Buildings and Hecke Algebras

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Dedicated to the memory of Geoff Coales,
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Contents

Abstract

We establish a strong connection between buildings and Hecke algebras by studying two algebras of averaging operators on buildings. To each locally finite regular building $\mathscr{X}$ we associate a natural algebra $\mathscr{B}$ of chamber set averaging operators, and when the building is affine we also define an algebra $\mathscr{A}$ of vertex set averaging operators. We show that for appropriately parametrised Hecke algebras $\mathscr{H}$ and $\tilde{\mathscr{H}}$, the algebra $\mathscr{B}$ is isomorphic to $\mathscr{H}$ and the algebra $\mathscr{A}$ is isomorphic to the center of $\tilde{\mathscr{H}}$. On the one hand these results give a thorough understanding of the algebras $\mathscr{A}$ and $\mathscr{B}$. On the other hand they give a nice geometric and combinatorial understanding of Hecke algebras, and in particular of the Macdonald spherical functions and the center of affine Hecke algebras. Our results also produce interesting examples of association schemes and polynomial hypergroups.

It is shown that all algebra homomorphisms $h:\mathscr{A}\to\mathbb{C}$ may be expressed in terms of the Macdonald spherical functions. We also provide a second formula for these homomorphisms in terms of an integral over the boundary of $\mathscr{X}$. The algebra $\mathscr{A}$ may be regarded as a subalgebra of the C^*-algebra of bounded linear operators on the Hilbert space $\ell^2(V_P)$ of square summable functions $f:V_P\to\mathbb{C}$, where V_P is in most cases the set of special vertices of $\mathscr{X}$. We write $\mathscr{A}_2$ for the closure of $\mathscr{A}$ in this algebra. The Gelfand map $\mathscr{A}_2\to\mathscr{C}(M_2)$, where $M_2=\mathrm{Hom}(\mathscr{A}_2,\mathbb{C})$, is studied, and we compute M_2 and the Plancherel measure of $\mathscr{A}_2$. This 'spherical harmonic analysis' is applied to give a local limit theorem for radial random walks on affine buildings.

In an appendix we discuss an alternative approach to the study of the algebra $\mathscr{A}$ and the algebra homomorphisms $h:\mathscr{A}\to\mathbb{C}$ in the 'low dimension' cases, using more elementary techniques (that is, without the machinery of Hecke algebras).

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This thesis contains no material which has been accepted for the award of any other degree or diploma. All work in this thesis, except where duly attributed to another person, is believed to be original.

Introduction

Buildings are certain geometric objects, initially studied by Jacques Tits in the mid 1950's to give a systematic geometric interpretation of the semi-simple Lie groups (see the introduction to [**40**]). Since then, the theory of buildings has enjoyed a rich and rapid development in many directions, with Tits being a main contributor. To a large extent this activity has been motivated by the useful and interesting connections between buildings and other branches of mathematics (for example, p-adic and arithmetic groups). One aim of this thesis is to describe some close connections between buildings and Hecke algebras, through the combinatorial study of two algebras of averaging operators associated to (regular) buildings.

Often we are interested in *affine buildings*, of which *homogeneous trees* are the simplest examples. Indeed, a motivation for our work is the existing theory of 'harmonic analysis on homogeneous trees' (see [**14**] for example). There is also an extensive literature in the higher rank cases too (see [**23**]), but under the assumption that the building is constructed from a group. It is another aim of this thesis to lay the foundations for 'harmonic analysis on affine buildings', using building theoretic, rather than group theoretic, techniques.

We note that there exist buildings that do not arise from the standard group constructions, even in the spherical and affine cases. For example, in a locally finite $\tilde{A}_2$ building, the vertices of distance 1 from a given vertex have the structure of a projective plane (which might not be Desarguesian). This plane may vary from vertex to vertex (with the order fixed) [**34**], resulting in a building with trivial automorphism group. The situation for $\tilde{C}_2$ and $\tilde{G}_2$ buildings is similar.

For a simpler example, notice that in a homogeneous tree in which each vertex has valency $q+1$, it is only when q is a prime power that the tree arises as the Bruhat-Tits building of a group.

Our methods deal with buildings in a uniform, axiomatic manner. We do not assume that our buildings arise from groups, and so it is the building theoretic aspects which determine our results.

Let us give a brief description of the structure and main results of this thesis.

Buildings and Regularity. In Chapter 1 we collect some basic facts regarding Coxeter groups, chamber systems, labelled simplicial complexes, Coxeter complexes and buildings. There are two main definitions of buildings in the literature: one in terms of *simplicial* *complexes* ([**40**],[**7**]), and a second (more recent) definition in terms of *chamber systems* ([**41**],[**35**]). In this chapter we discuss both definitions. This is all standard material, and serves as a rather rapid introduction to buildings.

Section 1.7 contains a discussion of *regularity* in buildings, a concept which will be important throughout. To discuss regularity, it is convenient to regard buildings as certain *chamber systems*. Thus a building $\mathscr{X}$ is a set $\mathcal{C}$ of *chambers* with an associated Coxeter system (W, S), and a function $\delta : \mathcal{C} \times \mathcal{C} \to W$ such that $\delta(c, d)$ defines a W-valued distance between c and d (in some sense that will be made clear in Chapter 1).

For each $c \in \mathcal{C}$ and $w \in W$, define $\mathcal{C}_w(c) = \{d \in \mathcal{C} \mid \delta(c, d) = w\}$. We always assume *local finiteness*, by which we mean $|\mathcal{C}_s(c)| < \infty$ for all $c \in \mathcal{C}$ and $s \in S$. We call $\mathscr{X}$ *regular* if for each $s \in S$ we have $|\mathcal{C}_s(c)| = |\mathcal{C}_s(d)|$ for all $c, d \in \mathcal{C}$. In a regular building we write $q_s = |\mathcal{C}_s(c)|$, and we call the set $\{q_s\}_{s\in S}$ the *parameter system* of the building. In Proposition 1.7.1 we show that regularity implies the stronger result that $|\mathcal{C}_w(c)| = |\mathcal{C}_w(d)|$ for all $c, d \in \mathcal{C}$ and $w \in W$, and as such we define $q_w = |\mathcal{C}_w(c)|$. In Theorem 1.7.4 we prove that all locally finite *thick* buildings with no rank 2 residues of type $\tilde{A}_1$ are regular, showing that regularity is a very weak hypothesis. Note that *thin* buildings are also regular.

After Chapter 1, all buildings in this thesis are locally finite and regular.

The Algebra $\mathscr{B}$. In Chapter 2 we discuss *chamber set averaging operators* associated to arbitrary locally finite regular buildings. For each $w \in W$ we define an operator B_w, acting on the space of functions $f : \mathcal{C} \to \mathbb{C}$, by

$$(B_w f)(c) = \frac{1}{q_w} \sum_{d\in\mathcal{C}_w(c)} f(d) \qquad \text{for all } c \in \mathcal{C}.$$

In Theorem 2.2.1 we show that the linear span of these operators over $\mathbb{C}$ forms an associative algebra $\mathscr{B}$, which is isomorphic to a suitably parametrised Hecke algebra (put very briefly, Hecke algebras may be considered as certain deformations of the group algebra of a Coxeter group). This generalises results in [**15**, Chapter 6], where an analogous algebra is studied under the assumption that there is a group G (of label preserving simplicial complex automorphisms) acting *strongly transitively* on the building. We note that it is simple to see that all buildings admitting such a group are regular. However not all regular buildings admit such a group, for example, the $\tilde{A}_2$ buildings with trivial automorphism group. Our results uniformly cover all regular buildings.

We note that some of our results in Chapter 2 are proved in [**46**] in the quite different context of *association schemes*.

The Algebra $\mathscr{A}$. A main part of this thesis is the study of an algebra $\mathscr{A}$ of *vertex set averaging operators* (described below), and the homomorphisms $h : \mathscr{A} \to \mathbb{C}$. Chapter 3 gives some preliminary material for this study. We give a brief discussion of root systems, hyperplane arrangements and Weyl groups. We describe how a root system can be associated to each locally finite regular affine building (the non-reduced root systems of type BC_n play a role here). All of the material of this chapter is known.

Chapters 4 and 5 are devoted to the study of the algebra $\mathscr{A}$. Let us give an historical discussion to motivate this study. Let $G = PGL(n+1, F)$ where F is a local field, and let $K = PGL(n+1, \mathcal{O})$, where $\mathcal{O}$ is the ring of integers in F. The space of bi-K-invariant compactly supported functions on G forms a commutative convolution algebra (see [**23**, Corollary 3.3.7] for example). Associated to G there is a building $\mathscr{X}$ (of type $\widetilde{A}_n$), and the above algebra is isomorphic to an algebra $\mathscr{A}$ of averaging operators defined on the space of all functions $G/K \to \mathbb{C}$. In [**10**] it was shown that these averaging operators may be defined in a natural way using only the geometric and combinatorial properties of $\mathscr{X}$, hence removing the group G entirely from the discussion. For example, in the case $n = 1$, $\mathscr{X}$ is a homogeneous tree and $\mathscr{A}$ is the algebra generated by the operator A_1, where for each vertex x, $(A_1 f)(x)$ is the average value of f over the neighbours of x.

In [**10**], using this geometric approach, Cartwright showed that $\mathscr{A}$ is a commutative algebra, and that the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ can be expressed in terms of the classical Hall-Littlewood polynomials of [**25**, III, §2]. It was not assumed that $\mathscr{X}$ was constructed from a group G (although there always is such a group when $n \geq 3$). Although not entirely realised in [**10**], as a consequence of our work here we see that the commutativity of the algebra $\mathscr{A}$ and the description of the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ follow from the fact that $\mathscr{A}$ is isomorphic to the center of an appropriately parametrised affine Hecke algebra.

One aim of this thesis is to generalise the above results to arbitrary affine buildings. In the main body of text we will assume that our buildings are of irreducible type, although the general case is dealt with in Appendix A.

Let $\mathscr{X}$ be a regular affine building, and let V denote the vertex set of $\mathscr{X}$. In Definition 3.8.1 we define a subset $V_P \subseteq V$ of *good* vertices, which, for the sake of this simplified description, can be thought of as the *special vertices* of $\mathscr{X}$.

Let R be the root system associated to $\mathscr{X}$ (as in Chapter 3), and let P be the *coweight* lattice of R, and write P^+ for the set of dominant coweights. For each $x \in V_P$ and $\lambda \in P^+$ we define (Definition 4.2.2) sets $V_\lambda(x)$ in such a way that $\{V_\lambda(x)\}_{\lambda \in P^+}$ forms a partition of V_P. In Theorem 4.3.4 we show that regularity implies that the cardinalities $|V_\lambda(x)|$, $\lambda \in P^+$, are independent of the particular $x \in V_P$, and as such we write $N_\lambda = |V_\lambda(x)|$. For each $\lambda \in P^+$ we define an operator A_λ, acting on the space of functions $f : V_P \to \mathbb{C}$, by

$$(A_\lambda f)(x) = \frac{1}{N_\lambda} \sum_{y \in V_\lambda(x)} f(y) \qquad \text{for all } x \in V_P.$$

These operators generalise the operators studied in [**10**].

Let $\mathscr{A}$ be the linear span of $\{A_\lambda\}_{\lambda\in P^+}$ over $\mathbb{C}$. In Theorem 4.4.8 we show (using only regularity) that $\mathscr{A}$ is a commutative algebra.

To give a more thorough description of the algebra $\mathscr{A}$ we need some affine Hecke algebra theory, and this is given in Chapter 5. In Section 5.1 we give the definition of affine Hecke algebras (we use $\tilde{\mathscr{H}}$ to denote such an algebra), and provide some very basic properties of these algebras. In Section 5.2 we describe the center $Z(\tilde{\mathscr{H}})$ of $\tilde{\mathscr{H}}$, and discuss the *Macdonald spherical functions* $P_\lambda(x)$, $\lambda\in P^+$, which are certain special elements of $Z(\tilde{\mathscr{H}})$ which arise naturally in connection with the *Satake isomorphism*.

In Theorem 5.3.5 we prove the important result that $\mathscr{A}$ is isomorphic to $Z(\tilde{\mathscr{H}})$, with the isomorphism determined by $A_\lambda\mapsto P_\lambda(x)$. It is a standard fact that $Z(\tilde{\mathscr{H}})\cong\mathbb{C}[P]^{W_0}$ (here $\mathbb{C}[P]^{W_0}$ is the algebra of W_0-invariant elements of the group algebra of the abelian group P, and W_0 is the Weyl group of R), and so $\mathscr{A}\cong\mathbb{C}[P]^{W_0}$, giving a very concrete description of the algebra $\mathscr{A}$.

This isomorphism serves two purposes. Firstly it gives us an essentially complete understanding of the algebra $\mathscr{A}$. For example, in Theorem 5.3.6 we use rather simple facts about the Macdonald spherical functions to show that $\mathscr{A}$ is generated by $\{A_{\lambda_i}\}_{i\in I_0}$ where $\{\lambda_i\}_{i\in I_0}$ is a set of fundamental coweights of R. On the other hand, since $\mathscr{A}$ is a purely combinatorial object, the above isomorphism gives a nice combinatorial description of $Z(\tilde{\mathscr{H}})$ when a suitable building exists. In particular the *structure constants* $c_{\lambda,\mu;\nu}$ that appear in

$$P_\lambda(x)P_\mu(x)=\sum_{\nu\in P^+}c_{\lambda,\mu;\nu}P_\nu(x)\qquad\text{are}\qquad c_{\lambda,\mu;\nu}=\frac{N_\nu}{N_\lambda N_\mu}|V_\lambda(x)\cap V_{\mu^*}(y)|,$$

for some $\mu^*\in P^+$ (depending only on μ in a simple way). This shows that (when a suitable building exists) $c_{\lambda,\mu;\nu}\geq 0$.

In Theorem 5.4.2 we extend this result by showing that the $c_{\lambda,\mu;\nu}$'s are (up to positive normalisation factors) polynomials in the variables $\{q_s-1\}_{s\in S}$ with nonnegative integer coefficients (even when no building exists). This generalises the main theorem in [**30**], where the corresponding result for the A_n case (where the $c_{\lambda,\mu,\nu}$'s are certain *Hall polynomials*) is proved. Thus we see how to construct a polynomial hypergroup from the structure constants $c_{\lambda,\mu,\nu}$ as in [**4**] (see also [**22**]).

Since the submission of this thesis we have learnt that Theorem 5.4.2 has been proved independently by Schwer in [**38**], where a formula for $c_{\lambda,\mu;\nu}$ is given.

We note that our results concerning the algebra $\mathscr{A}$ give interesting examples of *association schemes* (see Remark 3.8.3 and Remark 4.4.9), generalising the well known construction of association schemes from *infinite distance regular graphs*.

The Algebra Homomorphisms $h:\mathscr{A}\to\mathbb{C}$. Chapters 6 and 7 study the algebra homomorphisms $h:\mathscr{A}\to\mathbb{C}$. In Chapter 6 we use the isomorphism $\mathscr{A}\cong\mathbb{C}[P]^{W_0}$ to give the first of two formulae for these homomorphisms. This formula is in terms of the Macdonald spherical functions of [**23**], and so we shall call it the *Macdonald formula*.

We may regard $\mathscr{A}$ as a subalgebra of the C^*-algebra of bounded linear operators on the Hilbert space $\ell^2(V_P)$ of square summable functions $f : V_P \to \mathbb{C}$, and we write $\mathscr{A}_2$ for the closure of $\mathscr{A}$ in this algebra. In Section 6.2 we study the Gelfand map $\mathscr{A}_2 \to \mathscr{C}(M_2)$, where $M_2 = \mathrm{Hom}(\mathscr{A}_2, \mathbb{C})$, and discuss the Plancherel measure of $\mathscr{A}_2$. In Section 6.3 we compute the Plancherel measure, using results from [**23**, Chapter V]. We then apply the theory from Section 6.2 to compute M_2. This *spherical harmonic analysis* is used in the important Theorem 7.7.2 (see below), as well as in Chapter 8, where we study random walks on buildings.

A sector in an affine building is the analogue of a *ray* in a (homogeneous) tree. The *boundary* Ω of an affine building is the set of equivalence classes of sectors, where two sectors are declared to be equivalent if and only if their intersection contains a sector. In Chapter 7 we show that there is a natural topology on Ω, making it into a totally disconnected compact Hausdorff space. In Theorem 7.6.4 we give a second formula for the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ in terms of an integral over Ω. This formula is an analogue of the formula in [**23**, Proposition 3.3.1], which expresses the *zonal spherical functions on a group G of p-adic type* as an integral over K, where K is a certain compact subgroup of G. For example, when $G = SL(n+1, F)$, where F is a p-adic field, K is $SL(n+1, \mathcal{O})$, where $\mathcal{O}$ is the ring of integers in F. In Theorem 7.7.2 we show that the Macdonald and integral formulae for the algebra homomorphisms agree.

Random Walks on Affine Buildings. In Chapter 8 we study *radial random walks* on affine buildings. These are random walks in which the transition probabilities satisfy $p(x, y) = p(x', y')$ whenever $y \in V_\lambda(x)$ and $y' \in V_\lambda(x')$ for some $\lambda \in P^+$. We apply the results obtained concerning the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ to prove a *local limit theorem* for these walks (that is, we give an asymptotic formula for the k-step transition probabilities $p^{(k)}(x, y)$). This generalises results in [**12**].

Appendices. The main body of text is followed by a series of appendices. The study of the algebra $\mathscr{A}$ assumed irreducibility, and in Appendix A we demonstrate that this assumption can be removed without too much difficulty. In Appendix B we provide the proofs of some results whose proofs were omitted from the main body of text.

Appendix C gives an interesting alternative proof of the Macdonald formula in the dimension 1 and 2 cases, using elementary methods (that is, without the Hecke algebra machinery). The calculations in this appendix follow [**11**], where $\tilde{A}_2$ buildings are studied. Here we give the calculations for affine buildings of types BC_1, A_1, BC_2, C_2 and G_2. We will also list the relevant results (from [**11**]) for affine buildings of type A_2 for completeness.

In Appendix D we list some relevant root data for the irreducible root systems, and in Appendix E we describe the parameter systems of locally finite regular affine buildings of irreducible type.

Comparison with results of Macdonald. Let us conclude this introduction with a comparison between our results and those in [**23**]. Let G be a group of p-adic type, with a maximal compact subgroup K, as in [**23**, §2.7]. Associated to G there is an irreducible (but not necessarily reduced) root system R, as in [**23**, Chapter II]. As mentioned above, the typical example here is $G = SL(n+1, F)$ and $K = SL(n+1, \mathcal{O})$, where F is a p-adic field and $\mathcal{O}$ is the ring of integers in F. In this case R is a root system of type A_n.

A function $f : G \to \mathbb{C}$ is called *bi-K-invariant* if $f(gk) = f(kg) = f(g)$ for all $g \in G$ and $k \in K$. Let $\mathscr{L}(G, K)$ be the space of continuous, compactly supported bi-K-invariant functions on G. In [**23**, Theorem 3.3.6], Macdonald shows that $\mathscr{L}(G, K) \cong \mathbb{C}[Q]^{W_0}$ (the subalgebra of W_0-invariant elements of the group algebra of the coroot lattice Q of R), and thus $\mathscr{L}(G, K)$ is a commutative convolution algebra.

A function $\phi : G \to \mathbb{C}$ is called a *zonal spherical function relative to K* if

(i) $\phi(1) = 1$,
(ii) ϕ is bi-K-invariant and continuous, and
(iii) $f * \phi = \lambda_f \phi$ for all $f \in \mathscr{L}(G, K)$, where λ_f is a scalar

(see [**23**, Proposition 1.2.5]). In [**23**, Proposition 3.3.1] Macdonald gives a formula for the zonal spherical functions in terms of an integral over K, and in [**23**, Theorem 4.1.2] he uses this integral formula to obtain a second 'summation' formula for the spherical functions in terms of a sum over W_0 of rational functions.

The group G acts strongly transitively on its Bruhat-Tits building $\mathscr{X}$ [**23**, Lemma 2.4.4], which is locally finite, regular and affine, although Macdonald makes little use of this building. The contents of this thesis lead us to the conclusion that, rather than playing a relatively minor role, the building theoretic elements alone determine the nature of the algebra $\mathscr{L}(G, K)$ and the zonal spherical functions. Moreover, since there are (locally finite regular affine) buildings that are not the Bruhat-Tits buildings of any group, our building approach puts the results of [**23**] into a more general setting.

Let us see how our results relate to Macdonald's. Firstly, the algebra $\mathscr{L}(G, K)$ is isomorphic to a subalgebra $\mathscr{A}_Q$ of $\mathscr{A}$, spanned by the operators $\{A_\lambda \mid \lambda \in Q \cap P^+\}$. The reason that this smaller algebra occurs here is that Macdonald supposes that his groups of p-adic type are simply connected, and so for him the coweight lattice has to be replaced by the coroot lattice, and thus the set V_P is replaced by the smaller set V_Q consisting of all those vertices of one special type. See [**11**, Propositions 2.4 and 2.5] for details in the $R = A_2$ case.

In Theorem 5.3.5 we show that $\mathscr{A} \cong \mathbb{C}[P]^{W_0}$, and in Proposition 5.3.7 we deduce that $\mathscr{A}_Q \cong \mathbb{C}[Q]^{W_0}$, thus proving the analogue of [**23**, Theorem 3.3.6].

The zonal spherical functions on G correspond to the spherical functions on V_Q (see Definition 7.6.2), which in turn correspond to the algebra homomorphisms $h : \mathscr{A}_Q \to \mathbb{C}$ (see Proposition 7.6.3). Our analogue of Macdonald's integral formula is Theorem 7.6.4 (see also Corollary 7.6.5), and our analogue of his summation formula is (6.1.1). Both of our formulae are proved in the context of the larger algebra $\mathscr{A}$. We discuss a method for deducing the corresponding results for certain subalgebras $\mathscr{A}_L$ (including $\mathscr{A}_Q$) in Section 4.5 and Proposition 5.3.7.

CHAPTER 1

General Building Theory

In this chapter we give a discussion of buildings, both from the chamber system point of view, and from the simplicial complex point of view.

1.1. Coxeter Groups

Let I be an index set, which we assume throughout is finite, and for each pair $i, j \in I$ let $m_{i,j}$ be an integer or ∞ such that $m_{i,j} = m_{j,i} \geq 2$ for all $i \neq j$, and $m_{i,i} = 1$ for all $i \in I$. We call $M = (m_{i,j})_{i,j \in I}$ a *Coxeter matrix*. The *Coxeter group* of type M is the group

$$W = \langle \{s_i\}_{i \in I} \mid (s_i s_j)^{m_{i,j}} = 1 \text{ for all } i, j \in I \rangle, \tag{1.1.1}$$

where the relation $(s_i s_j)^{m_{i,j}} = 1$ is omitted if $m_{i,j} = \infty$. Writing $S = \{s_i \mid i \in I\}$, it is more precise to call (W, S) a *Coxeter system*, although we shall rarely do so.

For subsets $J \subset I$ we write W_J for the subgroup of W generated by $\{s_i\}_{i \in J}$. Given $w \in W$, we define the *length* $\ell(w)$ of w to be smallest $n \in \mathbb{N}$ such that $w = s_{i_1} \ldots s_{i_n}$, with $i_1, \ldots, i_n \in I$.

It will be useful on occasion to work with I^*, the free monoid on I. Thus elements of I^* are *words* $f = i_1 \cdots i_n$ where $i_1, \ldots, i_n \in I$, and we write $s_f = s_{i_1} \cdots s_{i_n} \in W$. An *elementary homotopy* [**35**, §2.1] is an alteration from a word of the form $f_1 p(i,j) f_2$ to a word of the form $f_1 p(j,i) f_2$, where $p(i,j) = \cdots ijij$ ($m_{i,j}$ terms). We say that the words f and f' are *homotopic* if f can be transformed into f' by a sequence of elementary homotopies, in which case we write $f \sim f'$. A word f is said to be *reduced* if it is not homotopic to a word of the form $f_1 i i f_2$ for any $i \in I$. By [**35**, Theorem 2.11] we have that $f = i_1 \cdots i_n \in I^*$ is reduced if and only if $s_f = s_{i_1} \cdots s_{i_n}$ is a reduced expression in W (that is, $\ell(s_f) = n$).

The *Coxeter graph* of W is the graph $D = D(W)$ with vertex set I, such that vertices $i, j \in I$ are joined by an edge if and only if $m_{i,j} \geq 3$. If $m_{i,j} \geq 4$ then the edge $\{i, j\}$ is labelled by $m_{i,j}$. By an *automorphism of* D we mean a permutation σ of I such that $m_{\sigma(i),\sigma(j)} = m_{i,j}$ for all $i, j \in I$. Write $\mathrm{Aut}(D)$ for the group of all automorphisms of D. An automorphism $\sigma \in \mathrm{Aut}(D)$ induces a group automorphism of W, which we also denote by σ, by

$$\sigma(w) = s_{\sigma(i_1)} \cdots s_{\sigma(i_n)} \tag{1.1.2}$$

whenever $s_{i_1} \cdots s_{i_n}$ is an expression for W (not necessarily reduced).

1.2. Chamber Systems

A set $\mathcal{C}$ is called a *chamber system over I* if each $i \in I$ determines a partition of $\mathcal{C}$, two elements in the same block of this partition being called *i-adjacent*. The elements of $\mathcal{C}$ are called *chambers*, and we write $c \sim_i d$ to mean that the chambers c and d are i-adjacent. By a *gallery* of type $i_1 \cdots i_n \in I^*$ in $\mathcal{C}$ we mean a sequence $c_0, \ldots, c_n$ of chambers such that $c_{k-1} \sim_{i_k} c_k$ and $c_{k-1} \neq c_k$ for $1 \leq k \leq n$. If we remove the condition that $c_{k-1} \neq c_k$ for all $1 \leq k \leq n$, we call the sequence $c_0, \ldots, c_n$ a *pre-gallery*.

A gallery of type $j_1 \cdots j_n$ with $j_1, \ldots, j_n \in J \subset I$ is called a *J-gallery*. For $c \in \mathcal{C}$ we write $R_J(c)$ for the *J-residue* of c, that is,

$$R_J(c) = \{d \in \mathcal{C} \mid \text{there exists a } J\text{-gallery from } c \text{ to } d\}. \tag{1.2.1}$$

A chamber system $\mathcal{C}$ over I is said to be *thick* if for each $c \in \mathcal{C}$ and $i \in I$ there exist at least two distinct chambers $d \neq c$ such that $d \sim_i c$, and $\mathcal{C}$ is called *thin* if for each $c \in \mathcal{C}$ and $i \in I$ there exists exactly one chamber $d \neq c$ such that $d \sim_i c$.

If $\mathcal{C}$ and $\mathcal{D}$ are chamber systems over a common index set I, we call a map $\psi : \mathcal{C} \to \mathcal{D}$ an *isomorphism of chamber systems*, or simply an *isomorphism*, if ψ is a bijection such that $c \sim_i d$ if and only if $\psi(c) \sim_i \psi(d)$.

1.3. Simplicial Complexes

A *simplicial complex* with *vertex set* X is a collection Σ of finite subsets of X (called *simplices*) such that the singleton $\{x\}$ is a simplex for each $x \in X$, and every subset of a simplex σ is a simplex (called a *face of* σ). If σ is a simplex which is not a proper subset of any other simplex, then we call σ a *maximal simplex* of Σ. The *dimension* of a simplex σ is $|\sigma| - 1$, where $|\sigma|$ denotes the cardinality of the set σ. We will always assume that the simplices of a simplicial complex have bounded dimension, and so every simplex is contained in a maximal simplex.

A *labelled simplicial complex* with vertex set X is a simplicial complex equipped with a set I of *types*, and a *type map* $\tau : X \to I$ such that the restriction of τ to any maximal simplex is a bijection. The *type* of a simplex $\sigma = \{x_1, \ldots, x_k\}$ is $\{\tau(x_1), \ldots, \tau(x_k)\}$, and the *cotype* of σ is $I \backslash \{\tau(x_1), \ldots, \tau(x_k)\}$.

It is clear that each maximal simplex of a labelled simplicial complex must have the same dimension, d, say. In this case we say that Σ has *dimension* d, and we call the maximal simplices *chambers*. We write $\mathcal{C}(\Sigma)$ for the set of all chambers of Σ. A *panel* of a d-dimensional labelled simplicial complex Σ is a simplex π of dimension $d - 1$. It is clear that the cotype of π is i (more accurately, $\{i\}$) for some $i \in I$.

An *isomorphism of simplicial complexes* is a bijection of the vertex sets that maps simplices, and only simplices, to simplices. If both simplicial complexes are labelled by the same set, then an isomorphism which preserves types is said to be *type preserving*.

1.4. Connection Between Chamber Systems and Simplicial Complexes

Given a chamber system $\mathcal{C}$ over I we can construct a labelled simplicial complex as follows. For each $i \in I$, form the set

$$X_i = \{R_{I\backslash\{i\}}(c) \mid c \in \mathcal{C}\},$$

and let X be the disjoint union over $i \in I$ of these sets. Call elements of X *vertices*, and define a type map $\tau : X \to I$ by $\tau(x) = i$ if $x \in X_i$. Declare simplices to be the subsets of the sets

$$\{R_{I\backslash\{i\}}(c) \mid i \in I\}, \qquad c \in \mathcal{C}. \tag{1.4.1}$$

This produces a labelled simplicial complex, with maximal simplices as in (1.4.1).

Conversely, given a labelled simplicial complex Σ with vertex set X and type map $\tau : X \to I$, we can construct a chamber system by declaring maximal simplices C and D of Σ to be i-adjacent (where $i \in I$) if either $C = D$ or if all the vertices of C and D are the same except for those of type i.

Under certain weak hypotheses the above operations are mutually inverse, up to isomorphisms of chamber systems and type preserving isomorphisms of labelled simplicial complexes. We refer the reader to [**9**, Proposition 1.4] for details. Put briefly, it suffices to assume that

(i) our labelled simplicial complexes satisfy the condition that if $C, D \in \mathcal{C}(\Sigma)$ and $\sigma \subset C \cap D$, then there exists a gallery $C = C_0, \ldots, C_n = D$ with $\sigma \subset C_k$ for each $0 \leq k \leq n$, and

(ii) that our chamber systems satisfy the condition that for all $c, d \in \mathcal{C}$ and $J \subset I$, if $R_{I\backslash\{i\}}(c) = R_{I\backslash\{i\}}(d)$ for each $i \in I\backslash J$, then $R_J(c) = R_J(d)$.

We remark that these conditions are satisfied for Coxeter complexes (Section 1.5) and buildings (Section 1.6) (see [**9**]).

1.5. Coxeter Complexes

To each Coxeter group W over I we associate a (thin) chamber system $\mathcal{C}(W)$, called the *Coxeter complex* of W, by taking the elements $w \in W$ as chambers, and for each $i \in I$ define i-adjacency by declaring $w \sim_i w$ and $w \sim_i ws_i$.

By the discussion in Section 1.4, we may also describe $\mathcal{C}(W)$ as a labelled simplicial complex $\Sigma(W)$, whose vertex set is the (automatically disjoint) union over $i \in I$ of the sets $X_i = \{wW_{I\backslash\{i\}} \mid w \in W\}$. If $x \in X_i$ we say that x has type i, and write $\tau(x) = i$. Simplices of $\Sigma(W)$ are subsets of maximal simplices, which are defined to be sets of the form $\{wW_{I\backslash\{i\}} \mid i \in I\}$, $w \in W$. It is not difficult to see that $\Sigma(W)$ and $\mathcal{C}(W)$ satisfy conditions (i) and (ii) of Section 1.4.

Example 1.5.1. Let W be a Coxeter group of type $\tilde{A}_2$, that is,

$$W = \langle s_0, s_1, s_2 \mid s_0^2 = s_1^2 = s_2^2 = (s_0s_1)^3 = (s_0s_2)^3 = (s_1s_2)^3 = 1\rangle.$$

The Coxeter complex of W is shown in Figure 1.5.1.

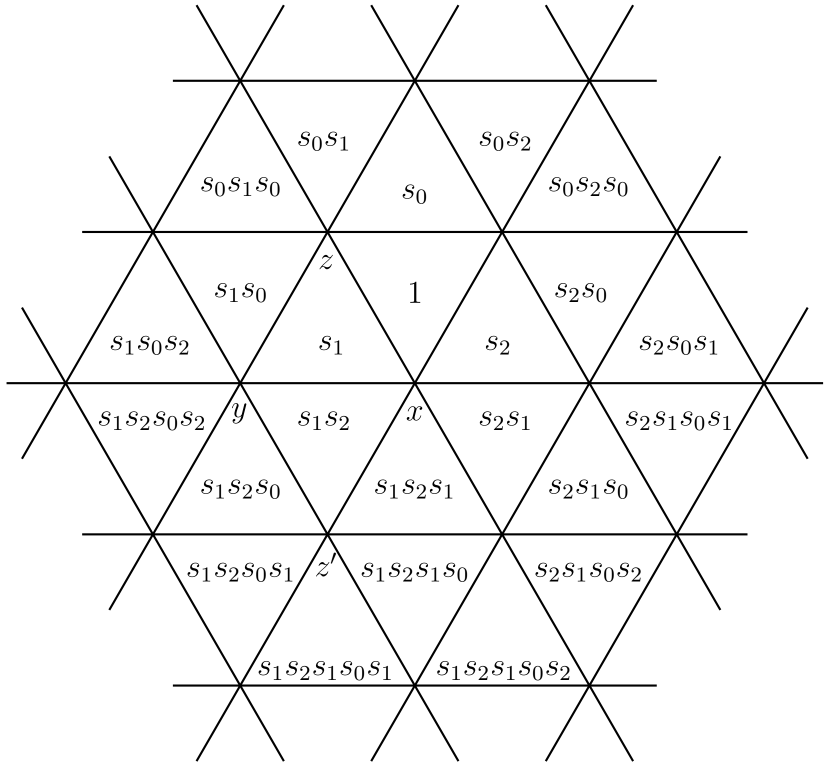


Figure 1.5.1

As a *chamber system*, $\mathcal{C}(W)$ consists of the elements of W, which are shown in Figure 1.5.1 as triangles. To consider $\mathcal{C}(W)$ as a *labelled simplicial complex* $\Sigma(W)$, we define vertices to be sets of the form $R_{I\setminus\{i\}}(w) = wW_{I\setminus\{i\}}$, $w \in W$ and $i \in I = \{0, 1, 2\}$. For example, the *vertices* x, y and z in Figure 1.5.1 are really the sets

$$\begin{aligned} x &= s_1W_{I\setminus\{0\}} = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1 = s_2s_1s_2\} \\ y &= s_1W_{I\setminus\{1\}} = \{s_1, s_1s_0, s_1s_2, s_1s_0s_2, s_1s_2s_0, s_1s_2s_0s_2 = s_1s_0s_2s_0\} \\ z &= s_1W_{I\setminus\{2\}} = \{1, s_1, s_0, s_1s_0, s_0s_1, s_0s_1s_0 = s_1s_0s_1\}, \end{aligned} \tag{1.5.1}$$

and have types 0, 1 and 2 respectively. Note that we could write $x = wW_{I\setminus\{0\}}$ for any $w \in s_1W_{I\setminus\{0\}}$, and similarly for y and z. We have chosen the representations in (1.5.1) to make it clear that

$$\{x, y, z\} = \{s_1W_{I\setminus\{i\}} \mid i \in I\}$$

is a maximal simplex.

Finally, note how adjacency works in the simplicial complex context. The maximal simplices $\{x, y, z\}$ and $\{x, y, z'\}$ are 2-adjacent, for they have all vertices in common except for those of type 2. That is, they share a panel $\{x, y\} = s_1 W_{I\setminus\{0,1\}} = \{s_1, s_1 s_2\}$ of cotype 2.

1.6. Building Definitions

We now give two definitions of buildings. The first definition is in terms of chamber systems, and the second is in terms of simplicial complexes. Although it is certainly not obvious, the two definitions are equivalent, via the conversion between chamber systems and labelled simplicial complexes described in Section 1.4 (see [**9**] for details).

Definition 1.6.1 ([**40**],[**35**]). Let M be the Coxeter matrix of a Coxeter group W over I. Then $\mathscr{X}$ is a *building of type* M if

(i) $\mathscr{X}$ is a chamber system over I such that for each $c \in \mathscr{X}$ and $i \in I$, there is a chamber $d \neq c$ in $\mathscr{X}$ such that $d \sim_i c$, and
(ii) there exists a W*-distance function* $\delta : \mathscr{X} \times \mathscr{X} \to W$ such that if f is a reduced word then $\delta(c, d) = s_f$ if and only if c and d can be joined by a gallery of type f.

Definition 1.6.2 ([**41**],[**7**]). Let W be a Coxeter group of type M. A *building of type* M is a nonempty simplicial complex $\mathscr{X}$ which contains a family of subcomplexes called *apartments* such that

(i) each apartment is isomorphic to $\Sigma(W)$,
(ii) given any two maximal simplices of $\mathscr{X}$ there is an apartment containing both,
(iii) given any two apartments $\mathcal{A}$ and $\mathcal{A}'$ that contain a common maximal simplex, there exists an isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ fixing $\mathcal{A} \cap \mathcal{A}'$ pointwise.

We remark that Definition 1.6.2(iii) can be replaced with the following (see [**7**, p.76]).

(iii)′ If $\mathcal{A}$ and $\mathcal{A}'$ are apartments both containing simplices ρ and σ, then there is an isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ fixing ρ and σ pointwise.

We will always use the symbol $\mathscr{X}$ to denote a building, and it will be clear from the context if $\mathscr{X}$ is to be regarded as a chamber system, or as a simplicial complex. We write $\mathcal{C} = \mathcal{C}(\mathscr{X})$ for the set of all chambers of $\mathscr{X}$, and $V = V(\mathscr{X})$ for the set of all vertices of $\mathscr{X}$. We often say that $\mathscr{X}$ is a *building of type* W rather than a building of type M. The *rank* of a building of type M is the cardinality of the index set I.

It is clear that Coxeter complexes are (thin) buildings. Indeed, a building is thin if and only if it is isomorphic to a Coxeter complex.

1.7. Regularity and Parameter Systems

In this section we write $\mathscr{X}$ for a building of type M, with associated Coxeter group W over index set I. We will assume that $\mathscr{X}$ is *locally finite*, by which we mean that $|\{d \in \mathcal{C} \mid d \sim_i c\}| < \infty$ for all $i \in I$ and $c \in \mathcal{C}$.

For each $c \in \mathcal{C}$ and $w \in W$, let

$$\mathcal{C}_w(c) = \{d \in \mathcal{C} \mid \delta(c,d) = w\}\,. \tag{1.7.1}$$

Observe that for each $c \in \mathcal{C}$, the family $\{\mathcal{C}_w(c)\}_{w\in W}$ forms a partition of $\mathcal{C}$, and for $s = s_i$,

$$\mathcal{C}_s(c) = \{d \in \mathcal{C} \mid d \sim_i c \text{ and } d \neq c\},$$

as illustrated in Figure 1.7.1, where $\mathcal{C}_s(c) = \{d_1, d_2, d_3, d_4\}$.

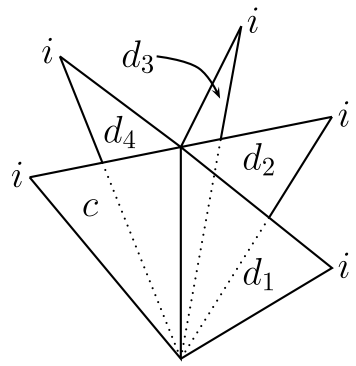


FIGURE 1.7.1

We say that $\mathscr{X}$ is *regular* if for each $s \in S$, $|\mathcal{C}_s(c)|$ is independent of $c \in \mathcal{C}$. If $\mathscr{X}$ is a regular building we define $q_s = |\mathcal{C}_s(c)|$ for each $s \in S$ (this is independent of $c \in \mathcal{C}$ by definition), and we call $\{q_s\}_{s\in S}$ the *parameter system of the building*. Local finiteness implies that $q_s < \infty$ for all $s \in S$. We write q_i in place of q_{s_i} for $i \in I$. In Figure 1.7.1, $q_i = 4$.

The two main results of this section are Proposition 1.7.1(ii), where we give a method for finding relationships that must hold between the parameters of buildings, and Theorem 1.7.4, where we generalise [**37**, Proposition 3.4.2] and show that all thick buildings with no rank 2 residues of type $\widetilde{A}_1$ are regular.

PROPOSITION 1.7.1. *Let $\mathscr{X}$ be a locally finite regular building.*

(i) $|\mathcal{C}_w(c)| = q_{i_1}q_{i_2}\cdots q_{i_n}$ *whenever* $w = s_{i_1}\cdots s_{i_n}$ *is a reduced expression, and*
(ii) $q_i = q_j$ *whenever* $m_{i,j} < \infty$ *is odd.*

PROOF. We first prove (i). The result is true when $\ell(w) = 1$ by regularity. We claim that whenever $s = s_i \in S$ and $\ell(ws) = \ell(w) + 1$,

$$\mathcal{C}_{ws}(c) = \bigcup_{d\in\mathcal{C}_w(c)} \mathcal{C}_s(d) \tag{1.7.2}$$

where the union is disjoint, from which the result follows by induction.

First suppose that $a \in \mathcal{C}_{ws}(c)$ where $\ell(ws) = \ell(w) + 1$. Then there exists a minimal gallery $c = c_0, \ldots, c_k = a$ of type fi (where $w = s_f$ with $f \in I^*$ reduced) from c to a, and in particular $a \in \mathcal{C}_s(c_{k-1})$ where $c_{k-1} \in \mathcal{C}_w(c)$. On the other hand, if $a \in \mathcal{C}_s(d)$ for some $d \in \mathcal{C}_w(c)$ then $a \in \mathcal{C}_{ws}(c)$ since $\ell(ws) = \ell(w) + 1$, and so equality holds in (1.7.2). To see that the union is disjoint, suppose that $d, d' \in \mathcal{C}_w(c)$ and that $\mathcal{C}_s(d) \cap \mathcal{C}_s(d') \neq \emptyset$. Then if $d' \neq d$ we have $d' \in \mathcal{C}_s(d)$, and thus $d' \in \mathcal{C}_{ws}(c)$, a contradiction.

To prove (ii), suppose $m_{i,j} < \infty$ is odd. Since $s_i s_j s_i \cdots = s_j s_i s_j \cdots$ ($m_{i,j}$ factors on each side), by (i) we have $q_i q_j q_i \cdots = q_j q_i q_j \cdots$ ($m_{i,j}$ factors on each side), and the result follows. □

Corollary 1.7.2. *Let $\mathscr{X}$ be a locally finite regular building of type W. If $s_j = w s_i w^{-1}$ for some $w \in W$ then $q_i = q_j$.*

Proof. By [**5**, IV, §1, No.3, Proposition 3], $s_j = w s_i w^{-1}$ for some $w \in W$ if and only if there exists a sequence $s_{i_1}, \ldots, s_{i_p}$ such that $i_1 = i$, $i_p = j$, and $m_{i_k, i_{k+1}}$ is finite and odd for each $1 \leq k < p$. The result now follows from Proposition 1.7.1(ii). □

Proposition 1.7.1(i) justifies the notation $q_w = q_{i_1} \cdots q_{i_n}$ whenever $s_{i_1} \cdots s_{i_n}$ is a reduced expression for w; it is independent of the particular reduced expression chosen. Clearly we have $q_{w^{-1}} = q_w$ for all $w \in W$.

Example 1.7.3. Using Proposition 1.7.1(ii) it is now a simple exercise to describe the relations between the parameters of any given (locally finite) regular building. For example, in a building of type $\bullet\overset{4}{\text{——}}\bullet\text{——}\bullet$ (with the nodes labelled $0, 1$ and 2 from left to right) we must have $q_1 = q_2$ since $m_{1,2} = 3$ is odd. Note that we cannot relate q_0 to q_1 since $m_{0,1} = 4$ is even.

The following theorem seems to be well known (see [**37**, Proposition 3.4.2] for the case $|W| < \infty$), but we have been unable to find a direct proof in the literature. For the sake of completeness we will provide a proof here.

Theorem 1.7.4. *Let $\mathscr{X}$ be a thick building such that $m_{i,j} < \infty$ for each pair $i, j \in I$. Then $\mathscr{X}$ is regular.*

Before giving the proof of Theorem 1.7.4 we make some preliminary observations. First we note that the assumption that $m_{i,j} < \infty$ in Theorem 1.7.4 is essential, for $\widetilde{A}_1$ buildings are not in general regular, as they are just trees with no end vertices. Secondly we note that Theorem 1.7.4 shows that most 'interesting' buildings are regular, for examining the Coxeter graphs of the (irreducible) affine Coxeter groups, for example, we see that $m_{i,j} = \infty$ only occurs in $\widetilde{A}_1$ buildings. Thus regularity is not a very restrictive hypothesis.

Recall that for $m \geq 2$ or $m = \infty$ a *generalised m-gon* is a connected bipartite graph with diameter m and girth $2m$. By [**35**, Proposition 3.2], a building of type $\bullet\overset{m}{\text{——}}\bullet$ is a generalised m-gon, and vice versa (where the edge set of the m-gon is taken to be the chamber set of the building, and vice versa).

In a generalised m-gon we define the *valency* of a vertex v to be the number of edges that contain v, and we call the generalised m-gon *thick* if every vertex has valency at least 3. By [**35**, Proposition 3.3], in a thick generalised m-gon with $m < \infty$, vertices in the same partition have the same valency. In the statement of [**35**, Proposition 3.3], the assumption $m < \infty$ is inadvertently omitted. The result is in fact false if $m = \infty$, for a thick generalised ∞-gon is simply a tree in which each vertex has valency at least 3.

Proof of Theorem 1.7.4. For each $c \in \mathcal{C}$ and each $i \in I$, let $q_i(c) = |\mathcal{C}_{s_i}(c)|$, so by thickness, $q_i(c) > 1$. We will show that $q_i(c) = q_i(d)$ for all $c, d \in \mathcal{C}$ and for all $i \in I$.

Let $c \in \mathcal{C}$ and $i \in I$ be fixed. By [**35**, Theorem 3.5], for $j \in I$ the residue $R_{\{i,j\}}(c)$ is a thick building of type $M_{\{i,j\}}$ which is in turn a thick generalised $m_{i,j}$-gon, by [**35**, Proposition 3.2]. Thus, since $m_{i,j} < \infty$ by assumption, [**35**, Proposition 3.3] implies that

$$q_i(d) = q_i(c) \qquad\qquad \text{for all } d \in R_{\{i,j\}}(c)\,. \tag{1.7.3}$$

Now, with c and i fixed as above, let $d \in \mathcal{C}$ be any other chamber. Suppose firstly that $d \sim_k c$ for some $k \in I$. If $k = i$, then $q_i(d) = q_i(c)$ since $\sim_i$ is an equivalence relation. So suppose that $k \neq i$. Then

$$\begin{aligned} q_i(d) + 1 &= |\{a \in \mathcal{C} : a \sim_i d\}| \\ &= |\{a \in R_{\{i,k\}}(d) : a \sim_i d\}| \\ &= |\{a \in R_{\{i,k\}}(d) : a \sim_i c\}| && \text{by (1.7.3)} \\ &= |\{a \in R_{\{i,k\}}(c) : a \sim_i c\}| && \text{since } R_{\{i,k\}}(d) = R_{\{i,k\}}(c) \\ &= |\{a \in \mathcal{C} : a \sim_i c\}| = q_i(c) + 1\,, \end{aligned}$$

and so $q_i(d) = q_i(c)$. Induction now shows that $q_i(d)$ is independent of the particular $d \in \mathcal{C}$, and so the building is regular. □

Remark 1.7.5. The description of parameter systems given in this section by no means comes close to *classifying* the parameter systems of buildings. For example, it is an open question as to whether thick $\widetilde{A}_2$ buildings exist with parameters that are not prime powers. By the free construction of certain buildings given in [**34**] this is equivalent to the corresponding question concerning the parameters of projective planes (generalised 3-gons). See [**3**, Section 6.2] for a discussion of the known parameter systems of generalised 4-gons.

We conclude this chapter by recording a definition for later reference.

Definition 1.7.6. Let $\{q_s\}_{s\in S}$ be a set of indeterminates such that $q_{s'} = q_s$ whenever $s' = wsw^{-1}$ for some $w \in W$. Then [**5**, IV, §1, No.5, Proposition 5] implies that for $w \in W$, the monomial $q_w = q_{s_{i_1}} \cdots q_{s_{i_n}}$ is independent of the particular reduced decomposition $w = s_{i_1} \cdots s_{i_n}$ of w. If U is a finite subset of W, the *Poincaré polynomial* $U(q)$ of U is

$$U(q) = \sum_{w \in U} q_w\,.$$

Usually the set $\{q_s\}_{s\in S}$ will be the parameters of a building (see Corollary 1.7.2).

CHAPTER 2

Chamber Set Averaging Operators

Let $\mathscr{X}$ be a locally finite regular building, considered as a chamber system, as in Definition 1.6.1. In this chapter we define *chamber set averaging operators*, acting on the space of all functions $f : \mathcal{C} \to \mathbb{C}$, and study an associated algebra $\mathscr{B}$. Our results here generalise the results in [**15**, Chapter 6], where it is assumed that there is a group G of type preserving simplicial complex automorphisms acting *strongly transitively* on $\mathscr{X}$. This means that G acts transitively on the set of pairs $(\mathcal{A}, c)$ of apartments $\mathcal{A}$ and chambers c with $c \subset \mathcal{A}$. All buildings admitting such a group action are necessarily regular, whereas the converse is not true. Our proofs work for all locally finite regular buildings, which, by Theorem 1.7.4, includes all thick buildings with no rank 2 residues of type $\widetilde{A}_1$. It should be noted that our results also apply to thin buildings (where $q_i = 1$ for all $i \in I$), as well as to regular buildings that are neither thick nor thin (that is, buildings that have $q_i = 1$ for some but not all $i \in I$). We note that some of the results of this section are proved in [**46**] in the context of *association schemes*.

2.1. The Algebra $\mathscr{B}$

Definition 2.1.1. Recall the definition of the sets $\mathcal{C}_w(c)$ from (1.7.1). For each $w \in W$, define an operator B_w, acting on the space of all functions $f : \mathcal{C} \to \mathbb{C}$ by

$$(B_w f)(c) = \frac{1}{q_w} \sum_{d \in \mathcal{C}_w(c)} f(d) \quad \text{for all } c \in \mathcal{C}. \tag{2.1.1}$$

Since $|\mathcal{C}_w(c)| = q_w$, the operator B_w truly is an *averaging operator* in the usual sense.

Definition 2.1.2. Let $\mathscr{B}$ be the linear span over $\mathbb{C}$ of the set $\{B_w \mid w \in W\}$.

In Proposition 2.1.9 we show that $\mathscr{B}$ is an associative algebra. To do so we need to understand products $B_{w_1} B_{w_2}$ of the averaging operators.

If $\mathcal{C}' \subseteq \mathcal{C}$, write $1_{\mathcal{C}'} : \mathcal{C} \to \{0,1\}$ for the characteristic function on $\mathcal{C}'$. Since $b \in \mathcal{C}_w(a)$ if and only if $a \in \mathcal{C}_{w^{-1}}(b)$, for $w_1, w_2 \in W$ we have

$$\begin{aligned}
(B_{w_1}B_{w_2}f)(a) &= \frac{1}{q_{w_1}} \sum_{b \in \mathcal{C}_{w_1}(a)} (B_{w_2}f)(b) \\
&= \frac{1}{q_{w_1}q_{w_2}} \sum_{b \in \mathcal{C}_{w_1}(a)} \sum_{c \in \mathcal{C}_{w_2}(b)} f(c) \\
&= \frac{1}{q_{w_1}q_{w_2}} \sum_{b \in \mathcal{C}} \sum_{c \in \mathcal{C}} 1_{\mathcal{C}_{w_1}(a)}(b) 1_{\mathcal{C}_{w_2}(b)}(c) f(c) \qquad (2.1.2) \\
&= \frac{1}{q_{w_1}q_{w_2}} \sum_{c \in \mathcal{C}} \left(\sum_{b \in \mathcal{C}} 1_{\mathcal{C}_{w_1}(a)}(b) 1_{\mathcal{C}_{w_2^{-1}}(c)}(b) \right) f(c) \\
&= \frac{1}{q_{w_1}q_{w_2}} \sum_{c \in \mathcal{C}} |\mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2^{-1}}(c)|\, f(c)\,.
\end{aligned}$$

We wish to explicitly compute the above when $w_2 = s \in S$ (and so $w_2^{-1} = w_2$). Thus we have the following lemmas.

