$ residue. Finally, in $A_n^1 \times E_m^m$, $m = 7, 8$, the 0 node is noncommutative in an $A_1^1 \times E_m^m$ residue, which for both values of m is in the table.

Suppose that $I_1 = A_n$ and $I_2 = E_8$. By symmetry we can suppose that 0 is connected to the 1 node of A_n, and so the possibilities are $A_n^1 \times E_8^k$ with $k = 1, 2, 8$. In $A_n^1 \times E_8^1$ the 0 node is noncommutative in an E_8 residue, and in $A_n^1 \times E_8^2$ the 0 node is noncommutative in an E_7 residue. In $A_n^1 \times E_8^8$ the 0 node is noncommutative in an $A_1^1 \times E_8^8$ residue.

Case 6: *The valency of* 0 *is* 2 *with* $(p, q) = (3, 4)$, *and* I *has* 2 *connected components*. Let I_1 and I_2 be the connected components. Suppose that 0 is connected to $i \in I_1$ by a single bond, and to $j \in I_2$ by a double bond. If $|I_2| > 1$ then 0 is noncommutative in an $F_{4,2}$ residue (or can be compared to such a vertex using Lemma 3.1). Thus $I_2 = \{j\}$. If i is not an end node of I_1 then 0 is noncommutative in a $\tilde{B}_{4,2}$ residue (or can be compared to such a vertex). Thus $I_2 = \{j\}$ and i is an end node of I_1. So we need to consider each diagram $A_1^{1,1} \times X_n^k$ for each spherical type X_n and end vertex k of X_n.

If X_n contains a bond with bond number $\geqslant 4$, then 0 is noncommutative in a $\tilde{C}_k$ residue (for appropriate k). If X_n contains a vertex with degree $\geqslant 3$, then 0 is noncommutative in a $\tilde{B}_k$ residue (for appropriate k). Hence $X_n = A_n$ and we get a B_{n+2} diagram.

Thus all infinite non-affine cases are noncommutative, and the proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. The 'if' part is Lemma 2.5, and the 'only if' part is because, as we have seen, there are no other commutative cases other than the listed spherical cases (in which case $\pi = \mathrm{id}$) and the listed affine cases (in which case π is opposition in the spherical residue). □

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Appendix A

The appendix has 2 sections. The first section illustrates a technique that can be used to determine if the minimal length double coset representatives of a spherical Coxeter group are involutions (this was used in Claim 2 of the proof of Theorem 2.1). The second section gives the tables of words that were used in the text to prove noncommutativity.

A.1. Involutions

There are various ways to determine whether the minimal length double coset representatives of a spherical Coxeter group are involutions. For example [8, Theorem 3.1] gives a method using the representation theory of the Coxeter group. It is also possible to determine if the double coset representatives are involutions by a direct, elementary argument. Let us outline this in the most involved example $E_{8,1}$.

Proposition A.1. *Let (W, S) be the Coxeter system of type E_8 and let $I = S\backslash\{1\}$. Each element of R_I is an involution.*

Proof. Let Σ be the Coxeter complex of (W, S) with usual W-distance function $\delta(u, v) = u^{-1}v$. Let X be the set of vertices of type 1 in Σ. If $x \in X$ let $C(x)$ denote the set of all chambers of Σ containing x. For $x, y \in X$ the set $\delta(C(x), C(y))$ is a double coset $W_I z W_I$, and the W-distance $\delta(x, y)$ between x and y is defined to be the minimal length representative of this double coset. If $w \in R_I$ then (see the proof of Proposition 1.3)

$$\#\{y \in X \mid \delta(x, y) = w\} = \frac{|W_I w W_I|}{|W_I|} = \frac{|M_{I,w}||W_I|}{|W_I|} = \frac{|W_I|}{|W_{I\cap wIw^{-1}}|}. \tag{A.1}$$

It is known that there are exactly 10 double cosets $W_I w W_I$ in E_8 (see [6, Table 10.5]). Let $w_0, w_1, \ldots, w_9$ be the minimal length double coset representatives. Fix the vertex $x_0 \in X$ of type 1 contained in the chamber of Σ corresponding to the identity element of W. Let $i \in \{0, 1, \ldots, 9\}$ be arbitrary. Put $S_i = I \cap w_i S w_i^{-1}$, and let $W_i = W_{S_i} = \langle S_i \rangle$. By (A.1) the number of vertices $x \in X$ with $\delta(x_0, x) = w_i$ is equal to the quotient $|W_I|/|W_i|$. The total number of vertices of type 1 is equal to $|X| = |W|/|W_I| = 2160$. Denote by X_i the set of vertices $x \in X$ with $\delta(x_0, x) = w_i$. Thus $|X_0| + |X_1| + \cdots + |X_9| = |X| = 2160$.