LEMMA 2.1.3. *Let $w \in W$ and $s \in S$, and fix $a \in \mathcal{C}$. Then*

$$\mathcal{C}_w(a) \cap \mathcal{C}_s(b) \neq \emptyset \Rightarrow \begin{cases} b \in \mathcal{C}_{ws}(a) & \text{if } \ell(ws) = \ell(w)+1, \text{ and} \\ b \in \mathcal{C}_w(a) \cup \mathcal{C}_{ws}(a) & \text{if } \ell(ws) = \ell(w)-1. \end{cases}$$

PROOF. Let $s = s_i$ where $i \in I$. Suppose first that $\ell(ws) = \ell(w)+1$ and that $c \in \mathcal{C}_w(a) \cap \mathcal{C}_s(b)$. Let f be a reduced word in I^* so that $s_f = w$, and so there exists a gallery from a to c of type f. Since $b \in \mathcal{C}_s(c)$, there is a gallery of type fi from a to b, which is a reduced word by hypothesis. It follows that $b \in \mathcal{C}_{ws}(a)$.

Suppose now that $\ell(ws) = \ell(w)-1$, and that $c \in \mathcal{C}_w(a) \cap \mathcal{C}_s(b)$. Since ws is not reduced, there exists a reduced word f' such that $f'i$ is a reduced word for w. This shows that there exist a minimal gallery $a = a_0, \ldots, a_m = c$ such that $a_{m-1} \in \mathcal{C}_s(c)$. Since $b \in \mathcal{C}_s(c)$ too, it follows that either $b = a_{m-1}$ or $b \in \mathcal{C}_s(a_{m-1})$. In the former case we have $b \in \mathcal{C}_{ws}(a)$ and in the latter we have $b \in \mathcal{C}_w(a)$. □

REMARK 2.1.4. The above lemma is essentially [**42**, §2.1, Axiom Bu2], where an alternative (equivalent) definition of buildings is adopted.

LEMMA 2.1.5. *Let $w \in W$ and $s \in S$. Fix $a, b \in \mathcal{C}$. Then*

$$|\mathcal{C}_w(a) \cap \mathcal{C}_s(b)| = \begin{cases} 1 & \text{if } \ell(ws) = \ell(w)+1 \text{ and } b \in \mathcal{C}_{ws}(a), \\ q_s & \text{if } \ell(ws) = \ell(w)-1 \text{ and } b \in \mathcal{C}_{ws}(a), \text{ and} \\ q_s - 1 & \text{if } \ell(ws) = \ell(w)-1 \text{ and } b \in \mathcal{C}_w(a). \end{cases}$$

PROOF. Suppose first that $\ell(ws) = \ell(w) + 1$ and that $b \in \mathcal{C}_{ws}(a)$. Thus there is a minimal gallery $a = a_0, \ldots, a_m = b$ such that $a_{m-1} \in \mathcal{C}_s(b)$. There are q_s chambers c in $\mathcal{C}_s(b)$. One of these chambers is a_{m-1}, which lies in $\mathcal{C}_w(a)$, and the remaining $q_s - 1$ lie in $\mathcal{C}_{ws}(a)$, so a_{m-1} is the only element of $\mathcal{C}_w(a) \cap \mathcal{C}_s(b)$. Thus $|\mathcal{C}_w(a) \cap \mathcal{C}_s(b)| = 1$ as claimed in this case.

Suppose now that $\ell(ws) = \ell(w) - 1$ and that $b \in \mathcal{C}_{ws}(a)$. Write $s = s_i$. Let $w = s_f$ where $f \in I^*$ is reduced. Since $\ell(ws) = \ell(w) - 1$, there exists a reduced word f' such that $f'i$ is a reduced word for w, and thus there exists a minimal gallery of type f' from a to b. Thus each $c \in \mathcal{C}_s(b)$ can be joined to a by a gallery of type $f'i \sim f$, and hence $c \in \mathcal{C}_w(a)$, verifying the count in this case.

Finally, suppose that $\ell(ws) = \ell(w) - 1$ and $b \in \mathcal{C}_w(a)$. Then, as in the proof of Lemma 2.1.3, there exists a minimal gallery $a = a_0, \ldots, a_m = b$ such that $b \in \mathcal{C}_s(a_{m-1})$. Exactly one of the q_s chambers $c \in \mathcal{C}_s(b)$ equals a_{m-1}, and thus lies in $\mathcal{C}_{ws}(a)$. For the remaining $q_s - 1$ chambers we have $c \in \mathcal{C}_s(a_{m-1})$, and thus $c \in \mathcal{C}_w(a)$, completing the proof. $\square$

THEOREM 2.1.6. *Let $w \in W$ and $s \in S$. Then*

$$B_w B_s = \begin{cases} B_{ws} & \text{when } \ell(ws) = \ell(w) + 1\,, \\ \frac{1}{q_s} B_{ws} + \left(1 - \frac{1}{q_s}\right) B_w & \text{when } \ell(ws) = \ell(w) - 1\,. \end{cases}$$

PROOF. Let us look at the case $\ell(ws) = \ell(w) - 1$. The case $\ell(ws) = \ell(w) + 1$ is similar. By (2.1.2) and Lemma 2.1.5 we have

$$B_w B_s = \frac{q_{ws}}{q_w} B_{ws} + \left(1 - \frac{1}{q_s}\right) B_w\,.$$

All that remains is to show that $\frac{q_{ws}}{q_w} = \frac{1}{q_s}$. If f is a reduced word with $s_f = w$ and $s = s_i$, the hypothesis that $\ell(ws) = \ell(w) - 1$ implies that there exists a reduced word f' such that $f'i$ is a reduced word for w. The result now follows. $\square$

COROLLARY 2.1.7. *$B_{w_1} B_{w_2} = B_{w_1 w_2}$ whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.*

COROLLARY 2.1.8. *Let $w_1, w_2 \in W$. There exist numbers $b_{w_1,w_2;w_3} \in \mathbb{Q}^+$ such that*

$$B_{w_1} B_{w_2} = \sum_{w_3 \in W} b_{w_1,w_2;w_3} B_{w_3} \qquad \text{and} \qquad \sum_{w_3 \in W} b_{w_1,w_2;w_3} = 1\,.$$

Moreover, $|\{w_3 \in W \mid b_{w_1,w_2;w_3} \neq 0\}|$ is finite for all $w_1, w_2 \in W$.

PROOF. An induction on $\ell(w_2)$ shows existence of the numbers $b_{w_1,w_2;w_3} \in \mathbb{Q}^+$ such that $B_{w_1} B_{w_2} = \sum_{w_3} b_{w_1,w_2;w_3} B_{w_3}$, and shows that only finitely many of the $b_{w_1,w_2;w_3}$'s are nonzero for fixed w_1 and w_2. Evaluating both sides at the constant function $1_{\mathcal{C}} : \mathcal{C} \to \{1\}$ shows that $\sum_{w_3} b_{w_1,w_2;w_3} = 1$. $\square$

Recall the definition of $\mathscr{B}$ from Definition 2.1.2.

PROPOSITION 2.1.9. *$\mathscr{B}$ is an associative algebra, generated by $\{B_s\}_{s\in S}$, with vector space basis $\{B_w\}_{w\in W}$.*

PROOF. The first statement follows from Corollary 2.1.8 ($\mathscr{B}$ is associative since multiplication is given by composition of maps). Suppose we have a relation $\sum_{k=1}^n b_k B_{w_k} = 0$, and fix $a, b \in \mathcal{C}$ with $\delta(a,b) = w_j$ with $1 \le j \le n$. Then writing $\delta_b = 1_{\{b\}}$ we have

$$0 = \sum_{k=1}^n b_k (B_{w_k}\delta_b)(a) = \sum_{k=1}^n b_k q_{w_k}^{-1}\delta_{k,j} = b_j q_{w_j}^{-1},$$

and so $b_j = 0$. From Corollary 2.1.7 we see that $\{B_s \mid s \in S\}$ generates $\mathscr{B}$. □

We refer to the numbers $b_{w_1,w_2;w_3}$ from Corollary 2.1.8 as the *structure constants* of the algebra $\mathscr{B}$ (with respect to the natural basis $\{B_w \mid w \in W\}$).

We say that $\mathscr{X}$ is *chamber regular* if for all w_1 and w_2 in W,

$$|\mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2}(b)| = |\mathcal{C}_{w_1}(c) \cap \mathcal{C}_{w_2}(d)| \qquad \text{whenever } \delta(a,b) = \delta(c,d).$$

The following proposition shows that all regular buildings are chamber regular.

PROPOSITION 2.1.10. *Let $\mathscr{X}$ be a regular building of type W, and let $w_1, w_2, w_3 \in W$. For any pair $a, b \in \mathcal{C}$ with $b \in \mathcal{C}_{w_3}(a)$ we have*

$$|\mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2^{-1}}(b)| = \frac{q_{w_1}q_{w_2}}{q_{w_3}} b_{w_1,w_2;w_3}\,,$$

and so $\mathscr{X}$ is chamber regular.

PROOF. By (2.1.2) we have $(B_{w_1}B_{w_2}\delta_b)(a) = q_{w_1}^{-1}q_{w_2}^{-1}|\mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2^{-1}}(b)|$, whereas by Corollary 2.1.8 we have $(B_{w_1}B_{w_2}\delta_b)(a) = q_{w_3}^{-1}b_{w_1,w_2;w_3}$. □

DEFINITION 2.1.11. Let $\{q_i\}_{i\in I}$ be the parameter system of a locally finite regular building of type W. Define $\mathrm{Aut}_q(D) = \{\sigma \in \mathrm{Aut}(D) \mid q_{\sigma(i)} = q_i \text{ for all } i \in I\}$, where D is the Coxeter diagram of W.

The following is stronger than chamber regularity, and will be used in Chapter 4. Recall the notation of (1.1.2).

LEMMA 2.1.12. *For all $w_1, w_2 \in W$ and $\sigma \in \mathrm{Aut}_q(D)$ we have*

$$|\mathcal{C}_{\sigma(w_1)}(a') \cap \mathcal{C}_{\sigma(w_2)}(b')| = |\mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2}(b)|\,,$$

whenever $a, b, a', b' \in \mathcal{C}$ are chambers with $\delta(a', b') = \sigma(\delta(a,b))$.

PROOF. We first show that, in the notation of Corollary 2.1.8,

$$b_{w_1,w_2;w_3} = b_{\sigma(w_1),\sigma(w_2);\sigma(w_3)} \tag{2.1.3}$$

for all $w_1, w_2, w_3 \in W$.

Theorem 2.1.6, the definition of $\mathrm{Aut}_q(D)$ and the fact that $\ell(\sigma(w)) = \ell(w)$ for all $w \in W$ show that this is true when $\ell(w_2) = 1$, beginning an induction. Suppose (2.1.3) holds whenever $\ell(w_2) < n$, and suppose $w = s_{i_1} \cdots s_{i_{n-1}} s_{i_n}$ has length n. Write $w' = s_{i_1} \cdots s_{i_{n-1}}$ and $s = s_{i_n}$. Observe that $\sigma(w) = \sigma(w')\sigma(s)$, so that $B_{\sigma(w)} = B_{\sigma(w')} B_{\sigma(s)}$ by Theorem 2.1.6, and so

$$\begin{aligned}
B_{\sigma(w_1)} B_{\sigma(w)} &= (B_{\sigma(w_1)} B_{\sigma(w')}) B_{\sigma(s)} \\
&= \sum_{w_3 \in W} b_{\sigma(w_1),\sigma(w');\sigma(w_3)} B_{\sigma(w_3)} B_{\sigma(s)} \\
&= \sum_{w_3 \in W} \left(b_{w_1,w';w_3} \sum_{w_4 \in W} b_{\sigma(w_3),\sigma(s);\sigma(w_4)} B_{\sigma(w_4)} \right) \\
&= \sum_{w_4 \in W} \left(\sum_{w_3 \in W} b_{w_1,w';w_3} b_{w_3,s;w_4} \right) B_{\sigma(w_4)} \,.
\end{aligned}$$

Thus

$$b_{\sigma(w_1),\sigma(w);\sigma(w_4)} = \sum_{w_3 \in W} b_{w_1,w';w_3} b_{w_3,s;w_4} \qquad \text{for all } w_4 \in W \,. \tag{2.1.4}$$

The same calculation without the σ's shows that this is also $b_{w_1,w;w_4}$. This completes the induction step, and so (2.1.3) holds for all w_1, w_2 and w_3 in W.

Thus for any chambers a, b, a', b' with $\delta(a,b) = w_3$, and $\delta(a',b') = \sigma(w_3)$ we have (using Proposition 2.1.10)

$$\begin{aligned}
|\mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2}(b)| &= \frac{q_{w_1} q_{w_2^{-1}}}{q_{w_3}} b_{w_1,w_2^{-1};w_3} \\
&= \frac{q_{\sigma(w_1)} q_{\sigma(w_2^{-1})}}{q_{\sigma(w_3)}} b_{\sigma(w_1),\sigma(w_2^{-1});\sigma(w_3)} \\
&= |\mathcal{C}_{\sigma(w_1)}(a') \cap \mathcal{C}_{\sigma(w_2)}(b')| \,. \qquad \square
\end{aligned}$$

2.2. Connections with Hecke Algebras

Those readers familiar with Hecke algebras will notice immediately from Theorem 2.1.6 the connection between $\mathscr{B}$ and Hecke algebras. Let us briefly describe this connection. We will have much more to say about Hecke algebras in Chapter 5.

For our purposes we define Hecke algebras as follows (see [**19**, Chapter 7]). For each $s \in S$, let a_s and b_s be complex numbers such that $a_{s'} = a_s$ and $b_{s'} = b_s$ whenever $s' = wsw^{-1}$ for some $w \in W$. The *Hecke algebra* $\mathscr{H}(a_s, b_s)$ is the algebra over $\mathbb{C}$ with presentation given by basis elements T_w, $w \in W$, and relations

$$T_w T_s = \begin{cases} T_{ws} & \text{when } \ell(ws) = \ell(w) + 1 \,, \\ a_s T_{ws} + b_s T_w & \text{when } \ell(ws) = \ell(w) - 1 \,. \end{cases} \tag{2.2.1}$$

THEOREM 2.2.1. *Suppose a building $\mathscr{X}$ of type W exists with parameters $\{q_s\}_{s\in S}$. Then $\mathscr{B} \cong \mathscr{H}(q_s^{-1}, 1-q_s^{-1})$.*

PROOF. We note first that by Corollary 1.7.2, the numbers $a_s = q_s^{-1}$ and $b_s = 1 - q_s^{-1}$ satisfy the condition $a_{s'} = a_s$ and $b_{s'} = b_s$ whenever $s' = wsw^{-1}$ for some $w \in W$.

Since $\{T_w\}_{w\in W}$ is a vector space basis of $\mathscr{H}(q_s^{-1}, 1-q_s^{-1})$ and $\{B_w\}_{w\in W}$ is a vector space basis of $\mathscr{B}$ (see Proposition 2.1.9) there exists a unique vector space isomorphism $\Phi : \mathscr{H}(q_s^{-1}, 1-q_s^{-1}) \to \mathscr{B}$ such that $\Phi(T_w) = B_w$ for all $w \in W$. By (2.2.1) and Theorem 2.1.6 we have $\Phi(T_wT_s) = \Phi(T_w)\Phi(T_s)$ for all $w \in W$ and $s \in S$, and so Φ is an algebra homomorphism. It follows that Φ is an algebra isomorphism. $\square$

CHAPTER 3

Affine Coxeter Complexes and Affine Buildings

This chapter is preparation for our study of the algebra $\mathscr{A}$ of Chapter 4.

3.1. Root Systems

Let E be an n-dimensional vector space over $\mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle$. For $\alpha \in E \backslash \{0\}$ define $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$, and let $H_\alpha = \{x \in E \mid \langle x, \alpha \rangle = 0\}$. The *orthogonal reflection in H_α* is the map $s_\alpha : E \to E$, $s_\alpha(x) = x - \langle x, \alpha \rangle \alpha^\vee$ for all $x \in E$.

Definition 3.1.1. A subset R of E is called a *root system* in E if

(R1) R is finite, R spans E and $0 \notin R$, and
(R2) if $\alpha \in R$ then $s_\alpha(R) = R$, and
(R3) if $\alpha, \beta \in R$ then $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$.

A root system is said to be *reduced* if in addition to (R1), (R2) and (R3) it satisfies

(R4) if $\alpha \in R$ then the only other multiple of α in R is $-\alpha$,

and *irreducible* if in addition to (R1), (R2) and (R3) it satisfies

(R5) R cannot be partitioned into two proper subsets R_1 and R_2 such that $\langle \alpha, \beta \rangle = 0$ for all $\alpha \in R_1$ and $\beta \in R_2$.

The elements of R are called *roots*, and the *rank* of R is n, the dimension of E. A root system that is not reduced is said to be *non-reduced* and a root system that is not irreducible is said to be *reducible*.

We will assume that R is irreducible, but not necessarily reduced. We discuss the general case in Appendix A.

Let $B = \{\alpha_i \mid i \in I_0\}$ be a *base* of R, where $I_0 = \{1, 2, \ldots, n\}$. Thus B is a subset of R such that (i) B is a vector space basis of E, and (ii) each root in R can be written as a linear combination of elements of B with integer coefficients which are either all nonnegative or all nonpositive. We say that $\alpha \in R$ is *positive* (respectively *negative*) if the expression for α from (ii) has only nonnegative (respectively nonpositive) coefficients. Let R^+ (respectively R^-) be the set of all positive (respectively negative) roots. Thus $R^- = -R^+$ and $R = R^+ \cup R^-$, where the union is disjoint.

Define the *height (with respect to B)* of $\alpha = \sum_{i \in I_0} k_i \alpha_i \in R$ by $\mathrm{ht}(\alpha) = \sum_{i \in I_0} k_i$. By [**5**, VI, §1 No.8, Proposition 25] there exists a unique root $\tilde{\alpha} \in R$ whose height is maximal, and defining numbers m_i by

$$\tilde{\alpha} = \sum_{i \in I_0} m_i \alpha_i \tag{3.1.1}$$

we have $m_i \geq 1$ for all $i \in I_0$. To complete the notation we define $m_0 = 1$.

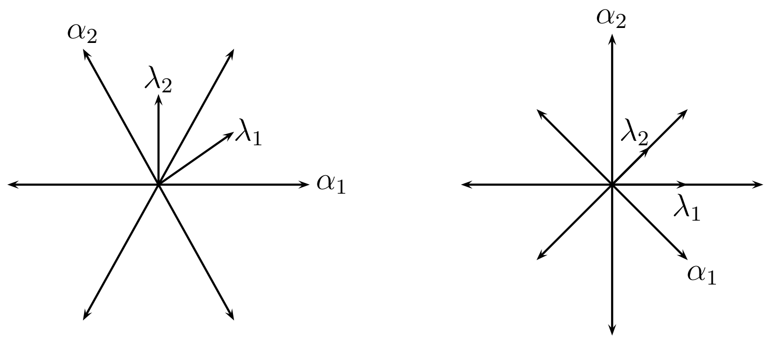
The *dual* (or *inverse*) of R is $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$. The elements of $R^\vee$ are called *coroots of* R. By [**5**, VI, §1, No.1, Proposition 2] $R^\vee$ is an irreducible root system which is reduced if and only if R is.

We define a dual basis $\{\lambda_i\}_{i \in I_0}$ of E by $\langle \lambda_i, \alpha_j \rangle = \delta_{i,j}$. Recall that the *coroot lattice* Q of R is the $\mathbb{Z}$-span of $R^\vee$, and the *coweight lattice* P of R is the $\mathbb{Z}$-span of $\{\lambda_i\}_{i \in I_0}$. Elements of P are called *coweights* (of R), and the vectors λ_i, $i \in I_0$, are called *fundamental coweights*. It is clear that $Q \subseteq P$. We call a coweight $\lambda = \sum_{i \in I_0} a_i \lambda_i$ *dominant* if $a_i \geq 0$ for all $i \in I_0$, and we write P^+ for the set of all dominant coweights.

Let R and R' be root systems in vector spaces E and E', respectively. We call R and R' *isomorphic* if there exists a vector space isomorphism $\phi : E \to E'$ mapping R to R', such that $\langle \phi(\alpha), \phi(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle$ for all $\alpha, \beta \in R$. The irreducible root systems have been classified up to isomorphism [**5**, VI, §4]. The irreducible reduced systems are labelled by symbols A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 2)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4 and G_2. No two systems in the above list are isomorphic, except that B_2 is isomorphic to C_2 (we keep both systems to maintain certain dualities between R and $R^\vee$).

Example 3.1.2. For the A_2 root system we may take $E = \{\xi \in \mathbb{R}^3 \mid \xi_1 + \xi_2 + \xi_3 = 0\}$ and $R = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 3\}$. Let $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - e_3$. Then $B = \{\alpha_1, \alpha_2\}$ is a base of R. We compute $\lambda_1 = \frac{2}{3}e_1 - \frac{1}{3}e_2 - \frac{1}{3}e_3$ and $\lambda_2 = \frac{1}{3}e_1 + \frac{1}{3}e_2 - \frac{2}{3}e_3$.

For the C_2 root system we may take $E = \mathbb{R}^2$ and $R = \pm\{e_1 - e_2, e_1 + e_2, 2e_1, 2e_2\}$. Let $\alpha_1 = e_1 - e_2$ and $\alpha_2 = 2e_2$. Then $B = \{\alpha_1, \alpha_2\}$ is a base of R, and we have $\lambda_1 = e_1$ and $\lambda_2 = \frac{1}{2}e_1 + \frac{1}{2}e_2$. These systems are shown in Figure 3.1.1.



The root system A_2 The root system C_2

Figure 3.1.1

For each $n \geq 1$, there is exactly one irreducible non-reduced root system (up to isomorphism) of rank n, denoted by BC_n. To describe this system, we may take $E = \mathbb{R}^n$ with the usual inner product, and let $\alpha_j = e_j - e_{j+1}$ for $1 \leq j < n$ and $\alpha_n = e_n$. Then $B = \{\alpha_j\}_{j=1}^n$, and $R^+ = \{e_k, 2e_k, e_i + e_j, e_i - e_j \mid 1 \leq k \leq n, 1 \leq i < j \leq n\}$. Notice that $R^\vee = R$, and one easily sees that $Q = P$.

3.2. Hyperplane Arrangements and Reflection Groups

Let R be an irreducible (but not necessarily reduced) root system, and for each $\alpha \in R$ and $k \in \mathbb{Z}$ let $H_{\alpha;k} = \{x \in E \mid \langle x, \alpha \rangle = k\}$. Let $\mathcal{H}$ denote the family of these (affine) *hyperplanes* $H_{\alpha;k}$, $\alpha \in R$, $k \in \mathbb{Z}$. Note that $H_{\alpha;0} = H_\alpha$ for all $\alpha \in R$. We denote by $\mathcal{H}_0$ the family of these hyperplanes H_α, $\alpha \in R$.

Given $H_{\alpha;k} \in \mathcal{H}$, the associated *orthogonal reflection* is the map $s_{\alpha;k} : E \to E$ given by $s_{\alpha;k}(x) = x - (\langle x, \alpha \rangle - k)\alpha^\vee$ for all $x \in E$. Note that $s_{\alpha;0} = s_\alpha$ for all $\alpha \in R$. We write s_i in place of s_{α_i}. The *Weyl group of* R, denoted $W_0(R)$, or simply W_0, is the subgroup of $\mathrm{GL}(E)$ generated by the reflections s_α, $\alpha \in R$, and the *affine Weyl group of* R, denoted $W(R)$, or simply W, is the subgroup of $\mathrm{Aff}(E)$ generated by the reflections $s_{\alpha;k}$, $\alpha \in R$, $k \in \mathbb{Z}$. Here $\mathrm{Aff}(E)$ is the set of maps $x \mapsto Tx + v$, $T \in \mathrm{GL}(E)$, $v \in E$. Writing t_v for the translation $x \mapsto x + v$, we consider E as a subgroup of $\mathrm{Aff}(E)$ by identifying v and t_v. We have $\mathrm{Aff}(E) = \mathrm{GL}(E) \ltimes E$, and $W \cong W_0 \ltimes Q$. Note that $W_0(R^\vee) = W_0(R)$ [**5**, VI, §1, No.1].

Let $s_0 = s_{\tilde{\alpha};1}$, define $I = I_0 \cup \{0\}$, and let $S_0 = \{s_i \mid i \in I_0\}$ and $S = \{s_i \mid i \in I\}$. The group W_0 (respectively W) is a Coxeter group over I_0 (respectively I) generated by S_0 (respectively S).

We write $\Sigma = \Sigma(R)$ for the vector space E equipped with the sectors, chambers and vertices as defined below. The open connected components of $E \setminus \bigcup_{H \in \mathcal{H}} H$ are called the *chambers* of Σ (this terminology is motivated by building theory, and differs from that used in [**5**] where there are *chambers* and *alcoves*), and we write $\mathcal{C}(\Sigma)$ for the set of chambers of Σ. Since R is irreducible, each $C \in \mathcal{C}(\Sigma)$ is an open (geometric) simplex [**5**, V, §3, No.9, Proposition 8]. Call the extreme points of the sets $\overline{C}$, $C \in \mathcal{C}(\Sigma)$, *vertices* of Σ, and write $V(\Sigma)$ for the set of all vertices of Σ.

The choice of B gives a natural *fundamental chamber* C_0.

$$C_0 = \{x \in E \mid \langle x, \alpha_i \rangle > 0 \text{ for all } i \in I_0 \text{ and } \langle x, \tilde{\alpha} \rangle < 1\}, \tag{3.2.1}$$

where we use the notation of (3.1.1).

The *fundamental sector* of Σ is

$$\mathcal{S}_0 = \{x \in E \mid \langle x, \alpha_i \rangle > 0 \text{ for all } i \in I_0\}, \tag{3.2.2}$$

and the *sectors* of Σ are the sets $\lambda + w\mathcal{S}_0$, where $\lambda \in P$ and $w \in W_0$. The sector $\mathcal{S} = \lambda + w\mathcal{S}_0$ is said to have *base vertex* λ (we will see in Section 3.4 that λ is indeed a vertex of Σ).

The group W_0 acts simply transitively on the set of sectors based at 0, and $\overline{\mathcal{S}}_0$ is a fundamental domain for the action of W_0 on E. Similarly, W acts simply transitively on $\mathcal{C}(\Sigma)$, and $\overline{C}_0$ is a fundamental domain for the action of W on E [**5**, VI, §1-3]. It follows easily from [**5**, VI, §2, No.2, Proposition 4(ii)] that W_0 acts simply transitively on the set of $C \in \mathcal{C}(\Sigma)$ with $0 \in \overline{C}$.

3.3. A Geometric Realisation of the Coxeter Complex

The set $\mathcal{C}(\Sigma)$ from Section 3.2 forms a chamber system over I if we declare $wC_0 \sim_i wC_0$ and $wC_0 \sim_i ws_iC_0$ for each $w \in W$ and each $i \in I$. The map $w \mapsto wC_0$ is an isomorphism of the Coxeter complex $\mathcal{C}(W)$ of Section 1.5 onto this chamber system, and so Σ may be regarded as a *geometric realisation* of $\mathcal{C}(W)$ (see Example 1.5.1).

Furthermore, Σ may be naturally regarded as a labelled simplicial complex with vertex set $V(\Sigma)$ by taking maximal simplices to be the sets $V(C)$, $C \in \mathcal{C}(\Sigma)$, where $V(C)$ denotes the vertex set of C. The type map on Σ is constructed as follows. The vertices of C_0 are $\{0\} \cup \{\lambda_i/m_i \mid i \in I_0\}$ (see [**5**, VI, §2, No.2]), and we declare $\tau(0) = 0$ and $\tau(\lambda_i/m_i) = i$ for $i \in I_0$. This extends to a unique labelling $\tau : V(\Sigma) \to I$ (see [**9**, Lemma 1.5]).

Recall the definition of $\Sigma(W)$ from Section 1.5. The map $\psi : \Sigma \to \Sigma(W)$ of simplicial complexes given by $\psi(x) = wW_{I\setminus\{i\}}$ if x is the type i vertex of wC_0 is a (well defined) type preserving isomorphism of simplicial complexes. Thus Σ may also be regarded as a geometric realisation of $\Sigma(W)$ (see Example 1.5.1). The natural action of W on Σ is type preserving [**7**, Theorem, p.58].

3.4. Special and Good Vertices of Σ

Following [**5**, V, §3, No.10], a point $v \in E$ is said to be *special* if for every $H \in \mathcal{H}$ there exists a hyperplane $H' \in \mathcal{H}$ parallel to H such that $v \in H'$. Note that in our set-up $0 \in E$ is special. Each special point is a vertex of Σ [**5**, V, §3, No.10], and thus we will call the special points *special vertices*. Note that in general not all vertices are special (for example, in the $\widetilde{C}_2$ and $\widetilde{G}_2$ complexes). When R is reduced P is the set of special vertices of Σ [**5**, VI, §2, No.2, Proposition 3]. When R is non-reduced then P is a proper subset of the special vertices of Σ (see Example 3.4.3).

To deal with the reduced and non-reduced cases simultaneously, we define the *good vertices* of Σ to be the elements of P. On the first reading the reader is encouraged to think of P as the set of all special vertices, for this is true unless R is of type BC_n. Note that, according to our definitions, every sector of Σ is based at a good vertex of Σ.

LEMMA 3.4.1. *Let* $I_P = \{\tau(\lambda) \mid \lambda \in P\}$. *Then* $I_P = \{i \in I \mid m_i = 1\}$.

PROOF. We have already noted that the vertices of C_0 are $\{0\} \cup \{\lambda_i/m_i \mid i \in I_0\}$. The good vertices of C_0 are those in P, and thus have type 0 or i for some i with $m_i = 1$. ☐

Example 3.4.2 ($R = C_2$). Take $E = \mathbb{R}^2$, $\alpha_1 = e_1 - e_2$ and $\alpha_2 = 2e_2$. Then $B = \{\alpha_1, \alpha_2\}$ and $R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$ (see Example 3.1.2).

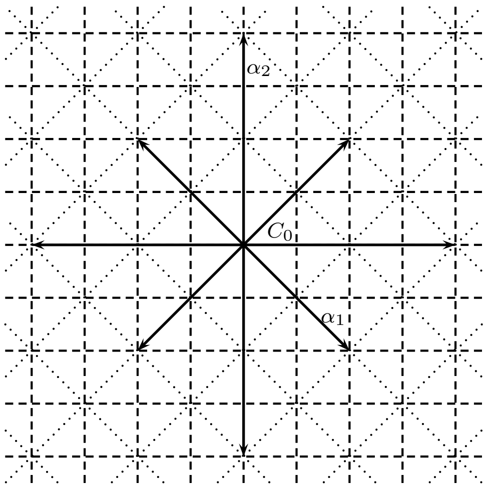


Figure 3.4.1

The dotted lines in Figure 3.4.1 are the hyperplanes $\{H_{w\alpha_1;k} \mid w \in W_0, k \in \mathbb{Z}\}$, and the dashed lines are the hyperplanes $\{H_{w\alpha_2;k} \mid w \in W_0, k \in \mathbb{Z}\}$. We have $\lambda_1 = e_1$ and $\lambda_2 = \frac{1}{2}(e_1 + e_2)$, and $\tau(0) = 0$, $\tau(\frac{1}{2}e_1) = 1$ and $\tau(\frac{1}{2}(e_1 + e_2)) = 2$. We have $P = \{(x, y) \in (\frac{1}{2}\mathbb{Z})^2 \mid x + y \in \mathbb{Z}\}$, which coincides with the set of all special vertices (as expected, since R is reduced here). Thus $I_P = \{0, 2\}$.

Example 3.4.3 ($R = BC_2$). Take $E = \mathbb{R}^2$, $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2$. Then $B = \{\alpha_1, \alpha_2\}$ and $R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_2, 2\alpha_1 + 2\alpha_2\}$.

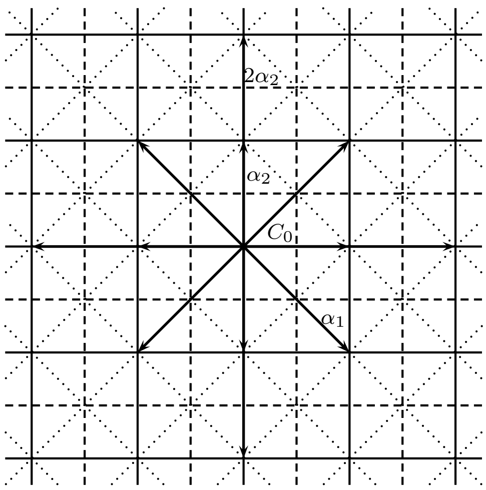


Figure 3.4.2

The dotted lines in Figure 3.4.2 represent the hyperplanes in $\{H_{w\alpha_1;k} \mid w \in W_0, k \in \mathbb{Z}\}$, and solid lines represent the hyperplanes in $\{H_{w\alpha_2;k} \mid w \in W_0, k \in \mathbb{Z}\}$. The union of the dashed and solid lines represent the hyperplanes in $\{H_{w(2\alpha_2);k} \mid w \in W_0, k \in \mathbb{Z}\}$.

In contrast to the previous example, here we have $\lambda_1 = e_1$ and $\lambda_2 = e_1 + e_2$. The set of special vertices and the vertex types are as in Example 3.4.2, but here $P = \mathbb{Z}^2$ (and so $I_P = \{0\}$, as expected from Lemma 3.4.1).

3.5. The Extended Affine Weyl Group

The *extended affine Weyl group of* R, denoted $\tilde{W}(R)$ or simply $\tilde{W}$, is $\tilde{W} = W_0 \ltimes P$. In particular, notice that for each $\lambda \in P$, the translation $t_\lambda : E \to E$, $t_\lambda(x) = x + \lambda$, is in $\tilde{W}$.

In general $\tilde{W}$ is larger than W. In fact, $\tilde{W}/W \cong P/Q$ [**5**, VI, §2, No.3]. We note that while $W(C_n) = W(BC_n)$, $\tilde{W}(C_n)$ is not isomorphic to $\tilde{W}(BC_n)$.

The group $\tilde{W}$ permutes the chambers of Σ, but in general does not act simply transitively. Recall [**27**, §2.2] that for $w \in \tilde{W}$, the *length* of w is defined by

$$\ell(w) = |\{H \in \mathcal{H} \mid H \text{ separates } C_0 \text{ and } w^{-1}C_0\}|\,. \tag{3.5.1}$$

We have $\ell(w) = \ell(w^{-1})$ ([**27**, §2.2]), and so

$$\ell(w) = |\{H \in \mathcal{H} \mid H \text{ separates } C_0 \text{ and } wC_0\}|.$$

When $w \in W$, (3.5.1) agrees with the definition of $\ell(w)$ given previously for Coxeter groups.

The subgroup $G = \{g \in \tilde{W} \mid \ell(g) = 0\}$ will play an important role; it is the stabiliser of C_0 in $\tilde{W}$. By [**5**, VI, §2, No.3] we have $\tilde{W} \cong W \ltimes G$, and furthermore, $G \cong P/Q$, and so G is a finite abelian group. Let w_0 and $w_{0\lambda}$ denote the longest elements of W_0 and $W_{0\lambda}$ respectively, where

$$W_{0\lambda} = \{w \in W_0 \mid w\lambda = \lambda\}. \tag{3.5.2}$$

Recall the definition of the numbers m_i (with $m_0 = 1$) from (3.1.1). Then

$$G = \{g_i \mid m_i = 1\} \tag{3.5.3}$$

where $g_0 = 1$ and $g_i = t_{\lambda_i} w_{0\lambda_i} w_0$ for $i \in I_P \backslash \{0\}$ (see [**5**, VI, §2, No.3] in the reduced case and note that $G = \{1\}$ in the non-reduced case since $G \cong P/Q$).

3.6. Automorphisms of Σ and D

An *automorphism* of Σ is a bijection ψ of E that maps chambers, and only chambers, to chambers with the property that $C \sim_i D$ if and only if $\psi(C) \sim_{i'} \psi(D)$ for some $i' \in I$ (depending on C, D and i). Let $\mathrm{Aut}(\Sigma)$ denote the automorphism group of Σ. Clearly W_0, W and $\tilde{W}$ can be considered as subgroups of $\mathrm{Aut}(\Sigma)$, and we have $W_0 \leq W \leq \tilde{W} \leq \mathrm{Aut}(\Sigma)$. Note that in some cases $\tilde{W}$ is a proper subgroup of $\mathrm{Aut}(\Sigma)$. For example, if R is of type A_2, then the map $a_1\lambda_1 + a_2\lambda_2 \mapsto a_1\lambda_2 + a_2\lambda_1$ is in $\mathrm{Aut}(\Sigma)$ but is not in $\tilde{W}$.

Write D for the Coxeter graph of W. Recall the definition of the type map $\tau : V(\Sigma) \to I$ from Section 3.3.

PROPOSITION 3.6.1. *Let $\psi \in \mathrm{Aut}(\Sigma)$. Then there exists an automorphism $\sigma \in \mathrm{Aut}(D)$ such that $(\tau \circ \psi)(v) = (\sigma \circ \tau)(v)$ for all $v \in V(\Sigma)$. If $C \sim_i D$, then $\psi(C) \sim_{\sigma(i)} \psi(D)$.*

PROOF. The result follows from [**7**, p.64–65]. $\square$

For each $g_i \in G$ (see (3.5.3)), let $\sigma_i \in \mathrm{Aut}(D)$ be the automorphism induced as in Proposition 3.6.1. We call the automorphisms $\sigma_i \in \mathrm{Aut}(D)$ *type-rotating* (for in the $\widetilde{A}_n$ case they are the permutations $k \mapsto k+i \mod n+1$), and we write $\mathrm{Aut}_{\mathrm{tr}}(D)$ for the group of all type-rotating automorphisms of D. Thus

$$\mathrm{Aut}_{\mathrm{tr}}(D) = \{\sigma_i \mid i \in I_P\} . \tag{3.6.1}$$

Note that since $g_0 = 1$, $\sigma_0 = \mathrm{id}$.

Let D_0 be the Coxeter graph of W_0. We have [**5**, VI, §4, No.3]

$$\mathrm{Aut}(D) = \mathrm{Aut}(D_0) \ltimes \mathrm{Aut}_{\mathrm{tr}}(D) . \tag{3.6.2}$$

The group $\tilde{W}$ has a presentation with generators s_i, $i \in I$, and g_j, $j \in I_P$, and relations (see [**31**, (1.20)])

$$\begin{aligned} (s_i s_j)^{m_{i,j}} &= 1 && \text{for all } i, j \in I, \text{ and} \\ g_j s_i g_j^{-1} &= s_{\sigma_j(i)} && \text{for all } i \in I \text{ and } j \in I_P . \end{aligned} \tag{3.6.3}$$

PROPOSITION 3.6.2. *Let $i \in I_P$ and $\sigma \in \mathrm{Aut}_{\mathrm{tr}}(D)$.*

(i) *$\sigma_i(0) = i$.*
(ii) *If $\sigma(i) = i$, then $\sigma = \sigma_0 = \mathrm{id}$.*
(iii) *$\mathrm{Aut}_{\mathrm{tr}}(D)$ acts simply transitively on the good types of D.*

PROOF. (i) follows from the formula $g_i = t_{\lambda_i} w_{0\lambda_i} w_0$ ($i \in I_0$) given in Section 3.5. By (i) we have $(\sigma_i^{-1} \circ \sigma \circ \sigma_i)(0) = 0$, and so $\sigma_i^{-1} \circ \sigma \circ \sigma_i = \sigma_0 = \mathrm{id}$. Thus (ii) holds, and (iii) is now clear. $\square$

PROPOSITION 3.6.3. *Let $\psi \in \mathrm{Aut}(\Sigma)$.*

(i) *The image under ψ of a gallery in Σ is again a gallery in Σ.*
(ii) *A gallery in Σ is minimal if and only if its image under ψ is minimal.*
(iii) *There exists a unique $\sigma \in \mathrm{Aut}(D)$ so that ψ maps galleries of type f to galleries of type $\sigma(f)$. If $\psi = w \in \tilde{W}$ then $\sigma \in \mathrm{Aut}_{\mathrm{tr}}(D)$. If $w = w' g_i$, where $w' \in W$, then $\sigma = \sigma_i$.*
(iv) *If $\psi \in \tilde{W}$ maps $\lambda \in P$ to $\mu \in P$, then the induced automorphism from (iii) is $\sigma = \sigma_m \circ \sigma_l^{-1}$, where $l = \tau(\lambda)$ and $m = \tau(\mu)$.*

Proof. (i) and (ii) are obvious.

(iii) The first statement follows easily from Proposition 3.6.1, and the remaining statements follow from the definition of $\mathrm{Aut}_{\mathrm{tr}}(D)$.

(iv) Since $\sigma(l) = m$, we have $(\sigma \circ \sigma_l)(0) = \sigma_m(0)$, and so $\sigma = \sigma_m \circ \sigma_l^{-1}$ by Proposition 3.6.2. $\square$

Proposition 3.6.4. *$x \mapsto -x$ is an automorphism of Σ.*

Proof. The map $x \mapsto -x$ maps $\mathcal{H}$ to itself and is continuous, and so maps chambers to chambers. If $C \sim_i D$ and $C \neq D$ then there is only one $H \in \mathcal{H}$ separating C and D, and then $-H$ is the only hyperplane in $\mathcal{H}$ separating $-C$ and $-D$, and so $-C \sim_{i'} -D$ for some $i' \in I$. $\square$

Definition 3.6.5. Let $\sigma_* \in \mathrm{Aut}(D)$ be the automorphism of D induced by the automorphism $x \mapsto -x$ of Σ (see Proposition 3.6.4). Furthermore, for $\lambda \in P$ let $\lambda^* = w_0(-\lambda)$, where w_0 is the longest element of W_0. Finally, for $l \in I_P$ let $l^* = \tau(\lambda^*)$, where $\lambda \in P$ is any vertex with $\tau(\lambda) = l$.

We need to check that the definition of l^* is unambiguous. If $\tau(\lambda) = \tau(\mu)$, then $\lambda = w\mu$ for some $w \in W$. Since $W = W_0 \ltimes Q$ we have $w = w't_\gamma$ for some $w' \in W_0$ and $\gamma \in Q$, and so $-\lambda = -w'(\gamma + \mu) = w't_{-\gamma}(-\mu) = w''(-\mu)$ for some $w'' \in W$. Thus $\tau(-\lambda) = \tau(-\mu)$, and so $\tau(\lambda^*) = \tau(\mu^*)$.

Note that in general σ_* is not an element of $\mathrm{Aut}_{\mathrm{tr}}(D)$. In the BC_n case, σ_* is the identity, for the map $x \mapsto -x$ fixes the good type 0, implying that $\sigma_* = \mathrm{id}$ by direct consideration of the Coxeter graph.

Proposition 3.6.6. *If $\lambda \in P^+$, then $\lambda^* \in P^+$.*

Proof. Observe that $w_0(-\mathcal{S}_0) = \mathcal{S}_0$ since $-\mathcal{S}_0$ is a sector that lies on the opposite side of every wall to $\mathcal{S}_0$. Thus $w_0(-\lambda) \in P^+$. $\square$

3.7. Special Group Elements and Technical Results

For $i \in I$, let $W_i = W_{I\setminus\{i\}}$ (this extends our notation for W_0). Given $\lambda \in P^+$, define t'_λ to be the unique element of W such that $t_\lambda = t'_\lambda g$ for some $g \in G$, and, using [**5**, VI, §1, Exercise 3], let w_λ be the unique minimum length representative of the double coset $W_0 t'_\lambda W_l$, where $l = \tau(\lambda)$. Fix a reduced word $f_\lambda \in I^*$ such that $s_{f_\lambda} = w_\lambda$.

Proposition 3.7.1. *Let $\lambda \in P^+$ and $i \in I_P$. Suppose that $\tau(\lambda) = l$, and write $j = \sigma_i(l)$. Then $g_j = g_i g_l$ and $t_\lambda = t'_\lambda g_l$.*

Proof. We see that $g_j = g_i g_l$ since the image of 0 under both functions is the same. Temporarily write $t_\lambda = t'_\lambda g_\lambda$, and so $g_\lambda = {t'_\lambda}^{-1} t_\lambda$. Observe that $g_\lambda(0) = v_k$ for some $k \in I_P$

(here v_k is the type k vertex of C_0). But $({t'_\lambda}^{-1}t_\lambda)(0) = {t'_\lambda}^{-1}(\lambda) = v_l$, since t'_λ is type preserving. Thus $v_k = v_l$, so $k = l$, and so $g_\lambda = g_l$. □

Recall that $\sigma \in \mathrm{Aut}(D)$ induces an automorphism (which we also denote by σ) of W as in (1.1.2). From (3.6.3) we have the following.

LEMMA 3.7.2. *Let $\lambda \in P$ and $l = \tau(\lambda)$. Then $g_l W_0 g_l^{-1} = W_l = \sigma_l(W_0)$, and so W_l is the stabiliser of the type l vertex v_l of C_0.*

PROPOSITION 3.7.3. *Let $\lambda \in P^+$. Then*

(i) *$w_\lambda = t_\lambda w_{0\lambda} w_0 g_l^{-1} = t'_\lambda \sigma_l(w_{0\lambda} w_0)$, where $l = \tau(\lambda)$, and $w_{0\lambda}$ and w_0 are the longest elements of $W_{0\lambda}$ and W_0 respectively.*

(ii) *$\lambda \in w_\lambda \overline{C}_0$, and $w_\lambda C_0$ is the unique chamber nearest C_0 with this property,*

(iii) *$w_\lambda C_0 \subseteq \mathcal{S}_0$.*

PROOF. (i) By Proposition 3.7.1 and Lemma 3.7.2 we have $W_0 t_\lambda W_0 = W_0 t'_\lambda g_l W_0 = W_0 t'_\lambda W_l g_l$, and so it follows that the double coset $W_0 t_\lambda W_0$ has unique minimal length representative $m_\lambda = w_\lambda g_l$. By [**27**, (2.4.5)] (see also [**31**, (2.16)]) we have $m_\lambda = t_\lambda w_{0\lambda} w_0$, proving the first equality in (i). Then

$$w_\lambda = m_\lambda g_l^{-1} = t_\lambda w_{0\lambda} w_0 g_l^{-1} = t'_\lambda g_l w_{0\lambda} w_0 g_l^{-1} = t'_\lambda \sigma_l(w_{0\lambda} w_0).$$

(ii) With m_λ as above we have $m_\lambda(0) = (t_\lambda w_{0\lambda} w_0)(0) = \lambda$, so $\lambda \in m_\lambda \overline{C}_0$. Now $w_\lambda = m_\lambda g_l^{-1}$, and since $g_l^{-1} \in G$ fixes C_0 we have $\lambda \in w_\lambda \overline{C}_0$.

To see that w_λ is the unique chamber nearest C_0 that contains λ in its closure, notice that by Lemma 3.7.2 the stabiliser of λ in W is $t'_\lambda W_l t'^{-1}_\lambda$, which acts simply transitively on the set of chambers containing λ in their closure. So if wC_0 is a chamber containing λ in its closure, then $wC_0 = (t'_\lambda w_l t'^{-1}_\lambda) t'_\lambda(C_0) = t'_\lambda w_l C_0$ for some $w_l \in W_l$. Thus we have $w = t'_\lambda w_l \in t'_\lambda W_l \subset W_0 t'_\lambda W_l$, and so $\ell(w_\lambda) \leq \ell(w)$. The uniqueness follows from [**35**, Theorem 2.9].

We now prove (iii). The result is clear if $\lambda = 0$, so let $\lambda \in P^+\backslash\{0\}$. If $\lambda \in \mathcal{S}_0$ then $\mathcal{S}_0 \cap w_\lambda C_0 \neq \emptyset$, and so $w_\lambda C_0 \subseteq \mathcal{S}_0$ since $w_\lambda C_0$ is connected and contained in $E\backslash \bigcup_{H \in \mathcal{H}_0} H$.

Now suppose that $\lambda \in \overline{\mathcal{S}}_0 \backslash \mathcal{S}_0$, so $\lambda \in H_\alpha$ for some $\alpha \in B$. Let $C_0, C_1, \ldots, C_m = w_\lambda C_0$ be the gallery of type f_λ from C_0 to $w_\lambda C_0$. If $w_\lambda C_0 \nsubseteq \mathcal{S}_0$ then this gallery crosses the wall H_α, so let C_k be the first chamber on the opposite side of H_α to C_0. The sequence $C_0, \ldots, C_{k-1}, s_\alpha(C_k), \ldots, s_\alpha(w_\lambda C_0)$ joins 0 to λ as $s_\alpha(\lambda) = \lambda$. Since $C_{k-1} = s_\alpha(C_k)$, there exists a gallery joining 0 to λ of length strictly less than m, a contradiction. □

Each coset $wW_{0\lambda}$, $w \in W_0$, has a unique minimal length representative. To see this, by Lemma 4.2.1 $W_{0\lambda}$ is the subgroup of W_0 generated by $S_{0\lambda} = \{s \in S_0 \mid s\lambda = \lambda\}$, and the result follows by applying [**5**, IV, §1, Exercise 3]. We write W_0^λ for the set of minimal length representatives of elements of $W_0/W_{0\lambda}$. The following proposition records some simple facts.

PROPOSITION 3.7.4. *Let $\lambda \in P^+$ and write $l = \tau(\lambda)$. Then*

(i) $t'_\lambda = w_\lambda w_l$ *for some* $w_l \in W_l$, *and* $\ell(t'_\lambda) = \ell(w_\lambda) + \ell(w_l)$.

(ii) *Each* $w \in W_0$ *can be written uniquely as* $w = uv$ *with* $u \in W_0^\lambda$ *and* $v \in W_{0\lambda}$, *and moreover* $\ell(w) = \ell(u) + \ell(v)$.

(iii) *For* $v \in W_{0\lambda}$, $vw_\lambda = w_\lambda w_l \sigma_l(v) w_l^{-1}$ *where* $w_l \in W_l$ *is as in (i). Moreover*

$$\ell(vw_\lambda) = \ell(v) + \ell(w_\lambda) = \ell(w_\lambda) + \ell(w_l\sigma_l(v)w_l^{-1}).$$

(iv) *Each* $w \in W_0 w_\lambda W_l$ *can be written uniquely as* $w = uw_\lambda w'$ *for some* $u \in W_0^\lambda$ *and* $w' \in W_l$, *and moreover* $\ell(w) = \ell(u) + \ell(w_\lambda) + \ell(w')$.

PROOF. (i) follows from the proof of Proposition 3.7.3 and [**5**, VI, §1, Exercise 3].

(ii) is immediate from the definition of W_0^λ, and [**5**, VI, §1, Exercise 3].

(iii) Observe first that $vt_\lambda = t_\lambda v$ in the extended affine Weyl group, for $vt_\lambda v^{-1} = t_{v\lambda}$ for all $v \in W_0$, and $t_{v\lambda} = t_\lambda$ if $v \in W_{0\lambda}$. Since $t_\lambda = t'_\lambda g_l$ (see Proposition 3.7.1) we have

$$vt'_\lambda = vt_\lambda g_l^{-1} = t_\lambda v g_l^{-1} = t'_\lambda(g_l v g_l^{-1}) = t'_\lambda \sigma_l(v),$$

and so from (i), $vw_\lambda = w_\lambda w_l \sigma_l(v) w_l^{-1}$. By [**5**, IV, §1, Exercise 3] we have $\ell(vw_\lambda) = \ell(v) + \ell(w_\lambda)$; in fact, $\ell(ww_\lambda) = \ell(w) + \ell(w_\lambda)$ for all $w \in W_0$. Observe now that $ws_\alpha w^{-1} = s_{w\alpha}$ for $w \in W_0$, and it follows that $\ell(w_l\sigma_l(v)w_l^{-1}) = \ell(v)$.

(iv) By [**5**, IV, §1, Exercise 3] each $w \in W_0 w_\lambda W_l$ can be written as $w = w_1 w_\lambda w_2$ for some $w_1 \in W_0$ and $w_2 \in W_l$ with $\ell(w) = \ell(w_1) + \ell(w_\lambda) + \ell(w_2)$. Write $w_1 = uv$ where $u \in W_0^\lambda$ and $v \in W_{0\lambda}$ as in (ii). Then by (iii)

$$w_1 w_\lambda w_2 = uvw_\lambda w_2 = uw_\lambda(w_l\sigma_l(v)w_l^{-1}w_2),$$

and so each $w \in W_0 w_\lambda W_l$ can be written as $w = uw_\lambda w'$ for some $u \in W_0^\lambda$ and $w' \in W_l$ with $\ell(w) = \ell(u) + \ell(w_\lambda) + \ell(w')$. Suppose that we have two such expressions $w = u_1 w_\lambda w'_1 = u_2 w_\lambda w'_2$ where $u_1, u_2 \in W_0^\lambda$ and $w'_1, w'_2 \in W_l$. Write v_l for the type l vertex of C_0. Then $(u_1 w_\lambda w'_l)(v_l) = (u_1 w_\lambda)(v_l) = u_1\lambda$, and similarly $(u_2 w_\lambda w'_2)(v_l) = u_2\lambda$. Thus $u_1^{-1}u_2 \in W_{0\lambda}$, and so $u_1 W_{0\lambda} = u_2 W_{0\lambda}$, forcing $u_1 = u_2$. This clearly implies that $w'_1 = w'_2$ too, completing the proof. □

Recall the definitions of σ_*, λ^* and l^* from Definition 3.6.5.

PROPOSITION 3.7.5. *Let $\lambda \in P^+$ (so $\lambda^* \in P^+$ too), and write $\tau(\lambda) = l$.*

(i) $\sigma_*^2 = \mathrm{id}$ *and* $\sigma_*(0) = 0$.

(ii) $\sigma_*(w_\lambda) = w_{\lambda^*}$ *and* $\sigma_*(l) = l^*$.

(iii) $\sigma_* \circ \sigma_i \circ \sigma_*^{-1} = \sigma_{i^*}$ *for all* $i \in I_P$.

(iv) $w_{\lambda^*} = \sigma_l^{-1}(w_\lambda^{-1})$.

PROOF. (i) is clear, since $-(-x) = x$ for all $x \in E$.

(ii) Let ψ be the automorphism of Σ given by $\psi(x) = w_0(-x)$ for all $x \in E$. Then the automorphism of D induced by ψ is σ_* (see Proposition 3.6.1). Let $C_0, \ldots, C_m = w_\lambda C_0$ be the gallery of type f_λ in Σ starting at C_0, and so $\psi(C_0), \ldots, \psi(C_m)$ is a minimal gallery of type $\sigma_*(f_\lambda)$ (see Proposition 3.6.3). Observe that $\psi(C_0) = C_0$ and $\lambda^* \in \psi(\overline{C}_m)$. The gallery $\psi(C_0), \ldots, \psi(C_m)$ from C_0 to λ^* cannot be replaced by any shorter gallery joining C_0 and λ^*, for if so, by applying ψ^{-1} we could obtain a gallery from C_0 to λ of length $< \ell(w_\lambda)$. Thus $\psi(C_m) = C_{\lambda^*}$ by Proposition 3.7.3, and so $\sigma_*(f_\lambda) \sim f_{\lambda^*}$. Therefore $\sigma_*(w_\lambda) = w_{\lambda^*}$, and so $\sigma_*(l) = l^*$.

(iii) Since $\mathrm{Aut}_{\mathrm{tr}}(D)$ is normal in $\mathrm{Aut}(D)$ (see (3.6.2)) we know that $\sigma_* \circ \sigma_i \circ \sigma_*^{-1} = \sigma_k$ for some $k \in I_P$. By (i) and (ii) we have $(\sigma_* \circ \sigma_i \circ \sigma_*^{-1})(0) = i^*$ and the result follows.

(iv) Let $C_0, \ldots, C_m$ be the gallery from (ii) and write $f_\lambda = i_1 \cdots i_m$. Then $C_m, \ldots, C_0$ is a gallery of type $\mathrm{rev}(f_\lambda) = i_m \cdots i_1$ joining λ to 0. Let $\psi = w_0 \circ w_{0\lambda}^{-1} \circ t_{-\lambda} : \Sigma \to \Sigma$ where $w_{0\lambda}$ is the longest element of $W_{0\lambda}$. By Proposition 3.7.3(i) we have

$$\psi(C_m) = (w_0 \circ w_{0\lambda}^{-1} \circ t_{-\lambda} \circ w_\lambda)(C_0) = C_0.$$

Thus by Proposition 3.6.3 $C_0 = \psi(C_m), \ldots, \psi(C_0)$ is a gallery of type $\sigma_l^{-1}(\mathrm{rev}(f_\lambda))$ joining 0 to λ^* (since $\lambda^* \in \psi(\overline{C}_0)$). Since no shorter gallery joining 0 to λ^* exists (for if so apply ψ^{-1} to obtain a contradiction) it follows that

$$w_{\lambda^*} = \sigma_l^{-1}(s_{\mathrm{rev}(f_\lambda)}) = \sigma_l^{-1}(s_{f_\lambda}^{-1}) = \sigma_l^{-1}(w_\lambda^{-1})\,. \qquad \square$$

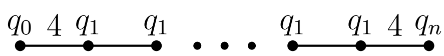
3.8. Affine Buildings

Here we consider buildings as simplicial complexes, as in Definition 1.6.2.

A building $\mathscr{X}$ is called *affine* if the associated Coxeter group W is an affine Weyl group. To study the algebra $\mathscr{A}$ of the next chapter, it is convenient to associate a root system R to each irreducible locally finite regular affine building. If $\mathscr{X}$ is of type W, we wish to choose R so that (among other things) (i) the affine Weyl group of R is isomorphic to W, and (ii) $q_{\sigma(i)} = q_i$ for all $i \in I$ and $\sigma \in \mathrm{Aut}_{\mathrm{tr}}(D)$ (note that $\mathrm{Aut}_{\mathrm{tr}}(D)$ depends on the choice of R, see (3.6.1)).

It turns out (as should be expected) that the choice of R is in most cases straight forward; for example, if $\mathscr{X}$ is of type $\widetilde{F}_4$ then choose R to be a root system of type F_4 (and call $\mathscr{X}$ an *affine building of type* F_4). The regular buildings of types $\widetilde{A}_1$ and $\widetilde{C}_n$ ($n \geq 2$) are the only exceptions to this rule, and in these cases the non-reduced root systems BC_n ($n \geq 1$) play an important role. Let us briefly describe why.

Using Proposition 1.7.1(ii) we see that the parameters of a regular $\widetilde{C}_n$ ($n \geq 2$) building must be as follows:



If we choose R to be a C_n root system then the automorphism $\sigma_n \in \mathrm{Aut}_{\mathrm{tr}}(D)$ interchanges the left most and right most nodes, and so condition (ii) is not satisfied (unless $q_0 = q_n$). If, however, we take R to be a BC_n root system, then $\mathrm{Aut}_{\mathrm{tr}}(D) = \{\mathrm{id}\}$, and so both conditions (i) and (ii) are satisfied.

Thus, in order to facilitate the statements of later results, we rename regular $\widetilde{C}_n$ ($n \geq 2$) buildings, and call them *affine buildings of type* BC_n (or $\widetilde{BC}_n$ ($n \geq 2$) buildings). We reserve the name '$\widetilde{C}_n$ building' for the special case when $q_0 = q_n$ in the above parameter system. For a similar reason we rename regular $\widetilde{A}_1$ buildings (which are *semi-homogeneous trees*) and call them *affine buildings of type* BC_1 (or $\widetilde{BC}_1$ buildings), and reserve the name '$\widetilde{A}_1$ building' for homogeneous trees. With these conventions we make the following definitions.

Definition 3.8.1. Let $\mathscr{X}$ be an affine building of type R with vertex set V, and let $\Sigma = \Sigma(R)$. Let $V_{\mathrm{sp}}(\Sigma)$ denote the set of all special vertices of Σ (see Section 3.4), and let $I_{\mathrm{sp}} = \{\tau(\lambda) \mid \lambda \in V_{\mathrm{sp}}(\Sigma)\}$.

(i) A vertex $x \in V$ is said to be *special* if $\tau(x) \in I_{\mathrm{sp}}$. We write V_{sp} for the set of all special vertices of $\mathscr{X}$.
(ii) A vertex $x \in V$ is said to be *good* if $\tau(x) \in I_P$, where I_P is as in Section 3.4. We write V_P for the set of all good vertices of $\mathscr{X}$.

Clearly $V_P \subset V_{\mathrm{sp}}$. In fact if R is reduced, then by the comments made in Section 3.4, $V_P = V_{\mathrm{sp}}$. If R is non-reduced (so R is of type BC_n for some $n \geq 1$), then V_P is the set of all type 0 vertices of $\mathscr{X}$ (for $I_P = \{0\}$ by Lemma 3.4.1), whereas V_{sp} is the set of all type 0 and type n vertices of $\mathscr{X}$.

Proposition 3.8.2. *A vertex $x \in V$ is good if and only if there exists an apartment $\mathcal{A}$ containing x and a type preserving isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(x) \in P$.*

Proof. Let $x \in V_P$, and choose any apartment $\mathscr{A}$ containing x. Let $\psi : \mathcal{A} \to \Sigma$ be a type preserving isomorphism (from the building axioms). Then $\psi(x)$ is a vertex in Σ with type $\tau(x) \in I_P$, and so $\psi(x) \in P$. The converse is obvious. □

Remark 3.8.3. We note that *infinite distance regular graphs* are just $\widetilde{BC}_1$ buildings in very thin disguise. To see the connection, given any $p, q \geq 1$, construct a $\widetilde{BC}_1$ building (that is, a semi-homogeneous tree) with parameters $q_0 = p$ and $q_1 = q$. Construct a new graph $\Gamma_{p,q}$ with vertex set V_P and vertices $x, y \in V_P$ connected by an edge if and only if $d(x, y) = 2$. It is simple to see that $\Gamma_{p,q}$ is the (graph) free product $\mathbb{K}_q * \cdots * \mathbb{K}_q$ (p copies) where $\mathbb{K}_q$ is the complete graph on q letters. By the classification ([**20**], [**28**]) $\Gamma_{p,q}$ is infinite distance regular, and all infinite distance regular graphs occur in this way.

Recall the definition of $\mathrm{Aut}_q(D)$ from (2.1.11).

THEOREM 3.8.4. *The diagrams in Appendix E characterise the parameter systems of the locally finite regular affine buildings. In each case* $\mathrm{Aut}_{\mathrm{tr}}(D) \cup \{\sigma_*\} \subseteq \mathrm{Aut}_q(D)$.

PROOF. These parameter systems are found case by case using Proposition 1.7.1(ii) and the classification of the irreducible affine Coxeter graphs. Note that $\mathrm{Aut}_{\mathrm{tr}}(D) \cup \{\sigma_*\} = \{\mathrm{id}\}$ if $\mathscr{X}$ is a $\widetilde{BC}_n$ building. Thus the final result follows by considering each Coxeter graph. $\square$

CHAPTER 4

Vertex Set Averaging Operators

Let $\mathscr{X}$ be a regular affine building of type R (see Section 3.8). We will assume that R is irreducible (see Appendix A for the general case).

For each $\lambda \in P^+$ we will define an averaging operator A_λ, acting on the space of all functions $f : V_P \to \mathbb{C}$. These operators A_λ were defined in [**39**, II, §1.1.2] for homogeneous trees, [**11**] and [**29**] for $\widetilde{A}_2$ buildings, and [**10**] for $\widetilde{A}_n$ buildings.

We will study an associated algebra $\mathscr{A}$, and show that it is commutative. A more thorough description of $\mathscr{A}$ requires some Hecke algebra theory, and so will be discussed in Chapter 5, where we show that there is an *affine Hecke algebra* $\tilde{\mathscr{H}}$ such that $\mathscr{B} \cong \mathscr{H}$, a large subalgebra of $\tilde{\mathscr{H}}$, and $\mathscr{A} \cong Z(\tilde{\mathscr{H}})$, the center of $\tilde{\mathscr{H}}$.

4.1. Initial Observations

Recall the definition of type preserving isomorphisms of labelled simplicial complexes from Section 1.3.

Definition 4.1.1. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be apartments of $\mathscr{X}$.

(i) An isomorphism $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ is called *type-rotating* if and only if it is of the form $\psi = \psi_2^{-1} \circ w \circ \psi_1$ where $\psi_1 : \mathcal{A}_1 \to \Sigma$ and $\psi_2 : \mathcal{A}_2 \to \Sigma$ are type preserving isomorphisms, and $w \in \tilde{W}$.

(ii) We have an analogous definition for isomorphisms $\psi : \mathcal{A}_1 \to \Sigma$ by omitting ψ_2.

Proposition 4.1.2. *Let $\mathcal{A}, \mathcal{A}'$ be any apartments and suppose that $\psi : \mathcal{A} \to \mathcal{A}'$ is an isomorphism. Then*

(i) *The image under ψ of a gallery in $\mathcal{A}$ is a gallery in $\mathcal{A}'$.*

(ii) *A gallery in $\mathcal{A}$ is minimal if and only if its image under ψ is minimal.*

(iii) *There exists a unique automorphism $\sigma \in \mathrm{Aut}(D)$ so that ψ maps galleries of type f in $\mathcal{A}$ to galleries of type $\sigma(f)$ in $\mathcal{A}'$. If ψ is type-rotating, then $\sigma \in \mathrm{Aut}_{\mathrm{tr}}(D)$, and $(\tau \circ \psi)(x) = (\sigma \circ \tau)(x)$ for all vertices x of $\mathcal{A}$.*

(iv) *If ψ is type-rotating and maps a type $i \in I_P$ vertex in $\mathcal{A}$ to a type $j \in I_P$ vertex in $\mathcal{A}'$, then the induced automorphism from (iii) is $\sigma = \sigma_j \circ \sigma_i^{-1}$.*

Proof. This follows from Proposition 3.6.3 and the definition of type-rotating isomorphisms. □

LEMMA 4.1.3. *Suppose $x \in V_P$ is contained in the apartments $\mathcal{A}$ and $\mathcal{A}'$ of $\mathscr{X}$, and suppose that $\psi : \mathcal{A} \to \Sigma$ and $\psi' : \mathcal{A}' \to \Sigma$ are type-rotating isomorphisms such that $\psi(x) = 0 = \psi'(x)$. Let $\psi'' : \mathcal{A} \to \mathcal{A}'$ be a type preserving isomorphism mapping x to x (the existence of which is guaranteed by Definition 1.6.2). Then $\phi = \psi' \circ \psi'' \circ \psi^{-1}$ is in W_0.*

PROOF. Observe that $\phi : \Sigma \to \Sigma$ has $\phi(0) = 0$. Since ψ and ψ' are type-rotating isomorphisms we have $\psi = w \circ \psi_1$ and $\psi' = w' \circ \psi_1'$ for some $w, w' \in \tilde{W}$ and $\psi_1 : \mathcal{A} \to \Sigma$, $\psi_1' : \mathcal{A}' \to \Sigma$ type preserving isomorphisms. Therefore

$$\phi = w' \circ \psi_1' \circ \psi'' \circ \psi_1^{-1} \circ w^{-1} = w' \circ \phi' \circ w^{-1}, \quad \text{say}.$$

Now $\phi' = \psi_1' \circ \psi'' \circ \psi_1^{-1} : \Sigma \to \Sigma$ is a type preserving automorphism, as it is a composition of type preserving isomorphisms. By [**35**, Lemma 2.2] we have $\phi' = v$ for some $v \in W$, and hence $\phi = w' \circ v \circ w^{-1} \in \tilde{W}$. Since $\phi(0) = 0$ and $\tilde{W} = W_0 \ltimes P$ we in fact have $\phi \in W_0$, completing the proof. $\square$

4.2. The Sets $V_\lambda(x)$

The following definition gives the analogue of the partition $\{\mathcal{C}_w(a)\}_{w \in W}$ used for the chamber set of $\mathscr{X}$. Let us first record the following lemma from [**7**, p.24] (or [**18**, §10.3, Lemma B]). Recall the definition of the fundamental sector $\mathcal{S}_0$ from (3.2.2).

LEMMA 4.2.1. *Let $w \in W_0$ and $\lambda \in E$. If $\lambda' = w\lambda \in \overline{\mathcal{S}}_0 \cap w\overline{\mathcal{S}}_0$ then $\lambda' = \lambda$, and $w \in \langle \{ s_i \mid s_i\lambda = \lambda \} \rangle$.*

DEFINITION 4.2.2. Given $x \in V_P$ and $\lambda \in P^+$, we define $V_\lambda(x)$ to be the set of all $y \in V_P$ such that there exists an apartment $\mathcal{A}$ containing x and y and a type-rotating isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(x) = 0$ and $\psi(y) = \lambda$.

Notice that for all $x \in V_P$ and $\lambda \in P^+$ we have $V_\lambda(x) \neq \emptyset$.

PROPOSITION 4.2.3. *Let $V_\lambda(x)$ be as in Definition 4.2.2.*

(i) *Given $x, y \in V_P$, there exists some $\lambda \in P^+$ such that $y \in V_\lambda(x)$.*
(ii) *If $y \in V_\lambda(x) \cap V_{\lambda'}(x)$ then $\lambda = \lambda'$.*
(iii) *Let $y \in V_\lambda(x)$. If $\mathcal{A}$ is any apartment containing x and y, then there exists a type-rotating isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(x) = 0$ and $\psi(y) = \lambda$.*

PROOF. First we prove (i). By Definition 1.6.2 there exists an apartment $\mathcal{A}$ containing x and y and a type preserving isomorphism $\psi_1 : \mathcal{A} \to \Sigma$. Let $\mu = \psi_1(x)$ and $\nu = \psi_1(y)$, so $\mu, \nu \in P$. There exists a $w \in W_0$ such that $w(\nu - \mu) \in \overline{\mathcal{S}}_0 \cap P$ [**18**, p.55, exercise 14], and so the isomorphism $\psi = w \circ t_{-\mu} \circ \psi_1$ satisfies $\psi(x) = 0$ and $\psi(y) = w(\nu - \mu) \in P^+$, proving (i).

We now prove (ii). Suppose that there are apartments $\mathcal{A}$ and $\mathcal{A}'$ containing x and y, and type-rotating isomorphisms $\psi : \mathcal{A} \to \Sigma$ and $\psi' : \mathcal{A}' \to \Sigma$ such that $\psi(x) = \psi'(x) = 0$ and $\psi(y) = \lambda \in P^+$ and $\psi'(y) = \lambda' \in P^+$. We claim that $\lambda = \lambda'$.

By Definition 1.6.2(iii)$'$ there exists a type preserving isomorphism $\psi'' : \mathcal{A} \to \mathcal{A}'$ which fixes x and y. Then $\phi = \psi' \circ \psi'' \circ \psi^{-1} : \Sigma \to \Sigma$ is a type-rotating automorphism of Σ that fixes 0 and maps λ to λ'. By Lemma 4.1.3 we have $\phi = w$ for some $w \in W_0$, and so we have $\lambda' = w\lambda \in \overline{\mathcal{S}}_0 \cap w\overline{\mathcal{S}}_0$. Thus by Lemma 4.2.1 we have $\lambda' = \lambda$.

Note first that (iii) is not immediate from the definition of $V_\lambda(x)$. To prove (iii), by the definition of $V_\lambda(x)$ there exists an apartment $\mathcal{A}'$ containing x and y, and a type-rotating isomorphism $\psi' : \mathcal{A}' \to \Sigma$ such that $\psi'(x) = 0$ and $\psi'(y) = \lambda$. Then by Definition 1.6.2(iii)$'$ there is a type preserving isomorphism $\phi : \mathcal{A} \to \mathcal{A}'$ fixing x and y. Then $\psi = \psi' \circ \phi : \mathcal{A} \to \Sigma$ has the required properties. $\square$

Remark 4.2.4. Note that the assumption that ψ is type-rotating in Definition 4.2.2 is essential for Proposition 4.2.3(ii) to hold. To see this we only need to look at an apartment of an $\widetilde{A}_2$ building. The map $a_1\lambda_1 + a_2\lambda_2 \mapsto a_1\lambda_2 + a_2\lambda_1$, shown in Figure 4.2.1, is an automorphism which maps λ_1 to λ_2. Thus if we omitted the hypothesis that ψ is type-rotating in Definition 4.2.2, part (ii) of Proposition 4.2.3 would be false.

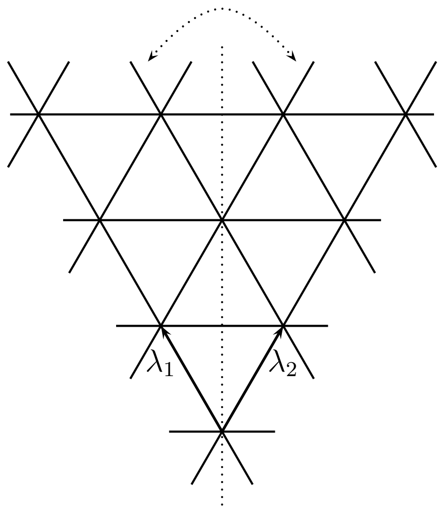


Figure 4.2.1

Proposition 4.2.5. *If $y \in V_\lambda(x)$, then $x \in V_{\lambda^*}(y)$ where λ^* is as in Definition 3.6.5.*

Proof. If $\psi : \mathcal{A} \to \Sigma$ is a type-rotating isomorphism mapping x to 0 and y to λ, then $w_0 \circ t_{-\lambda} \circ \psi : \mathcal{A} \to \Sigma$ is a type-rotating isomorphism mapping y to 0 and x to $\lambda^* = w_0(-\lambda) \in P^+$ (see Proposition 3.6.6). $\square$

LEMMA 4.2.6. *Let $x \in V_P$ and $\lambda \in P^+$. If $y, y' \in V_\lambda(x)$ then $\tau(y) = \tau(y')$.*

PROOF. Let $\mathcal{A}$ be an apartment containing x and y, and let $\mathcal{A}'$ be an apartment containing x and y'. Let $\psi : \mathcal{A} \to \Sigma$ and $\psi' : \mathcal{A}' \to \Sigma$ be type-rotating isomorphisms with $\psi(x) = \psi'(x) = 0$ and $\psi(y) = \psi'(y') = \lambda$. Thus $\chi = \psi'^{-1} \circ \psi : \mathcal{A} \to \mathcal{A}'$ is a type preserving automorphism since $\chi(x) = x$ (see Proposition 3.6.2). Since $\chi(y) = y'$ we have $\tau(y) = \tau(y')$. □

In light of the above lemma we define $\tau(V_\lambda(x)) = \tau(y)$ for any $y \in V_\lambda(x)$.

Clearly the sets $V_\lambda(x)$ are considerably more complicated objects than the sets $\mathcal{C}_w(a)$. The following theorem provides an important connection between the sets $V_\lambda(x)$ and $\mathcal{C}_w(a)$ that will be relied on heavily in subsequent work. For $c \in \mathcal{C}$ and $i \in I$, let $\pi_i(c)$ be the type i vertex of c. For the following theorem the reader is reminded of the definition of $w_\lambda \in W$ and $f_\lambda \in I^*$ from Section 3.7.

THEOREM 4.2.7. *Let $x \in V_P$ and $\lambda \in P^+$. Suppose $\tau(x) = i$ and $\tau(V_\lambda(x)) = j$, and let $a \in \mathcal{C}$ be any chamber with $\pi_i(a) = x$. Then*

$$\{b \in \mathcal{C} | \pi_j(b) \in V_\lambda(x)\} = \bigcup_{w \in W_i \sigma_i(w_\lambda) W_j} \mathcal{C}_w(a)\,,$$

where the union is disjoint.

PROOF. Suppose first that $y = \pi_j(b) \in V_\lambda(x)$. Let $a = c_0, c_1, \ldots, c_n = b$ be a minimal gallery from a to b of type f, say. By [**35**, Theorem 3.8], all the c_k lie in some apartment, $\mathcal{A}$, say. Let $\psi : \mathcal{A} \to \Sigma$ be a type-rotating isomorphism such that $\psi(x) = 0$ and $\psi(y) = \lambda$ (see Proposition 4.2.3(iii)). Then $\psi(c_0), \psi(c_1), \ldots, \psi(c_n)$ is a minimal gallery of type $\sigma_i^{-1}(f)$ by Proposition 4.1.2.

Recall the definition of the fundamental chamber C_0 from (3.2.1). Since 0 is a vertex of $\psi(c_0)$, we can construct a gallery from $\psi(c_0)$ to C_0 of type e_1, say, where $s_{e_1} \in W_0$. Similarly there is a gallery from $w_\lambda C_0$ to $\psi(c_n)$ of type e_2, where $s_{e_2} \in W_{\sigma_i^{-1}(j)}$. Thus we have a gallery

$$\psi(c_0) \xrightarrow{e_1} C_0 \xrightarrow{f_\lambda} w_\lambda C_0 \xrightarrow{e_2} \psi(c_n)$$

of type $e_1 f_\lambda e_2$. Since Σ is a Coxeter complex, galleries (reduced or not) from one chamber to another of types f_1 and f_2, say, satisfy $s_{f_1} = s_{f_2}$ [**35**, p.12], so $s_{\sigma_i^{-1}(f)} = s_{e_1 f_\lambda e_2}$. Thus

$$\delta(a, b) = s_f = \sigma_i(s_{\sigma_i^{-1}(f)}) = \sigma_i(s_{e_1 f_\lambda e_2}) = s_{e_1'} s_{\sigma_i(f_\lambda)} s_{e_2'}$$

where $e_1' \in W_i$ and $e_2' \in W_j$. Thus $b \in \mathcal{C}_w(a)$ for some $w \in W_i \sigma_i(w_\lambda) W_j$.

Now suppose that $b \in \mathcal{C}_w(a)$ for some $w \in W_i \sigma_i(w_\lambda) W_j$. Let $y = \pi_j(b)$. By [**35**, p.35, Exercise 1], there exists a gallery of type $e_1' \sigma_i(f_\lambda) e_2'$ from a to b where $e_1' \in W_i$ and $e_2' \in W_j$. Let $c_k, c_{k+1}, \ldots, c_l$ be the subgallery of type $\sigma_i(f_\lambda)$. Note that $\pi_i(c_k) = x$ and $\pi_j(c_l) = y$. Observe that $\sigma_i(f_\lambda)$ is reduced since $\sigma_i \in \mathrm{Aut}(D)$, and so all of the chambers c_m,

$k \le m \le l$, lie in an apartment $\mathcal{A}$, say. Let $\psi : \mathcal{A} \to \Sigma$ be a type-rotating isomorphism such that $\psi(x) = 0$. Thus $\psi(c_k), \ldots, \psi(c_l)$ is a gallery of type f_λ in Σ (Proposition 4.1.2). Since W_0 acts transitively on the chambers $C \in \mathcal{C}(\Sigma)$ with $0 \in \overline{C}$ (Section 3.2) there exists $w \in W_0$ such that $w(\psi(c_k)) = C_0$. Then $\psi' = w \circ \psi : \mathcal{A} \to \Sigma$ is a type-rotating isomorphism that takes the gallery $c_k, \ldots, c_l$ in $\mathcal{A}$ of type $\sigma_i(f_\lambda)$ to a gallery $C_0, \ldots, \psi'(c_l)$ of type f_λ. But in a Coxeter complex there is only one gallery of each type. So $\psi'(c_l)$ must be $w_\lambda(C_0)$, and by considering types $\psi'(y) = \lambda$, and so $y \in V_\lambda(x)$. □

For $x \in V$ we write $\mathrm{st}(x)$ for the set of all chambers that have x as a vertex. Recall the definition of Poincaré polynomials from Definition 1.7.6.

Lemma 4.2.8. *Let $x \in V_P$. Then $|\mathrm{st}(x)| = W_0(q)$. In particular, this value is independent of the particular $x \in V_P$.*

Proof. Suppose $\tau(x) = i \in I_P$ and let c_0 be any chamber that has x as a vertex. Then

$$\mathrm{st}(x) = \{c \in \mathcal{C} \mid \delta(c_0, c) \in W_i\} = \bigcup_{w \in W_i} \mathcal{C}_w(c_0)$$

where the union is disjoint, and so $|\mathrm{st}(x)| = \sum_{w \in W_i} q_w$. Theorem 3.8.4 now shows that

$$|\mathrm{st}(x)| = \sum_{w \in W_0} q_{\sigma_i(w)} = \sum_{w \in W_0} q_w = W_0(q). \qquad \square$$

Note that if the hypothesis 'let $x \in V_P$' in Lemma 4.2.8 is replaced by the hypothesis 'let x be a special vertex', then in the non-reduced case it is no longer true in general that $|\mathrm{st}(x)| = W_0(q)$.

4.3. The Cardinalities $|V_\lambda(x)|$

In this section we will find a closed form for $|V_\lambda(x)|$. We need to return to the operators B_w introduced in Chapter 2.

For each $i \in I$ define an element $\mathbb{1}_i \in \mathscr{B}$ by

$$\mathbb{1}_i = \frac{1}{W_i(q)} \sum_{w \in W_i} q_w B_w\,. \tag{4.3.1}$$

Lemma 4.3.1. *Let $i \in I$. Then $\mathbb{1}_i B_w = B_w \mathbb{1}_i = \mathbb{1}_i$ for all $w \in W_i$, and $\mathbb{1}_i^2 = \mathbb{1}_i$.*

PROOF. Suppose s is a generator of W_i and set $W_i^{\pm} = \{w \in W_i \mid \ell(ws) = \ell(w) \pm 1\}$. Then

$$\begin{aligned} W_i(q)\mathbb{1}_i B_s &= \sum_{w\in W_i^+} q_w B_{ws} + \sum_{w'\in W_i^-} q_w \left(\frac{1}{q_s}B_{ws} + \left(1 - \frac{1}{q_s}\right)B_w\right) \\ &= \sum_{w\in W_i^-} \frac{q_w}{q_s}B_w + \sum_{w'\in W_i^-} q_w \left(\frac{1}{q_s}B_{ws} + \left(1 - \frac{1}{q_s}\right)B_w\right) \\ &= \sum_{w\in W_i^-} \left(\frac{q_w}{q_s}B_{ws} + q_w B_w\right) \\ &= \sum_{w\in W_i^+} q_w B_w + \sum_{w\in W_i^-} q_w B_w = W_i(q)\mathbb{1}_i\,. \end{aligned}$$

A similar calculation works for $B_s\mathbb{1}_i$ too. It follows that $\mathbb{1}_i B_w = B_w \mathbb{1}_i = \mathbb{1}_i$ for all $w \in W_i$ and so $\mathbb{1}_i^2 = \mathbb{1}_i$. □

Recall the definition of $W_{0\lambda}$ from (3.5.2).

THEOREM 4.3.2. *Let $\lambda \in P^+$ and write $l = \tau(\lambda)$. Then*

$$\sum_{w\in W_0 w_\lambda W_l} q_w B_w = \frac{W_0^2(q)}{W_{0\lambda}(q)} q_{w_\lambda} \mathbb{1}_0 B_{w_\lambda} \mathbb{1}_l.$$

PROOF. Recall from Corollary 2.1.7 that $B_v B_w = B_{vw}$ whenever $\ell(vw) = \ell(v) + \ell(w)$. Then by Proposition 3.7.4(ii), Proposition 3.7.4(iii), Lemma 4.3.1 and Proposition 3.7.4(iv) (in that order)

$$\begin{aligned} \mathbb{1}_0 B_{w_\lambda} \mathbb{1}_l &= \frac{1}{W_0(q)} \sum_{u\in W_0^\lambda} \sum_{v\in W_{0\lambda}} q_u q_v B_u B_v B_{w_\lambda} \mathbb{1}_l \\ &= \frac{1}{W_0(q)} \sum_{u\in W_0^\lambda} \sum_{v\in W_{0\lambda}} q_u q_v B_u B_{w_\lambda} B_{w_l \sigma_l(v) w_l^{-1}} \mathbb{1}_l \\ &= \frac{1}{W_0(q)} \sum_{u\in W_0^\lambda} \sum_{v\in W_{0\lambda}} q_u q_v B_u B_{w_\lambda} \mathbb{1}_l \\ &= \frac{W_{0\lambda}(q)}{W_0(q) W_l(q)} q_{w_\lambda}^{-1} \sum_{w\in W_0 w_\lambda W_l} q_w B_w, \end{aligned}$$

and the result follows, since

$$W_l(q) = \sum_{w\in W_l} q_w = \sum_{w\in W_0} q_{\sigma_l(w)} = W_0(q)$$

by Proposition 3.8.4. □

LEMMA 4.3.3. *Let $\lambda \in P^+$, $x \in V_P$, and $y \in V_\lambda(x)$. Write $\tau(x) = i$, $\tau(y) = j$ and $\tau(\lambda) = l$. Then $\sigma_i^{-1}(j) = l$, and so $\sigma_j = \sigma_i \circ \sigma_l$.*

PROOF. Since $y \in V_\lambda(x)$, there exists an apartment $\mathcal{A}$ containing x and y and a type-rotating isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(x) = 0$ and $\psi(y) = \lambda$. Since $\psi(x) = 0$, the σ from Proposition 4.1.2(iii) maps i to 0 and so is σ_i^{-1}. Thus $\lambda = \psi(y)$ has type $\sigma(j) = \sigma_i^{-1}(j)$ and so $l = \sigma_i^{-1}(j)$. Thus $\sigma_j(0) = (\sigma_i \circ \sigma_l)(0)$, and so $\sigma_j = \sigma_i \circ \sigma_l$. $\square$

THEOREM 4.3.4. *Let $x \in V_P$ and $\lambda \in P^+$ with $\tau(\lambda) = l \in I_P$. Then*

$$|V_\lambda(x)| = \frac{1}{W_0(q)} \sum_{w \in W_0 w_\lambda W_l} q_w = \frac{W_0(q)}{W_{0\lambda}(q)} q_{w_\lambda} = |V_{\lambda^*}(x)|\,.$$

PROOF. Let $i = \tau(x)$ and $j = \tau(V_\lambda(x))$. Let $\mathcal{C}_\lambda(x) = \{c \in \mathcal{C} \mid \pi_j(c) \in V_\lambda(x)\}$ and construct a map $\psi : \mathcal{C}_\lambda(x) \to V_\lambda(x)$ by $c \mapsto \pi_j(c)$ for all $c \in \mathcal{C}_\lambda(x)$. Clearly ψ is surjective.

Observe that for each $y \in V_\lambda(x)$ the set $\{c \in \mathcal{C}_\lambda(x) \mid \psi(c) = y\}$ has $|\mathrm{st}(y)|$ distinct elements, and so by Lemma 4.2.8 we see that $\psi : \mathcal{C}_\lambda(x) \to V_\lambda(x)$ is a $W_0(q)$-to-one surjection. Let $c_0 \in \mathcal{C}$ be any chamber that has x as a vertex. Then by the above and Theorem 4.2.7 we have

$$|V_\lambda(x)| = \frac{|\mathcal{C}_\lambda(x)|}{W_0(q)} = \frac{1}{W_0(q)} \sum_{w \in W_i \sigma_i(w_\lambda) W_j} |\mathcal{C}_w(c_0)| = \frac{1}{W_0(q)} \sum_{w \in W_i \sigma_i(w_\lambda) W_j} q_w\,.$$

Since $\sigma_i^{-1}(j) = l$ (Lemma 4.3.3) we have $W_i \sigma_i(w_\lambda) W_j = \sigma_i(W_0 w_\lambda W_l)$, and so by Theorem 3.8.4

$$|V_\lambda(x)| = \frac{1}{W_0(q)} \sum_{w \in W_0 w_\lambda W_l} q_{\sigma_i(w)} = \frac{1}{W_0(q)} \sum_{w \in W_0 w_\lambda W_l} q_w\,.$$

Let $1_\mathcal{C} : \mathcal{C} \to \{1\}$ be the constant function. Then $(B_w 1_\mathcal{C})(c) = 1$ for all $c \in \mathcal{C}$, and so we compute $(\mathbb{1}_l 1_\mathcal{C})(c) = 1$ for all $c \in \mathcal{C}$. Thus by Theorem 4.3.2

$$\sum_{w \in W_0 w_\lambda W_l} q_w = \frac{W_0^2(q)}{W_{0\lambda}(q)} q_{w_\lambda}.$$

Now, by Proposition 3.7.5 and Theorem 3.8.4 we have

$$|V_{\lambda^*}(x)| = \frac{1}{W_0(q)} \sum_{w \in \sigma_*(W_0 w_\lambda W_l)} q_w = \frac{1}{W_0(q)} \sum_{w \in W_0 w_\lambda W_l} q_w = |V_\lambda(x)|. \qquad \square$$

DEFINITION 4.3.5. For $\lambda \in P^+$ we define $N_\lambda = |V_\lambda(x)|$, which is independent of $x \in V_P$ by Theorem 4.3.4.

Note that $N_\lambda = N_{\lambda^*}$, and since $V_\lambda(x) \neq \emptyset$ for all $x \in V_P$ and $\lambda \in P^+$, we have $N_\lambda > 0$ for all $\lambda \in P^+$.

We conclude this section by giving an alternative formula for N_λ.

By [**5**, VI, §1, No.6, Corollary 3 to Proposition 17] we have $\ell(w_0w)=\ell(w_0)-\ell(w)$ for all $w\in W_0$, and so writing $W_0(q^{-1})=\sum_{w\in W_0}q_w^{-1}$ we have

$$W_0(q)=\sum_{w\in W_0}q_{w_0w}=q_{w_0}W_0(q^{-1}). \tag{4.3.2}$$

Each $\tilde{w}\in\tilde{W}$ can be written uniquely as $\tilde{w}=wg$ where $w\in W$ and $g\in G$, and we define $q_{\tilde{w}}=q_w$. In particular, $q_g=1$ for all $g\in G$, and it is clear that $q_{uv}=q_uq_v$ whenever $u,v\in\tilde{W}$ satisfy $\ell(uv)=\ell(u)+\ell(v)$. See Appendix B.1 for a calculation of q_{t_λ}.

Proposition 4.3.6. *Let $\lambda\in P^+$. Then*

$$N_\lambda=\frac{W_0(q^{-1})}{W_{0\lambda}(q^{-1})}q_{t_\lambda}.$$

Proof. Let $l=\tau(\lambda)$, and let w_0 and $w_{0\lambda}$ be the longest elements of W_0 and $W_{0\lambda}$, respectively. Since $t_\lambda=w_\lambda g_l w_0 w_{0\lambda}$ and $\ell(w_\lambda g_l w_0 w_{0\lambda})=\ell(w_\lambda g_l)+\ell(w_0w_{0\lambda})$ (see Proposition 3.7.3) we have $q_{t_\lambda}=q_{w_\lambda}q_{w_0w_{0\lambda}}$. Since $\ell(w_0w)=\ell(w_0)-\ell(w)$ for all $w\in W_0$ it follows that $q_{w_0w_{0\lambda}}=q_{w_0}q_{w_{0\lambda}}^{-1}$. Thus

$$q_{w_\lambda}=q_{t_\lambda}q_{w_0w_{0\lambda}}^{-1}=q_{t_\lambda}q_{w_0}^{-1}q_{w_{0\lambda}}.$$

Since $W_{0\lambda}$ is a Coxeter group we have $W_{0\lambda}(q)=q_{w_{0\lambda}}W_{0\lambda}(q^{-1})$, as in (4.3.2). Putting all of this together the result follows. □

We say that $\lambda\in P$ is *strongly dominant* if $\langle\lambda,\alpha_i\rangle>0$ for all $i=1,\ldots,n$, and we write P^{++} for the set of all strongly dominant coweights. Note that when $\lambda\in P^{++}$, $W_{0\lambda}=\{1\}$, and so by Proposition 4.3.6

$$N_\lambda=W_0(q^{-1})q_{t_\lambda}\quad\text{for all }\lambda\in P^{++}. \tag{4.3.3}$$

4.4. The Operators A_λ and the Algebra $\mathscr{A}$

We now define the *vertex set averaging operators* on $\mathscr{X}$.

Definition 4.4.1. For each $\lambda\in P^+$, define an operator A_λ, acting on the space of all functions $f:V_P\to\mathbb{C}$, by

$$(A_\lambda f)(x)=\frac{1}{N_\lambda}\sum_{y\in V_\lambda(x)}f(y)\quad\text{for all }x\in V_P.$$

LEMMA 4.4.2. *The operators A_λ are linearly independent.*

PROOF. Suppose we have a relation $\sum_{\lambda\in P^+} a_\lambda A_\lambda = 0$, and fix $x, y \in V_P$ with $y \in V_\mu(x)$. Then writing δ_y for the function taking the value 1 at y and 0 elsewhere,

$$0 = \sum_{\lambda\in P^+} a_\lambda (A_\lambda \delta_y)(x) = \sum_{\lambda\in P^+} a_\lambda N_\lambda^{-1}\delta_{\lambda,\mu} = a_\mu N_\mu^{-1}\,,$$

and so $a_\mu = 0$. □

Following the same technique used in (2.1.2) for the chamber set averaging operators, we have

$$(A_\lambda A_\mu f)(x) = \frac{1}{N_\lambda N_\mu}\sum_{y\in V_P} |V_\lambda(x)\cap V_{\mu^*}(y)|\, f(y) \qquad \text{for all } x \in V_P\,. \tag{4.4.1}$$

Our immediate goal now is to understand the cardinalities $|V_\lambda(x)\cap V_{\mu^*}(y)|$.

DEFINITION 4.4.3. We say that $\mathscr{X}$ is *vertex regular* if, for all $\lambda,\mu,\nu \in P^+$,

$$|V_\lambda(x)\cap V_{\mu^*}(y)| = |V_\lambda(x')\cap V_{\mu^*}(y')| \qquad \text{whenever } y \in V_\nu(x) \text{ and } y' \in V_\nu(x')\,,$$

and *strongly vertex regular* if for all $\lambda,\mu,\nu \in P^+$

$$|V_\lambda(x)\cap V_{\mu^*}(y)| = |V_{\lambda^*}(x')\cap V_{\mu}(y')| \qquad \text{whenever } y \in V_\nu(x) \text{ and } y' \in V_{\nu^*}(x')\,.$$

Strong vertex regularity implies vertex regularity. To see this, suppose we are given $x, y, x', y' \in V_P$ with $y \in V_\nu(x)$ and $y' \in V_\nu(x')$, and choose any pair $x'', y'' \in V_P$ with $y'' \in V_{\nu^*}(x'')$. Then if strong vertex regularity holds, we have

$$|V_\lambda(x)\cap V_{\mu^*}(y)| = |V_{\lambda^*}(x'')\cap V_{\mu}(y'')| = |V_\lambda(x')\cap V_{\mu^*}(y')|\,.$$

LEMMA 4.4.4. *Let $y \in V_\nu(x)$ and suppose that $z \in V_\lambda(x)\cap V_{\mu^*}(y)$. Write $\tau(x) = i$, $\tau(y) = j$, $\tau(z) = k$, $\tau(\lambda) = l$, $\tau(\mu) = m$, and $\tau(\nu) = n$.*

(i) *$\sigma_i^{-1}(k) = l$, $\sigma_k^{-1}(j) = m$ and $\sigma_i^{-1}(j) = n$. Thus $\sigma_i^{-1}\circ\sigma_k = \sigma_l$, $\sigma_k^{-1}\circ\sigma_j = \sigma_m$ and $\sigma_i^{-1}\circ\sigma_j = \sigma_n$.*

(ii) *$\sigma_n = \sigma_l\circ\sigma_m$.*

PROOF. (i) follows immediately from Lemma 4.3.3. To prove (ii), we have

$$\sigma_l\circ\sigma_m = \sigma_i^{-1}\circ\sigma_k\circ\sigma_k^{-1}\circ\sigma_j = \sigma_i^{-1}\circ\sigma_j = \sigma_n.$$ □

Recall the definition of the automorphism $\sigma_* \in \mathrm{Aut}(D)$ from Section 3.6.

THEOREM 4.4.5. *$\mathscr{X}$ is strongly vertex regular.*

PROOF. Let $x, y \in V_P$ with $y \in V_\nu(x)$ and suppose that $z \in V_\lambda(x) \cap V_{\mu^*}(y)$. Let $\tau(x) = i$, $\tau(y) = j$ and $\tau(z) = k$. With the notation used in the proof of Theorem 4.3.4, define a map $\psi : \mathcal{C}_\lambda(x) \cap \mathcal{C}_{\mu^*}(y) \to V_\lambda(x) \cap V_{\mu^*}(y)$ by the rule $\psi(c) = \pi_k(c)$. As in the proof of Theorem 4.3.4 we see that this is a $W_0(q)$-to-one surjection, and thus by Theorem 4.2.7

$$|V_\lambda(x) \cap V_{\mu^*}(y)| = \frac{1}{W_0(q)} \sum_{\substack{w_1 \in W_i \sigma_i(w_\lambda) W_k \\ w_2 \in W_j \sigma_j(w_{\mu^*}) W_k}} |\mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2}(b)|$$

where a and b are any chambers with $\pi_i(a) = x$ and $\pi_j(b) = y$. Notice that this implies that $\delta(a, b) \in W_i \sigma_i(w_\nu) W_j$, by Theorem 4.2.7.

Writing $\tau(\lambda) = l$ and $\tau(\nu) = n$, Lemma 4.4.4(i) implies that

$$W_i \sigma_i(w_\lambda) W_k = \sigma_i(W_0 w_\lambda \sigma_i^{-1}(W_k)) = \sigma_i(W_0 w_\lambda W_{\sigma_i^{-1}(k)}) = \sigma_i(W_0 w_\lambda W_l)\,,$$

$$W_j \sigma_j(w_{\mu^*}) W_k = \sigma_i(W_{\sigma_i^{-1}(j)} (\sigma_i^{-1} \circ \sigma_j)(w_{\mu^*}) W_{\sigma_i^{-1}(k)}) = \sigma_i(W_n \sigma_n(w_{\mu^*}) W_l)$$

and similarly $W_i \sigma_i(w_\nu) W_j = \sigma_i(W_0 w_\nu W_n)$. Applying Lemma 2.1.12 (with $\sigma = \sigma_i$) we therefore have

$$|V_\lambda(x) \cap V_{\mu^*}(y)| = \frac{1}{W_0(q)} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_n \sigma_n(w_{\mu^*}) W_l}} |\mathcal{C}_{w_1}(a') \cap \mathcal{C}_{w_2}(b')| \tag{4.4.2}$$

where a', b' are any chambers with $\delta(a', b') \in W_0 w_\nu W_n$.

Vertex regularity follows from (4.4.2), for the value of $|V_\lambda(x) \cap V_{\mu^*}(y)|$ is seen to only depend on λ, μ and ν. To see that strong vertex regularity holds, we use Proposition 3.7.5 to see that

$$W_0 w_\lambda W_l = \sigma_*(W_{\sigma_*^{-1}(0)} \sigma_*^{-1}(w_\lambda) W_{\sigma_*^{-1}(l)}) = \sigma_*(W_0 w_{\lambda^*} W_{l^*})\,,$$

$$W_n \sigma_n(w_{\mu^*}) W_l = \sigma_*(W_{n^*} (\sigma_*^{-1} \circ \sigma_n \circ \sigma_*)(w_\mu) W_{l^*}) = \sigma_*(W_{n^*} \sigma_{n^*}(w_\mu) W_{l^*})\,,$$

and similarly $W_0 w_\nu W_n = \sigma_*(W_0 w_{\nu^*} W_{n^*})$. A further application of Lemma 2.1.12 (with $\sigma = \sigma_*$) implies that

$$|V_\lambda(x) \cap V_{\mu^*}(y)| = \frac{1}{W_0(q)} \sum_{\substack{w_1 \in W_0 w_{\lambda^*} W_{l^*} \\ w_2 \in W_{n^*} \sigma_{n^*}(w_\mu) W_{l^*}}} |\mathcal{C}_{w_1}(a'') \cap \mathcal{C}_{w_2}(b'')|$$

where a'', b'' are any chambers with $\delta(a'', b'') \in W_0 w_{\nu^*} W_{n^*}$. Thus by comparison with (4.4.2) we have

$$|V_\lambda(x) \cap V_{\mu^*}(y)| = |V_{\lambda^*}(x') \cap V_\mu(y')|,$$

where $x', y' \in V_P$ are any vertices with $y' \in V_{\nu^*}(x')$; that is, strong vertex regularity holds. $\square$

COROLLARY 4.4.6. *Let $\lambda, \mu \in P^+$. There exist numbers $a_{\lambda,\mu;\nu} \in \mathbb{Q}^+$ such that*

$$A_\lambda A_\mu = \sum_{\nu \in P^+} a_{\lambda,\mu;\nu} A_\nu \qquad \text{and} \qquad \sum_{\nu \in P^+} a_{\lambda,\mu;\nu} = 1\,.$$

Moreover, $|\{\nu \in P^+ \mid a_{\lambda,\mu;\nu} \neq 0\}|$ is finite for all $\lambda, \mu \in P^+$.

PROOF. Let $v \in V_\nu(u)$ and set

$$a_{\lambda,\mu;\nu} = \frac{N_\nu}{N_\lambda N_\mu} |V_\lambda(u) \cap V_{\mu^*}(v)|\,, \tag{4.4.3}$$

which is independent of the particular pair u, v by vertex regularity. The numbers $a_{\lambda,\mu;\nu}$ are clearly nonnegative and rational, and from (4.4.1) we have

$$\begin{aligned}(A_\lambda A_\mu f)(x) &= \sum_{\nu \in P^+} \Bigg(\sum_{y \in V_\nu(x)} \frac{|V_\lambda(x) \cap V_{\mu^*}(y)|}{N_\lambda N_\mu} f(y) \Bigg) \\ &= \sum_{\nu \in P^+} a_{\lambda,\mu;\nu} \Bigg(\frac{1}{N_\nu} \sum_{y \in V_\nu(x)} f(y) \Bigg) \\ &= \sum_{\nu \in P^+} a_{\lambda,\mu;\nu} (A_\nu f)(x)\,.\end{aligned}$$

When $f = 1_{V_P} : V_P \to \{1\}$ we see that $\sum a_{\lambda,\mu;\nu} = 1$.

We now show that only finitely many of the $a_{\lambda,\mu;\nu}$'s are nonzero for each fixed pair $\lambda, \mu \in P^+$. Fix $x \in V_P$ and observe that $a_{\lambda,\mu;\nu} \neq 0$ if and only if $V_\lambda(x) \cap V_{\mu^*}(y) \neq \emptyset$ for each $y \in V_\nu(x)$. Applying $(N_\lambda A_\lambda)(N_\mu A_\mu)$ to the constant function $1_{V_P} : V_P \to \{1\}$, we obtain

$$\sum_{y \in V_P} |V_\lambda(x) \cap V_{\mu^*}(y)| = N_\lambda N_\mu,$$

and hence $V_\lambda(x) \cap V_{\mu^*}(y) \neq \emptyset$ for only finitely many $y \in V_P$. $\square$

DEFINITION 4.4.7. Let $\mathscr{A}$ be the linear span of $\{A_\lambda \mid \lambda \in P^+\}$ over $\mathbb{C}$. The previous corollary shows that $\mathscr{A}$ is an associative algebra.