Let w be the longest element in W. Since the opposition relation in Σ induces the trivial permutation on S (and this permutation is given by conjugation with w), w is central in W. Hence if w_i is an involution, then so is ww_i, and it interchanges x_0 with the unique vertex x_i' opposite x_i, where $x_i = w_i x_0$ is the image of x_0 under w_i. Consequently if w_i is an involution and if $\delta(x_0, x_i') = w_j$ then w_j is also an involution. In this case we say that w_j is *complementary* to w_i. Of course it could happen that $i = j$. In this case, x_i and x_i' are contained in opposite chambers, and so the longest element w of W belongs to $w_i W_I w_i W_I$. Since the length of the longest element in W_I is 42 and since $\ell(w) = 120$ this implies that $\ell(w_i) \geqslant 18$.

We now apply the above to some specific values of w_i. We take $w_0 = e$, the identity, and $w_1 = s_1 = 1$. Thus $|X_0| = 1$ and $|X_1| = |W(D_7)|/|W(A_6)| = 64$, and since $\ell(w_0), \ell(w_1) < 18$ we obtain complementary involutions w_9 and w_8, respectively, with $|X_9| = 1$ and $|X_8| = 64$. Now put $w_2 = 13425431$ (which is obtained by considering the residue of a vertex of type 6). The element w_2 maps the generators $(3, 4, 2, 5, 7, 8)$ to $(3, 4, 5, 2, 7, 8)$, and so one calculates that $|X_2| = |W(D_7)|/|W(D_4 \times A_2)| = 280$. Since $\ell(w_2) = 8 < 18$ we have a complementary involution $w_7 \neq w_2$ with $|X_7| = 280$. So far we have accounted for $2(1 + 64 + 280) = 690$ of the total 2160 type 1 vertices.

In the residue of an element of type 8 (which is a Coxeter system of type E_7) we find the involutive minimal length double coset representative $w_3 = 13425463576452431$, which maps the generators $(2, 4, 5, 6, 7)$ to $(7, 6, 5, 4, 2)$. Consequently $|X_3| = |W(D_7)|/|W(A_5)| = 448$. As $\ell(w_3) = 17 < 18$ we have another involution w_6 with $|X_6| = 448$, accounting for $690 + 2 \times 448 = 1586$ of the 2160 vertices. Finally we can consider, in each of the 14 residues of type E_7 through x_0, the element of type 1 opposite x_0. This gives rise to another involutive double coset representative w_4, with $|X_4| = |W(D_7)|/|W(D_6)| = 14$. This one must be self-complementary, as otherwise the unique missing class X_5 would also contain 14 elements and the total number of vertices does not add up to 2160. Indeed we calculate that $|X_5| = 560$. Hence w_5 is also self-complementary. But what is more important, it must also be an involution as otherwise w_5^{-1} is a different minimal double coset representative, contradicting the fact that we only have 10 of these. Hence all minimal coset representatives are involutions. □

A.2. Tables of words to prove noncommutativity

Conventions: We use standard Bourbaki labelling for the spherical and affine types [4, Plates I–IX]. The cases H_3 and H_4 are not given an explicit labelling in Bourbaki. We adopt the labelling of H_3 with $m_{12} = 3$ and $m_{23} = 5$, and of H_4 with $m_{12} = m_{23} = 3$ and $m_{34} = 5$.

Each word is of the form $w = uw_I s_i$, where $I = S\backslash\{i\}$. We also list the index k used in the argument of Proposition 2.8. The cases where a slight modification of Proposition 2.8 is required are labelled by $(*)$. The precise details for these cases are given in Claim 4 of Section 3.

The $D_{n,i}$ word (with $n/2 < i < n-1$) is

$$u = \big[i(i-1)\cdots(2i-n+1)\big]\big[(i+1)i\cdots(2i-n+2)\big]\cdots\big[(n-1)(n-2)\cdots i\big],$$
$$w_I = \begin{cases} [n(n-2)(n-3)\cdots(i+1)][12\cdots(i-1)] & \text{if } i/2 < i < n-2, \\ n12\cdots(n-3) & \text{if } i = n-2. \end{cases}$$

Table 1
The spherical cases.

	u	w_I	k
$D_{n,i}$, $\frac{n}{2} < i < n-1$	see below	see below	n
$E_{6,5}$	542345	1634	6
$E_{7,6}$	65423456	17345	7
$E_{8,7}$	7654234567	183456	8
$E_{8,2}$	245678345672	456345134	1
$F_{4,2}$	232	431	1
$H_{3,2}$	232	31	1
$H_{4,2}$	23432	431	1
$H_{4,4}$	434323434	123	1(∗)

Table 2
The affine cases.

	u	w_I	k
$\tilde{B}_{n,i}$, $1 < i < n-1$	$i\cdots 320123\cdots i$	$(i+1)\cdots n(n-1)\cdots(i+1)$	$i+1$
$\tilde{B}_{n,n}$	$[n\cdots 1][n\cdots 2]\cdots[n(n-1)][n]$	$023\cdots(n-2)(n-1)$	0
$\tilde{C}_{n,i}$, $1 \leqslant i < n$	$i\cdots 3210123\cdots i$	$(i+1)\cdots n(n-1)\cdots(i+1)$	$i+1$
$\tilde{D}_{n,i}$, $1 < i < n-1$	$i\cdots 320123\cdots i$	$(i+1)\cdots n(n-2)\cdots(i+1)$	$i+1$
$\tilde{E}_{7,2}$	245341031245342	65764534	7
$\tilde{E}_{8,1}$	134562453413245676805432456781	345672456345243	0(∗)
$\tilde{E}_{8,8}$	876542345678	1034567	0
$\tilde{F}_{4,1}$	12321	4320	0
$\tilde{F}_{4,4}$	43231234	3231230123	0(∗)
$\tilde{G}_{2,1}$	212	01	0
$\tilde{G}_{2,2}$	12121	02	0

Table 3
The infinite non-affine cases.

	u	w_I	k
$A_2^{1,1,2}$	010	21	2
$A_3^{1,2}$	0210	2312	3
$A_4^{1,3}$	032430	123	1
$B_2^{1,2}$	01210	212	2(∗)
$B_2^{1,1,1}$	010	121	2
$B_3^{1,3}$	03230	12321	1
$B_3^{3,3}$	0323032303230	1323	1
B_4^{3}	03430	234123	1(∗)

(continued on next page)

Table 3 (*continued*)

	u	w_I	k
B_5^3	032430	1234543	5
B_6^3	032430	123456543	5
$D_4^{1,4}$	01240	123421	3
D_5^3	03243120	3543	5
$D_5^{1,5}$	05342350	12345321	1
D_6^3	03243120	346543	6
$D_6^{1,6}$	06453460	1234564321	1
D_7^3	03243120	34576543	7
E_6^3	0345243013452430	61345243	6
E_7^2	02435420	65431243524654376542	7
E_7^6	06543245607654324560	1765432456	1
E_8^1	0134254310	654234567813425436542765431	$2(*)$
E_8^7	076543245670876543245670	187654324567	1
F_4^2	02320	1234232	4
$F_4^{1,1}$	0123210123210	412321	4
$F_4^{1,4}$	0432340	12321	1
H_3^2	02320	32132	1
H_3^3	03230	2321323	1
$H_3^{1,1}$	010	232132321	$2(*)$
H_4^1	012343210	43423412324341234321	$4(*)$
H_4^2	0234320	4342312	1
$I_2(5)^{1,1}$	010	2121	$2(*)$
$I_2(7)^1$	012120	12121	$1(*)$
$A_1^1 \times A_2^{1,2}$	0120	$11'$	$1'$
$A_1^1 \times F_4^1$	0123210	$1'4321$	$1'$
$A_1^1 \times H_3^1$	0123210	$1'321$	$1'$
$A_1^1 \times B_3^2$	02320	$121'$	$1'$
$A_1^1 \times D_4^2$	023420	$1'12$	$1'$
$A_1^1 \times E_7^7$	076542345670	$11'34567$	$1'$
$A_1^1 \times E_8^8$	08765423456780	$11'345678$	$1'$

The node of the A_1 component in the final 7 composite cases is labelled by $1'$.

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