We refer to the numbers $a_{\lambda,\mu;\nu}$ in Corollary 4.4.6 as the *structure constants* of the algebra $\mathscr{A}$.

THEOREM 4.4.8. *The algebra $\mathscr{A}$ is commutative.*

PROOF. We need to show that $a_{\lambda,\mu;\nu} = a_{\mu,\lambda,\nu}$ for all $\lambda, \mu, \nu \in P^+$. Fixing any pair u, v in V_P with $v \in V_\nu(u)$, strong vertex regularity implies that

$$a_{\lambda,\mu;\nu} = \frac{N_\nu}{N_\lambda N_\mu} |V_\lambda(u) \cap V_{\mu^*}(v)| = \frac{N_\nu}{N_\lambda N_\mu} |V_{\lambda^*}(v) \cap V_\mu(u)| = a_{\mu,\lambda;\nu}$$

completing the proof. $\square$

Note that a similar calculation using Theorem 4.3.4 (specifically the fact that $N_\lambda = N_{\lambda^*}$) shows that $a_{\lambda,\mu;\nu} = a_{\lambda^*,\mu^*;\nu^*}$ for all $\lambda, \mu, \nu \in P^+$.

REMARK 4.4.9. Let X be a set and let K be a partition of $X \times X$ such that $\emptyset \notin K$ and $\{(x,x) \mid x \in X\} \in K$. For $k \in K$, define $k^* = \{(y,x) \mid (x,y) \in k\}$, and for each $x \in X$ and $k \in K$ define $k(x) = \{y \in X \mid (x,y) \in k\}$. Recall [**46**] that an *association scheme* is a pair (X,K) as above such that (i) $k \in K$ implies that $k^* \in K$, and (ii) for each $k,l,m \in K$ there exists a cardinal number $e_{k,l;m}$ such that

$$(x,y) \in m \quad \text{implies that} \quad |k(x) \cap l^*(y)| = e_{k,l;m}\,.$$

Let $X = V_P$, and for each $\lambda \in P^+$ let $\lambda' = \{(x,y) \mid y \in V_\lambda(x)\}$. Then $L = \{\lambda' \mid \lambda \in P^+\}$ forms a partition of $V_P \times V_P$, and $\lambda'(x) = V_\lambda(x)$ for $x \in V_P$.

By vertex regularity it follows that the pair (V_P, L) forms an association scheme, and the cardinal numbers $e_{\lambda',\mu';\nu'}$ are simply $N_\lambda N_\mu N_\nu^{-1} a_{\lambda,\mu;\nu}$. By strong vertex regularity this association scheme also satisfies the condition $e_{\lambda',\mu';\nu'} = e_{\mu',\lambda';\nu'}$ for all $\lambda,\mu,\nu \in P^+$ (see [**46**, p.1, footnote]).

Note that the algebra $\mathscr{A}$ is essentially the *Bose-Mesner algebra* of the association scheme (V_P, L) (see [**2**, Chapter 2]). With reference to Remark 3.8.3, the above construction generalises the familiar construction of association schemes from infinite distance regular graphs (see [**2**, §1.4.4] for the case of *finite* distance regular graphs).

Recall the definition of the numbers $b_{w_1,w_2;w_3}$ given in Corollary 2.1.8.

PROPOSITION 4.4.10. *Let $\tau(\lambda) = l$ and $\tau(\nu) = n$. Suppose that $y \in V_\nu(x)$ and that $V_\lambda(x) \cap V_{\mu^*}(y) \neq \emptyset$. Then*

$$a_{\lambda,\mu;\nu} = \frac{W_{0\lambda}(q)W_{0\mu}(q)}{W_{0\nu}(q)W_0^2(q)q_{w_\lambda}q_{w_\mu}} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} q_{w_1}q_{w_2}b_{w_1,w_2;w_\nu}$$

PROOF. By Lemma 4.4.4(ii) we have $\sigma_n = \sigma_l \circ \sigma_m$. Thus by Proposition 3.7.5(iv) we have $W_n\sigma_n(w_{\mu^*})W_l = (W_l\sigma_l(w_\mu)W_n)^{-1}$, and so by (4.4.2) we see that

$$|V_\lambda(x) \cap V_{\mu^*}(y)| = \frac{1}{W_0(q)} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} |\mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2^{-1}}(b)| \tag{4.4.4}$$

whenever $\delta(a,b) \in W_0 w_\nu W_n$.

By Proposition 2.1.10 (and the proof thereof) we have

$$|\mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2^{-1}}(b)| = q_{w_1}q_{w_2}(B_{w_1}B_{w_2}\delta_b)(a)\,,$$

and the result now follows from (4.4.4) by using Theorem 4.3.4 and the definitions of $a_{\lambda,\mu;\nu}$ and $b_{w_1,w_2;w_3}$, by choosing $b \in \mathcal{C}_{w_\nu}(a)$. □

We conclude this section by recording an identity for later use.

PROPOSITION 4.4.11. *Let $\lambda,\mu,\nu\in P^+$. Then*
$$\frac{a_{\lambda,\mu;\nu^*}}{N_\nu}=\frac{a_{\nu,\mu;\lambda^*}}{N_\lambda}=\frac{a_{\lambda,\nu;\mu^*}}{N_\mu}.$$

PROOF. Since
$$(A_\lambda A_\mu)A_\nu=\sum_{\zeta\in P^+}\bigg(\sum_{\eta\in P^+}a_{\lambda,\mu;\eta}a_{\eta,\nu;\zeta}\bigg)A_\zeta$$
and
$$A_\lambda(A_\mu A_\nu)=\sum_{\zeta\in P^+}\bigg(\sum_{\eta\in P^+}a_{\mu,\nu;\eta}a_{\lambda,\eta;\zeta}\bigg)A_\zeta,$$
we have
$$\sum_{\eta\in P^+}a_{\lambda,\mu;\eta}a_{\eta,\nu;\zeta}=\sum_{\eta\in P^+}a_{\mu,\nu;\eta}a_{\lambda,\eta;\zeta}\quad\text{for all }\zeta\in P^+.$$
Since $a_{\lambda,\mu;0}=N_\lambda^{-1}\delta_{\lambda,\mu^*}$ we see that $a_{\lambda,\mu;\nu^*}/N_\nu=a_{\mu,\nu;\lambda^*}/N_\lambda$, and the result follows by commutativity. □

REMARK 4.4.12. There is a similar identity to that in Proposition 4.4.11 for the algebra $\mathscr{B}$. Indeed, there is a similar identity for any association scheme (see [**46**, Lemma 1.1.3]).

4.5. Subalgebras of $\mathscr{A}$

Let L be a lattice satisfying
$$Q\subseteq L\subseteq P. \tag{4.5.1}$$

LEMMA 4.5.1. *With L as above, if $\lambda\in L$ and $\mu\in P$ satisfy $\tau(\lambda)=\tau(\mu)$, then $\mu\in L$. Thus L is stable under W_0.*

PROOF. If $\tau(\lambda)=\tau(\mu)$, then $\mu-\lambda\in Q$, proving the first statement since $Q\subseteq L$. Thus if $\lambda\in L$ and $w\in W_0$, we have $w\lambda\in L$ since $\tau(w\lambda)=\tau(\lambda)$. Hence $W_0L=L$. □

Let $I_L=\{\tau(\lambda)\mid\lambda\in L\}$ and $V_L=\{x\in V\mid\tau(x)\in I_L\}$ (here V is the vertex set of $\mathscr{X}$). Clearly $\{0\}=I_Q\subseteq I_L\subseteq I_P$ and $V_Q\subseteq V_L\subseteq V_P$.

Let $\mathscr{A}_L$ be the linear span of $\{A_\lambda\mid\lambda\in L\}$ over $\mathbb{C}$. Thus $\mathscr{A}_P=\mathscr{A}$.

PROPOSITION 4.5.2. *For all lattices L as in (4.5.1), $\mathscr{A}_L$ is a subalgebra of $\mathscr{A}$.*

PROOF. Let $\lambda,\mu\in L$ and let $l=\tau(\lambda)$ and $m=\tau(\mu)$. By Corollary 4.4.6 and (4.4.3), $A_\lambda A_\mu=\sum_{\nu\in P^+}a_{\lambda,\mu;\nu}A_\nu$ where $a_{\lambda,\mu;\nu}\neq 0$ if and only if there is a vertex $z\in V_\lambda(x)\cap V_{\mu^*}(y)$, where $x,y\in V_P$ are any vertices with $y\in V_\nu(x)$. Suppose we have such a vertex z. By choosing $x\in V_Q$ (so $\tau(x)=0$), since $z\in V_\lambda(x)$ we have $\tau(z)=l$. Since $y\in V_\mu(z)$ we have $\tau(y)=\sigma_l(m)$. Thus $\tau(y)=\tau(\lambda+\mu)$, and since $\lambda+\mu\in L$ we have $y\in V_L$. Thus $A_\lambda A_\mu$ is a finite linear combination of $\{A_\nu\mid\nu\in L\}$, and the result follows. □

In particular, $\mathscr{A}_Q$ is a subalgebra of $\mathscr{A}$. In the case where $\mathscr{X}$ is the Bruhat-tits building of a group G of p-adic type with maximal compact subgroup K (as in [**23**, §2.4–2.7]), $\mathscr{A}_Q$ is isomorphic to $\mathscr{L}(G, K)$, the space of continuous, compactly supported bi-K-invariant functions on G.

CHAPTER 5

Affine Hecke Algebras, and the Isomorphism $\mathscr{A} \to \mathbb{C}[P]^{W_0}$

In this chapter we make an important connection between the algebra $\mathscr{A}$ and affine Hecke algebras. In particular, in Theorem 5.3.5 we show that $\mathscr{A}$ is isomorphic to $Z(\tilde{\mathscr{H}})$, the center of an appropriately parametrised affine Hecke algebra $\tilde{\mathscr{H}}$.

In Sections 5.1 and 5.2 we give an outline of some known results regarding affine Hecke algebras. The main references for this material are [**27**] and [**31**]. Note that in [**31**] there is only one parameter q, although the results there go through without any serious difficulty in the more general case of multiple parameters $\{q_s\}_{s\in S}$. Note also that in [**31**] $Q = Q(R)$ and $P = P(R)$, whereas for us $Q = Q(R^\vee)$ and $P = P(R^\vee)$.

In Section 5.4 we give a positivity result for certain structure constants in $Z(\tilde{\mathscr{H}})$, generalising a result of Miller Malley [**30**].

5.1. Affine Hecke Algebras

Let R be an irreducible (but not necessarily reduced) root system, and let $\tilde{W}$ be the extended affine Weyl group of R.

Let $\{q_s\}_{s\in S}$ be a set of positive real numbers with $q_{s_i} = q_{s_j}$ whenever s_i and s_j are conjugate in $\tilde{W}$. The *affine Hecke algebra* $\tilde{\mathscr{H}}$ with parameters $\{q_s\}_{s\in S}$ is the algebra over $\mathbb{C}$ with presentation given by the generators T_w, $w \in \tilde{W}$, and relations

$$T_{w_1}T_{w_2} = T_{w_1w_2} \qquad \text{if } \ell(w_1w_2) = \ell(w_1) + \ell(w_2)\,, \tag{5.1.1}$$

$$T_wT_s = \frac{1}{q_s}T_{ws} + \left(1 - \frac{1}{q_s}\right)T_w \qquad \text{if } \ell(ws) < \ell(w) \text{ and } s \in S\,. \tag{5.1.2}$$

By (5.1.1), $T_1T_w = T_wT_1 = T_w$ for all $w \in \tilde{W}$, and hence $T_1 = I$ since $\{T_w\}_{w\in\tilde{W}}$ generates $\tilde{\mathscr{H}}$. Then (5.1.2) implies that each T_s, $s \in S$, is invertible, and from (5.1.1) we see that each T_g, $g \in G$, is invertible, with inverse $T_{g^{-1}}$ (recall the definition of G from Section 3.5). Since each $w \in \tilde{W}$ can be written as $w = w'g$ for some $w' \in W$ and $g \in G$, it follows that each T_w, $w \in \tilde{W}$, is invertible.

Remark 5.1.1. (i) In [**27**] the numbers $\{q_s\}_{s\in S}$ are taken as positive real variables. Our choice to fix the numbers $\{q_s\}_{s\in S}$ does not change the algebraic structure of $\tilde{\mathscr{H}}$ in any serious way (for our purposes, at least).

(ii) The condition that $q_{s_i} = q_{s_j}$ whenever $s_i = ws_jw^{-1}$ for some $w \in \tilde{W}$ is equivalent to the condition that $q_{s_i} = q_{s_j}$ whenever $s_i = us_{\sigma(j)}u^{-1}$ for some $\sigma \in \mathrm{Aut_{tr}}(D)$ and $u \in W$.

This condition is quite restrictive, and it is easy to see that we obtain the parameter systems given in Appendix E. Thus connections with our earlier results on the algebra $\mathscr{A}$ will become apparent when we take the numbers $\{q_s\}_{s\in S}$ to be the parameters of a locally finite regular affine building.

DEFINITION 5.1.2. (i) We write $q_w = q_{s_{i_1}} \cdots q_{s_{i_m}}$ if $s_{i_1} \cdots s_{i_m}$ is a reduced expression for $w \in W$. This is easily seen to be independent of the particular reduced expression (see [**5**, IV, §1, No.5, Proposition 5]). Each $\tilde{w} \in \tilde{W}$ can be written uniquely as $\tilde{w} = wg$ for $w \in W$ and $g \in G$, and we define $q_{\tilde{w}} = q_w$ (cf. Section 4.3). In particular $q_g = 1$ for all $g \in G$. Furthermore, if $s = s_i$ we write $q_s = q_i$.

(ii) To conveniently state later results we make the following definitions. Let $R_1 = \{\alpha \in R \mid 2\alpha \notin R\}$, $R_2 = \{\alpha \in R \mid \frac{1}{2}\alpha \notin R\}$ and $R_3 = R_1 \cap R_2$ (so $R_1 = R_2 = R_3 = R$ if R is reduced). For $\alpha \in R_2$, write $q_\alpha = q_i$ if $\alpha \in W_0\alpha_i$ (note that if $\alpha \in W_0\alpha_i$ then necessarily $\alpha \in R_2$). It follows easily from Corollary 1.7.2 that this definition is unambiguous.

Note that R is the disjoint union of R_3, $R_1\backslash R_3$ and $R_2\backslash R_3$, and define set of numbers $\{\tau_\alpha\}_{\alpha\in R}$ by

$$\tau_\alpha = \begin{cases} q_\alpha & \text{if } \alpha \in R_3 \\ q_0 & \text{if } \alpha \in R_1\backslash R_3 \\ q_\alpha q_0^{-1} & \text{if } \alpha \in R_2\backslash R_3, \end{cases}$$

where $q_0 = q_{s_0}$ (with $s_0 = s_{\tilde{\alpha};1}$ and $\tilde{\alpha}$ is as in (3.1.1)). It is convenient to also define $\tau_\alpha = 1$ if $\alpha \notin R$. The reader only interested in the reduced case can simply read τ_α as q_α. Note that $\tau_{w\alpha} = \tau_\alpha$ for all $\alpha \in R$ and $w \in W_0$.

REMARK 5.1.3. We have chosen a slight distortion of the usual definition of the algebra $\tilde{\mathscr{H}}$. This choice has been made so as to make the connection between the algebras $\mathscr{A}$ and $\tilde{\mathscr{H}}$ more transparent, as the reader will shortly see. To allow the reader to convert between our notation and that in [**27**], we provide the following instructions. With reference to our presentation for $\tilde{\mathscr{H}}$ given above, let $\tau_i = \sqrt{q_i}$ and $T_w' = \sqrt{q_w}\,T_w$ (these τ's are unrelated to those in Definition 5.1.2(ii)). Our presentation then transforms into that given in [**27**, 4.1.2] (with the T's there replaced by T''s). This transformation also makes it clear why the $\sqrt{q_w}$'s appear in the following discussion.

If $\lambda \in P^+$ let $x^\lambda = \sqrt{q_{t_\lambda}}\,T_{t_\lambda}$, and if $\lambda = \mu - \nu$ with $\mu, \nu \in P^+$ let $x^\lambda = x^\mu(x^\nu)^{-1}$. This is well defined by [**27**, p.40], and for all $\lambda, \mu \in P$ we have $x^\lambda x^\mu = x^{\lambda+\mu} = x^\mu x^\lambda$.

We write $\mathbb{C}[P]$ for the $\mathbb{C}$-span of $\{x^\lambda \mid \lambda \in P\}$. The group W_0 acts on $\mathbb{C}[P]$ by linearly extending the action $wx^\lambda = x^{w\lambda}$. We write $\mathbb{C}[P]^{W_0}$ for the set of elements of $\mathbb{C}[P]$ that are invariant under the action of W_0. By Corollary 5.2.2, the center $Z(\tilde{\mathscr{H}})$ of $\tilde{\mathscr{H}}$ is $\mathbb{C}[P]^{W_0}$.

Let $\mathscr{H}$ be the subalgebra of $\tilde{\mathscr{H}}$ generated by $\{T_s \mid s \in S\}$. The following relates the algebra $\mathscr{H}$ to the algebra $\mathscr{B}$ of chamber set averaging operators on an irreducible affine building.

PROPOSITION 5.1.4. *Suppose a building $\mathscr{X}$ of type R exists with parameters $\{q_s\}_{s\in S}$. Then $\mathscr{H} \cong \mathscr{B}$.*

PROOF. This follows in the same way as Theorem 2.2.1. □

We make the following parallel definition to (4.3.1). Recall the definition of Poincaré polynomials from Definition 1.7.6. For each $i \in I$, let

$$\mathbb{1}_i = \frac{1}{W_i(q)} \sum_{w\in W_i} q_w T_w, \tag{5.1.3}$$

where $W_i = W_{I\backslash\{i\}}$ (as before). Thus $\mathbb{1}_i$ is an element of $\mathscr{H}$. As a word of warning, we have used the same notation as in (4.3.1) where we defined the analogous element in $\mathscr{B}$. There should be no confusion caused by this decision.

The following lemma follows in exactly the same way as Lemma 4.3.1.

LEMMA 5.1.5. *$\mathbb{1}_i T_w = T_w \mathbb{1}_i = \mathbb{1}_i$ for all $w \in W_i$ and $i \in I$. Furthermore $\mathbb{1}_i^2 = \mathbb{1}_i$.*

5.2. The Macdonald Spherical Functions

The following relations are of fundamental significance.

THEOREM 5.2.1. *Let $\lambda \in P$ and $i \in I_0$.*

(i) *If $(R, i) \neq (BC_n, n)$ for any $n \geq 1$, then*

$$x^\lambda T_{s_i} - T_{s_i} x^{s_i\lambda} = (1 - q_i^{-1}) \frac{x^\lambda - x^{s_i\lambda}}{1 - x^{-\alpha_i^\vee}}.$$

(ii) *If $R = BC_n$ for some $n \geq 1$ and $i = n$, then*

$$x^\lambda T_{s_n} - T_{s_n} x^{s_n\lambda} = \left[1 - q_n^{-1} + q_n^{-1/2}\big(q_0^{1/2} - q_0^{-1/2}\big)x^{-(2\alpha_n)^\vee}\right] \frac{x^\lambda - x^{s_n\lambda}}{1 - x^{-2(2\alpha_n)^\vee}}.$$

PROOF. This follows from [**27**, (4.2.4)] (see Remark 5.1.3), taking into account [**27**, (1.4.3) and (2.1.6)] in case(ii). □

We note that the fractions appearing in Theorem 5.2.1 are in fact finite linear combinations of the x^μ's [**27**, (4.2.5)]. We refer to the relations in Theorem 5.2.1 as the *Bernstein relations*, for they are a crucial ingredient in the so-called *Bernstein presentation* of the Hecke algebra.

COROLLARY 5.2.2. *The center $Z(\tilde{\mathscr{H}})$ of $\tilde{\mathscr{H}}$ is $\mathbb{C}[P]^{W_0}$.*

PROOF. This well known fact can be proved using the Bernstein relations, exactly as in [**27**, (4.2.10)]. □

For each $\lambda \in P^+$, define an element $P_\lambda(x) \in \mathbb{C}[P]^{W_0}$ by

$$P_\lambda(x) = \frac{q_{t_\lambda}^{-1/2}}{W_0(q)} \sum_{w \in W_0} w\left(x^\lambda \prod_{\alpha \in R^+} \frac{\tau_\alpha \tau_{\alpha/2}^{1/2} x^{\alpha^\vee} - 1}{\tau_{\alpha/2}^{1/2} x^{\alpha^\vee} - 1}\right). \tag{5.2.1}$$

We call the elements $P_\lambda(x)$ the *Macdonald spherical functions* of $\tilde{\mathscr{H}}$. See (5.2.9) and (5.2.11) for some alternative formulae for $P_\lambda(x)$.

Remark 5.2.3. (i) We have chosen a slightly different normalisation of the Macdonald spherical function from that in [**27**]. Our formula uses the normalisation of [**23**, Theorem 4.1.2] (see (5.2.9)).

(ii) Notice that the formula simplifies in the reduced case (namely, $\tau_{\alpha/2} = 1$).

(iii) It is not immediately clear that $P_\lambda(x)$ as defined in (5.2.1) is in $\mathbb{C}[P]^{W_0}$, although this is a consequence of [**5**, VI, §3, No.3, Proposition 2] (see also the proof of Theorem 5.2.7).

The proof of Theorem 5.2.4 below follows [**31**, Theorem 2.9] closely.

Theorem 5.2.4. [**31**, Theorem 2.9]. *For $\lambda \in P^+$ we have $q_{t_\lambda}^{1/2} P_\lambda(x)\mathbb{1}_0 = \mathbb{1}_0 x^\lambda \mathbb{1}_0$.*

Proof. By the Satake isomorphism (see [**31**, Theorem 2.4] or [**21**, 5.2] for example) there exists some $P'_\lambda(x) \in \mathbb{C}[P]^{W_0}$ such that $P'_\lambda(x)\mathbb{1}_0 = \mathbb{1}_0 x^\lambda \mathbb{1}_0$. If $(R, i) \neq (BC_n, n)$, then by Theorem 5.2.1(i) (and using Lemma 5.1.5) we have

$$\begin{aligned}(1 + q_i T_{s_i}) x^\lambda \mathbb{1}_0 &= x^\lambda \mathbb{1}_0 + q_i x^{s_i\lambda} T_{s_i} \mathbb{1}_0 + (q_i - 1)\frac{x^\lambda - x^{s_i\lambda}}{1 - x^{-\alpha_i^\vee}} \mathbb{1}_0 \\ &= \frac{q_i x^\lambda - x^{\lambda - \alpha_i^\vee} - q_i x^{s_i\lambda - \alpha_i^\vee} + x^{s_i\lambda}}{1 - x^{-\alpha_i^\vee}} \mathbb{1}_0 \\ &= \left(\frac{q_i x^{\alpha_i^\vee} - 1}{x^{\alpha_i^\vee} - 1} x^\lambda + \frac{q_i x^{-\alpha_i^\vee} - 1}{x^{-\alpha_i^\vee} - 1} x^{s_i\lambda}\right)\mathbb{1}_0 \\ &= (1 + s_i)\frac{q_i x^{\alpha_i^\vee} - 1}{x^{\alpha_i^\vee} - 1} x^\lambda \mathbb{1}_0.\end{aligned} \tag{5.2.2}$$

A similar calculation, using Theorem 5.2.1(ii), shows that if $(R, i) = (BC_n, n)$, then

$$(1 + q_n T_{s_n}) x^\lambda \mathbb{1}_0 = (1 + s_n)\frac{\left(\sqrt{q_0 q_n}\, x^{(2\alpha_n)^\vee} - 1\right)\left(\sqrt{q_n/q_0}\, x^{(2\alpha_n)^\vee} + 1\right)}{x^{2(2\alpha_n)^\vee} - 1} x^\lambda \mathbb{1}_0. \tag{5.2.3}$$

It will be convenient to write (5.2.2) and (5.2.3) as one equation, as follows. In the reduced case, let $\beta_i = \alpha_i$ for all $i \in I_0$, and in the non-reduced case (so $R = BC_n$ for some $n \geq 1$) let $\beta_i = \alpha_i$ for $1 \leq i \leq n-1$ and let $\beta_n = 2\alpha_n$. For $\alpha \in R$ and $i \in I_0$, write

$$a_i(x^{\alpha^\vee}) = \frac{(\tau_{\beta_i} \tau_{\beta_i/2}^{1/2} x^{\alpha^\vee} - 1)(\tau_{\beta_i/2}^{1/2} x^{\alpha^\vee} + 1)}{x^{2\alpha^\vee} - 1},$$

and so in all cases

$$(1 + q_i T_{s_i}) x^\lambda \mathbb{1}_0 = (1 + s_i) a_i(x^{\beta_i^\vee}) x^\lambda \mathbb{1}_0\,. \tag{5.2.4}$$

By induction we see that (writing T_i for T_{s_i})

$$\left[\prod_{k=1}^{m}(1+q_{i_k}T_{i_k})\right]x^\lambda\mathbb{1}_0=\left[\prod_{k=1}^{m}(1+s_{i_k})a_{i_k}(x^{\beta_{i_k}^\vee})\right]x^\lambda\mathbb{1}_0, \tag{5.2.5}$$

where we write $\prod_{k=1}^m x_k$ for the ordered product $x_1\cdots x_m$. Therefore $\mathbb{1}_0x^\lambda\mathbb{1}_0$ can be written as $fx^\lambda\mathbb{1}_0$, where f is independent of λ and is a finite linear combination of terms of the form

$$(1+s_{i_1})a_{i_1}(x^{\beta_{i_1}^\vee})\cdots(1+s_{i_m})a_{i_m}(x^{\beta_{i_m}^\vee})$$

where $i_1,\ldots,i_m\in I_0$.

Thus we have

$$P'_\lambda(x)=\sum_{w\in W_0}w\left(b_w(x)x^\lambda\right)$$

where each $b_w(x)$ is a linear combination of products of terms $a_i(x^{\beta_i^\vee})$ and is independent of $\lambda\in P^+$. It is easily seen that this expression is unique, and since $P'_\lambda(x)\in\mathbb{C}[P]^{W_0}$ it follows that $b_w(x)=b_{w'}(x)$ for all $w,w'\in W_0$, and we write $b(x)$ for this common value. Thus

$$P'_\lambda(x)=\sum_{w\in W_0}w\left(b(x)x^\lambda\right)=\sum_{w\in W_0}w\left(x^{w_0\lambda}w_0b(x)\right)$$

where w_0 is the longest element of W_0.

We now compute the coefficient of $x^{w_0\lambda}$ in the above expression. Since this coefficient is independent of $\lambda\in P^+$ we may assume that $\langle\lambda,\alpha_i\rangle>0$ for all $i\in I_0$ and so $w\lambda\neq w_0\lambda$ for all $w\in W_0\backslash\{w_0\}$.

If $w_0=s_{i_1}\cdots s_{i_m}$ is a reduced expression, then

$$\begin{aligned}\mathbb{1}_0=\frac{1}{W_0(q)}\bigg(&(1+q_{i_1}T_{i_1})\cdots(1+q_{i_m}T_{i_m})\\&+\text{terms }(1+q_{j_1}T_{j_1})\cdots(1+q_{j_l}T_{j_l})\text{ with }j_k\in I_0\text{ and }l<m\bigg).\end{aligned}$$

Thus, by (5.2.5)

$$\begin{aligned}\mathbb{1}_0x^\lambda\mathbb{1}_0=\frac{1}{W_0(q)}\Bigg[&\left(\prod_{k=1}^{m}s_{i_k}a_{i_k}(x^{\beta_{i_k}^\vee})\right)x^\lambda\mathbb{1}_0\\&+\text{terms }\left(\prod_{k=1}^{l}s_{j_k}a_{j_k}(x^{\beta_{j_k}^\vee})\right)x^\lambda\mathbb{1}_0\text{ with }j_k\in I_0\text{ and }l<m\Bigg].\end{aligned}$$

Thus the coefficient of $x^{w_0\lambda}$ is

$$\begin{aligned} w_0 b(x) &= \frac{1}{W_0(q)} s_{i_1} a_{i_1}(x^{\beta_{i_1}^\vee}) \cdots s_{i_m} a_{i_m}(x^{\beta_{i_m}^\vee}) \\ &= \frac{1}{W_0(q)} w_0 \prod_{\beta \in R_1^+} \frac{(\tau_\beta \tau_{\beta/2}^{1/2} x^{\beta^\vee} - 1)(\tau_{\beta/2}^{1/2} x^{\beta^\vee} + 1)}{x^{2\beta^\vee} - 1}, \end{aligned}$$

where we have used the fact that

$$\{\beta_{i_m}^\vee, s_{i_m}\beta_{i_{m-1}}^\vee, \cdots, s_{i_m}s_{i_{m-1}}\cdots s_{i_2}\beta_{i_1}^\vee\} = (R_1^+)^\vee,$$

(see [**27**, (2.2.9)]) and the fact that $\tau_{w\alpha} = \tau_\alpha$ for all $w \in W_0$ and $\alpha \in R$.

Finally, let us demonstrate that

$$\prod_{\beta \in R_1^+} \frac{(\tau_\beta \tau_{\beta/2}^{1/2} x^{\beta^\vee} - 1)(\tau_{\beta/2}^{1/2} x^{\beta^\vee} + 1)}{x^{2\beta^\vee} - 1} = \prod_{\alpha \in R^+} \frac{\tau_\alpha \tau_{\alpha/2}^{1/2} x^{\alpha^\vee} - 1}{\tau_{\alpha/2}^{1/2} x^{\alpha^\vee} - 1}. \tag{5.2.6}$$

To see this we start with the right hand side of (5.2.6). Observe that each $\alpha \in R_2^+ \backslash R_3^+$ is equal to $\alpha'/2$ for some $\alpha' \in R_1^+ \backslash R_3^+$, and conversely each $\alpha \in R_1^+ \backslash R_3^+$ is equal to $2\alpha'$ for some $\alpha' \in R_2^+ \backslash R_3^+$ (this can be seen without the classification theorem). The factors of the right hand side of (5.2.6) corresponding to the roots $\alpha \in R_1^+ \backslash R_3^+$ and $\alpha/2 \in R_2^+ \backslash R_3^+$ combine to give

$$\frac{\tau_\alpha \tau_{\alpha/2}^{1/2} x^{\alpha^\vee} - 1}{\tau_{\alpha/2}^{1/2} x^{\alpha^\vee} - 1} \cdot \frac{\tau_{\alpha/2} x^{2\alpha^\vee} - 1}{x^{2\alpha^\vee} - 1} = \frac{(\tau_\alpha \tau_{\alpha/2}^{1/2} x^{\alpha^\vee} - 1)(\tau_{\alpha/2}^{1/2} x^{\alpha^\vee} + 1)}{x^{2\alpha^\vee} - 1},$$

since $\tau_{\alpha/4} = 1$. Furthermore, the factor corresponding to $\alpha \in R_3^+$ is

$$\frac{\tau_\alpha \tau_{\alpha/2}^{1/2} x^{\alpha^\vee} - 1}{\tau_{\alpha/2}^{1/2} x^{\alpha^\vee} - 1} = \frac{(\tau_\alpha \tau_{\alpha/2}^{1/2} x^{\alpha^\vee} - 1)(\tau_{\alpha/2}^{1/2} x^{\alpha^\vee} + 1)}{x^{2\alpha^\vee} - 1}$$

since $\tau_{\alpha/2} = 1$ if $\alpha \in R_3$. The result follows. □

Corollary 5.2.5. *For $\lambda \in P^+$ we have*

$$\mathbb{1}_0 T_{t_\lambda} \mathbb{1}_0 = P_\lambda(x) \mathbb{1}_0 .$$

Proof. This is clear, since $x^\lambda = q_{t_\lambda}^{1/2} T_{t_\lambda}$ by definition. □

It will be useful to have some alternative formulae for $P_\lambda(x)$. The *inversion set* of $w \in W_0$ is

$$R_2(w) = \{\alpha \in R_2^+ \mid H_\alpha \text{ is between } C_0 \text{ and } w^{-1}C_0\},$$

and we write $R(w) = \{\alpha \in R^+ \mid H_\alpha \text{ is between } C_0 \text{ and } w^{-1}C_0\}$. For $w \in W_0$ we have $R_2(w) = \{\alpha \in R_2^+ \mid w\alpha \in R^-\}$, and if $w = s_{i_1} \cdots s_{i_p}$ is a reduced expression, then by [**5**, VI, §1, No.6, Corollary 2 to Proposition 17],

$$R_2(w) = \{\alpha_{i_p}, s_{i_p}\alpha_{i_{p-1}}, \ldots, s_{i_p} \cdots s_{i_2}\alpha_{i_1}\}.$$

It follows that for $w \in W_0$,

$$q_w = \prod_{\alpha \in R_2(w^{-1})} q_\alpha = \prod_{\alpha \in R(w^{-1})} \tau_\alpha \tag{5.2.7}$$

(compare this with [**26**, (3.8)], noting that $q_w = q_{w^{-1}}$). The second equality in (5.2.7) is clear if R is reduced (for $R_2 = R$ and $\tau_\alpha = q_\alpha$). If R is of type BC_n, and if $\alpha \in R_2(w^{-1}) \backslash R_3$, then $q_\alpha = q_n = \tau_\alpha \tau_{2\alpha}$, verifying the result in this case too. In particular,

$$q_{w_0} = \prod_{\alpha \in R^+} \tau_\alpha. \tag{5.2.8}$$

Using (5.2.1), (5.2.8) and (4.3.2) we see that $P_\lambda(x)$ may be written as

$$P_\lambda(x) = \frac{q_{t_\lambda}^{-1/2}}{W_0(q^{-1})} \sum_{w \in W_0} w\big(x^\lambda c(x)\big), \tag{5.2.9}$$

where

$$c(x) = \prod_{\alpha \in R^+} \frac{1 - \tau_\alpha^{-1} \tau_{\alpha/2}^{-1/2}\, x^{-\alpha^\vee}}{1 - \tau_{\alpha/2}^{-1/2}\, x^{-\alpha^\vee}}. \tag{5.2.10}$$

PROPOSITION 5.2.6. *Let $c(x)$ be as in (5.2.10). Then*

$$c(x) = \prod_{\alpha \in R_2^+} \frac{(1 - \tau_{2\alpha}^{-1} \tau_\alpha^{-1/2} x^{-\alpha^\vee/2})(1 + \tau_\alpha^{-1/2} x^{-\alpha^\vee/2})}{1 - x^{-\alpha^\vee}}. \tag{5.2.11}$$

PROOF. See (5.2.6), and the paragraph immediately after it. ☐

THEOREM 5.2.7. *$\{P_\lambda(x) \mid \lambda \in P^+\}$ is a basis of $\mathbb{C}[P]^{W_0}$. Furthermore, the Macdonald spherical functions satisfy*

$$P_\lambda(x) P_\mu(x) = \sum_{\nu \preceq \lambda + \mu} c_{\lambda,\mu;\nu} P_\nu(x)$$

for some numbers $c_{\lambda,\mu;\nu}$, with

$$c_{\lambda,\mu;\lambda+\mu} = \frac{W_{0\lambda}(q^{-1}) W_{0\mu}(q^{-1})}{W_{0(\lambda+\mu)}(q^{-1}) W_0(q^{-1})},$$

where for $U \subset W$, $U(q^{-1}) = \sum_{w \in U} q_w^{-1}$.

PROOF. For $\lambda \in P^+$, let

$$\tilde{P}_\lambda(x) = \frac{W_0(q^{-1})}{W_{0\lambda}(q^{-1})} q_{t_\lambda}^{1/2} P_\lambda(x),$$

and define the *monomial symmetric function* $m_\lambda(x) \in \mathbb{C}[P]^{W_0}$ by

$$m_\lambda(x) = \sum_{\mu \in W_0 \lambda} x^\mu. \tag{5.2.12}$$

The set $\{m_\lambda(x)\}_{\lambda\in P^+}$ forms a basis for $\mathbb{C}[P]^{W_0}$. Using the formula (5.2.11) and the calculations made in [**26**, §10] we have

$$\tilde{P}_\lambda(x) = \sum_{\mu\preceq\lambda} c_{\lambda,\mu} m_\mu(x), \quad \text{where } c_{\lambda,\lambda} = 1, \tag{5.2.13}$$

which shows that $\{P_\lambda(x)\}_{\lambda\in P^+}$ forms a basis for $\mathbb{C}[P]^{W_0}$. Equation (5.2.13) is the so called *triangularity condition* of the Macdonald spherical functions.

It is clear that $m_\lambda(x)m_\mu(x) = \sum_{\nu\preceq\lambda+\mu} d_{\lambda,\mu;\nu} m_\nu(x)$ where $d_{\lambda,\mu;\lambda+\mu} = 1$, and so it follows that

$$\tilde{P}_\lambda(x)\tilde{P}_\mu(x) = \sum_{\nu\preceq\lambda+\mu} e_{\lambda,\mu;\nu}\tilde{P}_\nu(x)$$

for some numbers $e_{\lambda,\mu;\nu}$ with $e_{\lambda,\mu;\lambda+\mu} = 1$. It follows that for all $\lambda,\mu,\nu\in P^+$,

$$c_{\lambda,\mu;\nu} = \frac{W_{0\lambda}(q^{-1})W_{0\mu}(q^{-1})}{W_{0\nu}(q^{-1})W_0(q^{-1})} q_{t_\lambda}^{1/2} q_{t_\mu}^{1/2} q_{t_\nu}^{-1/2} e_{\lambda,\mu;\nu},$$

and the result follows. $\square$

5.3. The Isomorphism $\mathscr{A} \to \mathbb{C}[P]^{W_0}$

We can now see how to relate the vertex set averaging operators A_λ to the algebra elements $P_\lambda(x)$. Let us recall (and make) some definitions. For $\lambda,\mu,\nu\in P^+$ and $w_1,w_2,w_3\in W$, define numbers $a_{\lambda,\mu;\nu}$, $b_{w_1,w_2;w_3}$, $c_{\lambda,\mu;\nu}$ and $d_{w_1,w_2;w_3}$ by

$$A_\lambda A_\mu = \sum_{\nu\in P^+} a_{\lambda,\mu;\nu} A_\nu \qquad B_{w_1}B_{w_2} = \sum_{w_3\in W} b_{w_1,w_2;w_3} B_{w_3}$$
$$P_\lambda(x)P_\mu(x) = \sum_{\nu\in P^+} c_{\lambda,\mu;\nu} P_\nu(x) \qquad T_{w_1}T_{w_2} = \sum_{w_3\in W} d_{w_1,w_2;w_3} T_{w_3}\,.$$

Thus the numbers are the structure constants of the algebras $\mathscr{A}$, $\mathscr{B}$, $\mathbb{C}[P]^{W_0}$ and $\mathscr{H}$ with respect to the bases $\{A_\lambda \mid \lambda\in P^+\}$, $\{B_w \mid w\in W\}$, $\{P_\lambda(x) \mid \lambda\in P^+\}$ and $\{T_w \mid w\in W\}$ respectively.

Note that by Proposition 5.1.4 we have $b_{w_1,w_2;w_3} = d_{w_1,w_2;w_3}$ whenever a building with parameter system $\{q_s\}_{s\in S}$ exists. We stress that $d_{w_1,w_2;w_3}$ is a more general object, for it makes sense for a much more general set of q_s's.

Recall the definition of w_λ from Section 3.7, and recall the definition of $W_{0\lambda}$ from (3.5.2). We give the following lemma linking double cosets in W with double cosets in $\tilde{W}$.

LEMMA 5.3.1. *Let $\lambda\in P^+$ and $i\in I_P$. Suppose that $\tau(\lambda) = l$, and write $j = \sigma_i(l)$ (so $\sigma_j = \sigma_i\circ\sigma_l$). Then*

$$W_i\sigma_i(t'_\lambda)W_j = g_iW_0t_\lambda W_0 g_j^{-1}\,,$$

where the elements g_i are defined in (3.5.3).

PROOF. By Proposition 3.7.1, $g_j = g_i g_l$ and $t_\lambda = t'_\lambda g_l$, and by (3.6.3), $\sigma_k(w) = g_k w g_k^{-1}$ for all $w \in W$ and $k \in I_P$. Thus

$$W_i \sigma_i(t'_\lambda) W_j = (g_i W_0 g_i^{-1})(g_i t_\lambda g_l^{-1} g_i^{-1})(g_j W_0 g_j^{-1}) = g_i W_0 t_\lambda W_0 g_j^{-1} . \qquad \square$$

LEMMA 5.3.2. [**31**, Lemma 2.7]. *Let $\lambda \in P^+$. Then*

$$\sum_{w \in W_0 t_\lambda W_0} q_w T_w = \frac{W_0^2(q)}{W_{0\lambda}(q)} q_{w_\lambda} \mathbb{1}_0 T_{t_\lambda} \mathbb{1}_0 .$$

PROOF. This can be deduced from Theorem 4.3.2, or see the proof in [**31**]. $\square$

The following important theorem will be used (along with Proposition 4.4.10) to prove that $\mathscr{A} \cong Z(\tilde{\mathscr{H}})$.

THEOREM 5.3.3. *Let $\lambda, \mu, \nu \in P^+$ and write $\tau(\lambda) = l$, $\tau(\mu) = m$ and $\tau(\nu) = n$. Then if $c_{\lambda,\mu;\nu} \neq 0$ we have*

$$c_{\lambda,\mu;\nu} = \frac{W_{0\lambda}(q) W_{0\mu}(q)}{W_{0\nu}(q) W_0^2(q) q_{w_\lambda} q_{w_\mu}} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} q_{w_1} q_{w_2} d_{w_1,w_2;w_\nu} .$$

PROOF. To abbreviate notation we write $P_\lambda = P_\lambda(x)$. First observe that by Theorem 5.2.7 we have $c_{\lambda,\mu;\nu} = 0$ unless $\nu \preceq \lambda + \mu$. In particular we have $c_{\lambda,\mu;\nu} = 0$ when $\tau(\nu) \neq \tau(\lambda + \mu)$. It follows that $\sigma_n = \sigma_l \circ \sigma_m$, and so $g_n = g_l g_m$ (see Proposition 3.7.1). We will use this fact later.

By Corollary 5.2.5 and Lemma 5.3.2, for any $\lambda \in P^+$ we have

$$P_\lambda \mathbb{1}_0 = \mathbb{1}_0 T_{t_\lambda} \mathbb{1}_0 = \frac{W_{0\lambda}(q)}{W_0^2(q) q_{w_\lambda}} \sum_{w \in W_0 t_\lambda W_0} q_w T_w ,$$

and so if $i \in I_P$, $\tau(\lambda) = l$ and $j = \sigma_i(l)$ we have (see Lemma 5.3.1)

$$T_{g_i} P_\lambda \mathbb{1}_0 T_{g_j^{-1}} = \frac{W_{0\lambda}(q)}{W_0^2(q) q_{w_\lambda}} \sum_{w \in W_i \sigma_i(t'_\lambda) W_j} q_w T_w . \tag{5.3.1}$$

We can replace the t'_λ by w_λ in the above because $W_i \sigma_i(t'_\lambda) W_j = W_i \sigma_i(w_\lambda) W_j$ by Proposition 3.7.4(i) and the fact that $\sigma_i(W_l) = W_j$.

Using the fact that $g_n = g_l g_m$ if $c_{\lambda,\mu;\nu} \neq 0$ we have, by (5.3.1)

$$\begin{aligned}
P_\lambda \mathbb{1}_0 P_\mu \mathbb{1}_0 T_{g_n^{-1}} &= (P_\lambda \mathbb{1}_0 T_{g_l^{-1}})(T_{g_l} P_\mu \mathbb{1}_0 T_{g_n^{-1}}) \\
&= \frac{W_{0\lambda}(q) W_{0\mu}(q)}{W_0^4(q) q_{w_\lambda} q_{w_\mu}} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} q_{w_1} q_{w_2} T_{w_1} T_{w_2} \\
&= \frac{W_{0\lambda}(q) W_{0\mu}(q)}{W_0^4(q) q_{w_\lambda} q_{w_\mu}} \sum_{w_3 \in W} \left(\sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} q_{w_1} q_{w_2} d_{w_1,w_2;w_3} T_{w_3} \right) .
\end{aligned}$$

So the coefficient of T_{w_ν} in the expansion of $P_\lambda \mathbb{1}_0 P_\mu \mathbb{1}_0 T_{g_n^{-1}}$ in terms of the T_w's is

$$\frac{W_{0\lambda}(q)W_{0\mu}(q)}{W_0^4(q)q_{w_\lambda}q_{w_\mu}} \sum_{\substack{w_1\in W_0w_\lambda W_l \\ w_2\in W_l\sigma_l(w_\mu)W_n}} q_{w_1}q_{w_2}d_{w_1,w_2;w_\nu}\,. \tag{5.3.2}$$

On the other hand, by Theorem 5.2.7 we have

$$\begin{aligned} P_\lambda \mathbb{1}_0 P_\mu \mathbb{1}_0 T_{g_n^{-1}} &= \sum_{\eta \preceq \lambda+\mu} c_{\lambda,\mu;\eta} P_\eta \mathbb{1}_0 T_{g_n^{-1}} \\ &= \sum_{\eta\preceq\lambda+\mu}\left(\frac{W_{0\eta}(q)}{W_0^2(q)q_{w_\eta}}c_{\lambda,\mu;\eta}\sum_{w\in W_0w_\eta W_n} q_wT_w\right). \end{aligned}$$

Since the double cosets $W_0w_\eta W_n$ are disjoint over $\{\eta \in P^+ \mid \eta \preceq \lambda+\mu\}$ we see that the coefficient of T_{w_ν} is

$$\frac{W_{0\nu}(q)}{W_0^2(q)}c_{\lambda,\mu;\nu}\,. \tag{5.3.3}$$

The theorem now follows by equating (5.3.2) and (5.3.3). □

Corollary 5.3.4. *Suppose that an irreducible locally finite regular affine building exists with parameters $\{q_s\}_{s\in S}$. Then for all $\lambda,\mu,\nu \in P^+$ we have $a_{\lambda,\mu;\nu} = c_{\lambda,\mu;\nu}$.*

Proof. This follows from Theorem 5.3.3, (4.4.10) and Proposition 5.1.4. □

Theorem 5.3.5. *Suppose that an irreducible locally finite regular affine building exists with parameters $\{q_s\}_{s\in S}$. Then the map $P_\lambda(x) \mapsto A_\lambda$ determines an algebra isomorphism, and so $\mathscr{A} \cong Z(\tilde{\mathscr{H}}) = \mathbb{C}[P]^{W_0}$.*

Proof. Since $\{P_\lambda(x) \mid \lambda \in P^+\}$ is a basis of $\mathbb{C}[P]^{W_0}$ and $\{A_\lambda \mid \lambda \in P^+\}$ is a basis of $\mathscr{A}$, there exists a unique vector space isomorphism $\Phi : Z(\tilde{\mathscr{H}}) \to \mathscr{A}$ with $\Phi(P_\lambda) = A_\lambda$ for all $\lambda \in P^+$. Since $a_{\lambda,\mu;\nu} = c_{\lambda,\mu;\nu}$ by Corollary 5.3.4, we see that Φ is an algebra isomorphism. □

Theorem 5.3.6. *The algebra $Z(\tilde{\mathscr{H}})$ is generated by $\{P_{\lambda_i}(x) \mid i \in I_0\}$, and so $\mathscr{A}$ is generated by $\{A_{\lambda_i} \mid i \in I_0\}$.*

Proof. First we define a less restrictive partial order on P^+ than $\preceq$. For $\lambda,\mu \in P^+$ we define $\mu < \lambda$ if and only if $\lambda-\mu$ is an $\mathbb{R}^+$-linear combination of $(R^\vee)^+$ and $\lambda \neq \mu$. Clearly if $\mu \prec \lambda$ then $\mu < \lambda$. Observe also that $\lambda_i > 0$ for all $i \in I_0$ [**18**, p.72, Exercises 7–8]. Thus if $\lambda = \lambda' + \lambda_i$ for some $\lambda' \in P^+$ and $i \in I_0$, we have $\lambda - \lambda' = \lambda_i > 0$ and so $\lambda' < \lambda$.

Let $\mathcal{P}(\lambda)$ be the statement that P_λ is a polynomial in $P_{\lambda_1},\ldots,P_{\lambda_n}$ (and $P_0 = 1$). Suppose that $\mathcal{P}(\lambda)$ fails for some $\lambda \in P^+$. Since $\{\mu \in P^+ \mid \mu \leq \lambda\}$ is finite (by the proof of [**18**, Lemma 13.2B]) we can pick $\lambda \in P^+$ minimal with respect to $\leq$ such that $\mathcal{P}(\lambda)$ fails.

There is an i such that $\lambda - \lambda_i = \lambda'$ is in P^+. Then $\lambda' < \lambda$ and $P_\lambda = cP_{\lambda'}P_{\lambda_i} +$ a linear combination of P_μ's where $\mu < \lambda$, $\mu \neq \lambda$. Then $\mathcal{P}(\lambda')$ holds, as does $\mathcal{P}(\mu)$ for all these μ's. So $\mathcal{P}(\lambda)$ holds, a contradiction. $\square$

Let L be a lattice with $Q \subseteq L \subseteq P$, and let I_L, V_L and $\mathscr{A}_L$ be as in Section 4.5. Let $\mathbb{C}[L]$ denote the linear span of $\{x^\lambda \mid \lambda \in L\}$ over $\mathbb{C}$. It is clear that $\mathbb{C}[L]$ is a subalgebra of $\mathbb{C}[P]$, and by Lemma 4.5.1 $W_0\mathbb{C}[L] = \mathbb{C}[L]$. Let $\mathbb{C}[L]^{W_0}$ denote the W_0-invariant elements of $\mathbb{C}[L]$.

It is not difficult to see that $\{m_\lambda(x) \mid \lambda \in L\}$ forms a basis for $\mathbb{C}[L]^{W_0}$, and so by (5.2.13) we see that $\{P_\lambda(x) \mid \lambda \in L\}$ forms a basis for $\mathbb{C}[L]^{W_0}$.

PROPOSITION 5.3.7. *Suppose that an irredicible locally finite regular affine building exists with parameters $\{q_s\}_{s\in S}$. Then the map $P_\lambda(x) \mapsto A_\lambda$ for $\lambda \in L$ determines an algebra isomorphism $\mathbb{C}[L]^{W_0} \to \mathscr{A}_L$, and so $\mathscr{A}_L \cong \mathbb{C}[L]^{W_0}$.*

PROOF. See the proof of Theorem 5.3.5. $\square$

5.4. A Positivity Result and Hypergroups

Here we show that the structure constants $c_{\lambda,\mu;\nu}$ of the algebra $\mathbb{C}[P]^{W_0}$ are, up to positive normalisation factors, polynomials with nonnegative integer coefficients in the variables $\{q_s - 1 \mid s \in S\}$. This result has independently been obtained by Schwer in [**38**], where a formula for $c_{\lambda,\mu;\nu}$ is given (in the reduced case with $q_s = q$ for all $s \in S$).

Thus if $q_s \geq 1$ for all $s \in S$ then $c_{\lambda,\mu;\nu} \geq 0$ for all $\lambda, \mu, \nu \in P^+$. This result was proved for root systems of type A_n by Miller Malley in [**30**], where the numbers $c_{\lambda,\mu;\nu}$ are Hall polynomials (up to positive normalisation factors). Note that it is clear from (4.4.3) and Corollary 5.3.4 that $c_{\lambda,\mu;\nu} \geq 0$ when there exists a building with parameters $\{q_s\}_{s\in S}$.

In a recent series of papers ([**33**], [**17**], [**43**]) the numbers $a_{\lambda,\mu}$ appearing in the formula $P_\lambda(x) = \sum_\mu a_{\lambda,\mu} m_\mu(x)$ are studied. We will provide a connection with the results we prove here and the numbers $a_{\lambda,\mu}$ in Theorem 7.7.2.

The results of this section show how to construct a (commutative) polynomial hypergroup, in the sense of [**4**] (see also [**22**] where the A_2 case is discussed).

For each $w_1, w_2, w_3 \in W$, let $d'_{w_1,w_2;w_3} = q_{w_1}q_{w_2}q_{w_3}^{-1}d_{w_1,w_2;w_3}$.

LEMMA 5.4.1. *For all $w_1, w_2, w_3 \in W$, $d'_{w_1,w_2;w_3}$ is a polynomial with nonnegative integer coefficients in the variables $q_s - 1$, $s \in S$.*

PROOF. We prove the result by induction on $\ell(w_2)$. When $\ell(w_2) = 1$, so $w_2 = s$ for some $s \in S$, we have

$$d'_{w_1,s;w_3} = \begin{cases} 1 & \text{if } \ell(w_1 s) = \ell(w_1) + 1 \text{ and } w_3 = w_1 s\,, \\ q_s & \text{if } \ell(w_1 s) = \ell(w_1) - 1 \text{ and } w_3 = w_1 s\,, \\ q_s - 1 & \text{if } \ell(w_1 s) = \ell(w_1) - 1 \text{ and } w_3 = w_1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

proving the result in this case.

Suppose that $n \geq 2$ and that the result is true for $\ell(w_2) < n$. Then if $\ell(w_2) = n$, write $w_2 = ws$ with $\ell(w) = n - 1$. Thus

$$T_{w_1}T_{w_2} = (T_{w_1}T_w)T_s = \sum_{w' \in W} d_{w_1,w;w'}T_{w'}T_s = \sum_{w_3 \in W}\left(\sum_{w' \in W} d_{w_1,w;w'}d_{w',s;w_3}\right)T_{w_3}\,,$$

which implies that

$$d'_{w_1,w_2;w_3} = \sum_{w' \in W} d'_{w_1,w;w'}d'_{w',s;w_3}\,.$$

The result follows since $\ell(w) < n$ and $\ell(s) = 1$. □

For each $\lambda, \mu, \nu \in P^+$, let

$$c'_{\lambda,\mu;\nu} = \frac{W_0(q)W_{0\nu}(q)}{W_{0\lambda}(q)W_{0\mu}(q)}\frac{q_{w_\lambda}q_{w_\mu}}{q_{w_\nu}}c_{\lambda,\mu;\nu}. \tag{5.4.1}$$

THEOREM 5.4.2. *For all $\lambda, \mu, \nu \in P^+$, the structure constants $c'_{\lambda,\mu;\nu}$ are polynomials with nonnegative integer coefficients in the variables $q_s - 1$, $s \in S$.*

PROOF. We will use the same notation as in Theorem 5.3.3, so let $\tau(\lambda) = l$, $\tau(\mu) = m$ and $\tau(\nu) = n$. By Theorem 5.3.3 we have

$$c'_{\lambda,\mu;\nu} = \frac{1}{W_0(q)} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} d'_{w_1,w_2;w_\nu}\,,$$

and so it immediately follows from Lemma 5.4.1 that $W_0(q)c'_{\lambda,\mu;\nu}$ is a polynomial in the variables $q_s - 1$, $s \in S$, with nonnegative integer coefficients. The result stated in the theorem is stronger than this, and so we need to sharpen the methods used in the proof of Theorem 5.3.3.

In the notation of Proposition 3.7.4 we have the following (see Proposition 3.7.4 for proofs of similar facts). Firstly, each $w_1 \in W_0 w_\lambda W_l$ can be written uniquely as $w_1 = u_1 w_\lambda w_l$ for some $u_1 \in W_0^\lambda$ and $w_l \in W_l$, and moreover $\ell(w_1) = \ell(u_1) + \ell(w_\lambda) + \ell(w_l)$. Similarly, each $w_2 \in W_l \sigma_l(w_\mu) W_n$ can be written uniquely as $w_2 = w'_l \sigma_l(w_\mu) u_2$ for some $u_2 \in W_n^\mu$ and $w'_l \in W_l$, and moreover $\ell(w_2) = \ell(w'_l) + \ell(\sigma_l(w_\mu)) + \ell(u_2)$.

Secondly, each $w \in W_0 w_\lambda$ can be written uniquely as $w = uw_\lambda$ for some $u \in W_0^\lambda$, and moreover $\ell(w) = \ell(u) + \ell(w_\lambda)$. Similarly, each $w' \in \sigma_l(w_\mu)W_n$ can be written uniquely as $w' = \sigma_l(w_\mu)u'$ for some $u' \in W_n^\mu$, and moreover $\ell(w') = \ell(\sigma_l(w_\mu)) + \ell(u')$.

Using these facts, along with the facts that $\mathbb{1}_l^2 = \mathbb{1}_l$ and $W_l(q) = W_0(q)$, we have (compare with the proof of Theorem 5.3.3)

$$\begin{aligned}
P_\lambda \mathbb{1}_0 P_\mu \mathbb{1}_0 T_{g_n^{-1}} &= \frac{W_{0\lambda}(q)W_{0\mu}(q)}{W_0^4(q)q_{w_\lambda}q_{w_\mu}} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu)W_n}} q_{w_1}q_{w_2}T_{w_1}T_{w_2} \\
&= \frac{W_{0\lambda}(q)W_{0\mu}(q)W_l^2(q)}{W_0^4(q)q_{w_\lambda}q_{w_\mu}} \left(\sum_{u_1 \in W_0^\lambda} q_{u_1 w_\lambda}T_{u_1 w_\lambda}\right) \mathbb{1}_l^2 \left(\sum_{u_2 \in W_n^\mu} q_{\sigma_l(w_\mu)u_2}T_{\sigma_l(w_\mu)u_2}\right) \\
&= \frac{W_{0\lambda}(q)W_{0\mu}(q)}{W_0^2(q)q_{w_\lambda}q_{w_\mu}} \left(\sum_{w \in W_0 w_\lambda} q_w T_w\right) \mathbb{1}_l \left(\sum_{w' \in \sigma_l(w_\mu)W_n} q_{w'}T_{w'}\right) \\
&= \frac{W_{0\lambda}(q)W_{0\mu}(q)}{W_0^3(q)q_{w_\lambda}q_{w_\mu}} \sum_{\substack{w_1 \in W_0 w_\lambda,\, w_2 \in W_l \\ w_3 \in \sigma_l(w_\mu)W_n}} q_{w_1}q_{w_2}q_{w_3}T_{w_1}T_{w_2}T_{w_3}\,.
\end{aligned}$$

It is simple to see that

$$\sum_{\substack{w_1 \in W_0 w_\lambda,\, w_2 \in W_l \\ w_3 \in \sigma_l(w_\mu)W_n}} q_{w_1}q_{w_2}q_{w_3}T_{w_1}T_{w_2}T_{w_3} = \sum_{w \in W} d_w(\lambda,\mu)q_w T_w$$

where $d_w(\lambda,\mu)$ is a linear combination of products of $d'_{w_1,w_2;w_3}$'s with nonnegative integer coefficients, and so

$$P_\lambda \mathbb{1}_0 P_\mu \mathbb{1}_0 T_{g_n^{-1}} = \frac{W_{0\lambda}(q)W_{0\mu}(q)}{W_0^3(q)q_{w_\lambda}q_{w_\mu}} \sum_{w \in W} d_w(\lambda,\mu)q_w T_w.$$

So the coefficient of T_{w_ν} when $P_\lambda \mathbb{1}_0 P_\mu \mathbb{1}_0 T_{g_n^{-1}}$ is expanded in terms of the T_w's is

$$\frac{W_{0\lambda}(q)W_{0\mu}(q)}{W_0^3(q)} \frac{q_{w_\nu}}{q_{w_\lambda}q_{w_\mu}} d_{w_\nu}(\lambda,\mu). \tag{5.4.2}$$

Comparing (5.4.2) with (5.3.3) we see that $c'_{\lambda,\mu;\nu} = d_{w_\nu}(\lambda,\mu)$, and so the result follows from Lemma 5.4.1 and the fact that $d_{w_\nu}(\lambda,\mu)$ is a linear combination of products of $d'_{w_1,w_2;w_3}$'s with nonnegative integer coefficients. $\square$

CHAPTER 6

The Macdonald Formula

In the following chapters we describe the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$, along with some relevant *spherical harmonic analysis*. The isomorphism $\mathscr{A} \cong \mathbb{C}[P]^{W_0}$ gives one formula almost immediately in terms of the Macdonald spherical functions (see Section 6.1 below). In Chapter 7 we give a second formula in terms of an integral over the boundary of $\mathscr{X}$, which gives a 'building' analogue of [**23**, (4.2.1)].

The formulae in Theorems 6.3.2 and 6.3.7 for the *Plancherel measure* are essentially from [**23**, Chapter V].

6.1. The Macdonald Formula

Here we will use Theorem 5.3.5 to describe the *Macdonald formula* for the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$.

For $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, write u^λ in place of $u(\lambda)$. Each $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ induces a homomorphism, also called u, on $\mathbb{C}[P]$, and all homomorphisms $\mathbb{C}[P] \to \mathbb{C}$ arise in this way. For $w \in W_0$ and $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ we write $wu \in \mathrm{Hom}(P, \mathbb{C}^\times)$ for the homomorphism $(wu)^\lambda = u^{w\lambda}$. If $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, we write $P_\lambda(u)$ in place of $u(P_\lambda(x))$. Thus, by (5.2.9),

$$P_\lambda(u) = \frac{q_{t_\lambda}^{-1/2}}{W_0(q^{-1})} \sum_{w \in W_0} c(wu) u^{w\lambda}, \quad \text{where} \quad c(u) = \prod_{\alpha \in R^+} \frac{1 - \tau_\alpha^{-1} \tau_{\alpha/2}^{-1/2} u^{-\alpha^\vee}}{1 - \tau_{\alpha/2}^{-1/2} u^{-\alpha^\vee}}, \tag{6.1.1}$$

provided, of course, that the denominators of the $c(wu)$ functions do not vanish. Since $P_\lambda(u)$ is a Laurent polynomial, these *singular cases* can be obtained from the general formula by taking an appropriate limit (see [**23**, §4.6]). Finally, for $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, let $h_u : \mathscr{A} \to \mathbb{C}$ be the linear map with $h_u(A_\lambda) = P_\lambda(u)$ for each $\lambda \in P^+$.

PROPOSITION 6.1.1. *(cf.* [**23**, Theorem 3.3.12]*)*

(i) *Each homomorphism* $h : \mathscr{A} \to \mathbb{C}$ *is of the form* $h = h_u$ *for some* $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$.
(ii) $h_u = h_{u'}$ *if and only if* $u' = wu$ *for some* $w \in W_0$.

PROOF. Since $\mathbb{C}[P]$ is integral over $\mathbb{C}[P]^{W_0}$ [**1**, V, Exercise 12], every homomorphism $\mathbb{C}[P]^{W_0} \to \mathbb{C}$ is the restriction of a homomorphism $\mathbb{C}[P] \to \mathbb{C}$, and (i) follows.

It is clear that if $u' = wu$ for some $w \in W_0$, then $h_{u'} = h_u$. Suppose now that $h_u = h_{u'}$. Since $\{P_\lambda(x)\}_{\lambda \in P^+}$ forms a basis of $\mathbb{C}[P]^{W_0}$ we see that u and u' agree on $\mathbb{C}[P]^{W_0}$, and thus their kernels are maximal ideals of $\mathbb{C}[P]$ lying over the same maximal ideal of $\mathbb{C}[P]^{W_0}$. The result now follows by [**1**, V, Exercise 13]. □

We call the formula $h_u(A_\lambda) = P_\lambda(u)$ the *Macdonald formula for the algebra homomorphisms* $\mathscr{A} \to \mathbb{C}$. By comparing Proposition B.1.5 and [**23**, Corollary 3.2.5] we see that in the case when $P = Q$ (that is, when R is of type E_8, F_4, G_2 or BC_n for some $n \geq 1$) our formula $P_\lambda(u)$ agrees with the formula in [**23**, Theorem 4.1.2]. The reason we require $P = Q$ here is because in [**23**] $u \in \mathrm{Hom}(Q, \mathbb{C}^\times)$ (although u there is called s).

REMARK 6.1.2. We often think of $P_\lambda(u)$ as a function of the variables $u_i = u^{\lambda_i} \in \mathbb{C}^\times$, $i = 1, \ldots, n$. In general, the coroots $\alpha^\vee$, $\alpha \in R^+$, appearing in the formula for $c(u)$ do not have particularly neat expressions in terms of the basis $\{\lambda_i\}_{i=1}^n$. Thus in any given specific case it is often useful to work with numbers other than the $\{u_i\}_{i=1}^n$.

Let us illustrate this in the $R = D_n$ case, which may be described as follows (see Appendix D). Let $E = \mathbb{R}^n$ with standard orthonormal basis $\{e_i\}_{i=1}^n$, and take R to be the set of vectors $\pm e_i \pm e_j$, $1 \leq i < j \leq n$, where the $\pm$ signs may be taken independently. We have $R^\vee = R$, and the set $\{e_i - e_{i+1}, e_{n-1} + e_n\}_{1 \leq i \leq n-1}$ forms a base of R. The corresponding set of positive roots is $\{e_i - e_j, e_i + e_j\}_{1 \leq i < j \leq n}$. Observe that $e_1 = \lambda_1$, $e_i = \lambda_i - \lambda_{i-1}$ for $2 \leq i \leq n-2$, $e_{n-1} = \lambda_{n-1} + \lambda_n - \lambda_{n-2}$ and $e_n = \lambda_n - \lambda_{n-1}$. Thus, defining numbers $t_i \in \mathbb{C}^\times$, $i = 1, \ldots, n$, by $t_1 = u_1$, $t_i = u_i u_{i-1}^{-1}$ $(2 \leq i \leq n-2)$, $t_{n-1} = u_{n-1} u_n u_{n-2}^{-1}$ and $t_n = u_n u_{n-1}^{-1}$, we have

$$c(u) = \prod_{1 \leq i < j \leq n} \frac{(1 - q^{-1} t_i^{-1} t_j)(1 - q^{-1} t_i^{-1} t_j^{-1})}{(1 - t_i^{-1} t_j)(1 - t_i^{-1} t_j^{-1})}.$$

(Notice that $q_i = q$ for all $i \in I$, see Appendix E.)

6.2. The Plancherel Measure

Let $\ell^2(V_P)$ denote the Hilbert space of square summable functions $f : V_P \to \mathbb{C}$. Each $A \in \mathscr{A}$ maps $\ell^2(V_P)$ into itself, and for $\lambda \in P^+$ and $f \in \ell^2(V_P)$ we have $\|A_\lambda f\|_2 \leq \|f\|_2$ (see [**10**, Lemma 4.1] for a proof in a similar context). So we may regard $\mathscr{A}$ as a subalgebra of the C^*-algebra $\mathscr{L}(\ell^2(V_P))$ of bounded linear operators on $\ell^2(V_P)$. The facts that $y \in V_\lambda(x)$ if and only if $x \in V_{\lambda^*}(y)$, and $N_{\lambda^*} = N_\lambda$, imply that $A_\lambda^* = A_{\lambda^*}$, and so the adjoint A^* of any $A \in \mathscr{A}$ is also in $\mathscr{A}$.

Let $\mathscr{A}_2$ denote the completion of $\mathscr{A}$ with respect to $\|\cdot\|$, the ℓ^2-operator norm. So $\mathscr{A}_2$ is a commutative C^*-algebra. We write $M_2 = \mathrm{Hom}(\mathscr{A}_2, \mathbb{C})$ (this is the *maximal ideal space of* $\mathscr{A}_2$), and we denote the associated Gelfand transform $\mathscr{A}_2 \to \mathscr{C}(M_2)$ by $A \mapsto \widehat{A}$, where $\widehat{A}(h) = h(A)$. Here $\mathscr{C}(M_2)$ is the algebra of w*-continuous functions on M_2 with the sup norm. This map is an isometric isomorphism of C^*-algebras [**13**, Theorem I.3.1].

The algebra homomorphisms $\tilde{h} : \mathscr{A}_2 \to \mathbb{C}$ are precisely the extensions to $\mathscr{A}_2$ of the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ which are continuous with respect to the ℓ^2-operator norm. When there is no risk of ambiguity we will simply write h in place of $\tilde{h}$. If $h = h_u$ we write $\widehat{A}(u)$ in place of $\widehat{A}(h)$ (so $\widehat{A}_\lambda(u) = P_\lambda(u)$).

Let $\langle \cdot, \cdot \rangle$ be the usual inner product on $\ell^2(V_P)$ (this is not to be confused with the unrelated inner product on E). If $X \subset V_P$ we write 1_X for the characteristic function of X, and write δ_x for $1_{\{x\}}$.

Lemma 6.2.1. *Let $A \in \mathscr{A}_2$ and $o \in V_P$. Then $A\delta_o = 0$ implies that $A = 0$.*

Proof. Let $x \in V_P$. Observe that if $A \in \mathscr{A}$ then $A\delta_x$ is x-radial, for if $A = \sum_\lambda a_\lambda A_\lambda$ is a finite linear combination, then $A\delta_x = \sum_\lambda a_\lambda N_\lambda^{-1} 1_{V_{\lambda^*}(x)}$, which is x-radial. It follows that $A\delta_x$ is x-radial for all $A \in \mathscr{A}_2$. Now, given $A \in \mathscr{A}_2$ and $\mu \in P^+$, $\langle A\delta_x, 1_{V_{\mu^*}(x)} \rangle$ does not depend on $x \in V_P$, for if $A = \sum_\lambda a_\lambda A_\lambda \in \mathscr{A}$, then $\langle A\delta_x, 1_{V_{\mu^*}(x)} \rangle = a_\mu$. Thus if $A \in \mathscr{A}_2$ and $A\delta_o = 0$, then $\langle A\delta_o, 1_{V_{\mu^*}(o)} \rangle = 0$ for all $\mu \in P^+$, and so $\langle A\delta_x, 1_{V_{\mu^*}(x)} \rangle = 0$ for all $\mu \in P^+$ and for all $x \in V_P$. Since for any x, $A\delta_x$ is x-radial, it follows that $A\delta_x = 0$ for all x, and so $Af = 0$ for all finitely supported functions $f \in \ell^2(V_P)$. Thus by density the same is true for all $f \in \ell^2(V_P)$, completing the proof. $\square$

Since $A_\lambda \delta_o = N_\lambda^{-1} 1_{V_{\lambda^*}(o)}$ for each $o \in V_P$, we have $\langle A_\lambda \delta_o, A_\mu \delta_o \rangle = \delta_{\lambda,\mu} N_\lambda^{-1}$. Thus we can define an inner product on $\mathscr{A}_2$ (independent of $o \in V_P$) by $\langle A, B \rangle = \langle A\delta_o, B\delta_o \rangle$ (see Lemma 6.2.1).

For any fixed $o \in V_P$, the map $A \mapsto (A\delta_o)(o)$ maps the identity A_0 of $\mathscr{A}_2$ to 1 and satisfies $|(A\delta_o)(o)| \leq \|A\delta_o\|_2 \leq \|A\| = \|\widehat{A}\|_\infty$. Thus by the Riesz Representation Theorem there exists a unique regular Borel probability measure π on M_2 so that

$$(A\delta_o)(o) = \int_{M_2} \widehat{A}(h) d\pi(h) \quad \text{for all } A \in \mathscr{A}_2.$$

Hence, for all $A, B \in \mathscr{A}_2$,

$$\langle A, B \rangle = \langle A\delta_o, B\delta_o \rangle = (B^* A\delta_o)(o) = \int_{M_2} \widehat{A}(h)\overline{\widehat{B}(h)} d\pi(h). \tag{6.2.1}$$

We refer to π and M_2 as the *Plancherel measure* and *spectrum* of $\mathscr{A}_2$, respectively.

Proposition 6.2.2. $M_2 = \operatorname{supp}(\pi)$.

Proof. If $h_0 \in M_2 \backslash \operatorname{supp}(\pi)$, then by Urysohn's Lemma there is a $\varphi \in \mathscr{C}(M_2)$ so that $\varphi = 0$ on $\operatorname{supp}(\pi)$ and $\varphi(h_0) = 1$. Since $A \mapsto \widehat{A}$ is an isomorphism, there is an $A \in \mathscr{A}_2$ so that $\widehat{A} = \varphi$. Then by (6.2.1)

$$\|A\delta_o\|_2^2 = \langle A, A \rangle = \int_{\operatorname{supp}(\pi)} |\widehat{A}(h)|^2 d\pi(h) = 0,$$

and so $A = 0$ by Lemma 6.2.1, contradicting $\widehat{A} = \varphi \neq 0$. $\square$

6.3. Calculating the Plancherel Measure and the ℓ^2-spectrum

In this section we will calculate the Plancherel measure of $\mathscr{A}_2$. It turns out that there are two cases to consider. We will then use these results to compute the ℓ^2-spectrum of $\mathscr{A}$. The Plancherel measure will also be used in the proof of Theorem 7.7.2, where we show that $h_u = h'_u$ for all $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, as well as in the proof of the local limit theorem in Chapter 8.

LEMMA 6.3.1. *$\tau_\alpha < 1$ for some $\alpha \in R$ if and only if $R = BC_n$ and $q_n < q_0$.*

PROOF. If R is reduced we have $\tau_\alpha = q_\alpha$ for all $\alpha \in R$. Thus $\tau_\alpha < 1$ for some $\alpha \in R$ implies that $R = BC_n$ for some $n \geq 1$. Thus R may be described as follows (see Appendix D). Let $E = \mathbb{R}^n$ with standard basis $\{e_i\}_{i=1}^n$, and let R consist of the vectors $\pm e_i$, $\pm 2e_i$ and $\pm e_j \pm e_k$ for $1 \leq i \leq n$ and $1 \leq j < k \leq n$. Recall from Appendix E that in an affine building of type BC_n we have $q_1 = \cdots = q_{n-1}$. Thus by the definition of the numbers τ_α we have $\tau_{\pm e_i} = q_n q_0^{-1}$, $\tau_{\pm 2e_i} = q_0$ and $\tau_{\pm e_j \pm e_k} = q_1$ for $1 \leq i \leq n$ and $1 \leq j < k \leq n$. The result follows. $\square$

Following [**23**, Chapter V] we call the situation where $\tau_\alpha \geq 1$ for all $\alpha \in R$ the *standard case*, and we call the situation where $\tau_\alpha < 1$ for some $\alpha \in R$ the *exceptional case*.

6.3.1. The Standard Case. Let $\mathbb{U} = \{u \in \mathrm{Hom}(P, \mathbb{C}^\times) : |u^\lambda| = 1 \text{ for all } \lambda \in P\}$. Writing $u_i = u^{\lambda_i}$ for each $i = 1, \ldots, n$, we have $\mathbb{U} \cong \mathbb{T}^n$ where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

In the next theorem we introduce (following [**23**]) a measure π_0 which we will shortly see is closely related to the Plancherel measure π (in the standard case). We will write $\hat{A}(u)$ for $h_u(A)$ when $A \in \mathscr{A}$ and $u \in \mathbb{U}$. As we shall see in Corollary 6.3.4, each such h_u is continuous for the ℓ^2–operator norm, and so (6.3.1) will also be valid for $A, B \in \mathscr{A}_2$.

THEOREM 6.3.2. *(cf.* [**23**, Theorem 5.1.5]*) Let du denote the normalised Haar measure on $\mathbb{U}$, and let π_0 be the measure on $\mathbb{U}$ given by $d\pi_0(u) = \frac{W_0(q^{-1})}{|W_0|}|c(u)|^{-2}du$. Then*

$$\langle A, B \rangle = \int_{\mathbb{U}} \widehat{A}(u)\overline{\widehat{B}(u)}d\pi_0(u) \quad \textit{for all } A, B \in \mathscr{A}. \tag{6.3.1}$$

PROOF. We may assume that $A = A_\mu$ and $B = A_\nu$, where $\mu, \nu \in P^+$. Then the integrand in (6.3.1) is $\widehat{A_\mu A_{\nu^*}}(u)$. Now $A_\mu A_{\nu^*} = \sum_{\lambda \in P^+} a_{\mu,\nu^*;\lambda} A_\lambda$, and since $a_{\mu,\nu^*;\lambda} = \delta_{\mu,\nu}/N_\mu$, it suffices to show that $\int_{\mathbb{U}} \widehat{A}_\lambda(u) d\pi_0(u) = \delta_{\lambda,0}$ for each $\lambda \in P^+$. Notice that if $u \in \mathbb{U}$, then $\overline{u} = u^{-1}$, and so

$$|c(u)|^2 = c(u)c(u^{-1}) = \prod_{\alpha \in R} \frac{1 - \tau_\alpha^{-1}\tau_{\alpha/2}^{-1/2}u^{-\alpha^\vee}}{1 - \tau_{\alpha/2}^{-1/2}u^{-\alpha^\vee}}.$$

Thus $|c(wu)|^2 = |c(u)|^2$ for all $w \in W_0$. Furthermore, if $f(u) = \sum_\lambda a_\lambda u^\lambda$ is such that $\sum_\lambda |a_\lambda| < \infty$, then $\int_{\mathbb{U}} f(u)du = a_0$. It follows that $\int_{\mathbb{U}} f(wu)du = \int_{\mathbb{U}} f(u)du$ for all $w \in W_0$.

Using these facts we see that

$$\int_{\mathbb{U}} \widehat{A}_\lambda(u)d\pi_0(u) = q_{t_\lambda}^{-1/2}\int_{\mathbb{U}}\frac{u^\lambda}{c(u^{-1})}du. \tag{6.3.2}$$

Let $R_\tau^+ = \{\alpha \in R^+ \mid \tau_\alpha \neq 1\}$. Then it is clear that we can write

$$\frac{1}{c(u^{-1})} = \prod_{\alpha\in R_\tau^+}\frac{1-\tau_{\alpha/2}^{-1/2}u^{\alpha^\vee}}{1-\tau_\alpha^{-1}\tau_{\alpha/2}^{-1/2}u^{\alpha^\vee}} = \sum_{\gamma\in Q^+} a_\gamma u^\gamma$$

where $a_0 = 1$ and the series is uniformly convergent. Since $\{\lambda_i\}_{i=1}^n$ forms an *acute* basis of E [**5**, VI, §1, No.10] we have $\langle\lambda,\lambda_i\rangle \geq 0$ for all $\lambda \in P^+$ and for all $1 \leq i \leq n$. Thus each $\lambda \in P^+$ is a linear combination of $\{\alpha_i\}_{i=1}^n$ with nonnegative coefficients. It follows that if $\lambda \in P^+$, $\gamma \in Q^+$ and $\lambda+\gamma = 0$, then $\lambda = \gamma = 0$. Hence by (6.3.2) we have $\int_{\mathbb{U}}\widehat{A}_\lambda(u)d\pi_0(u) = \delta_{\lambda,0}$, completing the proof. □

Fix $o \in V_P$ and let $\ell_o^2(V_P)$ denote the space of all $f \in \ell^2(V_P)$ which are constant on each set $V_\lambda(o)$. For $A \in \mathscr{A}_2$ define $\|A\|_o$ by

$$\|A\|_o = \sup\{\|Af\|_2 \,:\, f \in \ell_o^2(V_P) \text{ and } \|f\|_2 \leq 1\},$$

which defines a norm on $\mathscr{A}_2$ (see Lemma 6.2.1), and clearly $\|A\|_o \leq \|A\|$ for all $A \in \mathscr{A}_2$.

Remark 6.3.3. In fact $\|A\|_o = \|A\|$ for all $A \in \mathscr{A}_2$ (in both the standard and exceptional cases). To see this, recall that an injective homomorphism between two C^*-algebras is an isometry [**13**, Theorem I.5.5]. Let $\Phi : \mathscr{A}_2 \to \mathscr{L}(\ell_o^2(V_P))$ be the linear map given by $A \mapsto A|_{\ell_o^2(V_P)}$. Since $\|A\|_o = \|A|_{\ell_o^2(V_P)}\|$, it suffices to show that Φ is an injection. This is clear from Lemma 6.2.1, for $\Phi(A) = 0$ implies that $A\delta_o = 0$, and so $A = 0$.

Corollary 6.3.4. *Each h_u, $u \in \mathbb{U}$, is continuous for the ℓ^2-operator norm.*

Proof. We show that in fact $|h_u(A)| \leq \|A\|_o$ for all $A \in \mathscr{A}$ and $u \in \mathbb{U}$. Suppose that this condition fails for some $u_0 \in \mathbb{U}$ and $A \in \mathscr{A}$. Then there exists $\delta > 0$ so that $|h_{u_0}(A)| > (1+\delta)\|A\|_o$. Since $h_u(A)$ is a Laurent polynomial in $u_1,\ldots,u_n$ there exists a neighbourhood $\mathcal{N}$ of u_0 in $\mathbb{U}$ such that $|h_u(A)| > (1+\delta)\|A\|_o$ for all $u \in \mathcal{N}$. Let $\mathcal{N}'$ denote the set of $u \in \mathbb{U}$ such that $|h_u(A)| > (1+\delta)\|A\|_o$, so $W_0\mathcal{N}' = \mathcal{N}'$. Let $\mathbb{U}/W_0$ denote the set of W_0 orbits in $\mathbb{U}$. It is compact Hausdorff with respect to the quotient topology, and

$$\mathscr{C}(\mathbb{U}/W_0) \cong \{\varphi \in \mathscr{C}(\mathbb{U}) \mid \varphi(wu) = \varphi(u) \text{ for all } w \in W_0 \text{ and } u \in \mathbb{U}\}.$$

Now there exists $\varphi \in \mathscr{C}(\mathbb{U}/W_0)$ such that $\varphi \neq 0$, but φ is 0 outside $\mathcal{N}'$. By the Stone-Weierstrass Theorem, for any given $\epsilon > 0$, there exists $B \in \mathscr{A}$ so that $\|\widehat{B}-\varphi\|_\infty < \epsilon$, and choosing ϵ suitably small we can ensure that

$$\int_{\mathcal{N}'}|\widehat{B}(u)|^2 d\pi_0(u) \geq \frac{1}{1+\delta}\int_{\mathbb{U}}|\widehat{B}(u)|^2 d\pi_0(u) > 0.$$

Thus by (6.3.1)

$$\begin{aligned}\|AB\delta_o\|_2^2 &= \int_{\mathbb{U}} |\widehat{A}(u)|^2|\widehat{B}(u)|^2 d\pi_0(u)\\ &\geq (1+\delta)^2\|A\|_o^2 \int_{\mathcal{N}'} |\widehat{B}(u)|^2 d\pi_0(u) \geq (1+\delta)\|A\|_o^2\|B\delta_o\|_2^2,\end{aligned}$$

and so $\|Af\|_2 \geq \sqrt{1+\delta}\|A\|_o\|f\|_2$ for $f = B\delta_o$, contrary to the definition of $\|A\|_o$. ☐

COROLLARY 6.3.5. *In the standard case, $M_2 = \{\tilde{h}_u \mid u \in \mathbb{U}\}$. Moreover, the map $\varpi : u \mapsto \tilde{h}_u$ induces a homeomorphism $\mathbb{U}/W_0 \to M_2$ (where $\mathbb{U}$ is given the Euclidean topology, $\mathbb{U}/W_0$ is given the quotient topology, and M_2 is given the w*-topology), and the Plancherel measure π is the image of the measure π_0 of Theorem 6.3.2 under ϖ.*

PROOF. The w*-topology on M_2, defined using the functionals $h \mapsto h(A)$, $A \in \mathscr{A}_2$, is compact, and so agrees with the topology defined using only the functionals $h \mapsto h(A)$ with $A \in \mathscr{A}$, since the latter is Hausdorff. Since each $h_u(A)$ ($A \in \mathscr{A}$ fixed) is a Laurent polynomial in $u_1, \ldots, u_n$, the map $\varpi : u \mapsto \tilde{h}_u$, defined from $\mathbb{U}$ to M_2 in light of Corollary 6.3.4, is continuous. Thus $\varpi(\mathbb{U})$ is closed in M_2, and ϖ induces a homeomorphism $\mathbb{U}/W_0 \to \varpi(\mathbb{U})$ since $\mathbb{U}/W_0$ is compact. The image π of π_0 under ϖ satisfies the defining properties of the Plancherel measure. Since $M_2 = \mathrm{supp}(\pi)$ by Proposition 6.2.2,

$$\pi(M_2\backslash\varpi(\mathbb{U})) = \pi_0(\varpi^{-1}(M_2\backslash\varpi(\mathbb{U}))) = 0,$$

and so $M_2 = \varpi(\mathbb{U})$. Thus ϖ is surjective, and it is injective by Proposition 6.1.1. ☐

6.3.2. The Exceptional Case. Let $R = BC_n$ for some $n \geq 1$ and suppose that $q_n < q_0$. Recall the description of R from Appendix D. Let $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and let $\alpha_n = e_n$. The set $B = \{\alpha_i\}_{i=1}^n$ is a base of R, and the set of positive roots with respect to B is

$$R^+ = \{e_i, 2e_i, e_j - e_k, e_j + e_k \mid 1 \leq i \leq n,\ 1 \leq j < k \leq n\}.$$

By Appendix E, $q_1 = \cdots = q_{n-1}$ in this case. Let $a = \sqrt{q_n q_0}$ and $b = \sqrt{q_n/q_0}$ (so $b < 1$).

Let $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$. Since $e_i \in P$ for each $i = 1, \ldots, n$, we may define numbers $t_i = t_i(u)$ by $t_i = u^{e_i}$. We will now give a formula for $c(u)$ in this case in terms of the numbers $\{t_i\}_{i=1}^n$ (see Remark 6.1.2 for a related discussion).

If $\alpha = 2e_i$, $1 \leq i \leq n$, then $\alpha \in R_1\backslash R_3$, and so $\tau_\alpha = q_0$. Now $\alpha/2 = e_i \in R_2\backslash R_3$, and so $\tau_{\alpha/2} = q_{\alpha/2}q_0^{-1}$. Since $|\alpha/2| = |\alpha_n|$ we have $q_{\alpha/2} = q_{\alpha_n} = q_n$, and thus $\tau_{\alpha/2} = q_n q_0^{-1}$. Now if $\alpha = e_i$, $1 \leq i \leq n$, then $\alpha \in R_2\backslash R_3$, and so by the above $\tau_\alpha = q_n q_0^{-1}$, and since $\alpha/2 \notin R$ we have $\tau_{\alpha/2} = 1$. Since $(2e_i)^\vee = e_i$ and $e_i^\vee = 2e_i$, the factors in $c(u)$ (see (6.1.1)) corresponding to the roots $\alpha = 2e_i$ and $\alpha = e_i$ are

$$\frac{1 - q_n^{-1/2}q_0^{-1/2}t_i^{-1}}{1 - q_n^{-1/2}q_0^{1/2}t_i^{-1}} \cdot \frac{1 - q_n^{-1}q_0 t_i^{-2}}{1 - t_i^{-2}} = \frac{(1 - a^{-1}t_i^{-1})(1 + b^{-1}t_i^{-1})}{1 - t_i^{-2}}.$$

If $\alpha = e_j \pm e_k$, $1 \le j < k \le n$, then $\alpha \in R_3$, and so $\tau_\alpha = q_\alpha$. Since $|\alpha| = |e_1 - e_2| = |\alpha_1|$ we have $q_\alpha = q_{\alpha_1} = q_1 (= q_2 = \cdots = q_{n-1})$, and so the product of the two factors of $c(u)$ corresponding to the roots $\alpha = e_j - e_k$ and $\alpha = e_j + e_k$ $(1 \le j < k \le n)$ is

$$\frac{(1 - q_1^{-1} t_j^{-1} t_k)}{(1 - t_j^{-1} t_k)} \cdot \frac{(1 - q_1^{-1} t_j^{-1} t_k^{-1})}{(1 - t_j^{-1} t_k^{-1})}.$$

Combining all these factors we see that $c(u)$ equals

$$\left\{ \prod_{i=1}^{n} \frac{(1 - a^{-1} t_i^{-1})(1 + b^{-1} t_i^{-1})}{1 - t_i^{-2}} \right\} \left\{ \prod_{1 \le j < k \le n} \frac{(1 - q_1^{-1} t_j^{-1} t_k)(1 - q_1^{-1} t_j^{-1} t_k^{-1})}{(1 - t_j^{-1} t_k)(1 - t_j^{-1} t_k^{-1})} \right\}. \tag{6.3.3}$$

Notice that when $R = BC_n$, $Q = P$, and so we will be able to apply the results of [**23**] (see the paragraph after the proof of Proposition 6.1.1). Thus when $q_1 > 1$ the Plancherel measure here depends on how many of the numbers $q_1^k b$, $k \in \mathbb{N}$, are less than 1 (see [**23**, page 70]). Since we have an underlying building we have the following simplification.

Lemma 6.3.6. *If $q_1 > 1$, then $q_1 b \ge 1$.*

Proof. By a well known theorem of D. Higman (see [**35**, page 30] for example), in a finite thick generalised 4-gon with parameters (k, l), we have $k \le l^2$ and $l \le k^2$. Thus by [**35**, Theorem 3.5 and Proposition 3.2] we have $q_1^2 \ge q_0$ (even if $q_0 = 1$), and so $q_1 b \ge \sqrt{q_n} \ge 1$. $\square$

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $dt = dt_1 \cdots dt_n$, where dt_i is normalised Haar measure on $\mathbb{T}$. Define $\phi_0(u) = c(u)c(u^{-1})$ and

$$\phi_1(u) = \lim_{t_1 \to -b} \frac{\phi_0(u)}{1 + b^{-1} t_1} \qquad \text{and} \qquad dt' = \delta_{-b}(t_1) dt_2 \cdots dt_n.$$

Note that the above limit exists since there is a factor $1 + b^{-1} t_1$ in $c(u^{-1})$ (see (6.3.3)).

We use the isomorphism $\mathbb{U} \to \mathbb{T}^n$, $u \mapsto (t_1, \ldots, t_n)$ to identify $\mathbb{U}$ with $\mathbb{T}^n$. Define $\mathbb{U}' = \{-b\} \times \mathbb{T}^{n-1}$, and write $U = \mathbb{U} \cup \mathbb{U}'$.

Theorem 6.3.7. *Let π_0 be the measure on $U = \mathbb{U} \cup \mathbb{U}'$ given by $d\pi_0(u) = \frac{W_0(q^{-1})}{|W_0|} \frac{dt}{\phi_0(u)}$ on $\mathbb{U}$ and $d\pi_0(u) = \frac{W_0(q^{-1})}{|W_0'|} \frac{dt'}{\phi_1(u)}$ on $\mathbb{U}'$, where W_0' is the Coxeter group C_{n-1} (with $C_1 = A_1$ and $C_0 = \{1\}$). Then (in the exceptional case)*

$$\langle A, B \rangle = \int_U \widehat{A}(u) \overline{\widehat{B}(u)} d\pi_0(u) \quad \text{for all } A, B \in \mathscr{A}. \tag{6.3.4}$$

Proof. When $q_1 > 1$ this follows from the 'group free' calculations made in [**23**, Theorem 5.2.10], taking into account Lemma 6.3.6. If $q_1 = 1$ the formula for $c(u)$ simplifies considerably, and a calculation similar to that in [**23**, Theorem 5.2.10] proves the result in this case too. $\square$

As in the standard case we have the following corollary (see Corollary 6.3.5).

COROLLARY 6.3.8. *In the exceptional case,* $M_2 = \{\tilde{h}_u \mid u \in U\}$. *Moreover, the map* $\varpi : u \mapsto \tilde{h}_u$ *induces a homeomorphism* $(\mathbb{U}/W_0) \cup (\mathbb{U}'/W_0') \to M_2$ *and the Plancherel measure* π *is the image of the measure* π_0 *of Theorem 6.3.7 under* ϖ.

CHAPTER 7

The Integral Formula

As usual, let $\mathscr{X}$ be a locally finite regular affine building of irreducible type. In this chapter we will provide a second formula for the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ in terms of an integral over the *boundary* of $\mathscr{X}$. In Theorem 7.7.2 we prove the non-trivial fact that this integral formula coincides with the Macdonald formula of Chapter 6, and in Theorem 7.7.3 we compute the norms $\|A_\lambda\|$.

Having both the integral and Macdonald formulae for the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ provides us with some very powerful machinery, which will be applied in Chapter 8 to study *radial random walks* on affine buildings.

7.1. Convex Hull

Let $H_{\alpha;k}$ be a hyperplane of Σ. The (closed) half-spaces of Σ associated to $H_{\alpha;k}$ are $H^+_{\alpha;k} = \{z \in E \mid \langle z, \alpha\rangle \geq k\}$ and $H^-_{\alpha;k} = \{z \in E \mid \langle z, \alpha\rangle \leq k\}$. The *walls* of an apartment $\mathcal{A}$ are the pre-images of the hyperplanes of Σ under an isomorphism $\psi : \mathcal{A} \to \Sigma$, and the *half-apartments* of $\mathcal{A}$ are the pre-images of the half-spaces of Σ under ψ.

Given a subset $X \subset V_P$, define the *convex hull of* X, or $\mathrm{conv}(X)$, to be the set of good vertices that lie in the intersection of all half-apartments that contain X.

We make the analogous definition of $\mathrm{conv}(X)$ for subsets $X \subset P$, with the word half-apartment replaced by half-space.

Let $\leq$ denote the partial order on P^+ given by $\mu \leq \lambda$ if and only if $\lambda - \mu \in P^+$. Note that this is quite different to the partial order $\preceq$ on P used earlier.

LEMMA 7.1.1. *Let* $\lambda \in P^+$. *Then* $\mathrm{conv}\{0, \lambda\} = \{\mu \in P^+ \mid \mu \leq \lambda\}$.

PROOF. Let $\mu \leq \lambda$ and write $\nu = \lambda - \mu \in P^+$. Suppose that $0, \lambda \in H^\pm_{\alpha;k}$. Since $H_{-\alpha;k} = H_{\alpha;-k}$ we may assume that $\alpha \in R^+$, and since $0 \in H^\pm_{\alpha;k}$ the only cases to consider are $H^-_{\alpha;k}$ with $k \geq 0$, and $H^+_{\alpha;k}$ with $k \leq 0$. In the case $k \geq 0$ we have $\langle \mu, \alpha\rangle = \langle \lambda - \nu, \alpha\rangle \leq \langle \lambda, \alpha\rangle \leq k$ and so $\mu \in H^-_{\alpha;k}$. In the case $k \leq 0$ we have $\langle \mu, \alpha\rangle \geq 0 \geq k$ and so $\mu \in H^+_{\alpha;k}$. Thus $\{\mu \in P^+ \mid \mu \leq \lambda\} \subseteq \mathrm{conv}\{0, \lambda\}$.

Now suppose that $\mu \in \mathrm{conv}\{0, \lambda\}$. Since $0, \lambda \in H^+_{\alpha_i;0}$ for each $i \in I_0$, we have $\mu \in H^+_{\alpha_i;0}$ for each $i \in I_0$ too, and so $\mu \in P^+$. Also, $0, \lambda \in H^-_{\alpha_i;\langle\lambda,\alpha_i\rangle}$ for each $i \in I_0$, and so $\langle \mu, \alpha_i\rangle \leq \langle \lambda, \alpha_i\rangle$ for each $i \in I_0$. That is, $\lambda - \mu \in P^+$, and so $\mu \leq \lambda$. $\square$

LEMMA 7.1.2. *Let $\lambda, \mu \in P^+$. Then $|V_\lambda(x) \cap V_{\mu^*}(y)| = 1$ whenever $y \in V_{\lambda+\mu}(x)$.*

PROOF. By Corollary 5.3.4, Theorem 5.2.7 and Proposition 4.3.6 we have

$$a_{\lambda,\mu;\lambda+\mu} = c_{\lambda,\mu;\lambda+\mu} = \frac{N_{\lambda+\mu}}{N_\lambda N_\mu},$$

and the result follows from (4.4.3) □

REMARK 7.1.3. A 'building theoretic' proof of Lemma 7.1.2 is given in Appendix B.4.

COROLLARY 7.1.4. *Let $\lambda \in P^+$ and $\mu \leq \lambda$, and let $x, y \in V_P$ be any vertices with $y \in V_\lambda(x)$. There exists a unique vertex, denoted $v_\mu(x,y)$, in the set $V_\mu(x) \cap V_{\nu^*}(y)$, where $\nu = \lambda - \mu \in P^+$.*

THEOREM 7.1.5. *Let $\lambda \in P^+$, $x \in V_P$, and $y \in V_\lambda(x)$. Then*

$$\mathrm{conv}\{x,y\} = \{v_\mu(x,y) \mid \mu \leq \lambda\}.$$

PROOF. Let H be a half-apartment of $\mathscr{X}$ containing $\{x,y\}$, and let $\mathcal{A}$ be any apartment containing H. It is easy to see (using Axiom (B2) of [**7**, p.76]) that there exists a type-rotating isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(x) = 0$ and $\psi(y) = \lambda$. Let $\mu \leq \lambda$ and write $\nu = \lambda - \mu \in P^+$. The vertex $v = \psi^{-1}(\mu)$ is in both $V_\mu(x)$ and $V_{\nu^*}(y)$ (as $y \in V_\nu(v)$, for $(t_{-\mu} \circ \psi)(v) = 0$ and $(t_{-\mu} \circ \psi)(y) = \nu \in P^+$), and so by Corollary 7.1.4 $v_\mu(x,y) = v \in \mathcal{A}$. Now $\psi(H)$ is a half-space of Σ which contains 0 and λ, and so by Lemma 7.1.1 $\mu \in \psi(H)$. Thus $v_\mu(x,y) = \psi^{-1}(\mu) \in H$, showing that

$$\{v_\mu(x,y) \mid \mu \leq \lambda\} \subseteq \mathrm{conv}\{x,y\}.$$

Suppose now that $v \in \mathrm{conv}\{x,y\}$. Thus there exists an apartment $\mathcal{A}$ containing x, y and v. Let $\psi : \mathcal{A} \to \Sigma$ be a type-rotating isomorphism such that $\psi(x) = 0$ and $\psi(y) = \lambda$, and write $\mu = \psi(v)$. If $\mu \notin \mathrm{conv}\{0,\lambda\}$ then there is a (closed) half-space of Σ which contains 0 and λ but not μ, and it follows that there exists a half-apartment of $\mathcal{A}$ which contains x and y but not v, a contradiction. Thus $\mu \in \mathrm{conv}\{0,\lambda\}$, and so by Lemma 7.1.1 $\mu \leq \lambda$. It follows that $v \in V_\mu(x) \cap V_{\nu^*}(y)$, where $\nu = \lambda - \mu \in P^+$, and so $v = v_\mu(x,y)$, completing the proof. □

Note that the above shows that $\mathrm{conv}\{x,y\}$ is a finite set for all $x, y \in V_P$. Indeed

$$|\mathrm{conv}\{x,y\}| = \prod_{i=1}^{n} (\langle \lambda, \alpha_i \rangle + 1)$$

if $y \in V_\lambda(x)$, which also shows that $|\mathrm{conv}\{x,y\}| = |\mathrm{conv}\{u,v\}|$ whenever $y \in V_\lambda(x)$ and $v \in V_\lambda(u)$, and that $|\mathrm{conv}\{x,y\}|$ does not depend on the parameters of the building.

7.2. Preliminary Results

We now give some background that involves only the root system R, which throughout is assumed to be irreducible. Recall that we write $\preceq$ for the partial order on P given by $\mu \preceq \lambda$ if and only if $\lambda - \mu \in Q^+$.

We say that a subset $X \subset P$ is *saturated* [**5**, VI, §1, Exercise 23] if $\mu - i\alpha^\vee \in X$ for all $\mu \in X$, $\alpha \in R$, and all i between 0 and $\langle \mu, \alpha \rangle$ inclusive. Every saturated set is stable under W_0, and for each $\lambda \in P^+$ there is a unique saturated set, denoted Π_λ, with highest coweight λ (that is, $\mu \preceq \lambda$ for all $\mu \in \Pi_\lambda$). Note that

$$\Pi_\lambda = \{w\mu \mid \mu \in P^+, \mu \preceq \lambda, w \in W_0\} \tag{7.2.1}$$

(see [**18**, Lemma 13.4B] for example).

We recall the definition of the *Bruhat order* on W [**19**, §5.9]. Let $v, w \in W$, and write $v \to w$ if $v = s_{\alpha;k}w$ for some $\alpha \in R$, $k \in \mathbb{Z}$, and $\ell(v) < \ell(w)$. Declare $v \leq w$ if and only if there exists a sequence $v = w_0 \to w_1 \to \cdots \to w_n = w$. This gives the Bruhat (partial) order on W. We extend the Bruhat order to $\tilde{W}$ as in [**27**, §2.3] by declaring $\tilde{v} \leq \tilde{w}$ if and only if $\tilde{v} = vg$ and $\tilde{w} = wg$ with $v \leq w$ in W and $g \in G$.

By a *sub-expression* of a fixed reduced expression $s_{i_1} \cdots s_{i_r} \in W$ we mean a product of the form $s_{i_{k_1}} \cdots s_{i_{k_q}}$ where $1 \leq k_1 < \cdots < k_q \leq r$. Let $w = s_{i_1} \cdots s_{i_r}$ be a fixed reduced expression for $w \in W$. By [**19**, Theorem 5.10], $v \leq w$ if and only v can be obtained as a sub-expression of this reduced expression.

Proposition 7.2.1. *Let $\tilde{v}, \tilde{w} \in \tilde{W}$ with $\tilde{v} \leq \tilde{w}$. If $\tilde{w}(0) \in \Pi_\lambda$, then $\tilde{v}(0) \in \Pi_\lambda$ too.*

Proof. Suppose first that $\tilde{v} = s_{\alpha;k}\tilde{w}$ with $\ell(\tilde{v}) < \ell(\tilde{w})$. Then by [**27**, (2.3.3)] $H_{\alpha;k}$ separates C_0 and $\tilde{w}C_0$, and thus $\langle \tilde{w}(0), \alpha \rangle - k$ is between 0 and $\langle \tilde{w}(0), \alpha \rangle$ (inclusive). Thus by the definition of saturated sets

$$\tilde{v}(0) = \tilde{w}(0) - (\langle \tilde{w}(0), \alpha \rangle - k)\alpha^\vee \in \Pi_\lambda,$$

and the result clearly follows by induction. □

7.3. Sectors

Let $\mathscr{X}$ be an irreducible regular affine building. A *sector* of $\mathscr{X}$ is a subcomplex $\mathcal{S} \subset \mathscr{X}$ such that there exists an apartment $\mathcal{A}$ such that $\mathcal{S} \subset \mathcal{A}$ and a type preserving isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(\mathcal{S})$ is a sector of Σ. The *base vertex* of $\mathcal{S}$ is $\psi^{-1}(\lambda)$, where $\lambda \in P$ is the base vertex of $\psi(\mathcal{S})$.

If $\mathcal{S}$ and $\mathcal{S}'$ are sectors of $\mathscr{X}$ with $\mathcal{S}' \subseteq \mathcal{S}$, then we say $\mathcal{S}'$ is a *subsector* of $\mathcal{S}$. The *boundary* Ω of $\mathscr{X}$ is the set of equivalence classes of sectors, where we declare two sectors equivalent if and only if they contain a common subsector. Given $x \in V_P$ and $\omega \in \Omega$, there exists a unique sector, denoted $\mathcal{S}^x(\omega)$, in the class ω with base vertex x [**35**, Lemma 9.7].

LEMMA 7.3.1. *Let $\mathcal{S}$ be a sector in an apartment $\mathcal{A}$ of $\mathscr{X}$. There exists a unique type-rotating isomorphism $\psi_{\mathcal{A},\mathcal{S}} : \mathcal{A} \to \Sigma$ such that $\psi_{\mathcal{A},\mathcal{S}}(\mathcal{S}) = \mathcal{S}_0$.*

PROOF. Let x be the base vertex of $\mathcal{S}$, and let $\psi' : \mathcal{A} \to \Sigma$ be a type preserving isomorphism. Writing $\lambda = \psi'(x)$ we see that $t_{-\lambda} \circ \psi' : \mathcal{A} \to \Sigma$ is a type-rotating isomorphism mapping $\mathcal{S}$ to a sector of Σ based at 0. Thus $(t_{-\lambda} \circ \psi')(\mathcal{S}) = w\mathcal{S}_0$ for some $w \in W_0$, and so $w^{-1} \circ t_{-\lambda} \circ \psi' : \mathcal{A} \to \Sigma$ is a type-rotating isomorphism satisfying the requirements of the lemma.

Let ψ and ψ' be two such isomorphisms. It follows from Lemma 4.1.3 that $\psi' \circ \psi^{-1} = w$ for some $w \in W_0$ (it is important that ψ and ψ' are type-rotating here). We have $w(C_0) = C_0$, and so $w = 1$ since W acts simply transitively on the chambers of Σ [**7**, p.142], and thus $\psi' = \psi$. □

7.4. Retractions and the Coweights $h(x, y; \omega)$

Given an apartment $\mathcal{A}$ and a sector $\mathcal{S}$ of $\mathcal{A}$, let $\rho_{\mathcal{A},\mathcal{S}} : \mathscr{X} \to \mathcal{A}$ be the *retraction onto $\mathcal{A}$ centered at $\mathcal{S}$* [**7**, pages 170–171]. This is defined as follows. Given any chamber c of $\mathscr{X}$, there exists a subsector $\mathcal{S}'$ of $\mathcal{S}$ and an apartment $\mathcal{A}'$ such that c and $\mathcal{S}'$ are contained in $\mathcal{A}'$ [**7**, page 170]. Writing $\psi_{\mathcal{A}'} : \mathcal{A}' \to \mathcal{A}$ for the isomorphism from building axiom (iii), we set $\rho_{\mathcal{A},\mathcal{S}}(c) = \psi_{\mathcal{A}'}(c)$, which is easily seen to be independent of the particular $\mathcal{S}'$ and $\mathcal{A}'$ chosen.

Recall that if $f = i_1 \cdots i_r \in I^*$ (the free monoid on I), we write s_f for the element $s_{i_1} \cdots s_{i_r} \in W$. For $\lambda \in P^+$ let w_λ and f_λ be as in Section 3.7.

THEOREM 7.4.1. *Let $x \in V_P$, $\omega \in \Omega$ and write $\mathcal{S} = \mathcal{S}^x(\omega)$. Let $\mathcal{A}$ be any apartment containing $\mathcal{S}$. Then $(\psi_{\mathcal{A},\mathcal{S}} \circ \rho_{\mathcal{A},\mathcal{S}})(y) \in \Pi_\lambda$ for all $y \in V_\lambda(x)$.*

PROOF. It follows easily from Proposition 4.1.2 that there exists a gallery $c_0, \ldots, c_n$ of type $\sigma_i(f_\lambda)$ from x to y, where $i = \tau(x)$. Write $\Phi = \psi_{\mathcal{A},\mathcal{S}} \circ \rho_{\mathcal{A},\mathcal{S}}$. Then $\Phi(c_0), \ldots, \Phi(c_n)$ is a pre-gallery of type f_λ from 0 to $\mu = \Phi(y)$. Removing any stutters, we have a gallery of type f'_λ, say, from 0 to μ, and $s_{f'_\lambda}$ is a sub-expression of s_{f_λ}. Thus $s_{f'_\lambda} \le s_{f_\lambda}$, and so $s_{f'_\lambda} g_l \le s_{f_\lambda} g_l$, where $l = \tau(\lambda)$. Since $(s_{f_\lambda} g_l)(0) = \lambda \in \Pi_\lambda$, it follows from Proposition 7.2.1 that $\mu' = (s_{f'_\lambda} g_l)(0) \in \Pi_\lambda$ too. Since $\Phi(c_0) = wC_0$ for some $w \in W_0$, we have $\Phi(c_n) = ws_{f'_\lambda} C_0$, and so by considering types of vertices we have

$$\mu = ws_{f'_\lambda}(g_l(0)) = w\mu'.$$

Thus $\mu \in \Pi_\lambda$ by (7.2.1). □

For each $x \in V_P$, $\omega \in \Omega$ and $\lambda \in P^+$, the intersection $V_\lambda(x) \cap \mathcal{S}^x(\omega)$ contains a unique vertex, denoted $v^x_\lambda(\omega) \in V_P$.

The coweights $h(x, y; \omega)$ in the next theorem are the analogs of the well studied *horocycle numbers* of homogeneous trees.

THEOREM 7.4.2. *Let $\omega \in \Omega$ and let $x, y \in V_P$.*

(i) *Let $z \in \mathcal{S}^x(\omega) \cap \mathcal{S}^y(\omega)$ and write $z = v^x_\nu(\omega) = v^y_\eta(\omega)$. The coweight $\nu - \eta$ is independent of the particular $z \in \mathcal{S}^x(\omega) \cap \mathcal{S}^y(\omega)$ chosen. We denote this common value by $h(x, y; \omega)$. If $\mu \in P^+$ and $\mu - \nu \in P^+$ then*

$$v^x_\mu(\omega) = v^y_{\mu - h(x,y;\omega)}(\omega).$$

(ii) *Suppose that $y \in V_\lambda(x)$. Write $\mathcal{S} = \mathcal{S}^x(\omega)$ and let $\mathcal{A}$ be any apartment containing $\mathcal{S}$. Then $h(x, y; \omega) = (\psi_{\mathcal{A},\mathcal{S}} \circ \rho_{\mathcal{A},\mathcal{S}})(y) \in \Pi_\lambda$.*

PROOF. (i) We have $v^x_{\nu+\mu'}(\omega) = v^y_{\eta+\mu'}(\omega)$ for all $\mu' \in P^+$, since both are equal to $v^z_{\mu'}(\omega)$. Thus, writing $\mu = \mu' + \nu$ we have

$$v^x_\mu(\omega) = v^y_{\mu-(\nu-\eta)}(\omega) \qquad \text{whenever } \mu - \nu \in P^+. \tag{7.4.1}$$

If we instead choose $z' \in \mathcal{S}^x(\omega) \cap \mathcal{S}^y(\omega)$, where $z' = v^x_{\nu'}(\omega) = v^y_{\eta'}(\omega)$, then following the above we have

$$v^x_\mu(\omega) = v^y_{\mu-(\nu'-\eta')}(\omega) \qquad \text{whenever } \mu - \nu' \in P^+. \tag{7.4.2}$$

By choosing $\mu \in P^+$ such that both $\mu - \nu$ and $\mu - \nu'$ are dominant, it follows from (7.4.1) and (7.4.2) that $\nu - \eta = \nu' - \eta'$. Then (7.4.1) proves the final claim.

(ii) Write $\mu = (\psi_{\mathcal{A},\mathcal{S}} \circ \rho_{\mathcal{A},\mathcal{S}})(y)$. By [**7**, page 170] there exists a subsector $\mathcal{S}'$ of $\mathcal{S} = \mathcal{S}^x(\omega)$ such that $\mathcal{S}'$ and y lie in a common apartment $\mathcal{A}'$. The restriction of $\rho_{\mathcal{A},\mathcal{S}}$ to $\mathcal{A}'$ is thus a type preserving isomorphism. Pick $\nu \in P^+$ such that $v^x_\nu(\omega) \in \mathcal{S}'$, $\nu - \mu \in P^+$ and $v^x_\nu(\omega) \in \mathcal{S}^y(\omega)$. The map $\psi = t_{-\mu} \circ \psi_{\mathcal{A},\mathcal{S}} \circ \rho_{\mathcal{A},\mathcal{S}} : \mathcal{A}' \to \Sigma$ is a type-rotating isomorphism such that $\psi(y) = 0$ and $\psi(v^x_\nu(\omega)) = \nu - \mu \in P^+$. Thus $v^x_\nu(\omega) \in V_{\nu-\mu}(y) \cap \mathcal{S}^y(\omega)$, and so $h(x, y; \omega) = \mu$. The fact that $h(x, y; \omega) \in \Pi_\lambda$ now follows from Theorem 7.4.1. □

PROPOSITION 7.4.3. *For all $x, y, z \in V_P$ and $\omega \in \Omega$ we have the 'cocycle relation' $h(x, y; \omega) = h(x, z; \omega) + h(z, y; \omega)$. Thus $h(x, x; \omega) = 0$ and $h(y, x; \omega) = -h(x, y; \omega)$.*

PROOF. For $\mu = k_1\lambda_1 + \cdots + k_n\lambda_n \in P^+$ with each k_i sufficiently large we have

$$v^x_\mu(\omega) = v^z_{\mu-h(x,z;\omega)}(\omega) = v^y_{\mu-h(x,z;\omega)-h(z,y;\omega)}(\omega)$$

and the result follows. □

The following theorem shows that if $y \in V_\lambda(x)$, then for any $\omega \in \Omega$, $\mathcal{S}^y(\omega)$ contains all vertices $v^x_\mu(\omega)$ for $\mu \in P^+$ large enough, where *large enough* depends only on λ, not on the particular x, y and ω.

For $\lambda \in P^+$, write $\mu \gg \lambda$ to mean that $\mu - \Pi_\lambda \subset P^+$ (in particular, $\mu \in P^+$).

THEOREM 7.4.4. *Let $x \in V_P$, $\lambda \in P^+$ and $y \in V_\lambda(x)$. Then $v^x_\mu(\omega) \in \mathcal{S}^y(\omega)$ for all $\omega \in \Omega$ and all $\mu \gg \lambda$, and so $v^x_\mu(\omega) = v^y_{\mu-h(x,y;\omega)}(\omega)$ for all $\omega \in \Omega$ and all $\mu \gg \lambda$.*

REMARK 7.4.5. The proof of Theorem 7.4.4 will be given at the end of this section. We are thankful to an anonymous referee of a paper drawn from this thesis for sketching this proof of the present form of Theorem 7.4.4, which replaces our less sharp version of this result.

LEMMA 7.4.6. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are apartments containing a common sector $\mathcal{S}$. Then the maps $\rho_{\mathcal{A}_1,\mathcal{S}}|_{\mathcal{A}_2} : \mathcal{A}_2 \to \mathcal{A}_1$ and $\rho_{\mathcal{A}_2,\mathcal{S}}|_{\mathcal{A}_1} : \mathcal{A}_1 \to \mathcal{A}_2$ are mutually inverse isomorphisms which fix $\mathcal{A}_1 \cap \mathcal{A}_2$ pointwise.*

PROOF. Fix any chamber $c \subset \mathcal{S}$, and let $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ be the unique isomorphism fixing $\mathcal{A}_1 \cap \mathcal{A}_2$ pointwise. Then by definition we have $\rho_{\mathcal{A}_2,\mathcal{S}}|_{\mathcal{A}_1} = \varphi$, and since $\varphi^{-1} : \mathcal{A}_2 \to \mathcal{A}_1$ is the unique isomorphism fixing $\mathcal{A}_1 \cap \mathcal{A}_2$ pointwise we have $\rho_{\mathcal{A}_1,\mathcal{S}}|_{\mathcal{A}_2} = \varphi^{-1}$. $\square$

LEMMA 7.4.7. *Let $\mathcal{A}$ be an apartment in $\mathscr{X}$, let $\mathcal{S}$ be a sector in $\mathcal{A}$, and let H be a wall in $\mathcal{A}$. Then exactly one of the two closed half-apartments in $\mathcal{A}$ determined by H, H^+ say, contains a subsector of $\mathcal{S}$. If $x \in V_P \cap H^+$, then the sector based at x and equivalent to $\mathcal{S}$ is contained in H^+.*

PROOF. Let $\psi = \psi_{\mathcal{A},\mathcal{S}}$ (see Lemma 7.3.1). Then $\psi(H) = H_{\alpha;k}$ for some $\alpha \in R$ and $k \in \mathbb{Z}$. Since $H_{-\alpha;k} = H_{\alpha;-k}$, we may suppose that $\alpha \in R^+$. If $k \leq 0$, then $\mathcal{S}_0 \subset H^+_{\alpha;k} = \{x : \langle x, \alpha\rangle \geq k\}$. If $k \geq 1$, then $\lambda + \mathcal{S}_0 \subset H^+_{\alpha;k}$ for $\lambda = k(\lambda_1 + \cdots + \lambda_n)$.

The final statement follows from [**35**, Lemma 9.1]. $\square$

PROPOSITION 7.4.8. *Let $c_0, ..., c_m$ be a gallery of type $j_1 \cdots j_m$, and let $\omega \in \Omega$. Let $\mathcal{A}$ be an apartment containing c_0 and a sector $\mathcal{S}$ in the class ω. Let $\rho = \rho_{\mathcal{S},\mathcal{A}}$, and let $e_k = \rho(c_k)$ for $k = 0, \ldots, m$. For $k = 1, \ldots, m$, let H_k denote the wall in $\mathcal{A}$ containing the panel in e_{k-1} and in e_k of type j_k. Let H_k^+ denote the half-apartment in $\mathcal{A}$ bounded by H_k which contains a subsector of $\mathcal{S}$ (see Lemma 7.4.7).*

Then there exists an apartment $\mathcal{B}$ containing c_m and

$$\bigcap_{k=1}^{m} H_k^+$$

(and therefore $\mathcal{B}$ contains a sector in the class ω).

PROOF. By induction on m. If $m = 1$, the panel of cotype j_1 common to $e_0 = c_0$ and e_1 is contained in c_1 and in H_1. So by the proof of [**35**, Lemma 9.4], there is an apartment $\mathcal{B}$ containing H_1^+ and c_1.

Now suppose that $m > 1$ and that there is an apartment $\mathcal{B}'$ containing c_{m-1} and $\cap_{i=1}^{m-1} H_k^+$. Let $\mathcal{S}'$ be any sector in the class ω contained in $\cap_{k=1}^{m-1} H_k^+$, and let $\rho' = \rho_{\mathcal{S}',\mathcal{B}'}$. If $\mathcal{T}$ is a common subsector of $\mathcal{S}$ and $\mathcal{S}'$, then $\rho = \rho_{\mathcal{T},\mathcal{A}}$ and $\rho' = \rho_{\mathcal{T},\mathcal{B}'}$. So by Lemma 7.4.6, the maps $\rho|_{\mathcal{B}'} : \mathcal{B}' \to \mathcal{A}$ and $\rho'|_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}'$ are mutually inverse isomorphisms.

Let H denote the wall in $\mathcal{B}'$ containing the panel of cotype j_m in c_{m-1} and in c_m, and let H^+ denote the half-apartment in $\mathcal{B}'$ bounded by H and containing a subsector of $\mathcal{S}$. The half-apartment $\rho'(H_m^+)$ in $\mathcal{B}'$ is bounded by H. To see this, let π denote the panel in c_{m-1} and c_m of cotype j_m, and let π' denote the panel in $e_{m-1} = \rho(c_{m-1})$ and $e_m = \rho(c_m)$ of cotype j_m. Then $\pi' = \rho(c_{m-1} \cap c_m)$, and so $\rho'(\pi') = c_{m-1} \cap c_m = \pi$. Now π' is in the wall of $\mathcal{A}$ bounding H_m^+, and so $\pi = \rho'(\pi')$ is in the wall of $\mathcal{B}$ bounding $\rho'(H_m^+)$. But π is in the wall H of $\mathcal{B}$ bounding H^+. Furthermore, $\rho'(H_m^+)$ contains a subsector of $\mathcal{S}$, because $H_m^+ \cap \mathcal{B}'$ contains such a subsector, and is fixed by ρ'. Thus

$$\rho'(H_m^+) = H^+. \tag{7.4.3}$$

By the proof of [**35**, Lemma 9.4], there is an apartment $\mathcal{B}$ containing c_m and H^+. Since $H_1^+ \cap \cdots \cap H_{m-1}^+$ is contained in $\mathcal{B}'$, it is fixed by ρ', and so

$$H_1^+ \cap \cdots \cap H_m^+ = \rho'((H_1^+ \cap \cdots \cap H_{m-1}^+) \cap H_m^+) \subset \rho'(H_m^+).$$

Thus by (7.4.3), $H_1^+ \cap \cdots \cap H_m^+ \subset H^+ \subset \mathcal{B}$, completing the induction step. $\square$

LEMMA 7.4.9. *Suppose that $C_0 = D_0, \ldots, D_m$ is a gallery in Σ joining 0 to λ of minimal length, and that $C_0 = E_0, \ldots, E_m$ is a pre-gallery in Σ. If both the gallery and the pre-gallery have the same type, then each type $l = \tau(\lambda)$ vertex of any of the D_i's and E_i's is in Π_λ.*

PROOF. For $k = 0, \ldots, m$, define $u_k, v_k \in W$ by $D_k = u_k C_0$ and $E_k = v_k C_0$. Then the type l vertices of D_k and E_k are $u_k g_l(0)$ and $v_k g_l(0)$ respectively (see the proof of Theorem 7.4.1). In particular, $u_m g_l(0) = \lambda \in \Pi_\lambda$.

Since the gallery $C_0 = D_0, \ldots, D_m$ and the pre-gallery $C_0 = E_0, \ldots, E_m$ have the same type, $j_1 \ldots j_m$, say, each u_k and each v_k is a subexpression of the reduced expression $u_m = s_{j_1} \cdots s_{j_m}$. Thus $u_k g_l, v_k g_l \le u_m g_l$ (with respect to the Bruhat order on $\tilde{W}$) for each $k = 0, \ldots, m$.

The result now follows from Proposition 7.2.1. $\square$

COROLLARY 7.4.10. *Suppose that $x, y \in V_P$, $\omega \in \Omega$, $\lambda \in P^+$, and $y \in V_\lambda(x)$. Consider a minimal gallery $c_0, ..., c_m$ from x to y and an apartment $\mathcal{A}$ containing c_0 and a sector in the class ω (and hence containing $\mathcal{S}^x(\omega)$). Let $\psi : \mathcal{A} \to \Sigma$ be the type-rotating isomorphism mapping $\mathcal{S}^x(\omega)$ to $\mathcal{S}_0$ (see Lemma 7.3.1). Finally, let H_i and H_i^+ be as in Proposition 7.4.8, and write $\rho = \rho_{\mathcal{A}, \mathcal{S}^x(\omega)}$. Then:*

(i) *$\psi^{-1}(\Pi_\lambda)$ contains all the type $j = \tau(y)$ vertices in each $\rho(c_i)$, and so in particular it contains $y' = \rho(y)$.*

(ii) *If $\mu \gg \lambda$, then $v_\mu^x(\omega) \in \bigcap_{i=1}^m H_i^+$.*

PROOF. (i) Let $e_i = \rho(c_i)$ and $E_i = \psi(e_i)$ for $i = 0, \ldots, m$. Let $\rho' = \rho_{c_0, \mathcal{A}}$ denote the retraction of center c_0 onto $\mathcal{A}$ (see [**7**, §IV.3]). Let $d_i = \rho'(c_i)$ and $D_i = \psi(d_i)$ for

$i = 0, \dots, m$. Then the gallery $C_0 = D_0, \dots, D_m$ and the pre-gallery $C_0 = E_0, \dots, E_m$ satisfy the hypotheses of Lemma 7.4.9, and the result follows.

(ii) For $i = 0, \dots, m$, let v_i be the type j vertex of e_i, and so $\{v_i\}_{i=0}^m \subset \psi^{-1}(\Pi_\lambda)$.

Let $1 \le i \le m$. If $v_{i-1} = v_i$, then $\psi(v_i) \in \Pi_\lambda \cap \psi(H_i)$, and so $\psi(v_i) + \mathcal{S}_0$ is contained in $\psi(H_i)^+$ by Lemma 7.4.7 (here $\psi(H_i)^+$ is the half-space of Σ bounded by $\psi(H_i)$ and containing a subsector of $\mathcal{S}_0 = \psi(\mathcal{S}^x(\omega)))$. Hence $\mathcal{S}^{v_i}(\omega)$ is contained in H_i^+. By our hypothesis, $\mu - \Pi_\lambda \subset P^+$, and so $\mu = \psi(v_i) + \nu_i$ for some $\nu_i \in P^+$. Hence $\psi(v_\mu^x(\omega)) = \mu \in \psi(v_i) + \mathcal{S}_0 \subset \psi(H_i)^+$. Hence $v_\mu^x(\omega) \in H_i^+$.

If $v_{i-1} \ne v_i$, then $\psi(v_{i-1})$ and $\psi(v_i)$ lie on opposite sides of $\psi(H_i)$. Then by Lemma 7.4.7, either $\psi(v_{i-1}) + \mathcal{S}_0$ or $\psi(v_i) + \mathcal{S}_0$ is contained in $\psi(H_i)^+$. Let us assume that $\psi(v_i) + \mathcal{S}_0 \subset \psi(H_i)^+$. Since $\mu - \Pi_\lambda \subset P^+$, we can write $\mu = \psi(v_i) + \nu_i$ for some $\nu_i \in P^+$. Hence $\psi(v_\mu^x(\omega)) = \mu \in \psi(v_i) + \mathcal{S}_0 \subset \psi(H_i)^+$, and so again $v_\mu^x(\omega) \in H_i^+$. □

Proof of Theorem 7.4.4. Let $c_0, \dots, c_m$ be a gallery of minimal length from x to y, and let $\mathcal{A}, \mathcal{B}$ and $H_1^+, \dots, H_m^+$ be as in Proposition 7.4.8. By Lemma 7.4.6, the map $\varphi = \rho_{\mathcal{A},\mathcal{S}^x(\omega)}|_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ is an isomorphism. It maps y to $y' = \rho_{\mathcal{A},\mathcal{S}^x(\omega)}(y)$ and fixes $\mathcal{A} \cap \mathcal{B} \supset \bigcap_{i=1}^m H_i^+$, which contains a sector in the class ω. Hence φ maps $\mathcal{S}^y(\omega)$ to $\mathcal{S}^{y'}(\omega)$. Moreover, if $\mu \gg \lambda$, then by Corollary 7.4.10(ii), φ fixes $v_\mu^x(\omega)$.

Let $\psi : \mathcal{A} \to \Sigma$ be the type-rotating isomorphism mapping $\mathcal{S}^x(\omega)$ to $\mathcal{S}_0$. Then by Corollary 7.4.10(i) we have $y' \in \psi^{-1}(\Pi_\lambda)$, and so we can write $\mu = \psi(y') + \nu$ for some $\nu \in P^+$. Therefore $v_\mu^x(\omega) \in \mathcal{S}^{y'}(\omega)$, and applying φ, we see that $v_\mu^x(\omega) \in \mathcal{S}^y(\omega)$. So $v_\mu^x(\omega) = v^y_{\mu - h(x,y;\omega)}(\omega)$ by the definition of $h(x,y;\omega)$. □

7.5. The Boundary of $\mathscr{X}$ and the Measures ν_x

Let $\le$ denote the partial order on P^+ given by $\mu \le \lambda$ if and only if $\lambda - \mu \in P^+$. Fixing $x \in V_P$, there is a natural map $\theta : \Omega \to \prod_{\lambda \in P^+} V_\lambda(x)$, where one maps ω to $(v_\lambda^x(\omega))_{\lambda \in P^+}$. For each pair $\lambda, \mu \in P^+$ with $\mu \le \lambda$, let $\varphi_{\mu,\lambda} : V_\lambda(x) \to V_\mu(x)$ be the map $y \mapsto v_\mu(x,y)$, where $v_\mu(x,y)$ is the unique vertex in $V_\mu(x) \cap \operatorname{conv}\{x,y\}$ (see Section 7.1). Then $(V_\lambda(x), \varphi_{\mu,\lambda})$ is an inverse system of topological spaces (where each finite set $V_\lambda(x)$ is given the discrete topology). The inverse limit $\varprojlim(V_\lambda(x), \varphi_{\mu,\lambda})$ is a compact Hausdorff topological space [**6**, I.9.6], and the map θ is a bijection of Ω onto this inverse limit, thus inducing a compact Hausdorff topology on Ω, which we show in Theorem 7.5.5 is independent of $x \in V_P$. See Appendix B.2 for a sketch of the proof that θ is a bijection of Ω onto $\varprojlim(V_\lambda(x), \varphi_{\mu,\lambda})$.

With $x \in V_P$ fixed as above, for each $y \in V_P$ let $\Omega_x(y) = \{\omega \in \Omega \mid y \in \mathcal{S}^x(\omega)\}$. The sets $\Omega_x(y)$, $y \in V_P$, form a basis of open and closed sets for the topology on Ω, and the functions $\omega \mapsto h(x,y;\omega)$ are locally constant on Ω, as we see in Lemma 7.5.1.

To each $x \in V_P$ there is a unique regular Borel probability measure ν_x on Ω such that $\nu_x(\Omega_x(y)) = N_\lambda^{-1}$ if $y \in V_\lambda(x)$. To see this, for each $\lambda \in P^+$ let $\mathscr{C}_\lambda(\Omega)$ be the space of all functions $f : \Omega \to \mathbb{C}$ which are constant on each set $\Omega_x(y)$, $y \in V_\lambda(x)$. For each $\lambda \in P^+$ define $J_\lambda : \mathscr{C}_\lambda(\Omega) \to \mathbb{C}$ by $J_\lambda(f) = \frac{1}{N_\lambda}\sum_{y\in V_\lambda(x)} c_y(f)$, where $c_y(f)$ is the constant value f takes on $\Omega_x(y)$. The space of all locally constant functions $f : \Omega \to \mathbb{C}$ is $\mathscr{C}_\infty(\Omega) = \bigcup_{\lambda\in P^+}\mathscr{C}_\lambda(\Omega)$. Define $J : \mathscr{C}_\infty(\Omega) \to \mathbb{C}$ by $J(f) = J_\lambda(f)$ if $f \in \mathscr{C}_\lambda(\Omega)$. The map J is linear, maps $1 \in \mathscr{C}_\infty(\Omega)$ to $1 \in \mathbb{C}$ and satisfies $|J(f)| \le \|f\|_\infty$ for all $f \in \mathscr{C}_\infty(\Omega)$. Since $\mathscr{C}_\infty(\Omega)$ is dense in $\mathscr{C}(\Omega)$, J extends uniquely to a linear map $\tilde{J} : \mathscr{C}(\Omega) \to \mathbb{C}$ such that $|\tilde{J}(f)| \le \|f\|_\infty$ for all $f \in \mathscr{C}(\Omega)$ (here $\mathscr{C}(\Omega)$ is the space of all continuous functions $f : \Omega \to \mathbb{C}$). Thus by the Riesz Representation Theorem there exists a unique regular Borel probability measure ν_x such that

$$\tilde{J}(f) = \int_\Omega f(w)d\nu_x(\omega) \quad \text{for all } f \in \mathscr{C}(\Omega).$$

In particular, $N_\lambda^{-1} = \nu_x(\Omega_x(y))$ if $y \in V_\lambda(x)$.

In Theorem 7.5.5(ii) we show that for $x, y \in V_P$, the measures ν_x and ν_y are mutually absolutely continuous, and we compute the *Radon-Nikodym derivative.*

LEMMA 7.5.1. *Let $y \in V_\nu(x)$ and suppose that $z \in V_\lambda(x) \cap V_\mu(y)$ with $\lambda \gg \nu$. Then*
 (i) $\Omega_x(z) \subset \Omega_y(z)$, and
 (ii) $h(x, y; \omega) = \lambda - \mu$ for all $\omega \in \Omega_x(z)$.

PROOF. (i) Let $\omega \in \Omega_x(z)$, and so $z = v^x_\lambda(\omega)$, and by Theorem 7.4.4 $z \in \mathcal{S}^y(\omega)$. Thus $\omega \in \Omega_y(z)$ and so $\Omega_x(z) \subset \Omega_y(z)$. Note that if $\mu \gg \nu^*$ too, then $\Omega_x(z) = \Omega_y(z)$.

Since $\Omega_x(z) \subset \Omega_y(z)$ and $z \in V_\lambda(x)\cap V_\mu(y)$ we have $v^x_\lambda(\omega) = z = v^y_\mu(\omega)$ for all $\omega \in \Omega_x(z)$, and so (ii) follows from Theorem 7.4.2(i). □

LEMMA 7.5.2. *Let $x, y \in V_P$. In the notation of Section 7.1, if $z \in \operatorname{conv}\{x, y\}$, then $\Omega_x(y) \subset \Omega_x(z)$.*

PROOF. Let $\omega \in \Omega_x(y)$. Then the sector $\mathcal{S}^x(\omega)$ contains x and y, and hence z. Thus $\omega \in \Omega_x(z)$. □

Let P^{++} denote the set of all *strongly dominant coweights of* R, that is, those $\lambda \in P$ such that $\langle\lambda, \alpha_i\rangle > 0$ for all $i \in I_0$.

We note that since $q_{uv} = q_u q_v$ whenever $\ell(uv) = \ell(u) + \ell(v)$, by [**27**, (2.4.1)] we have

$$q_{t_{\lambda+\mu}} = q_{t_\lambda} q_{t_\mu}, \quad \text{and so} \quad q_{t_\nu} = \prod_{i=1}^n q_{t_{\lambda_i}}^{\langle\nu,\alpha_i\rangle} \tag{7.5.1}$$

for all $\lambda, \mu, \nu \in P^+$.

PROPOSITION 7.5.3. *Let $\mu \in P$ be fixed. For all $\lambda \in P^{++}$ such that $\lambda - \mu \in P^{++}$ we have*

$$\frac{N_\lambda}{N_{\lambda-\mu}} = \prod_{i=1}^n q_{t_{\lambda_i}}^{\langle\mu,\alpha_i\rangle} = \prod_{\alpha\in R^+} \tau_\alpha^{\langle\mu,\alpha\rangle}$$

PROOF. By (4.3.3) we have $N_\lambda N_{\lambda-\mu}^{-1} = q_{t_\lambda} q_{t_{\lambda-\mu}}^{-1}$, and so by (7.5.1)

$$\frac{N_\lambda}{N_{\lambda-\mu}} = \frac{q_{t_\lambda}}{q_{t_{\lambda-\mu}}} = \prod_{i=1}^{n} q_{t_{\lambda_i}}^{\langle\lambda,\alpha_i\rangle - \langle\lambda-\mu,\alpha_i\rangle} = \prod_{i=1}^{n} q_{t_{\lambda_i}}^{\langle\mu,\alpha_i\rangle},$$

proving the first equality. On the other hand, by Proposition B.1.5,

$$\frac{N_\lambda}{N_{\lambda-\mu}} = \frac{q_{t_\lambda}}{q_{t_{\lambda-\mu}}} = \prod_{\alpha\in R^+} \tau_\alpha^{\langle\lambda,\alpha\rangle - \langle\lambda-\mu,\alpha\rangle} = \prod_{\alpha\in R^+} \tau_\alpha^{\langle\mu,\alpha\rangle}. \qquad \square$$

LEMMA 7.5.4. *Let* $\lambda, \mu \in P^+$. *Then* $\Pi_\lambda + \Pi_\mu \subset \Pi_{\lambda+\mu}$.

PROOF. Let $\lambda' \in \Pi_\lambda$ and $\mu' \in \Pi_\mu$. Then

$$w(\lambda' + \mu') = w\lambda' + w\mu' \preceq \lambda + \mu \quad \text{for all } w \in W_0.$$

By choosing $w \in W_0$ such that $w(\lambda' + \mu') \in P^+$ we have $w(\lambda' + \mu') \in \Pi_{\lambda+\mu}$ by (7.2.1), and so $\lambda' + \mu' \in \Pi_{\lambda+\mu}$. $\square$

THEOREM 7.5.5. *Consider the topologies and measures defined above on* Ω. *Then*

(i) *The topology on* Ω *does not depend on the particular* $x \in V_P$.

(ii) *For* $x, y \in V_P$, *the measures* ν_x *and* ν_y *are mutually absolutely continuous, and the Radon-Nikodym derivative is given by*

$$\frac{d\nu_y}{d\nu_x}(\omega) = \prod_{i=1}^{n} q_{t_{\lambda_i}}^{\langle h(x,y;\omega),\alpha_i\rangle} = \prod_{\alpha\in R^+} \tau_\alpha^{\langle h(x,y;\omega),\alpha\rangle}.$$

PROOF. (i) Let $x, y \in V_P$, with $y \in V_\nu(x)$, say, and choose $\lambda \gg \nu + \nu^*$. Notice that this implies that $\lambda \gg \nu$. Furthermore, for each $\nu' \in \Pi_\nu$, using Lemma 7.5.4 we have

$$(\lambda - \nu') - \Pi_{\nu^*} \subset \lambda - \Pi_\nu - \Pi_{\nu^*} \subset \lambda - \Pi_{\nu+\nu^*},$$

and so $\lambda - \nu' \gg \nu^*$.

Let $\omega_0 \in \Omega_x(v)$, a basic open set for the topology using x, where $v \in V_\eta(x)$, say. Since $\lambda \gg \nu$, by Theorem 7.4.4 we have $v_\lambda^x(\omega_0) \in \mathcal{S}^y(\omega_0)$. Let $z = v_\lambda^x(\omega_0) = v_{\lambda-h(x,y;\omega_0)}^y(\omega_0)$, and so $z \in V_\lambda(x) \cap V_{\lambda-h(x,y;\omega_0)}(y)$. Now $h(x, y; \omega_o) \in \Pi_\nu$ (see Theorem 7.4.2(ii)), and so by the above, $\lambda - h(x, y; \omega_0) \gg \nu^*$. Since $\lambda \gg \nu$ too, by Lemma 7.5.1 we have $\Omega_x(z) = \Omega_y(z)$. Choosing λ perhaps larger still so that $\lambda - \eta \in P^+$ we have $\Omega_x(z) \subset \Omega_x(v)$ by Lemma 7.5.2. Thus, since $z = v_{\lambda-h(x,y;\omega_0)}^y(\omega_0)$, we have $\omega_0 \in \Omega_y(z) = \Omega_x(z) \subset \Omega_x(v)$, which shows that the x-*open* sets are y-*open*, and the first statement follows.

(ii) With x, y, z and ω_0 as above, since $\Omega_x(z) = \Omega_y(z)$, and since $\nu_x(\Omega_x(z)) = N_\lambda^{-1}$ and $\nu_y(\Omega_y(z)) = N_{\lambda-h(x,y;\omega_0)}^{-1}$, we see that

$$\nu_y(\Omega_x(z)) = \nu_y(\Omega_y(z)) = N_{\lambda-h(x,y;\omega_0)}^{-1} = N_{\lambda-h(x,y;\omega_0)}^{-1} N_\lambda \nu_x(\Omega_x(z)),$$

and so the measures are mutually absolutely continuous, and the Radon-Nikodym derivative is given by

$$\frac{d\nu_y}{d\nu_x}(\omega) = \frac{N_\lambda}{N_{\lambda-h(x,y;\omega)}} \quad \text{for any } \lambda \in P^+ \text{ such that } \lambda \gg \nu + \nu^*.$$

The result follows from Proposition 7.5.3 by choosing λ perhaps larger still so that both λ and $\lambda - h(x,y;\omega)$ are strongly dominant. $\square$

Let $r \in \mathrm{Hom}(P, \mathbb{C}^\times)$ be the map

$$\mu \mapsto \prod_{i=1}^n q_{t_{\lambda_i}}^{\frac{1}{2}\langle\mu,\alpha_i\rangle} = \prod_{\alpha\in R^+} \tau_\alpha^{\frac{1}{2}\langle\mu,\alpha\rangle} \tag{7.5.2}$$

Following our usual convention we write r^μ in place of $r(\mu)$.

Proposition 7.5.3 immediately gives the following.

Corollary 7.5.6. *Let $\mu \in P$ be fixed. Then for any $\lambda \in P^{++}$ such that $\lambda - \mu \in P^{++}$,*

$$r^\mu = \left(\frac{N_\lambda}{N_{\lambda-\mu}}\right)^{1/2}.$$

7.6. The Integral Formula

For $\lambda \in P^+$, let us write $\mu \ggg \lambda$ to mean that $\mu - \Pi_\lambda \subset P^{++}$ (in particular, notice that if $\mu \ggg \lambda$, then $\mu \gg \lambda$ and $\mu \in P^{++}$). The reason for this is that we will want to ensure that the formula in Corollary 7.5.6 is applicable in the following results.

Recall the definition of the numbers $a_{\lambda,\mu;\nu}$ from (4.4.3).

Lemma 7.6.1. *Let $\lambda \in P^+$. For each $x \in V_P$, $\omega \in \Omega$, $\mu \in \Pi_\lambda$ and $\nu \ggg \lambda$,*

$$\frac{1}{N_\lambda}|\{y \in V_\lambda(x) \mid h(x,y;\omega) = \mu\}| = r^{-2\mu} a_{\lambda,\nu-\mu;\nu}.$$

In particular, the value of the left hand side is independent of $x \in V_P$ and $\omega \in \Omega$.

Proof. We will first show that whenever $\nu \gg \lambda$,

$$\{y \in V_\lambda(x) \mid h(x,y;\omega) = \mu\} = V_\lambda(x) \cap V_{(\nu-\mu)^*}(v_\nu^x(\omega)). \tag{7.6.1}$$

If $y \in V_\lambda(x) \cap V_{(\nu-\mu)^*}(v_\nu^x(\omega))$, then by Theorem 7.4.4, $v_\nu^x(\omega) \in \mathcal{S}^y(\omega) \cap V_{\nu-\mu}(y)$, and so $v_\nu^x(\omega) = v_{\nu-\mu}^y(\omega)$. Thus $h(x,y;\omega) = \mu$.

Conversely, if $y \in V_\lambda(x)$ and $h(x,y;\omega) = \mu$, then $v_\nu^x(\omega) = v_{\nu-\mu}^y(\omega)$ once $\nu \gg \lambda$ by Theorem 7.4.4. Thus $y \in V_\lambda(x) \cap V_{(\nu-\mu)^*}(v_\nu^x(\omega))$.

Now suppose that $\nu \ggg \lambda$. By (4.4.3), (7.6.1) and Corollary 7.5.6 we have

$$\frac{1}{N_\lambda}|\{y \in V_\lambda(x) \mid h(x,y;\omega) = \mu\}| = \frac{N_{\nu-\mu}}{N_\nu} a_{\lambda,\nu-\mu;\nu} = r^{-2\mu} a_{\lambda,\nu-\mu;\nu}. \qquad \square$$

We now describe the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ in terms of *zonal spherical functions*.

DEFINITION 7.6.2. Fix a vertex $x \in V_P$. A function $f : V_P \to \mathbb{C}$ is called *spherical with respect to* $x \in V_P$ if

(i) $f(x) = 1$,
(ii) f is x-radial (that is, $f(y) = f(y')$ whenever $y, y' \in V_\lambda(x)$), and
(iii) for each $A \in \mathscr{A}$ there is a number c_A such that $Af = c_A f$.

The following is proved in [**11**, Proposition 3.4] in the $\widetilde{A}_2$ case, and the proof there generalises immediately.

PROPOSITION 7.6.3. *An x-radial function $f : V_P \to \mathbb{C}$ is spherical if and only if the map $h : \mathscr{A} \to \mathbb{C}$ given by $h(A) = (Af)(x)$ defines an algebra homomorphism. Moreover, each $h \in \mathrm{Hom}(\mathscr{A}, \mathbb{C})$ arises in this way.*

Let $x \in V_P$ be fixed and let $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ and $y \in V_P$. Define

$$F_u^x(y) = \int_\Omega (ur)^{h(x,y;\omega)} d\nu_x(\omega),$$

where $(ur)^\lambda = u^\lambda r^\lambda$ for all $\lambda \in P$. The integral exists by Theorem 7.4.2(ii) and the fact that N_λ and $|\Pi_\lambda|$ are finite for each $\lambda \in P^+$.

In the following theorem we provide a second formula h'_u, $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, for the algebra homomorphisms $\mathscr{A} \to \mathbb{C}$. In Theorem 7.7.2 we will show that $h'_u = h_u$.

THEOREM 7.6.4. *Let $x, y \in V_P$ with $y \in V_\lambda(x)$. Then for all $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$*

(i) $F_u^x(x) = 1$,
(ii) $F_u^x(y) = F_u^{x'}(y')$ *whenever $x', y' \in V_P$ satisfy $y' \in V_\lambda(x')$, and*
(iii) $A_\lambda F_u^x = \varphi_\lambda(u) F_u^x$, *where for any $\nu \ggg \lambda$*

$$\varphi_\lambda(u) = \sum_{\mu \in \Pi_\lambda} r^{-\mu} a_{\lambda,\nu-\mu;\nu} u^\mu,$$

which is independent of $x \in V_P$.

Thus the map $h'_u : \mathscr{A} \to \mathbb{C}$ given by $h'_u(A) = (AF_u^x)(x)$ defines an algebra homomorphism (by Proposition 7.6.3).

PROOF. Since ν_x is a probability measure, (i) follows from Proposition 7.4.3.

We now prove (ii), which we note is stronger than the claim that F_u^x is x-radial. Let $\lambda \in P^+$. Since Ω is the union of the disjoint sets $\Omega_x(z)$ over $z \in V_\nu(x)$, we have

$$\begin{aligned} F_u^x(y) &= \sum_{z \in V_\nu(x)} \int_{\Omega_x(z)} (ur)^{h(x,y;\omega)} d\nu_x(\omega) \\ &= \sum_{\mu \in P^+} \sum_{z \in V_\nu(x) \cap V_\mu(y)} \int_{\Omega_x(z)} (ur)^{h(x,y;\omega)} d\nu_x(\omega). \end{aligned}$$

Now take $\nu \gg \lambda$, and so by Lemma 7.5.1 $h(x,y;\omega) = \nu - \mu$ for all $\omega \in \Omega_x(z)$ and $z \in V_\nu(x) \cap V_\mu(y)$. Since $\nu_x(\Omega_x(z)) = N_\nu^{-1}$ we have

$$F_u^x(y) = \sum_{\mu \in P^+} \frac{1}{N_\nu} |V_\nu(x) \cap V_\mu(y)| (ur)^{\nu-\mu},$$

and the result follows from (4.4.3).

We now prove (iii). Let $\nu \ggg \lambda$. By the cocycle relations (Proposition 7.4.3), Theorem 7.4.2(ii) and Lemma 7.6.1 we have

$$\begin{aligned}(A_\lambda F_u^x)(y) &= \frac{1}{N_\lambda} \sum_{z \in V_\lambda(y)} \int_\Omega (ur)^{h(x,z;\omega)} d\nu_x(\omega) \\ &= \int_\Omega \left(\frac{1}{N_\lambda} \sum_{z \in V_\lambda(y)} (ur)^{h(y,z;\omega)} \right) (ur)^{h(x,y;\omega)} d\nu_x(\omega) \\ &= \left(\sum_{\mu \in \Pi_\lambda} r^{-\mu} a_{\lambda,\nu-\mu;\nu} u^\mu \right) \int_\Omega (ur)^{h(x,y;\omega)} d\nu_x(\omega). \qquad \square\end{aligned}$$

COROLLARY 7.6.5. *Let $y \in V_\lambda(x)$. Then for any $\omega \in \Omega$,*

$$h'_u(A_\lambda) = F_u^x(y) = \frac{1}{N_\lambda} \sum_{z \in V_\lambda(x)} (ur)^{h(x,z;\omega)} = \varphi_\lambda(u).$$

PROOF. By Theorem 7.6.4(ii) and the definition of A_λ we have $(A_\lambda F_u^x)(x) = F_u^x(y)$, and by Theorem 7.6.4(i) we have $\varphi_\lambda(u) F_u^x(x) = \varphi_\lambda(u)$. The result now follows from Theorem 7.6.4(iii) and the proof thereof. $\square$

7.7. Equality of h_u and h'_u, and the Norms $\|A_\lambda\|$

In this section we show that $h_u = h'_u$ for all $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, where h'_u is as in Theorem 7.6.4 (see also Corollary 7.6.5). To conveniently state our results we will write $U = \mathbb{U}$ in the standard case. Thus $U = \mathbb{U}$ in the standard case and $U = \mathbb{U} \cup \mathbb{U}'$ in the exceptional case.

LEMMA 7.7.1. *Let $\lambda, \mu, \nu \in P^+$. Then*
(i) $\int_U P_\lambda(u) \overline{P_\mu(u)} d\pi_0(u) = \delta_{\lambda,\mu} N_\lambda^{-1}$, *and*
(ii) $a_{\lambda,\mu;\nu} = N_\nu \int_U P_\lambda(u) P_\mu(u) \overline{P_\nu(u)} d\pi_0(u)$.

PROOF. (i) Since each h_u, $u \in U$, is continuous with respect to the ℓ^2-operator norm (see Corollary 6.3.5 and Corollary 6.3.8) we have $P_\lambda(u) = \widehat{A}_\lambda(u)$ for all $\lambda \in P^+$ and all $u \in U$. Thus by (6.3.1) in the standard case, and (6.3.4) in the exceptional case,

$$\int_U P_\lambda(u) \overline{P_\mu(u)} d\pi_0(u) = \langle A_\lambda, A_\mu \rangle = N_\lambda^{-1} \delta_{\lambda,\mu}.$$

(ii) Using the previous part we have

$$\int_U P_\lambda(u)P_\mu(u)\overline{P_\nu(u)}d\pi_0(u) = \sum_{\eta\in P^+}\left(a_{\lambda,\mu;\eta}\int_U P_\eta(u)\overline{P_\nu(u)}d\pi_0(u)\right) = N_\nu^{-1}a_{\lambda,\mu;\nu},$$

completing the proof. □

THEOREM 7.7.2. *$h'_u = h_u$ for all $u\in\mathrm{Hom}(P,\mathbb{C}^\times)$.*

PROOF. From (5.2.13) we have

$$h_u(A_\lambda) = P_\lambda(u) = \sum_{\mu\in\Pi_\lambda} a_{\lambda,\mu}u^\mu \tag{7.7.1}$$

for some numbers $a_{\lambda,\mu}$. On the other hand, by Corollary 7.6.5 we have

$$h'_u(A_\lambda) = \sum_{\mu\in\Pi_\lambda} r^{-\mu}a_{\lambda,\nu-\mu;\nu}u^\mu \qquad \text{for any } \nu\ggg\lambda. \tag{7.7.2}$$

We will show that for all $\mu\in\Pi_\lambda$, $a_{\lambda,\mu} = r^{-\mu}a_{\lambda,\nu-\mu;\nu}$ provided that $\nu\ggg\lambda$. Comparing formulae (7.7.1) and (7.7.2) this evidently proves that $h_u = h'_u$ for all $u\in\mathrm{Hom}(P,\mathbb{C}^\times)$.

Let us first consider the standard case, so $U=\mathbb{U}$. Let $\mu\in\Pi_\lambda$ and $\nu\ggg\lambda$. By Corollary 7.5.6 we have $r^{-\mu} = \sqrt{N_{\nu-\mu}/N_\nu}$, and so by (4.3.3) we have $r^{-\mu}N_\nu = W_0(q^{-1})q_{t_\nu}^{1/2}q_{t_{\nu-\mu}}^{1/2}$. Thus by Lemma 7.7.1(ii),

$$\begin{aligned} r^{-\mu}a_{\lambda,\nu-\mu;\nu} &= r^{-\mu}N_\nu\int_{\mathbb{U}} P_\lambda(u)P_{\nu-\mu}(u)\overline{P_\nu(u)}d\pi_0(u) \\ &= W_0(q^{-1})q_{t_\nu}^{1/2}\int_{\mathbb{U}} P_\lambda(u)\overline{P_\nu(u)}\frac{u^{\nu-\mu}}{c(u^{-1})}du. \end{aligned} \tag{7.7.3}$$

Since $|c(wu^{-1})|^2 = |c(u^{-1})|^2$ for all $u\in\mathbb{U}$ we see that $c(wu^{-1})/c(u^{-1})\in L^1(\mathbb{U})$ for all $w\in W_0$, and so by (7.7.3) we have

$$r^{-\mu}a_{\lambda,\nu-\mu;\nu} = \sum_{w\in W_0}\int_{\mathbb{U}} u^{\nu-w\nu}\left(P_\lambda(u)u^{-\mu}\frac{c(wu^{-1})}{c(u^{-1})}\right)du. \tag{7.7.4}$$

We claim that the integral in (7.7.4) is 0 for all $w\neq 1$. To see this, notice that by Lemma 7.6.1, $r^{-\mu}a_{\lambda,\nu-\mu;\nu}$, $\mu\in\Pi_\lambda$, is independent of $\nu\ggg\lambda$, and so we may choose $\nu = N(\lambda_1+\cdots+\lambda_n)$ for suitably large $N\in\mathbb{N}$. Suppose $w\neq 1$. Since $w(\lambda_1+\cdots+\lambda_n)\neq\lambda_1+\cdots+\lambda_n$ we see that

$$|\langle\lambda_1+\cdots+\lambda_n - w(\lambda_1+\cdots+\lambda_n),\alpha_{i_0}\rangle|\geq 1$$

for at least one $i_0\in I_0$, and so $|\langle\nu-w\nu,\alpha_{i_0}\rangle|\geq N$. Thus by the Riemann-Lebesgue Lemma $\lim_{N\to\infty}\int_{\mathbb{U}} u^{\nu-w\nu}f(u)du = 0$ for all $f\in L^1(\mathbb{U})$. Thus using (7.7.1) we have

$$r^{-\mu}a_{\lambda,\nu-\mu;\nu} = \int_{\mathbb{U}} P_\lambda(u)u^{-\mu}du = a_{\lambda,\mu}.$$

Let us now prove the result in the exceptional case, where $U = \mathbb{U}\cup\mathbb{U}'$. Following the above we have

$$r^{-\mu}a_{\lambda,\nu-\mu;\nu} = a_{\lambda,\mu} + r^{-\mu}N_\nu\int_{\mathbb{U}'}P_\lambda(u)P_{\nu-\mu}(u)\overline{P_\nu(u)}d\pi_0(u) \tag{7.7.5}$$

whenever $\nu \ggg \lambda$. We will show that the integral in (7.7.5) is zero.

The group W_0 acts on $\{e_i\}_{i=1}^n$ as the group of all signed permutations. Since $t_i = u^{e_i}$, for each $w\in W_0$ we have $w(t_1,\ldots,t_n) = (t_{\sigma_w(1)}^{\epsilon_w(1)},\ldots,t_{\sigma_w(n)}^{\epsilon_w(n)})$ where σ_w is a permutation of $\{1,\ldots,n\}$ and $\epsilon_w:\{1,\ldots,n\}\to\{1,-1\}$. By directly examining the formula for $c(u)$ we see that if $\epsilon_w(\sigma_w(1)) = -1$ then $c(wu)|_{t_1=-b} = 0$. Write $W_0^+ = \{w\in W_0 \mid \epsilon_w(\sigma_w(1)) = 1\}$. Note that for all $\lambda\in P^+$, $\lambda^* = \lambda$. Thus $\overline{P_\lambda(u)} = P_\lambda(u)$ for all $u\in U$. Following the calculation in the standard case, and using the above observations, we see that

$$\begin{aligned} r^{-\mu}a_{\lambda,\nu-\mu;\nu} - a_{\lambda,\mu} &= r^{-\mu}N_\nu\int_{\mathbb{U}'}P_\lambda(u)P_{\nu-\mu}(u)P_\nu(u)d\pi_0(u)\\ &= \frac{1}{|W_0^+|}\sum_{w,w'\in W_0^+}\int_{\mathbb{U}'}u^{w\nu+w'\nu}\left(P_\lambda(u)u^{-w\mu}\frac{c(wu)c(w'u)}{\phi_1(u)}\right)dt'. \end{aligned}$$

Since $w_0 = -1$ it is clear that if $w\in W_0^+$ then $w_0w\notin W_0^+$. Take any $w,w'\in W_0^+$. As before, let $\nu = N(\lambda_1+\cdots+\lambda_n)$ for sufficiently large N. Since $w'\neq w_0w$ we have that $w\nu+w'\nu\neq 0$. The same argument as in the standard case now shows that $|\langle w\nu+w'\nu, e_{i_0}\rangle| \geq N$ for at least one $i_0\in I_0$. Furthermore, since $w,w'\in W_0^+$ we have $\langle w\nu+w'\nu, e_1\rangle \geq 0$. The result now follows by taking $N\to\infty$, noting that $b<1$, and using the Riemann-Lebesgue Lemma. $\square$

By Theorem 7.7.2, for $\lambda\in P^+$ we have

$$h_u(A_\lambda) = P_\lambda(u) = \int_\Omega (ur)^{h(x,y;\omega)}d\nu_x(\omega) \tag{7.7.6}$$

for any pair $x,y\in V_P$ with $y\in V_\lambda(x)$.

As an application of (7.7.6) we compute the norms $\|A_\lambda\|$, $\lambda\in P^+$.

THEOREM 7.7.3. *Let $\lambda\in P^+$. Then $\|A_\lambda\| = P_\lambda(1)$.*

PROOF. Since $A\mapsto\widehat{A}$ is an isometry, by Corollaries 6.3.5 and 6.3.8 we have

$$\|A_\lambda\| = \|\widehat{A}\|_\infty = \sup\{|h(A_\lambda)| : h\in M_2\} = \sup\{|h_u(A_\lambda)| : u\in U\},$$

where, as usual, $U = \mathbb{U}$ in the standard case and $U = \mathbb{U}\cup\mathbb{U}'$ in the exceptional case. In the standard case this implies that $\|A_\lambda\| = P_\lambda(1)$, for by (7.7.6) we have $P_\lambda(1) > 0$ and $|P_\lambda(u)| \leq P_\lambda(1)$ for all $u\in\mathbb{U}$ and $\lambda\in P^+$. In the exceptional case we only have $\|A_\lambda\| \geq P_\lambda(1)$, and so it remains to show that $\|A_\lambda\| \leq P_\lambda(1)$ in this case.

To see this, fix $o \in V_P$ and $\lambda \in P^+$. By Theorem 7.6.4(iii), Corollary 7.6.5 and (7.7.6) we have $(A_\lambda F_1^o)(x) = P_\lambda(1)F_1^o(x)$ for all $x \in V_P$. Similarly, since $\lambda^* = \lambda$ here,

$$(A_\lambda^* F_1^o)(x) = (A_{\lambda^*} F_1^o)(x) = (A_\lambda F_1^o)(x) = P_\lambda(1)F_1^o(x)$$

for all $x \in V_P$. Since $F_1^o > 0$ by (7.7.6), the Schur test (see [**32**, p.103] for example) implies that $\|A_\lambda\| \leq P_\lambda(1)$ (see also [**11**, Lemma 4.1]). □

Remark 7.7.4. See Lemma 8.2.7 for a description of $P_\lambda(1)$.

Remark 7.7.5. Observe that our proof of the integral formula for the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ did not depend on the isomorphism $\mathscr{A} \cong \mathbb{C}[P]^{W_0}$ (note that we give an alternate proof Lemma 7.1.2 in Appendix B.4). Thus Theorem 7.7.2 provides an alternative proof of the Macdonald formula for the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$. Such a proof is rather unsatisfactory though, for it requires the Macdonald formula to be 'guessed', and then verified (by Theorem 7.7.2).

CHAPTER 8

A Local Limit Theorem for Random Walks on Affine Buildings

8.1. Introduction

A *random walk* consists of a finite or countable *state space* X, and a *transition matrix* $(p(x,y))_{x,y\in X}$, where $p(x,y)\geq 0$ for all $x,y\in X$, and $\sum_{y\in X}p(x,y)=1$ for all $x\in X$. The functions $p(x,y)$ are called the *transition probabilities* of the random walk.

The natural interpretation of a random walk is that of a *random walker* taking discrete steps in X. Then for $x,y\in X$, $p(x,y)$ represents the probability that the walker, having started at site x, moves to site y in one step.

For $k\geq 0$, the *k-step transition probability* $p^{(k)}(x,y)$ is the probability that the walker, having started at site x, is at site y after k steps. A *local limit theorem* is any theorem giving an asymptotic formula for $p^{(k)}(x,y)$ as $k\to\infty$ (for fixed $x,y\in X$).

A random walk is called *irreducible* if for each pair $x,y\in X$ there exists $k=k(x,y)\in\mathbb{N}$ such that $p^{(k)}(x,y)>0$. The *period* of an irreducible random walk is

$$\mathfrak{p}=\gcd\{k\geq 1\mid p^{(k)}(x,x)>0\}.$$

This is easily seen to be independent of $x\in X$, by irreducibility (see [**45**]). An irreducible random walk is called *aperiodic* if $\mathfrak{p}=1$.

When $|X|=\infty$ it is often more desirable to regard a transition matrix $A=(p(x,y))_{x,y\in X}$ as an operator (also called A), acting on appropriate spaces of functions $f:X\to\mathbb{C}$, by

$$(Af)(x)=\sum_{y\in X}p(x,y)f(y)\quad\text{for all }x\in X.\tag{8.1.1}$$

We call the operator in (8.1.1) the *transition operator* for the random walk. Note that

$$p^{(k)}(x,y)=(A^k\delta_y)(x)\qquad\text{for all }x,y\in X\text{ and }k\in\mathbb{N}.\tag{8.1.2}$$

Let $\mathscr{X}$ be a *thick* (locally finite, regular) affine building (of irreducible type), and as usual write V_P for the set of all good vertices of $\mathscr{X}$. A random walk on V_P is called *radial* if it has a transition operator of the form

$$A=\sum_{\lambda\in P^+}a_\lambda A_\lambda\quad\text{where }a_\lambda\geq 0\text{ for all }\lambda\in P^+,\text{ and }\sum_{\lambda\in P^+}a_\lambda=1.\tag{8.1.3}$$

To avoid triviality we always assume $a_\lambda>0$ for at least one $\lambda\neq 0$. In this chapter we will prove a local limit theorem for such random walks, generalising the work of [**36**] (where homogeneous trees are studied) and [**12**] (where $\tilde{A}_n$ buildings are studied).

When $\mathscr{X}$ is a homogeneous tree, it is easily seen that a random walk on V_P is radial if and only if $p(x,y)$ depends only on the graph distance $d(x,y)$ between x and y; hence the word *radial*. In the general context, a random walk on V_P is radial if and only if $p(x,y)=p(x',y')$ whenever $y\in V_\lambda(x)$ and $y'\in V_\lambda(x')$ for some $\lambda\in P^+$.

The basic approach is as follows. Since $\|A\|\le 1$ (see the beginning of Section 6.2), we may regard A in (8.1.3) as in $\mathscr{A}_2$, and so $h_u(A)$, $u\in U$, is defined. Writing $\widehat{A}(u)=h_u(A)$ (as in Section 6.2) we have $\widehat{A}_\lambda(u)=P_\lambda(u)$, and so

$$\widehat{A}(u)=\sum_{\lambda\in P^+}a_\lambda P_\lambda(u). \tag{8.1.4}$$

By Corollaries 6.3.5 and 6.3.8, if $y\in V_\lambda(x)$ then $(A\delta_y)(x)=\int_U \widehat{A}(u)\overline{P_\lambda(u)}d\pi_0(u)$, and so by (8.1.2),

$$p^{(k)}(x,y)=\int_U \big(\widehat{A}(u)\big)^k\,\overline{P_\lambda(u)}d\pi_0(u). \tag{8.1.5}$$

We will prove the local limit theorem by determining the asymptotic behaviour of the integral in (8.1.5) (as $k\to\infty$).

8.2. Preliminary Results

LEMMA 8.2.1. *Let $\lambda\in P^+$, $\lambda\neq 0$, $x\in V_P$, and $y\in V_\lambda(x)$. Then*

(i) *there exists $z\in V_\lambda(x)\cap V_{\tilde\alpha^\vee}(y)$, and*
(ii) *with z as in* (i)*, there exists $\omega\in\Omega$ such that $h(y,z;\omega)=\tilde\alpha^\vee$.*

PROOF. Note first that if c and d are distinct i-adjacent chambers, $i\in I_P$, with type i vertices u and v respectively, then $v\in V_{\tilde\alpha^\vee}(u)$ (and $u\in V_{\tilde\alpha^\vee}(v)$). To see this, let $\mathcal{A}$ be any apartment containing c and d, and let $\psi:\mathcal{A}\to\Sigma$ be a type-rotating isomorphism such that $\psi(u)=0$ and $\psi(c)=C_0$. Since $\psi(d)$ is 0-adjacent to $\psi(c)$ we have $\psi(d)=s_{\tilde\alpha;1}(C_0)$, and so $\psi(v)=s_{\tilde\alpha;1}(0)=\tilde\alpha^\vee$. Thus $v\in V_{\tilde\alpha^\vee}(u)$.

Part (i) now follows exactly as in [**12**, Lemma 5.1], using thickness.

Part (ii) is a consequence of the following more general fact. Let $u,v\in V_P$ with $v\in V_\lambda(x)$. Then there exists $\omega\in\Omega$ such that $h(u,v;\omega)=\lambda$. To see this, let $\mathcal{A}$ be any apartment containing u and v, and let $\psi:\mathcal{A}\to\Sigma$ be a type-rotating isomorphism such that $\psi(u)=0$ and $\psi(v)=\lambda$. Let ω be the class of $\psi^{-1}(\mathcal{S}_0)$. Since $\psi^{-1}(\mathcal{S}_0)=\mathcal{S}^u(\omega)$ and $\psi^{-1}(\lambda+\mathcal{S}_0)=\mathcal{S}^v(\omega)$, we have $\psi^{-1}(\mu)=v^u_\mu(\omega)=v^v_{\mu-\lambda}(\omega)$ for sufficiently large $\mu\in P^+$, and so $h(u,v;\omega)=\lambda$. □

Recall that we write $\mathbb{U}=\{u\in\mathrm{Hom}(P,\mathbb{C}^\times):|u^\lambda|=1 \text{ for all } \lambda\in P\}$. Let

$$\mathbb{U}_Q=\{u\in\mathrm{Hom}(P,\mathbb{C}^\times)\mid u^\gamma=1\text{ for all }\gamma\in Q\}.$$

Then $\mathbb{U}_Q$ is isomorphic to the dual of the finite abelian group P/Q, and so $\mathbb{U}_Q\cong P/Q$. Thus $\mathbb{U}_Q$ is finite, and $\mathbb{U}_Q\subset\mathbb{U}$.

PROPOSITION 8.2.2. *Let $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, and suppose that $u^{w\tilde{\alpha}^\vee} = 1$ for all $w \in W_0$. Then $u \in \mathbb{U}_Q$.*

PROOF. Observe that if $\alpha, \beta \in R$ have the same length, then by [**5**, VI, §1, No.4, Proposition 11] there exists $w \in W_0$ such that $\beta = w\alpha$. Thus in the cases where there is only one root length (A_n, D_n, E_n), $W_0\tilde{\alpha}^\vee = R^\vee$, and the result follows. Consider the remaining cases (see Appendix D).

Let R be a root system of type B_n. Then $\tilde{\alpha}^\vee = e_1 + e_2$, and so $W_0\tilde{\alpha}^\vee$ contains the vectors $e_i \pm e_j$ for $1 \le i < j \le n$. Thus if $u^{w\tilde{\alpha}^\vee} = 1$ for all $w \in W_0$, it follows that $u^{e_i+e_j} = 1$ and $u^{e_i-e_j} = 1$ for all $1 \le i < j \le n$, and so $u^{2e_k} = 1$ for all $1 \le k \le n$. Thus $u^{\alpha^\vee} = 1$ for all $\alpha \in R$, and the result follows.

The cases C_n, F_4 and G_2 are similar. Finally, if R is of type BC_n then $\tilde{\alpha}^\vee = e_1$, and so if $u^{w\tilde{\alpha}^\vee} = 1$ for all $w \in W_0$ then $u^{e_i} = 1$ for all $1 \le i \le n$, completing the proof. □

As usual, if $u, v \in \mathrm{Hom}(P, \mathbb{C}^\times)$, define $uv \in \mathrm{Hom}(P, \mathbb{C}^\times)$ by $(uv)^\lambda = u^\lambda v^\lambda$ for all $\lambda \in P$.

LEMMA 8.2.3. *Let $u \in \mathbb{U}$ and $\lambda \in P^+$. Then $|P_\lambda(u)| \le P_\lambda(1)$, and equality holds for $\lambda \neq 0$ if and only if $u \in \mathbb{U}_Q$. Moreover, if $u_0 \in \mathbb{U}_Q$ then $P_\lambda(u_0 u) = u_0^\lambda P_\lambda(u)$ for all $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$.*

PROOF. (cf. [**12**, Lemma 5.3]) Let $x, y \in V_P$ be any vertices with $y \in V_\lambda(x)$. The inequality is clear from the integral formula (7.7.6). Suppose equality holds for some $\lambda \neq 0$. Write $f(\omega)$ for the integrand in (7.7.6). Then f is a continuous function on Ω and $f(\omega) \neq 0$ for all $\omega \in \Omega$. So $|\int_\Omega f(\omega)d\nu_x(\omega)| = \int_\Omega |f(\omega)|d\nu_x(\omega)$ implies that $f(\omega)/|f(\omega)|$ is constant, since $\nu_x(O) > 0$ for all non-empty open sets $O \subset \Omega$. Thus $u^{h(x,y;\omega)}$ takes the constant value $P_\lambda(u)/P_\lambda(1)$ for all $\omega \in \Omega$. Let z be as in Lemma 8.2.1(i). Since the value of the integral in (7.7.6) is unchanged if y is replaced by z it follows that $u^{h(x,y;\omega)} = u^{h(x,z;\omega)}$ for all $\omega \in \Omega$. Choosing $\omega \in \Omega$ as in Lemma 8.2.1(ii) and using the cocycle relations we have $u^{\tilde{\alpha}^\vee} = u^{h(y,z;\omega)} = 1$. Furthermore, since the value of the integral in (7.7.6) is unchanged if u is replaced by wu for any $w \in W_0$, then $u^{w\tilde{\alpha}^\vee} = 1$ for all $w \in W_0$. It follows from Proposition 8.2.2 that $u \in \mathbb{U}_Q$.

Conversely, if $u_0 \in \mathbb{U}_Q$ and $y \in V_\lambda(x)$, then $u_0^{h(x,y;\omega)} = u_0^\lambda$ for all $\omega \in \Omega$, because $h(x, y; \omega) \in \Pi_\lambda$ (Theorem 7.4.2(ii)), so that $\lambda - h(x, y; \omega) \in Q$. Thus it follows from (7.7.6) that $P_\lambda(u_0 u) = u_0^\lambda P_\lambda(u)$ for all $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$. In particular, $|P_\lambda(u_0)| = P_\lambda(1)$. □

For each $\omega \in \Omega$, $x, y \in V_P$ and $1 \le j \le n$, define $h_j(x, y; \omega) = \langle h(x, y; \omega), \alpha_j \rangle$. For $\lambda \in P^+$ write $\|\lambda\| = \sum_{j=1}^n \langle \lambda, \alpha_j \rangle$.

In the following series of estimates, we will write C for a positive constant, whose value may vary from line to line.

LEMMA 8.2.4. *Let $x \in V_P$ and $\lambda \in P^+$. Then $|h(x, y; \omega)| \le C\|\lambda\|$ and $|h_j(x, y; \omega)| \le C\|\lambda\|$ for all $\omega \in \Omega$, all $y \in V_\lambda(x)$, and all $j = 1, \ldots, n$.*

PROOF. Recall that $h(x,y;\omega)\in\Pi_\lambda$ for all $\omega\in\Omega$ and $y\in V_\lambda(x)$, and by [**27**, (2.6.2)] $\Pi_\lambda\subset\mathrm{conv}(W_0\lambda)$ (the usual convex hull in E here). Since $|w\lambda|=|\lambda|$ for all $w\in W_0$, this implies that $|h(x,y;\omega)|\leq|\lambda|$ for all $\omega\in\Omega$ and for all $y\in V_\lambda(x)$. Thus $|h(x,y;\omega)|\leq C\|\lambda\|$ (with $C=\max\{|\lambda_i|\}_{i=1}^n$).

We have $|\langle h(x,y;\omega),\alpha_j\rangle|\leq|h(x,y;\omega)||\alpha_j|$, proving the final claim. □

NOTATION. Let $\theta_1,\ldots,\theta_n\in\mathbb{R}$ and write $\theta=\theta_1\alpha_1+\cdots+\theta_n\alpha_n$ (so $\theta\in E$). Write $e^{i\theta}$ for the element of $\mathrm{Hom}(P,\mathbb{C}^\times)$ with $(e^{i\theta})^\lambda=e^{i\langle\lambda,\theta\rangle}$ for all $\lambda\in P^+$. With this notation (7.7.6) gives

$$P_\lambda(e^{i\theta})=\int_\Omega r^{h(x,y;\omega)}e^{i\langle h(x,y;\omega),\theta\rangle}d\nu_x(\omega)\quad\text{for all } y\in V_\lambda(x),\tag{8.2.1}$$

and since $P_\lambda(w^{-1}e^{i\theta})=P_\lambda(e^{i\theta})$ for all $w\in W_0$, it follows that

$$P_\lambda(e^{i\theta})=\int_\Omega r^{h(x,y;\omega)}e^{i\langle h(x,y;\omega),w\theta\rangle}d\nu_x(\omega)\quad\text{for all } w\in W_0,\ y\in V_\lambda(x).\tag{8.2.2}$$

COROLLARY 8.2.5. *$P_\lambda(e^{i\theta})=P_\lambda(1)(1+E_\lambda(\theta))$, where $|E_\lambda(\theta)|\leq C\|\lambda\||\theta|$.*

PROOF. We have

$$|P_\lambda(e^{i\theta})-P_\lambda(1)|\leq\int_\Omega r^{h(x,y;\omega)}|e^{i\langle h(x,y;\omega),\theta\rangle}-1|d\nu_x(\omega),$$

and the result follows from Lemma 8.2.4 since $|e^{iz}-1|\leq|z|$ for all $z\in\mathbb{R}$. □

Let $\lambda\in P^+$ and $y\in V_\lambda(x)$. For each $1\leq j,k\leq n$ define

$$b_{j,k}^\lambda=\frac{1}{2}\int_\Omega h_j(x,y;\omega)h_k(x,y;\omega)r^{h(x,y;\omega)}d\nu_x(\omega).\tag{8.2.3}$$

This is independent of the particular pair $x,y\in V_P$ with $y\in V_\lambda(x)$, for by (8.2.1)

$$\frac{\partial^2}{\partial\theta_j\partial\theta_k}P_\lambda(e^{i\theta})\bigg|_{\theta=0}=-\int_\Omega h_j(x,y;\omega)h_k(x,y;\omega)r^{h(x,y;\omega)}d\nu_x(\omega).$$

(Indeed any expression $\int_\Omega p(h_1(x,y;\omega),\ldots,h_n(x,y;\omega))r^{h(x,y;\omega)}d\nu_x(\omega)$, where p is a polynomial, is independent of the particular pair $x,y\in V_P$ with $y\in V_\lambda(x)$).

LEMMA 8.2.6. *Let $\lambda\in P^+$, and $\theta_1,\ldots,\theta_n\in\mathbb{R}$, and as usual write $\theta=\theta_1\alpha_1+\cdots+\theta_n\alpha_n$. Then*

$$P_\lambda(e^{i\theta})=P_\lambda(1)-\sum_{j,k=1}^n b_{j,k}^\lambda\theta_j\theta_k+R_\lambda(\theta)\tag{8.2.4}$$

where $|R_\lambda(\theta)|\leq C\|\lambda\|^3|\theta|^3P_\lambda(1)$. Furthermore, $\sum_{j,k=1}^n b_{j,k}^\lambda\theta_j\theta_k\geq0$, and when $\lambda\neq0$, equality holds if and only if $\theta=0$.

Proof. For $\varphi \in \mathbb{R}$ we have $e^{i\varphi} = 1 + i\varphi - \frac{1}{2}\varphi^2 + R(\varphi)$ where $|R(\varphi)| \leq \frac{1}{6}|\varphi|^3$. Applying this to $\varphi = \langle h(x,y;\omega), \theta\rangle$ and using (8.2.1) we have

$$\begin{aligned} P_\lambda(e^{i\theta}) &= P_\lambda(1) + i\int_\Omega \langle h(x,y;\omega),\theta\rangle r^{h(x,y;\omega)} d\nu_x(\omega) \\ &\quad - \frac{1}{2}\int_\Omega \langle h(x,y;\omega),\theta\rangle^2 r^{h(x,y;\omega)} d\nu_x(\omega) + R_\lambda(\theta), \end{aligned}$$

where $|R_\lambda(\theta)| \leq \frac{1}{6}|\langle h(x,y;\omega),\theta\rangle|^3 P_\lambda(1) \leq \frac{1}{6}|h(x,y;\omega)|^3|\theta|^3 P_\lambda(1)$. The bound for $|R_\lambda(\theta)|$ follows from Lemma 8.2.4.

We claim that for all $j = 1,\ldots,n$ and for all $y \in V_P$,

$$\int_\Omega h_j(x,y;\omega) r^{h(x,y;\omega)} d\nu_x(\omega) = 0.$$

To see this, let $j \in \{1,\ldots,n\}$ and set $\theta = \theta_j\alpha_j$ (that is, $\theta_k = 0$ for all $k \neq j$). By differentiating (8.2.2) with respect to θ_j, and then evaluating at $\theta_j = 0$, firstly with $w = 1$ and secondly with $w = s_j$, we see that

$$\int_\Omega h_j(x,y;\omega) r^{h(x,y;\omega)} d\nu_x(\omega) = -\int_\Omega h_j(x,y;\omega) r^{h(x,y;\omega)} d\nu_x(\omega),$$

proving the claim.

It is now clear that (8.2.4) holds, and that $\sum_{j,k=1}^n b_{j,k}^\lambda \theta_j\theta_k \geq 0$. If equality holds, then

$$\int_\Omega \langle h(x,y;\omega),\theta\rangle^2 r^{h(x,y;\omega)} d\nu_x(\omega) = 0.$$

Thus $\langle h(x,y;\omega),\theta\rangle = 0$ for almost all $\omega \in \Omega$, and thus for all $\omega \in \Omega$. Thus, since $\langle h(x,y;\omega), t\theta\rangle = 0$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$, we have $P_\lambda(e^{i(t\theta)}) = P_\lambda(1)$ for all $t \in \mathbb{R}$ by (8.2.1), and so $e^{i(t\theta)} \in \mathbb{U}_Q$ for all $t \in \mathbb{R}$ by Lemma 8.2.3. Hence $e^{i\langle\gamma,t\theta\rangle} = 1$ for all $\gamma \in Q$ and all $t \in \mathbb{R}$, and so $\langle\gamma,\theta\rangle = 0$ for all $\gamma \in Q$, implying that $\theta = 0$, since Q spans E. ☐

Lemma 8.2.7. *There exists a polynomial $p(x_1,\ldots,x_n)$ of degree at most M such that*

$$P_\lambda(1) = q_{t_\lambda}^{-1/2} p(\langle\lambda,\alpha_1\rangle,\ldots,\langle\lambda,\alpha_n\rangle) \tag{8.2.5}$$

for all $\lambda \in P^+$, where $M > 0$ is some integer depending only on the underlying root system. Furthermore, $q_{t_\lambda} \geq q^{\|\lambda\|}$ where $q = \min\{q_i\}_{i=0}^n > 1$, and so

$$P_\lambda(1) \leq C(\|\lambda\| + 1)^M q^{-\frac{1}{2}\|\lambda\|}. \tag{8.2.6}$$

Proof. Assuming that $u^{-\alpha^\vee} \neq 1$ for all $\alpha \in R_2^+$, by (5.2.11) we have

$$c(u) = \prod_{\alpha\in R_2^+} \frac{(1 - \tau_{2\alpha}^{-1}\tau_\alpha^{-1/2}u^{-\alpha^\vee/2})(1 + \tau_\alpha^{-1/2}u^{-\alpha^\vee/2})}{1 - u^{-\alpha^\vee}}. \tag{8.2.7}$$

Write $\sigma=\lambda_1+\cdots+\lambda_n$. It follows from [**5**, VI, §3, No.3, Proposition 2] that

$$\prod_{\alpha\in R_2^+}(1-u^{-w\alpha^\vee})=(-1)^{\ell(w)}u^{\sigma-w\sigma}\prod_{\alpha\in R_2^+}(1-u^{-\alpha^\vee})$$

for all $w\in W_0$, and so by (6.1.1) and (8.2.7) we have

$$P_\lambda(u)=q_{t_\lambda}^{-1/2}\frac{F(\lambda)}{\prod_{\alpha\in R_2^+}(1-u^{-\alpha^\vee})} \tag{8.2.8}$$

where $F(\lambda)$ equals

$$\frac{1}{W_0(q^{-1})}\sum_{w\in W_0}\bigg\{(-1)^{\ell(w)}u^{w\lambda+w\sigma-\sigma}\prod_{\alpha\in R_2^+}(1-\tau_{2\alpha}^{-1}\tau_\alpha^{-1/2}u^{-w\alpha^\vee/2})(1+\tau_\alpha^{-1/2}u^{-w\alpha^\vee/2})\bigg\}.$$

We know that $P_\lambda(u)$ is a Laurent polynomial in $u_1,\ldots,u_n$, and so (8.2.5) follows from (8.2.8) by repeated applications of L'Hôpital's rule.

To prove (8.2.6), recall from (7.5.1) that $q_{t_\lambda}=\prod_{i=1}^n q_{t_{\lambda_i}}^{\langle\lambda,\alpha_i\rangle}$. Now $q_{t_{\lambda_i}}\geq q$ since $\ell(t_\mu)>0$ for all $\mu\neq0$. Thus $q_{t_\lambda}\geq q^{\|\lambda\|}$, completing the proof. □

Let A be as in (8.1.3) and $\widehat{A}(u)=h_u(A)$ be as in (8.1.4).

It follows from Lemma 8.2.4 that $|b_{j,k}^\lambda|\leq C\|\lambda\|^2P_\lambda(1)$. Hence the inequality (8.2.6) implies that $\sum_{\lambda\in P^+}a_\lambda b_{j,k}^\lambda$ is absolutely convergent for each $1\leq j,k\leq n$. We define

$$b_{j,k}=\frac{1}{\widehat{A}(1)}\sum_{\lambda\in P^+}a_\lambda b_{j,k}^\lambda. \tag{8.2.9}$$

Corollary 8.2.8. *Let A be as in (8.1.3), and let $\theta_1,\ldots,\theta_n\in\mathbb{R}$. Then*

$$\widehat{A}(e^{i\theta})=\widehat{A}(1)\bigg(1-\sum_{j,k=1}^n b_{j,k}\theta_j\theta_k+R(\theta)\bigg),$$

where $\sum_{j,k=1}^n b_{j,k}\theta_j\theta_k>0$ unless $\theta=0$, and where $|R(\theta)|\leq C|\theta|^3$.

Proof. This follows from Lemma 8.2.6, using (8.2.6) to bound $R(\theta)$. □

Lemma 8.2.9. *Let $\theta_1,\ldots,\theta_n\in\mathbb{R}$. Then*

$$\frac{1}{|c(e^{i\theta})|^2}=\prod_{\alpha\in R_2^+}\frac{\langle\alpha^\vee,\theta\rangle^2}{\big(1-\tau_{2\alpha}^{-1}\tau_\alpha^{-1/2}\big)^2\big(1+\tau_\alpha^{-1/2}\big)^2}(1+E_\alpha(\theta))$$

where $|E_\alpha(\theta)|\leq C\langle\alpha^\vee,\theta\rangle^2$ for each $\alpha\in R_2^+$.

Proof. Observe that for $x\in\mathbb{R}$ and $p>1$

$$\left|\frac{1-e^{-ix}}{1-p^{-1}e^{-ix}}\right|^2=\frac{x^2}{(1-p^{-1})^2}(1+E_1(x)), \tag{8.2.10}$$

where $|E_1(x)| \leq Cx^2$, and for $p > 0$

$$\left|\frac{1+e^{-ix}}{1+p^{-1}e^{-ix}}\right|^2 = \frac{4}{(1+p^{-1})^2}(1+E_2(x)) \tag{8.2.11}$$

where $|E_2(x)| \leq Cx^2$ (indeed, we may take $C = 1/4$).

The result follows by using (8.2.10), (8.2.11) and the formula (8.2.7) for $c(e^{i\theta})$. $\square$

8.3. The Local Limit Theorem

Let A be as in (8.1.3), and write $\mathbb{U}_A = \{u \in \mathbb{U} : |\widehat{A}(u)| = \widehat{A}(1)\}$.

Lemma 8.3.1. *$\mathbb{U}_A = \{u \in \mathbb{U}_Q \mid u^\mu = u^\nu$ for all $\mu, \nu \in P^+$ with $a_\mu, a_\nu > 0\}$. If $u_0 \in \mathbb{U}_A$ then $\widehat{A}(u_0 u) = u_0^\mu \widehat{A}(u)$ for all $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, and all $\mu \in P^+$ such that $a_\mu > 0$.*

Proof. For $u \in \mathbb{U}$ we have

$$|\widehat{A}(u)| = \left|\sum_{\lambda\in P^+} a_\lambda P_\lambda(u)\right| \leq \sum_{\lambda\in P^+} a_\lambda |P_\lambda(u)| \leq \sum_{\lambda\in P^+} a_\lambda P_\lambda(1) = \widehat{A}(1). \tag{8.3.1}$$

If $u = u_0 \in \mathbb{U}_A$, then since equality must hold in the second inequality in (8.3.1) we have $|P_\lambda(u_0)| = P_\lambda(1)$ whenever $a_\lambda > 0$. Since we assume that $a_\lambda > 0$ for at least one nonzero $\lambda \in P^+$ we have $u_0 \in \mathbb{U}_Q$ by Lemma 8.2.3. Thus by Lemma 8.2.3 we have $P_\lambda(u_0) = u_0^\lambda P_\lambda(1)$ for all $\lambda \in P^+$, and so since equality must hold in the first inequality in (8.3.1) we have $u_0^\mu = u_0^\nu$ whenever $a_\mu, a_\nu > 0$, proving that

$$\mathbb{U}_A \subseteq \{u \in \mathbb{U}_Q \mid u^\mu = u^\nu \text{ for all } \mu, \nu \in P^+ \text{ with } a_\mu, a_\nu > 0\}.$$

Conversely, if $u_0 \in \mathbb{U}_Q$ and $u_0^\mu = u_0^\nu$ for all $\mu, \nu \in P^+$ with $a_\mu, a_\nu > 0$, then by Lemma 8.2.3 we see that $\widehat{A}(u_0 u) = u_0^\mu \widehat{A}(u)$ for all $u \in \mathbb{U}$ and any $\mu \in P^+$ such that $a_\mu > 0$, and so taking $u = 1$ we have $|\widehat{A}(u_0)| = \widehat{A}(1)$, so $u_0 \in \mathbb{U}_A$. $\square$

For $k \in \mathbb{N}$ and $\lambda \in P^+$ let

$$I_{k,\lambda} = \int_{\mathbb{U}} \left(\widehat{A}(u)\right)^k \overline{P_\lambda(u)} d\pi_0(u).$$

If $y \in V_\lambda(x)$, then by (8.1.5)

$$p^{(k)}(x,y) = \begin{cases} I_{k,\lambda} & \text{in the standard case, and} \\ I_{k,\lambda} + I'_{k,\lambda} & \text{in the exceptional case,} \end{cases} \tag{8.3.2}$$

where

$$I'_{k,\lambda} = \int_{\mathbb{U}'} \left(\widehat{A}(u)\right)^k \overline{P_\lambda(u)} d\pi_0(u). \tag{8.3.3}$$

Thus to give an asymptotic formula for $p^{(k)}(x,y)$ we need to give estimates for $I_{k,\lambda}$ and $I'_{k,\lambda}$.

Given $\epsilon > 0$ and $u_0 \in \mathbb{U}$, let $N_\epsilon(u_0) = \{u \in \mathbb{U} : |u^{\lambda_i} - u_0^{\lambda_i}| < \epsilon \text{ for all } i \in I_0\}$. Since $|\mathbb{U}_A| < \infty$ we may choose $\epsilon > 0$ sufficiently small so that

$$N_\epsilon(u_0) \cap N_\epsilon(u_0') = \emptyset \qquad \text{whenever } u_0, u_0' \in \mathbb{U}_A \text{ are distinct.} \tag{8.3.4}$$

Write $N_\epsilon = N_\epsilon(1)$ and $N_\epsilon(\mathbb{U}_A) = \bigcup_{u_0 \in \mathbb{U}_A} N_\epsilon(u_0)$.

Define $\rho_1 = \rho_1(\epsilon) = \max\{|\widehat{A}(u)|/\widehat{A}(1) : u \in \mathbb{U} \backslash N_\epsilon(\mathbb{U}_A)\}$ and let

$$I_{k,\lambda}^\epsilon = \int_{N_\epsilon} \big(\widehat{A}(u)\big)^k \, \overline{P_\lambda(u)} d\pi_0(u).$$

LEMMA 8.3.2. *Fix $\mu \in P^+$ such that $a_\mu > 0$, and let $\epsilon > 0$ satisfy (8.3.4). If $u_0^{k\mu} = u_0^\lambda$ for all $u_0 \in \mathbb{U}_A$, then*

$$I_{k,\lambda} = |\mathbb{U}_A| I_{k,\lambda}^\epsilon + \mathcal{O}\big(\rho_1^k \widehat{A}(1)^k\big).$$

Otherwise, $I_{k,\lambda} = 0$.

PROOF. It is clear from the formula for $c(u)$ (see (5.2.10)) that $c(u_0 u) = c(u)$ for all $u_0 \in \mathbb{U}_Q$ and $u \in \mathbb{U}$. Thus by Lemmas 8.2.3 and 8.3.1, if $u_0 \in \mathbb{U}_A$ we have

$$I_{k,\lambda} = u_0^{k\mu-\lambda} \int_{\mathbb{U}} \big(\widehat{A}(u_0^{-1}u)\big)^k \, \overline{P_\lambda(u_0^{-1}u)} d\pi_0(u_0^{-1}u) = u_0^{k\mu-\lambda} I_{k,\lambda}. \tag{8.3.5}$$

This shows that $I_{k,\lambda} = 0$ if there exists $u_0 \in \mathbb{U}_A$ such that $u_0^{k\mu-\lambda} \neq 1$.

Suppose now that $u_0^{k\mu-\lambda} = 1$ for all $u_0 \in \mathbb{U}_A$. It is clear that

$$I_{k,\lambda} = \int_{N_\epsilon(\mathbb{U}_A)} \big(\widehat{A}(u)\big)^k \, \overline{P_\lambda(u)} d\pi_0(u) + \mathcal{O}\big(\rho_1^k \widehat{A}(1)^k\big), \tag{8.3.6}$$

and since $N_\epsilon(u_0) = u_0 N_\epsilon$, the calculation in (8.3.5) shows that for each $u_0 \in \mathbb{U}_A$,

$$\int_{N_\epsilon(u_0)} \big(\widehat{A}(u)\big)^k \, \overline{P_\lambda(u)} d\pi_0(u) = u_0^{k\mu-\lambda} \int_{N_\epsilon} \big(\widehat{A}(u)\big)^k \overline{P_\lambda(u)} d\pi_0(u) = I_{k,\lambda}^\epsilon,$$

since $u_0^{k\mu-\lambda} = 1$. The result follows from (8.3.6) by the choice of ϵ. $\square$

It is clear from Corollary 8.2.8 that if each $|\theta_j|$, $j = 1, \ldots, n$, is sufficiently small, then

$$\widehat{A}(e^{i\theta}) = \widehat{A}(1) e^{-\sum_{i,j=1}^n b_{i,j}\theta_i\theta_j + G(\theta)} \quad \text{where} \quad G(\theta) = o\bigg(\sum_{i,j=1}^n b_{i,j}\theta_i\theta_j\bigg). \tag{8.3.7}$$

Writing $\delta = 2\sin^{-1}(\epsilon/2)$ we have $N_\epsilon = \{e^{i\theta} : |\theta_j| < \delta \text{ for } j = 1, \ldots, n\}$, and so we may choose $\epsilon > 0$ sufficiently small so that

$$|G(\theta)| \leq \frac{1}{2} \sum_{i,j=1}^n b_{i,j}\theta_i\theta_j \tag{8.3.8}$$

whenever $e^{i\theta} \in N_\epsilon$ and $|\theta_j| \leq \pi$ for $j = 1, \ldots, n$.

Define constants K_1, K_2 and K_3 by $K_1 = W_0(q^{-1})|W_0|^{-1}(2\pi)^{-n}$,

$$K_2 = \prod_{\alpha\in R_2^+} \frac{1}{\left(1-\tau_{2\alpha}^{-1}\tau_\alpha^{-1/2}\right)^2\left(1+\tau_\alpha^{-1/2}\right)^2}$$

$$K_3 = \int_{\mathbb{R}^n} e^{-\sum_{i,j=1}^n b_{i,j}\varphi_i\varphi_j} \prod_{\alpha\in R_2^+}\langle\alpha^\vee,\varphi\rangle^2 d\varphi_1\cdots d\varphi_n, \tag{8.3.9}$$

where $\varphi = \varphi_1\alpha_1 + \cdots + \varphi_n\alpha_n$.

Lemma 8.3.3. *Let $\epsilon > 0$ be such that (8.3.4) and (8.3.8) hold. Then*

$$I_{k,\lambda}^\epsilon = KP_\lambda(1)\widehat{A}(1)^k\, k^{-|R_2^+|-n/2}\left(1+\mathcal{O}\big(k^{-1/2}\big)\right),$$

where $K = K_1K_2K_3$.

Proof. Let $\delta = 2\sin^{-1}(\epsilon/2)$ as above. By the results of Section 6.2 we have

$$I_{k,\lambda}^\epsilon = K_1\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta}\big(\widehat{A}(e^{i\theta})\big)^k\frac{P_\lambda(e^{-i\theta})}{|c(e^{i\theta})|^2}\,d\theta_1\cdots d\theta_n,$$

and so by making the change of variable $\varphi_j = \sqrt{k}\theta_j$ for each $j = 1,\ldots,n$ we see that

$$I_{k,\lambda}^\epsilon = K_1k^{-n/2}\int_{-\sqrt{k}\delta}^{\sqrt{k}\delta}\cdots\int_{-\sqrt{k}\delta}^{\sqrt{k}\delta}\big(\widehat{A}(e^{i\varphi/\sqrt{k}})\big)^k\frac{P_\lambda(e^{-i\varphi/\sqrt{k}})}{|c(e^{i\varphi/\sqrt{k}})|^2}\,d\varphi_1\cdots d\varphi_n, \tag{8.3.10}$$

where $\varphi = \varphi_1\alpha_1 + \cdots + \varphi_n\alpha_n$.

By Corollary 8.2.5 we have

$$P_\lambda(e^{-i\varphi/\sqrt{k}}) = P_\lambda(1)(1+E_1(\varphi)) \quad\text{where}\quad |E_1(\varphi)| \le \frac{C\|\lambda\||\varphi|}{\sqrt{k}},$$

and it follows from Lemma 8.2.9 that

$$\frac{1}{|c(e^{i\varphi/\sqrt{k}})|^2} = K_2k^{-|R_2^+|}(1+E_2(\varphi))\prod_{\alpha\in R_2^+}\langle\alpha^\vee,\varphi\rangle^2,$$

where $|E_2(\varphi)| \le k^{-1}p(\varphi_1,\ldots,\varphi_n)$ for some polynomial $p(x_1,\ldots,x_n)$. Using these estimates (along with (8.3.7)) in (8.3.10), we see that $I_{k,\lambda}^\epsilon$ equals $K_1K_2P_\lambda(1)\widehat{A}(1)^k\,k^{-|R_2^+|-n/2}$ times

$$\int_{[-\sqrt{k}\delta,\sqrt{k}\delta]^n} e^{-\sum_{i,j=1}^n b_{i,j}\varphi_i\varphi_j+kG(\varphi/\sqrt{k})}\Bigg(\prod_{\alpha\in R_2^+}\langle\alpha^\vee,\varphi\rangle^2\Bigg)(1+E_1(\varphi))(1+E_2(\varphi))\,d\varphi_1\cdots d\varphi_n.$$

By (8.3.8), the above integrand is bounded by

$$e^{-\frac{1}{2}\sum_{i,j=1}^n b_{i,j}\varphi_i\varphi_j}\Bigg(\prod_{\alpha\in R_2^+}\langle\alpha^\vee,\varphi\rangle^2\Bigg)\Bigg(1+\frac{C\|\lambda\||\varphi|}{\sqrt{k}}\Bigg)\Bigg(1+\frac{p(\varphi_1,\ldots,\varphi_n)}{k}\Bigg),$$

and the lemma follows by the Dominated Convergence Theorem. $\square$

LEMMA 8.3.4. *Let $\lambda \in P^+$ and $k \in \mathbb{N}$. In the exceptional case, there exists $0 < \rho_2 < 1$ such that*

$$\int_{\mathbb{U}'} \big(\widehat{A}(u)\big)^k \overline{P_\lambda(u)} d\pi_0(u) = \mathcal{O}\big(\rho_2^k \widehat{A}(1)^k\big).$$

We prove Lemma 8.3.4 in Appendix B.3. We now give the local limit theorem.

THEOREM 8.3.5. *Let $y \in V_\lambda(x)$ and $k \in \mathbb{N}$, and suppose that $a_\mu > 0$. If $u_0^{k\mu} = u_0^\lambda$ for all $u_0 \in \mathbb{U}_A$, then*

$$p^{(k)}(x,y) = |\mathbb{U}_A| K P_\lambda(1) \widehat{A}(1)^k \, k^{-|R_2^+|-n/2}\big(1 + \mathcal{O}(k^{-1/2})\big),$$

where K is as in Lemma 8.3.3. If $u_0^{k\mu} \neq u_0^\lambda$ for some $u_0 \in \mathbb{U}_A$, then $p^{(k)}(x,y) = 0$.

PROOF. In the standard case the result follows from (8.3.2) and Lemmas 8.3.2 and 8.3.3. In the exceptional case, $Q = P$, and so $\mathbb{U}_Q = \{1\}$, and so $\mathbb{U}_A = \{1\}$. The result now follows from (8.3.2) and Lemmas 8.3.2, 8.3.3, and 8.3.4. □

COROLLARY 8.3.6. *Let A be as in (8.1.3), and suppose that $a_\mu > 0$. Then*

(i) *A is irreducible if and only if for each $\lambda \in P^+$ there exists $k = k(\lambda) \in \mathbb{N}$ such that $u_0^{k\mu} = u_0^\lambda$ for all $u_0 \in \mathbb{U}_A$, and*

(ii) *A is irreducible and aperiodic if and only if $|\mathbb{U}_A| = 1$.*

PROOF. First let us note that in the exeptional case it is easy to see that any walk with $a_\mu > 0$ for some $\mu \neq 0$ is both aperiodic and irreducible, and since $Q = P$ we have $\mathbb{U}_A = \{1\}$. Consider the standard case. Suppose that $a_\mu > 0$.

Let $y \in V_\lambda(x)$. If A is irreducible, then there exists $k \in \mathbb{N}$ such that $p^{(k)}(x,y) > 0$, and so $u_0^{k\mu} = u_0^\lambda$ for all $u_0 \in \mathbb{U}_A$, by (8.3.2) and Lemma 8.3.2. Conversely, if for each $\lambda \in P^+$ there exists $k_0 \in \mathbb{N}$ such that $u_0^{k_0\mu} = u_0^\lambda$ for all $u_0 \in \mathbb{U}_A$, then writing $r = |\mathbb{U}_A|$ we have $u_0^{(k_0+rl)\mu} = u_0^\lambda$ for all $u_0 \in \mathbb{U}_A$ and all $l \geq 0$. As $k \to \infty$ through the values $k_0 + rl$, Theorem 8.3.5 implies irreducibility.

If $|\mathbb{U}_A| = 1$ then A is clearly irreducible, and Theorem 8.3.5 shows that A is aperiodic. Conversely, if A is irreducible and aperiodic, then

$$1 = \gcd\{k \geq 1 \mid p^{(k)}(x,x) > 0\} = \gcd\{k \geq 1 \mid u_0^{k\mu} = 1 \text{ for all } u_0 \in \mathbb{U}_A\},$$

and so $\mathbb{U}_A = \{1\}$. □

8.4. An Explicit Evaluation of K_3

In this section we compute the integral K_3 from (8.3.9) in the B_n, C_n, D_n and BC_n cases (we have been unable to perform the calculations in the A_n case). The first step is to remove the $b_{j,k}$'s from the quadratic form $\sum_{j,k=1}^n b_{j,k}\varphi_j\varphi_k$. The key to this is the following proposition.

PROPOSITION 8.4.1. *Let $b^\lambda_{j,k}$ be as in (8.2.3). For each $\lambda \in P^+$ there exists a number $b^\lambda > 0$ such that $b^\lambda_{j,k} = \langle \alpha_j, \alpha_k\rangle b^\lambda$ for all $1 \le j,k \le n$.*

PROOF. Fix $x \in V_P$ and $y \in V_\lambda(x)$, and abbreviate $h(x,y;\omega)$ to h and $h_j(x,y;\omega)$ to h_j, where $y \in V_\lambda(x)$ is fixed. We first prove that

$$\frac{1}{|\alpha_j|^2} b^\lambda_{j,j} = \frac{1}{|\alpha_k|^2} b^\lambda_{k,k} \quad \text{for all } 1 \le j,k \le n. \tag{8.4.1}$$

To see this, observe that if $|\alpha_j| = |\alpha_k|$ then there exists $w \in W_0$ such that $w\alpha_j = \alpha_k$. Thus $P_\lambda(e^{i\varphi\alpha_j}) = P_\lambda(e^{i\varphi\alpha_k})$, and so differentiating twice with respect to φ, and then setting $\varphi = 0$, gives $-\int_\Omega h_j^2 r^h d\nu_x = -\int_\Omega h_k^2 r^h d\nu_x$, that is, $b^\lambda_{j,j} = b^\lambda_{k,k}$. Thus (8.4.1) holds when $|\alpha_j| = |\alpha_k|$. Suppose now that $|\alpha_j| \neq |\alpha_k|$, and write $c_{j,k} = \langle \alpha_j, \alpha_k^\vee\rangle$. Since $s_{\alpha_k}(\alpha_j) = \alpha_j - c_{j,k}\alpha_k$ we have $P_\lambda(e^{i\varphi\alpha_j}) = P_\lambda(e^{i\varphi(\alpha_j - c_{j,k}\alpha_k)})$, and so following the above we have $\int_\Omega h_j^2 r^h d\nu_x = \int_\Omega (h_j - c_{j,k}h_k)^2 r^h d\nu_x$. If $c_{j,k} \neq 0$ then this implies that $c_{j,k} b^\lambda_{k,k} = 2b^\lambda_{j,k}$. Since $b^\lambda_{j,k} = b^\lambda_{k,j}$, and since $c_{j,k} \neq 0$ implies that $c_{k,j} \neq 0$, we also have $c_{k,j} b^\lambda_{j,j} = 2b^\lambda_{j,k}$. Thus (if $c_{j,k} \neq 0$) $c_{j,k} b^\lambda_{k,k} = c_{k,j} b^\lambda_{j,j}$, that is, (8.4.1) holds. Finally, if $c_{j,k} = 0$, irreducibility implies that there exists j' and k' in $\{1,\ldots,n\}$ such that $|\alpha_j| = |\alpha_{j'}|$, $|\alpha_k| = |\alpha_{k'}|$, and $c_{j',k'} = \langle \alpha_{j'}, \alpha_{k'}^\vee\rangle \neq 0$. The result therefore follows in this case too by the above observations. Let b^λ be the constant value taken by $\frac{1}{|\alpha_j|^2} b^\lambda_{j,j}$ for $j = 1,\ldots,n$. Clearly $b^\lambda > 0$ and $b^\lambda_{j,j} = \langle \alpha_j, \alpha_j\rangle b^\lambda$ for all $j = 1,\ldots,n$.

We now show that $b^\lambda_{j,k} = \langle \alpha_j, \alpha_k\rangle b^\lambda$ for $j \neq k$ too. Suppose first that $c_{j,k} \neq 0$. Then the calculation made above shows that $b^\lambda_{j,k} = \frac{c_{j,k}}{2} b^\lambda_{k,k} = \langle \alpha_j, \alpha_k\rangle b^\lambda$. Finally, suppose that $c_{j,k} = 0$. In this case $s_{\alpha_k}(\alpha_j) = \alpha_j$ and $s_{\alpha_k}(\alpha_k) = -\alpha_k$, and so $P_\lambda(e^{i(\varphi_j\alpha_j + \varphi_k\alpha_k)}) = P_\lambda(e^{i(\varphi_j\alpha_j - \varphi_k\alpha_k)})$. By differentiating this once with respect to φ_j, then once with respect to φ_k, and then setting $\varphi_j = \varphi_k = 0$ we see that $-\int_\Omega h_j h_k r^h d\nu_x = \int_\Omega h_j h_k r^h d\nu_x$, that is, $b^\lambda_{j,k} = 0 = \langle \alpha_j, \alpha_k\rangle b^\lambda$. ☐

Recall the definition of $b_{j,k}$ from (8.2.9).

COROLLARY 8.4.2. *Let $b = \frac{1}{\hat{A}(1)} \sum_{\lambda \in P^+} a_\lambda b^\lambda$. Then $b > 0$, and $b_{j,k} = \langle \alpha_j, \alpha_k\rangle b$ for each $1 \le j,k \le n$.*

By making the change of variables $\theta_j = \sqrt{b}\varphi_j$ for each $j = 1,\ldots,n$ in (8.3.9) we have $K_3 = b^{-|R_2^+|-n/2} L$, where

$$L = \int_{\mathbb{R}^n} e^{-\sum_{j,k=1}^n \langle \alpha_j, \alpha_k\rangle \theta_j\theta_k} \prod_{\alpha \in R_2^+} \langle \alpha^\vee, \theta\rangle^2 d\theta_1 \cdots d\theta_n, \tag{8.4.2}$$

where $\theta = \theta_1\alpha_1 + \cdots + \theta_n\alpha_n$. The integral L depends only on the underlying root system, and not on the building parameters.

We now discuss a method of diagonalising the quadratic form in the integrand of (8.4.2). Let $\{v_i\}_{i=1}^n$ be any orthonormal basis for the underlying vector space E (see Appendix D). Let $A = (\langle \alpha_j, \alpha_k\rangle)_{j,k=1}^n$, and let $M = (\langle \lambda_i, v_j\rangle)_{i,j=1}^n$ (so M is invertible). Then

(i) $M^TAM = I$ (the $n \times n$ identity matrix), and
(ii) $\theta = \sum_{i=1}^n x_i v_i$.

Let $\mathbf{t}$ be the column vector $(\theta_i)_{i=1}^n$, and let $\mathbf{x} = M^{-1}\mathbf{t}$. Thus by (i) above, $\mathbf{t}^T A\mathbf{t} = \mathbf{x}^T\mathbf{x}$. By making the change of variable $\mathbf{t} = M\mathbf{x}$ we have

$$L = |\det(M)| \int_{\mathbb{R}^n} e^{-(x_1^2+\cdots+x_n^2)} \prod_{\alpha\in R_2^+} \langle \alpha^\vee, x\rangle^2 dx_1\cdots dx_n, \tag{8.4.3}$$

where $x = x_1v_1 + \cdots + x_nv_n$.

Of course we would like to choose the orthonormal basis $\{v_i\}_{i=1}^n$ such that the product in the integrand of (8.4.3) has a neat formula. Let us restrict ourselves to the infinite families of types B_n, C_n, D_n and BC_n. In these cases, let $v_i = e_i$ for $i = 1,\ldots,n$. Then $\det(M) = 1, \frac{1}{2}, \frac{1}{2}, 1$ in the B_n, C_n, D_n and BC_n cases respectively.

Writing

$$I_n = \int_{\mathbb{R}^n} x_1^2\cdots x_n^2 e^{-(x_1^2+\cdots+x_n^2)} \prod_{1\le i<j\le n} (x_i^2 - x_j^2)^2 dx_1\cdots dx_n$$
$$J_n = \int_{\mathbb{R}^n} e^{-(x_1^2+\cdots+x_n^2)} \prod_{1\le i<j\le n} (x_i^2 - x_j^2)^2 dx_1\cdots dx_n$$

we have $L = 2^{2n}I_n$ if R is of type B_n or BC_n, $L = \frac{1}{2}I_n$ if R is of type C_n, and $L = \frac{1}{2}J_n$ if R is of type D_n. Note that the integrals make sense for $n \ge 1$. We now show how to evaluate I_n and J_n. The trick is to convert the integral calculation into a determinant calculation, using *Gram's Identity* (see Proposition 8.4.5).

The following is an elementary calculation.

LEMMA 8.4.3. *Let $\alpha \in \mathbb{R}$, and suppose that $C = (c_{i,j})_{i,j=1}^n$ is a matrix such that $c_{i,j+1} = (i+j+\alpha)c_{i,j}$ for all $1 \le i \le n$ and $1 \le j \le n-1$. Then* $\det(C) = \prod_{i=1}^n (i-1)!c_{i,1}$.

COROLLARY 8.4.4. *For $1 \le i,j \le n$ let*

$$a_{i,j} = \frac{(2i+2j-2)!}{2^{2i+2j-2}(i+j-1)!} \quad \textit{and} \quad b_{i,j} = \frac{(2i+2j-4)!}{2^{2i+2j-4}(i+j-2)!},$$

and let $A = (a_{i,j})_{i,j=1}^n$ and $B = (b_{i,j})_{i,j=1}^n$. Then

$$\det(A) = \frac{1}{n!}2^{-n(n+1)}\prod_{i=1}^n (2i)! \quad \textit{and} \quad \det(B) = 2^{-n(n-1)}\prod_{i=1}^{n-1}(2i)!$$

PROOF. The numbers $a_{i,j}$ (respectively $b_{i,j}$) satisfy the conditions of Lemma 8.4.3 with $\alpha = -\frac{1}{2}$ (respectively $\alpha = -\frac{3}{2}$), and the result follows. □

The following is well known (in the context of *random matrix theory*). We provide a proof for completeness.

Proposition 8.4.5 (Gram's Identity). *For each $i = 1, \dots, n$ let $a_i(x)$ and $b_i(x)$ be real valued functions such that $c_{i,j} = \int_{-\infty}^{\infty} a_i(x)b_j(x)dx$ is finite for each i and j. Let $A(x) = (a_i(x_j))_{i,j=1}^n$ and $B(x) = (b_i(x_j))_{i,j=1}^n$. Then*

$$\int_{\mathbb{R}^n} \det(A(x)) \det(B(x))\, dx_1 \cdots dx_n = n! \det(C),$$

where $C = (c_{i,j})_{i,j=1}^n$.

Proof. Observe that

$$\begin{aligned}
\det(A(x)) \det(B(x)) &= \sum_{\sigma,\tau\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \prod_{i,j=1}^{n} a_i(x_{\sigma i}) b_j(x_{\tau j}) \\
&= \sum_{\sigma,\tau\in\mathfrak{S}_n} \operatorname{sgn}(\tau) \prod_{i,j=1}^{n} a_i(x_{\sigma i}) b_j(x_{\sigma\tau^{-1} j}) \\
&= \sum_{\sigma,\tau\in\mathfrak{S}_n} \operatorname{sgn}(\tau) \prod_{i,j=1}^{n} a_i(x_{\sigma i}) b_{\tau j}(x_{\sigma j}) \\
&= \sum_{\sigma,\tau\in\mathfrak{S}_n} \operatorname{sgn}(\tau) \prod_{i=1}^{n} a_i(x_{\sigma i}) b_{\tau i}(x_{\sigma i})
\end{aligned}$$

(on the second line we replace τ by $\sigma\tau^{-1}$ in the summation, on the third line we replace j by τj in the product). Thus

$$\int_{\mathbb{R}^n} \det(A(x)) \det(B(x))\, d\mathbf{x} = n! \sum_{\tau\in\mathfrak{S}_n} \operatorname{sgn}(\tau) \prod_{i=1}^{n} \int_{-\infty}^{\infty} a_i(x) b_{\tau i}(x) dx = n! \det(C). \qquad \square$$

Theorem 8.4.6. *For all $n \geq 1$ we have*

$$I_n = \pi^{n/2} 2^{-n(n+1)} \prod_{i=1}^{n} (2i)! \quad \textit{and} \quad J_n = \pi^{n/2} 2^{-n(n-1)} n! \prod_{i=1}^{n-1} (2i)!$$

Proof. We consider I_n first. Let $a_i(x) = x^{2i} e^{-x^2}$ and $b_i(x) = x^{2i-2}$, and write $A(x) = (a_i(x_j))_{i,j=1}^n$ and $B(x) = (b_i(x_j))_{i,j=1}^n$. By the Vandermonde determinant formula we see that

$$\det(A(x)) \det(B(x)) = x_1^2 \cdots x_n^2 e^{-(x_1^2 + \cdots + x_n^2)} \prod_{1\leq i<j\leq n} (x_i^2 - x_j^2)^2.$$

Now, in the notation of Corollary 8.4.4,

$$\int_{-\infty}^{\infty} a_i(x) b_j(x) dx = \sqrt{\pi} a_{i,j},$$

and so by Proposition 8.4.5 we have $I_n = n! \pi^{n/2} \det(A)$ (where $A = (a_{i,j})_{i,j=1}^n$), and the result follows (using Corollary 8.4.4).

To evaluate J_n, let $a_i(x) = x^{2i-2}e^{-x^2}$ and $b_i(x) = x^{2i-2}$. Then

$$\det(A(x))\det(B(x)) = e^{-(x_1^2+\cdots+x_n^2)} \prod_{1\le i<j\le n} (x_i^2 - x_j^2)^2,$$

and the result follows as above, using Corollary 8.4.4 and Proposition 8.4.5. $\square$

Remark 8.4.7. In the $\tilde{A}_n$ case it is not difficult to see (by taking M to be the $n\times(n+1)$ matrix $(\langle\lambda_i, e_j\rangle)$ and modifying the discussion after Corollary 8.4.2 accordingly) that the integral L from (8.4.2) may be written as

$$\int_{\mathbb{R}^n} e^{-(x_1^2+\cdots+x_{n+1}^2)} \prod_{1\le i<j\le n+1} (x_i - x_j)^2 dx_1\cdots dx_n$$

(up to some constant factors), where $x_{n+1} = -(x_1 + \cdots + x_n)$. We have been unable to compute this integral. In principle, the integrals for the E_6, E_7, E_8, F_4 and G_2 cases could be computed from (8.4.3).

APPENDIX A

The Reducible Case

In this appendix we will extend the results of Chapter 4 to reducible (locally finite regular) affine buildings.

A.1. Reducible Coxeter Groups

A Coxeter group is said to be *irreducible* if its Coxeter graph is connected, and *reducible* otherwise [**5**, IV, §2, No.9]. A reducible Coxeter group W decomposes naturally as a direct product $W^1 \times \cdots \times W^r$, where $\{D^k\}_{k=1}^r$ are the connected components of D, and for each $k = 1, \ldots, r$, W^k is the Coxeter group with Coxeter graph D^k. We call the groups W^k, $k = 1, \ldots, r$, the *irreducible components* of W. For each $k = 1, \ldots, r$, let I^k be the vertex set of D^k, so $I = \bigcup_{k=1}^r I^k$, where the union is disjoint.

In the reducible case, the group $\mathrm{Aut}(D)$ is often too large for our purposes. Thus we define $\mathrm{Aut}_{\mathrm{cp}}(D)$ to be the group of all *component preserving automorphisms of* D. That is, $\mathrm{Aut}_{\mathrm{cp}}(D)$ consists of all those $\sigma \in \mathrm{Aut}(D)$ such that $\sigma(D^k) = D^k$ for each $k = 1, \ldots, r$. Clearly we have $\mathrm{Aut}_{\mathrm{cp}}(D) = \mathrm{Aut}(D^1) \times \cdots \times \mathrm{Aut}(D^r)$.

A.2. Direct Products of Chamber Systems

Given chamber systems $\mathcal{C}^1, \ldots, \mathcal{C}^r$ over pairwise disjoint sets $I^1, \ldots, I^r$, the *direct product* $\mathcal{C}^1 \times \cdots \times \mathcal{C}^r$ is a chamber system over $I = \bigcup_{k=1}^r I^k$ with chambers given by r-tuples $(c^1, \ldots, c^r)$, $c^k \in \mathcal{C}^k$, and $(c^1, \ldots, c^r)$ is *i-adjacent* to $(d^1, \ldots, d^r)$ for $i \in I^k$ if $c^j = d^j$ for $j \neq k$ and $c^k \sim_i d^k$ in $\mathcal{C}^k$.

A.3. Reducible Buildings

We call a building $\mathscr{X}$ *irreducible* (respectively *reducible*) if the corresponding Coxeter group W is irreducible (respectively reducible).

Let $\mathscr{X}$ be a building as in Definition 1.6.1 of reducible type $W = W^1 \times \cdots \times W^r$, and write I^k for the vertex set of D^k for each $k = 1, \ldots, r$. Fix any chamber $a \in \mathscr{X}$, and for each $k = 1, \ldots, r$, let $\mathscr{X}^k$ denote the I^k-residue of a. The following theorem describes the structure of reducible buildings from the point of view of chamber systems.

THEOREM A.3.1. [**35**, Theorems 3.5 and 3.10]. *With the notation above,*

(i) $\mathscr{X}^j$ *is a building of type* W^j *for each* $j = 1, \ldots, d$, *and*
(ii) $\mathscr{X} \cong \mathscr{X}^1 \times \cdots \times \mathscr{X}^r$ *(direct product of chamber systems).*

The buildings $\mathscr{X}^1, \ldots, \mathscr{X}^r$ are called the *irreducible components of* $\mathscr{X}$. The isomorphism in Theorem A.3.1(ii) is constructed as follows. Given $c \in \mathcal{C}$, write $\delta(a, c) = w^1 \cdots w^r$ where $w^k \in W^k$ for each $k = 1, \ldots, r$. By commutativity (of elements of W^i with elements of W^j for $i \neq j$) and Definition 1.6.1(ii) it follows that for each $k = 1, \ldots, r$ there exists a chamber c^k such that c^k lies in a minimal gallery from a to c and $\delta(a, c^k) = w^k$. By [**35**, Chapter 3, Exercise 4] the chamber c^k is unique, and we set $\psi(c) = (c^1, \ldots, c^r)$. The map $\psi : \mathscr{X} \to \mathscr{X}^1 \times \cdots \times \mathscr{X}^r$ is then an isomorphism of chamber systems.

It should be noted that the particular choice of the fixed chamber $a \in \mathcal{C}$ above is immaterial. To see this, fix $b \in \mathcal{C}$. By Theorem A.3.1 we have $\mathscr{X} \cong \mathscr{Y}^1 \times \cdots \times \mathscr{Y}^r$, where $\mathscr{Y}^k = R_{I^k}(b)$ for each $k = 1, \ldots, r$. Suppose first that $b \sim_i a$ (and $b \neq a$) for $i \in I^j$, say. Then $\mathscr{X}^j = R_{I^j}(a) = R_{I^j}(b) = \mathscr{Y}^j$. If $k \neq j$ and $c \in \mathscr{X}^k$, it can be shown that there exists a unique chamber $\varphi(c) \in \mathscr{Y}^k$ such that $\delta(c, \varphi(c)) = s_i$ and $\delta(b, \varphi(c)) = \delta(a, c)$. The map $\varphi : \mathscr{X}^k \to \mathscr{Y}^k$ is a chamber system isomorphism for each $k \neq j$. It follows by induction on $\ell(\delta(a, b))$ that $\mathscr{X}^k \cong \mathscr{Y}^k$ for all $b \in \mathcal{C}$ and $k = 1, \ldots, r$.

We will discuss the structure of reducible buildings from the 'simplicial' point of view shortly. Let us first discuss the algebra $\mathscr{B}$ of Chapter 2 in the reducible case.

A.4. The Algebra $\mathscr{B}$

Recall that for algebras A_1 and A_2 over a ring $\mathcal{R}$, the *direct product* $A_1 \times A_2$ of A_1 and A_2 is the algebra over $\mathcal{R}$ with operations $a(x, y) + b(x', y') = (ax + bx', ay + by')$ and $(x, y)(x', y') = (xx', yy')$ for all $x, x' \in A_1$, $y, y' \in A_2$ and $a, b \in \mathcal{R}$.

Let $\mathscr{X}$ be a locally finite regular reducible building of type W. In the notation of Section A.3, each $\mathscr{X}^k$, $k = 1, \ldots, r$, is an irreducible regular building. For each $k = 1, \ldots, r$, let $\mathscr{B}^k$ be the algebra from Definition 2.1.2 associated to the building $\mathscr{X}^k$.

LEMMA A.4.1. *Let* $b^k_{u^k, v^k; w^k}$ $(u^k, v^k, w^k \in W^k)$ *be the structure constants of the algebra* $\mathscr{B}^k$, *and* $b_{u,v;w}$ $(u, v, w \in W)$ *be the structure constants of* $\mathscr{B}$. *Then*

$$b^k_{u^k, v^k; w^k} = b_{(1,\ldots,u^k,\ldots,1),(1,\ldots,v^k,\ldots,1);(1,\ldots,w^k,\ldots,1)}\,.$$

PROOF. Induction on $\ell^k(v^k)$. When $\ell^k(v^k) = 1$ the result is true by Theorem 2.1.6, since $q_{(1,\ldots,v^k,\ldots,1)} = q_{v^k}$. Suppose that the result is true whenever $v^k \in W^k$ satisfies $\ell^k(v^k) < n$, and suppose that $x^k = s^k_{i_1} \cdots s^k_{i_n} \in W^k$ has length n. Write $y^k = s^k_{i_1} \cdots s^k_{i_{n-1}}$ and $s^k = s^k_{i_n}$. Following the method used to derive (2.1.4) we have

$$b^k_{u^k,x^k;w^k} = \sum_{z^k\in W^k} b^k_{u^k,y^k;z^k} b^k_{z^k,s^k;w^k},$$

and the result follows by induction since $\ell^k(y^k) < n$ and $\ell^k(s^k) = 1$. $\square$

LEMMA A.4.2. *For all $u, v, w \in W$,*

$$b_{u,v;w} = \prod_{k=1}^{r} b^k_{u^k,v^k;w^k}\,.$$

PROOF. Note that $B_{(1,\ldots,w^j,\ldots,1)}$ and $B_{(1,\ldots,w^k,\ldots,1)}$ commute for $j \neq k$, and so

$$\begin{aligned} B_u B_v &= B_{(u^1,\ldots,1)} \cdots B_{(1,\ldots,u^r)} B_{(v^1,\ldots,1)} \cdots B_{(1,\ldots,v^r)} \\ &= \prod_{k=1}^{r} B_{(1,\ldots,u^k,\ldots,1)} B_{(1,\ldots,v^k,\ldots,1)}\,. \end{aligned}$$

By Lemma A.4.1

$$B_{(1,\ldots,u^k,\ldots,1)} B_{(1,\ldots,v^k,\ldots,1)} = \sum_{w^k\in W^k} b^k_{u^k,v^k;w^k} B_{(1,\ldots,w^k,\ldots,1)}\,,$$

and so it follows that

$$\begin{aligned} B_u B_v &= \sum_{w^1\in W^1} \cdots \sum_{w^r\in W^r} b^1_{u^1,v^1;w^1} \cdots b^r_{u^r,v^r;w^r} B_{(w^1,\ldots,w^r)} \\ &= \sum_{w\in W} \left(\prod_{k=1}^{r} b^k_{u^k,v^k;w^k} \right) B_w. \end{aligned}$$

$\square$

THEOREM A.4.3. $\mathscr{B} \cong \mathscr{B}^1 \times \cdots \times \mathscr{B}^r$.

PROOF. By the multiplication and addition laws in $\mathscr{B}^1 \times \cdots \times \mathscr{B}^r$ and the fact that $\sum_{w^k} b^k_{u^k,v^k;w^k} = 1$ for all $u^k, v^k \in W^k$ (see Corollary 2.1.8) we have

$$\begin{aligned} &(B^1_{u^1}, \ldots, B^r_{u^r})(B^1_{v^1}, \ldots, B^r_{v^r}) \\ &\quad = (B^1_{u^1} B^1_{v^1}, \ldots, B^r_{u^r} B^r_{v^r}) \\ &\quad = \left(\sum_{w^1\in W^1} b^1_{u^1,v^1;w^1} B^1_{w^1}\,, \ldots, \sum_{w^r\in W^r} b^r_{u^r,v^r;w^r} B^r_{w^r} \right) \\ &\quad = \sum_{w^1,\ldots,w^r} b^1_{u^1,v^1;w^1} \cdots b^r_{u^r,v^r;w^r} (B^1_{w^1}, \ldots, B^r_{w^r})\,. \end{aligned}$$

Thus by Lemma A.4.2

$$(B^1_{u^1}, \ldots, B^r_{u^r})(B^1_{v^1}, \ldots, B^r_{v^r}) = \sum_{w\in W} b_{u,v;w} (B^1_{w^1}, \ldots, B^r_{w^r})\,,$$

and so the homomorphism induced by $B_w \mapsto (B^1_{w^1}, \ldots, B^r_{w^r})$ is an isomorphism. $\square$

A.5. Polysimplicial Complexes

Unlike chamber systems, a (Cartesian) product of simplicial complexes is not necessarily a simplicial complex. Thus we make the following definition. Given simplicial complexes $\Sigma^1, \ldots, \Sigma^r$ with vertex sets $X^1, \ldots, X^r$, the *polysimplicial complex* $\Sigma = \Sigma^1 \times \cdots \times \Sigma^r$ with vertex set $X = X^1 \times \cdots \times X^r$ is the collection of all *polysimplices* $\sigma = \sigma^1 \times \cdots \times \sigma^r$, where σ^k is a simplex of Σ^k for each $1 \leq k \leq r$. If each Σ^k is a labelled simplicial complex with type map $\tau^k : V^k \to I^k$ then we say that $\Sigma = \Sigma^1 \times \cdots \times \Sigma^r$ is a *labelled polysimplicial complex*. The set of types of Σ is $J = I^1 \times \cdots \times I^r$, and the associated type map is given by $\tau(x) = (\tau^1(x^1), \ldots, \tau^r(x^r))$ for each $x = (x^1, \ldots, x^r) \in X$. Define *type preserving isomorphisms* of labelled polysimplicial complexes in the obvious way. We call maximal polysimplices of a labelled polysimplicial complex *chambers*. It is clear that each chamber of Σ is of the form $C^1 \times \cdots \times C^r$ where C^k is a chamber of Σ^k for each $1 \leq k \leq r$.

An *automorphism of a polysimplicial complex* $\Sigma = \Sigma^1 \times \cdots \times \Sigma^r$ is a bijection ψ such that

$$\psi(x) = (\psi^1(x^1), \ldots, \psi^r(x^r)),$$

where each $\psi^k : \Sigma^k \to \Sigma^k$ is an automorphism. In particular, automorphisms are assumed to be *component preserving*.

Given a direct product $\mathcal{C}^1 \times \cdots \times \mathcal{C}^r$ of chamber systems $\mathcal{C}^1, \ldots, \mathcal{C}^r$ over disjoint sets $I^1, \ldots, I^r$ one can construct a labelled polysimplicial complex. For each $(i^1, \ldots, i^r) \in J = I^1 \times \cdots \times I^r$ form the set

$$X_{i^1,\ldots,i^r} = \{(R_{I^1\setminus\{i^1\}}(c^1), \ldots, R_{I^r\setminus\{i^r\}}(c^r)) \mid (c^1, \ldots, c^r) \in \mathcal{C}^1 \times \cdots \times \mathcal{C}^r\},$$

and let X be the disjoint union over $(i^1, \ldots, i^r) \in J$ of these sets. Let $\tau(x) = (i^1, \ldots, i^r)$ if $x \in X_{i^1,\ldots,i^r}$. Declare polysimplices to be subsets of the maximal polysimplices

$$\{(R_{I^1\setminus\{i^1\}}(c^1), \ldots, R_{I^r\setminus\{i^r\}}(c^r)) \mid (i^1, \ldots, i^r) \in J\},$$

where $(c^1, \ldots, c^r) \in \mathcal{C}^1 \times \cdots \times \mathcal{C}^r$.

Conversely, given a labelled polysimplicial complex $\Sigma = \Sigma^1 \times \cdots \times \Sigma^r$ with vertex set $X = X^1 \times \cdots \times X^r$ and type map $\tau : X \to J = I^1 \times \cdots \times I^r$ we may construct a direct product of chamber systems as follows. Let $I = I^1 \cup \cdots \cup I^r$ (we assume the sets I^k are pairwise disjoint), and for $i \in I^k$, declare maximal polysimplices C and D to be i-adjacent if either $C = D$ or if all the vertices of C and D are the same except for those vertices $(x^1, \ldots, x^r)$ with $\tau^k(x^k) = i$.

With the conditions of Section 1.4 applied component-wise, the above operations are mutually inverse.

EXAMPLE A.5.1. Let Σ^1 be the labelled simplicial complex with vertex set $X^1 = \{x_1, x_2, x_3, x_4\}$, maximal simplices $\{\{x_1, x_3\}, \{x_2, x_3\}, \{x_3, x_4\}\}$, and type map $\tau^1 : X^1 \to I^1 = \{0, 1\}$ given by $\tau^1(x_1) = \tau^1(x_2) = \tau^1(x_4) = 0$ and $\tau^1(x_3) = 1$. Let Σ^2 be the labelled simplicial complex with vertex set $X^2 = \{y_1, y_2\}$, maximal simplices $\{\{y_1, y_2\}\}$, and type map $\tau^2 : X^2 \to I^2 = \{2, 3\}$ given by $\tau^2(y_1) = 2$ and $\tau^2(y_2) = 3$.

We can draw Σ^1 and Σ^2 as in Figure A.5.1.

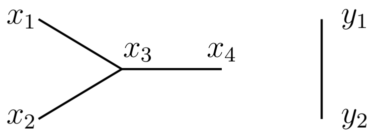


FIGURE A.5.1

Thus Σ can be drawn as in Figure A.5.2.

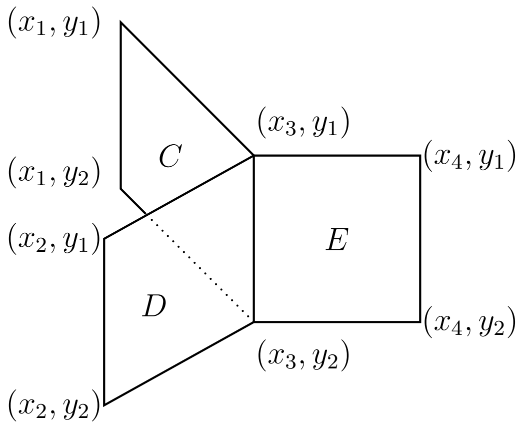


FIGURE A.5.2

The maximal polysimplices C, D and E are mutually 0-adjacent. For example, to see that $C \sim_0 D$, observe that C and D share all vertices, except for those vertices (x, y) with $\tau^1(x) = 0$.

In particular, if $W = W^1 \times \cdots \times W^r$, we have a natural description of the Coxeter complex as a polysimplicial complex, with a natural labelling $\tau : X \to J$, where $J = I^1 \times \cdots \times I^r$, and $\tau(x) = (\tau^1(x^1), \ldots, \tau^r(x^r))$, where $\tau^k : X^k \to I^k$.

A.6. Polysimplicial Buildings

Let us now consider reducible buildings from the polysimplicial point of view. Consider the building $\mathscr{X} \cong \mathscr{X}^1 \times \cdots \times \mathscr{X}^r$ of Theorem A.3.1(ii) as a polysimplicial complex, as in Section A.5. Call a sub-polysimplicial complex $\mathcal{A}$ of $\mathscr{X}$ an *apartment* if $\mathcal{A} = \mathcal{A}^1 \times \cdots \times \mathcal{A}^r$, where $\mathcal{A}^k$ is an apartment of $\mathscr{X}^k$ (as in Definition 1.6.2) for each $k = 1, \ldots, r$.

Write $\Sigma(W)$ for the polysimplicial Coxeter complex. The following Proposition follows from Definition 1.6.2.

PROPOSITION A.6.1. *Let $\mathscr{X}$ be a reducible building considered as a polysimplicial complex as above. Then (with the natural labelling $\tau : V \to J = I^1 \times \cdots \times I^r$)*

(i) *each apartment of $\mathscr{X}$ is isomorphic to $\Sigma(W)$ in a type preserving way,*
(ii) *given any two chambers of $\mathscr{X}$, there exists an apartment containing them both, and*
(iii) *given apartments $\mathcal{A}$ and $\mathcal{A}'$ of $\mathscr{X}$ containing a common chamber, there exists a type preserving isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ fixing $\mathcal{A} \cap \mathcal{A}'$ pointwise.*

A.7. Reducible Root Systems

Suppose $E = \bigoplus_{k=1}^r E^k$ is the direct sum of a family $\{E^k\}_{k=1}^r$ of vector spaces. For each $k = 1, \dots, r$, let R^k be a root system in E^k. Then $R = \bigcup_{k=1}^r R^k$ (disjoint union) is a root system in E with dual $R^\vee = \bigcup_{k=1}^r (R^k)^\vee$. We say that R is the *direct sum of the root systems R^k*.

Every root system R in E is the direct sum of a family $\{R^k\}_{k=1}^r$ of irreducible root systems [**5**, VI, §1, No.2, Proposition 6]. Furthermore, this decomposition is unique up to permutation of the index set $\{1, \dots, r\}$. We call the R^k the *irreducible components of R*.

For each irreducible component R^k of R choose a base $B^k = \{\alpha_i^k \mid i \in I_0^k\}$ (and we take the sets $I_0^1, \dots, I_0^r$ to be pairwise disjoint). Let Q^k and P^k be the coroot and coweight lattices of R^k. One easily verifies that $B = \bigcup_{k=1}^r B^k$ is a base of R. Furthermore, $Q = \bigoplus_{k=1}^r Q^k$, $P = \bigoplus_{k=1}^r P^k$ and $P^+ = \bigoplus_{k=1}^r (P^k)^+$.

We have the direct product decomposition $W_0 = W_0^1 \times \cdots \times W_0^r$, and similarly for the groups W, $\tilde{W}$, and G. We define

$$\mathrm{Aut}_{\mathrm{tr}}(D) = \mathrm{Aut}_{\mathrm{tr}}(D^1) \times \cdots \times \mathrm{Aut}_{\mathrm{tr}}(D^r).$$

In particular, $\mathrm{Aut}_{\mathrm{tr}}(D) \subset \mathrm{Aut}_{\mathrm{cp}}(D)$.

The hyperplane construction of Chapter 3 now yields a geometric realisation of the polysimplicial Coxeter complex. For example, in the $A_1 \times A_1$ case the hyperplanes $\mathcal{H}$ form a grid, and the chambers are represented as squares.

A.8. Reducible Affine Buildings

Suppose that $\mathscr{X}$ is a reducible regular locally finite affine building. Then the irreducible components $\mathscr{X}^1, \dots, \mathscr{X}^r$ are irreducible regular locally finite affine buildings. Let $R^1, \dots, R^r$ be the associated root systems, as in Section 3.8. Then we associate the direct sum R of the family $\{R^k\}_{k=1}^r$ to $\mathscr{X}$. We define the special and good vertices of $\mathscr{X}$ as in the irreducible case. Write J_P for the set of all good types (we use this notation instead of I_P, for in the reducible case, $J_P \subset J$).

A.9. The Algebra $\mathscr{A}$

Define operators A_λ, $\lambda \in P$, as in the irreducible case, and let $\mathscr{A}$ denote the linear span over $\mathbb{C}$ of $\{A_\lambda\}_{\lambda\in P^+}$. Following the proof of Theorem 4.4.8 we see that $\mathscr{A}$ is a commutative algebra.

Here we will show that $\mathscr{A} \cong \mathscr{A}^1 \times \cdots \times \mathscr{A}^r$, where $\mathscr{A}^k$, $k = 1, \ldots, r$, is the vertex operator algebra as in Definition 4.4.7 associated to the building $\mathscr{X}^k$.

Lemma A.9.1. *For each $k = 1, \ldots, r$, let $U^k \subseteq W^k$ and write $U = U^1 \times \cdots \times U^r$. Then the Poincaré polynomial of U satisfies $U(q) = \prod_{k=1}^r U^k(q)$.*

Proof. We have

$$U(q) = \sum_{u\in U} q_u = \prod_{k=1}^r \left(\sum_{u^k\in U^k} q_{(1,\ldots,u^k,\ldots,1)} \right) = \prod_{k=1}^r U^k(q)\,. \qquad \square$$

Lemma A.9.2. *Let $a^k_{\lambda^k,\mu^k;\nu^k}$, $\lambda^k, \mu^k, \nu^k \in (P^k)^+$, be the structure constants of the algebra $\mathscr{A}^k$, and $a_{\lambda,\mu;\nu}$, $\lambda, \mu, \nu \in P^+$, be the structure constants of $\mathscr{A}$. Then*

$$a_{\lambda,\mu;\nu} = \prod_{k=1}^r a^k_{\lambda^k,\mu^k;\nu^k}.$$

Proof. We prove the lemma using Proposition 4.4.10 (which also holds in the reducible case). Observe that

$$W_{0\lambda} = \{w \in W_0 \mid w^k\lambda^k = \lambda^k \text{ for all } k = 1, \ldots, r\},$$

and so by Lemma A.9.1 we have $W_{0\lambda}(q) = \prod_{k=1}^r W^k_{0\lambda^k}(q)$. Similarly the terms $W_{0\mu}(q)$, $W_{0\nu}(q)$ and $W_0(q)$ factorise. Let $l = (l^1, \ldots, l^r) = \tau(\lambda) \in J$, and write $W_l = W_{l^1} \times\cdots\times W_{l^r}$. Notice that

$$W_0 w_\lambda W_l = (W^1_{0^1} w_{\lambda^1} W^1_{l^1}, \ldots, W^r_{0^r} w_{\lambda^r} W^r_{l^r})$$

and similarly for the double cosets $W_l\sigma_l(w_\mu)W_n$ and $W_0w_\nu W_n$ (where $n = \tau(\nu)$). Putting all of this together, and using Lemma A.9.1, Lemma A.9.2 and Lemma A.4.2, we see that the entire expression for $a_{\lambda,\mu;\nu}$ factorises, and the result follows. $\square$

Theorem A.9.3. *$\mathscr{A} \cong \mathscr{A}^1 \times \cdots \times \mathscr{A}^r$.*

Proof. Recall that $P^+ = \bigoplus_{k=1}^r (P^k)^+$, and by Corollary 4.4.6 $\sum_{\nu^k} a^k_{\lambda^k,\mu^k;\nu^k} = 1$ for all $\lambda^k, \mu^k \in (P^k)^+$. The theorem now follows in the same way as Theorem A.4.3, bearing in mind Lemma A.9.2 above. $\square$

A.10. Algebra Homomorphisms

Theorem A.9.3 allows us to extend all of our results on the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ to reducible (locally finite regular) affine buildings.

Proposition A.10.1. *The algebra homomorphisms* $h : \mathscr{A} \to \mathbb{C}$ *are precisely the maps* $h : \mathscr{A} \to \mathbb{C}$ *with*

$$h(A) = h^1(A^1) \cdots h^r(A^r) \quad \textit{for all } A = (A^1, \ldots, A^r),$$

where each $h^k : \mathscr{A}^k \to \mathbb{C}$ *is an algebra homomorphism.*

Proof. First suppose that $h : \mathscr{A} \to \mathbb{C}$ is an algebra homomorphism. Then

$$\begin{aligned} h(A) &= h((A^1, \ldots, A^r)) \\ &= h((A^1, \ldots, 1) \cdots (1, \ldots, A^r)) \\ &= h((A^1, \ldots, 1)) \cdots h((1, \ldots, A^r)). \end{aligned}$$

For each $k = 1, \ldots, r$, define $h^k(A^k) = h(1, \ldots, A^k, \ldots, 1)$. This is clearly an algebra homomorphism, and h is of the prescribed form.

On the other hand, if for each $k = 1, \ldots, r$, $h^k : \mathscr{A}^k \to \mathbb{C}$ is an algebra homomorphism, then $h : \mathscr{A} \to \mathbb{C}$ defined by $h(A) = h^1(A^1) \cdots h^r(A^r)$ is an algebra homomorphism too. $\square$

APPENDIX B

Some Miscellaneous Results

As the title suggests, this appendix contains some results and formulae whose proofs were omitted from the main body of text.

B.1. Calculation of q_{t_λ}

To make the formula (5.2.1) completely explicit we need to compute q_{t_λ}.

LEMMA B.1.1. *Let H be a wall of $\mathscr{X}$. Suppose that π_1 is a cotype i panel of H and that π_2 is a cotype j panel of H. Then $q_i = q_j$.*

PROOF. If σ is any simplex of $\mathscr{X}$ and c any chamber of $\mathscr{X}$, then there is a unique chamber, denoted $\mathrm{proj}_\sigma(c)$, nearest c having σ as a face [**35**, Corollary 3.9]. We show that the map $\varphi : \mathrm{st}(\pi_1) \to \mathrm{st}(\pi_2)$ given by $\varphi(c) = \mathrm{proj}_{\pi_2}(c)$ is a bijection (here $\mathrm{st}(\pi_i)$, $i = 1, 2$, denotes the set of chambers of $\mathscr{X}$ having π_i as a face). Observe first that if $c \in \mathrm{st}(\pi_1)$ then $\mathrm{proj}_{\pi_1}(\mathrm{proj}_{\pi_2}(c)) = c$. To see this, let $\mathcal{A}$ be any apartment containing c and H (see [**35**, Theorem 3.6]), and let H^+ denote the half apartment of $\mathcal{A}$ containing c. Let d be the unique chamber in $\mathrm{st}(\pi_2) \cap H^+$. It follows from [**35**, Theorem 3.8] that $\mathrm{proj}_{\pi_2}(c) = d$, and so by symmetry $\mathrm{proj}_{\pi_1}(d) = c$. Similarly we have $\mathrm{proj}_{\pi_2}(\mathrm{proj}_{\pi_1}(d)) = d$ for all $d \in \mathrm{st}(\pi_2)$. So the map φ is bijective. $\square$

Lemma B.1.1 allows us to make the following (temporary) definitions. Given a wall H of $\mathscr{X}$, write $q_H = q_i$, where i is the cotype of any panel of H. Now choose any apartment $\mathcal{A}$ of $\mathscr{X}$, and let $\psi : \mathcal{A} \to \Sigma$ be a type-rotating isomorphism. For each $\alpha \in R$ and $k \in \mathbb{Z}$, write $q_{\alpha;k} = q_H$, where $H = \psi^{-1}(H_{\alpha;k})$. We must show that this definition is independent of the particular $\mathcal{A}$ and ψ chosen. To see this, let $\mathcal{A}'$ be any (perhaps different) apartment of $\mathscr{X}$, and let $\psi' : \mathcal{A}' \to \Sigma$ be a type-rotating isomorphism. Write $H' = \psi'^{-1}(H_{\alpha;k})$. With H as above, let π be a panel of H, with cotype i, say, and so $q_H = q_i$. The isomorphism $\psi'^{-1} \circ \psi : \mathcal{A} \to \mathcal{A}'$ is type-rotating and sends H to H'. Thus $(\psi'^{-1} \circ \psi)(\pi)$ is a panel of H' with cotype $\sigma(i)$ for some $\sigma \in \mathrm{Aut}_{\mathrm{tr}}(D)$, and so $q_{H'} = q_{\sigma(i)} = q_i = q_H$, by Theorem 3.8.4.

LEMMA B.1.2. *Let R be reduced. Then each wall of Σ contains an element of P.*

PROOF. Each panel of each wall $H_{\alpha;k}$, $\alpha \in R$, $k \in \mathbb{Z}$, contains $n-1$ vertices whose types are pairwise distinct. Since R is reduced, the good vertices are simply the *special* *vertices*, that is, the elements of P (see [**5**, VI, §2, No.2, Proposition 3]). Thus when there are two or more good types the result follows.

This leaves the cases E_8, F_4 and G_2. Since $H_{-\alpha;k} = H_{\alpha;-k}$ it suffices to prove the result when $\alpha \in R^+$. Using the data in [**5**, Plates VII-IV] we see that for each $\alpha = \sum_{i=1}^n a_i\alpha_i \in R^+$ there is an index i_1 such that $a_{i_1} = 1$, or a pair of indices (i_2, i_3) such that $a_{i_2} = 2$ and $a_{i_3} = 3$. In the former case $k\lambda_{i_1} \in H_{\alpha;k}$, and in the latter case $\frac{k}{2}\lambda_{i_2} \in H_{\alpha;k}$ if k is even, and $\frac{k-3}{2}\lambda_{i_2} + \lambda_{i_3} \in H_{\alpha;k}$ if k is odd. $\square$

PROPOSITION B.1.3. *If R is reduced, then $q_{\alpha;k} = q_\alpha$ for all $\alpha \in R$ and $k \in \mathbb{Z}$.*

PROOF. The proof consists of the following steps:

(i) $q_{w\alpha;0} = q_{\alpha;0}$ for all $\alpha \in R$ and $w \in W_0$.
(ii) $q_{\alpha_i;0} = q_{\alpha_i}$ for each $i = 1, \ldots, n$.
(iii) $q_{\alpha;0} = q_\alpha$ for all $\alpha \in R$.
(iv) $q_{\alpha;k} = q_{\alpha;0}$ for all $\alpha \in R$ and $k \in \mathbb{Z}$.

(i) Let $\mathcal{A}$ be an apartment of $\mathscr{X}$, and let $\psi : \mathcal{A} \to \Sigma$ be a type-rotating isomorphism. Write $H = \psi^{-1}(H_{\alpha;0})$, so that $q_{\alpha;0} = q_H$. Let $w \in W_0$. Now the isomorphism $\psi' = w \circ \psi : \mathcal{A} \to \Sigma$ is type-rotating, and $\psi'^{-1}(H_{w\alpha;0}) = \psi^{-1}(H_{\alpha;0}) = H$. Thus $q_{w\alpha;0} = q_H = q_{\alpha;0}$.

(ii) Let C_0 be the fundamental chamber of Σ, and for each $i = 1, \ldots, n$ let $C_i = s_iC_0$. Let $\mathcal{A}$ and $\psi : \mathcal{A} \to \Sigma$ be as in (i), and write $H = \psi^{-1}(H_{\alpha_i;0})$. Then $\delta(\psi^{-1}(C_0), \psi^{-1}(C_i)) = s_{\sigma(i)}$ for some $\sigma \in \mathrm{Aut}_{\mathrm{tr}}(D)$, and so $q_{\alpha_i;0} = q_H = q_{\sigma(i)}$, and so by Theorem 3.8.4 $q_{\alpha_i;0} = q_i = q_{\alpha_i}$.

(iii) Each $\alpha \in R$ is equal to $w\alpha_i$ for some $w \in W_0$ and some i, and so (iii) follows from (i) and (ii).

(iv) Let $\alpha \in R$ and $k \in \mathbb{Z}$. By Lemma B.1.2 there exists $\lambda \in H_{\alpha;k} \cap P$, and so $H_{\alpha;k} = t_\lambda(H_{\alpha;0})$. Let $\mathcal{A}$ and ψ be as in (i), and write $H = \psi^{-1}(H_{\alpha;k})$, so that $q_{\alpha;k} = q_H$. The map $\psi' = t_\lambda^{-1} \circ \psi : \mathcal{A} \to \Sigma$ is a type-rotating isomorphism, and $\psi'^{-1}(H_{\alpha;0}) = \psi^{-1}(H_{\alpha;k}) = H$. Thus $q_{\alpha;k} = q_H = q_{\alpha;0}$. $\square$

We need an analogue of Proposition B.1.3 when R is of type BC_n for some $n \geq 1$. Observe that if $\alpha \in R_1\backslash R_3$, then $\alpha/2 \in R_2\backslash R_3$, and $H_{\alpha;2k} = H_{\alpha/2;k}$ for all $k \in \mathbb{Z}$. Thus we define $\mathcal{H}_1 = \{H_{\alpha;k} \mid \alpha \in R_1\backslash R_3,\ k \text{ odd}\}$, $\mathcal{H}_2 = \{H_{\alpha;k} \mid \alpha \in R_2\backslash R_3,\ k \in \mathbb{Z}\}$ and $\mathcal{H}_3 = \{H_{\alpha;k} \mid \alpha \in R_3\}$. Then $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$, where the union is disjoint. We have

$$q_{\alpha;k} = \begin{cases} q_0 & \text{if } H_{\alpha;k} \in \mathcal{H}_1 \\ q_n & \text{if } H_{\alpha;k} \in \mathcal{H}_2 \\ q_\alpha & \text{if } H_{\alpha;k} \in \mathcal{H}_3 \end{cases} \tag{B.1.1}$$

We omit the details of this calculation.

REMARK B.1.4. Proposition B.1.3 and formula (B.1.1) give the connection between our definitions of R and τ_α and Macdonald's definitions of Σ_1 and q_a [**23**, §3.1].

Recall that for $\tilde{w} \in \tilde{W}$ we define $\mathcal{H}(\tilde{w}) = \{H \in \mathcal{H} \mid H$ separates C_0 and $\tilde{w}C_0\}$. Also, observe that each $H \in \mathcal{H}$ is equal to $H_{\alpha;k}$ for some $\alpha \in R_1^+$ and some $k \in \mathbb{Z}$, and if $H_{\alpha;k} = H_{\alpha';k'}$ with $\alpha, \alpha' \in R_1^+$ and $k, k' \in \mathbb{Z}$, then $\alpha = \alpha'$ and $k = k'$.

Proposition B.1.5. *Let $\lambda \in P^+$. Then*

$$q_{t_\lambda} = \prod_{\alpha \in R^+} \tau_\alpha^{\langle \lambda, \alpha \rangle}.$$

Proof. Write $t_\lambda = t'_\lambda g_l$, where $t'_\lambda \in W$ and $l = \tau(\lambda)$. Then $\mathcal{H}(t_\lambda) = \mathcal{H}(t'_\lambda)$ and $q_{t_\lambda} = q_{t'_\lambda}$. Suppose that $t'_\lambda = s_{i_1} \cdots s_{i_m}$ is a reduced expression for t'_λ. Writing $H_i = H_{\alpha_i;0}$ if $i = 1, \ldots, n$ and $H_0 = H_{\tilde{\alpha};1}$, we have

$$\begin{aligned} \mathcal{H}(t_\lambda) &= \{H_{i_1}, s_{i_1}H_{i_2}, \ldots, s_{i_1} \cdots s_{i_{m-1}} H_{i_m}\} \\ &= \{H_{\alpha;k_\alpha} \mid \alpha \in R_1^+ \text{ and } 0 < k_\alpha \leq \langle \lambda, \alpha \rangle\} \end{aligned} \tag{B.1.2}$$

where the hyperplanes in each set are pairwise distinct ([**19**, Theorem 4.5]). If $1 \leq r \leq m$ and if $s_{i_1} \cdots s_{i_{r-1}} H_{i_r} = H_{\alpha;k}$ then it is easy to see that $q_{\alpha;k} = q_{i_r}$. Then using (B.1.2), Proposition B.1.3, (B.1.2), and the fact that $\langle \lambda, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in R_1 \backslash R_3$, we have

$$q_{t_\lambda} = \prod_{r=1}^m q_{i_r} = \prod_{\alpha \in R_1^+} \prod_{k_\alpha = 1}^{\langle \lambda, \alpha \rangle} q_{\alpha;k_\alpha} = \Bigg[\prod_{\alpha \in R_3^+} q_\alpha^{\langle \lambda, \alpha \rangle}\Bigg]\Bigg[\prod_{\alpha \in R_1^+ \backslash R_3^+} (q_0 q_n)^{\langle \lambda, \alpha \rangle / 2}\Bigg],$$

and the result follows by direct calculation. $\square$

B.2. The Topology on Ω

Here we give a sketch of the following theorem, which was used in the construction of the topology on Ω. Recall the definition of the maps $\varphi_{\mu,\lambda}$ made in the opening paragraphs of Section 7.5.

Theorem B.2.1. *Fix $x \in V_P$ and define $\theta : \Omega \to \prod_{\lambda \in P^+} V_\lambda(x)$ by $\omega \mapsto (v_\lambda^x(\omega))_{\lambda \in P^+}$. Then θ is a bijection of Ω onto $\varprojlim(V_\lambda(x), \varphi_{\mu,\lambda})$.*

Proof. It is not too difficult to see that if $\mathcal{S}$ and $\mathcal{S}'$ are sectors with the same good vertices, then $\mathcal{S} = \mathcal{S}'$. Thus it is clear that θ is injective. To show that θ is surjective, let $(v_\nu)_{\nu \in P^+} \in \varprojlim(V_\lambda(x), \varphi_{\mu,\lambda})$. For each $m \geq 1$ let $\mu_m = m(\lambda_1 + \cdots + \lambda_n)$, and let $\mathcal{C}(x; m)$ denote the set of chambers contained in the intersection of all half-apartments containing x and v_{μ_m}. Since $\mu_m \in P^{++}$ the sets $\mathcal{C}(x; m)$ are nonempty for all $m \geq 1$, and $\mathcal{C}(x; m) \subset \mathcal{C}(x; k)$ whenever $m \leq k$. Furthermore, for $m \geq 1$ write $\mathcal{C}_m$ for the set of chambers of Σ contained in the intersection of all half-spaces containing 0 and μ_m.

For each $m \geq 1$ there exists an apartment $\mathcal{A}_m$ containing x and v_{μ_m}, and a type-rotating isomorphism $\psi_m : \mathcal{A}_m \to \Sigma$ such that $\psi_m(x) = 0$ and $\psi_m(v_{\mu_m}) = \mu_m$. Furthermore, if $\mathcal{A}'_m$ and ψ'_m also have these properties, then it is easy to see that $\psi_m|_{\mathcal{C}(x;m)} = \psi'_m|_{\mathcal{C}(x;m)}$. Also, $\psi_{m+1}|_{\mathcal{C}(x;m)} = \psi_m|_{\mathcal{C}(x;m)}$ for all $m \geq 1$.

For each $m \geq 1$ define $\xi_m : \mathcal{C}_m \to \mathscr{X}$ by $\xi_m = \psi_m^{-1}|_{\mathcal{C}_m}$. Since $\xi_{m+1}|_{\mathcal{C}_m} = \xi_m$ we have $\xi_k|_{\mathcal{C}_m} = \xi_m$ for all $k \geq m$. We may therefore define $\xi : \mathcal{C}(\mathcal{S}_0) \to \mathscr{X}$ by $\xi(C) = \xi_m(C)$ once $C \in \mathcal{C}_m$. By replacing the type map $\tau : V(\Sigma) \to I$ on Σ by $\sigma_i \circ \tau$ where $i = \tau(x)$, we may take all of the above isomorphisms to be type preserving, and so by [**35**, Theorem 3.6] we see that ξ extends to an isometry $\tilde{\xi} : \mathcal{C}(\Sigma) \to \mathscr{X}$. Then $\tilde{\xi}(\mathcal{C}(\Sigma))$ is an apartment of $\mathscr{X}$, and $\mathcal{S} = \tilde{\xi}(\mathcal{C}(\mathcal{S}_0))$ is a sector. Let ω be the class of $\mathcal{S}$. Then $\theta(\omega) = (v_\nu)_{\nu \in P^+}$. □

B.3. The Exceptional Case

In this section we prove Lemma 8.3.4. Let R be a root system of type BC_n for some $n \geq 1$, and suppose that $q_n < q_0$.

Lemma B.3.1. $N_{\lambda_1} = (1 + q_1^{n-1} q_n)(1 + q_1 + \cdots + q_1^{n-1}) q_0$.

Proof. Note that $\lambda_1 = e_1 = \tilde{\alpha}^\vee$, and it follows from Proposition 3.7.3(ii) that $w_{\lambda_1} = s_0$, and so $q_{w_{\lambda_1}} = q_0$. Thus by Theorem 4.3.4, $N_{\lambda_1} = \frac{W_0(q)}{W_{0\lambda_1}(q)} q_0$. By [**24**, §2.2] we have

$$W_0(q) = \prod_{i=0}^{n-1} (1 + q_1^i q_n)(1 + q_1 + \cdots + q_1^i),$$

and since $\lambda_1 = e_1$ we easily see (using Lemma 4.2.1 with $\lambda = \lambda' = \lambda_1$) that $W_{0\lambda_1}$ is a Coxeter group of type C_{n-1}, and so by the same formula

$$W_{0\lambda_1}(q) = \prod_{i=0}^{n-2} (1 + q_1^i q_n)(1 + q_1 + \cdots + q_1^i).$$

The result follows. □

Let $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ be parametrised by the numbers $t_j = u^{e_j}$, $j = 1, \ldots, n$, as in Section 6.3.2. In the following lemma we obtain a more explicit formula for $P_{\lambda_1}(u)$ (in terms of the numbers $\{t_j\}_{j=1}^n$).

Lemma B.3.2. *Let $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$. Then*

$$P_{\lambda_1}(u) = N_{\lambda_1}^{-1} \left((q_0 - 1)(1 + q_1 + \cdots + q_1^{n-1}) + \sqrt{q_0 q_n} q_1^{n-1} \sum_{j=1}^n (t_j + t_j^{-1}) \right).$$

Proof. Since $\lambda_1 = \tilde{\alpha}^\vee$ we have $\Pi_{\lambda_1} = \{0\} \cup W_0 \lambda_1$, and so by (7.7.2) (and W_0-invariance) we have

$$P_{\lambda_1}(u) = a_{\lambda_1,\nu;\nu} + r^{-\lambda_1} a_{\lambda_1,\nu-\lambda_1;\nu} \sum_{\mu \in W_0 \lambda_1} t_1^{\langle \mu, e_1 \rangle} \cdots t_n^{\langle \mu, e_n \rangle}$$

for suitably large $\nu \in P^+$. Since $W_0 \lambda_1 = \{\pm e_j \mid 1 \leq j \leq n\}$ we have

$$P_{\lambda_1}(u) = a_{\lambda_1,\nu;\nu} + r^{-\lambda_1} a_{\lambda_1,\nu-\lambda_1;\nu} \sum_{j=1}^n (t_j + t_j^{-1}).$$

It remains to compute the constants.

Let us first consider $r^{-\lambda_1}a_{\lambda_1,\nu-\lambda_1;\nu}$. Recall that $\lambda^*=\lambda$ for all $\lambda\in P^+$ in the BC_n case. By (4.4.3), Lemma 7.1.2, and Corollary 7.5.6, if $\nu\in P^+$ is suitably large, and if $y\in V_\nu(x)$, then

$$r^{-\lambda_1}a_{\lambda_1,\nu-\lambda_1;\nu}=r^{-\lambda_1}\frac{N_\nu}{N_{\lambda_1}N_{\nu-\lambda_1}}|V_{\lambda_1}(x)\cap V_{\nu-\lambda_1}(y)|=r^{\lambda_1}N_{\lambda_1}^{-1}.$$

Since $R^+=\{e_i,2e_i,e_j\pm e_k\mid 1\leq i\leq n,1\leq j<k\leq n\}$, by (7.5.2) we compute

$$r^{\lambda_1}=\prod_{\alpha\in R^+}\tau_\alpha^{\frac{1}{2}\langle\lambda_1,\alpha\rangle}=\sqrt{q_0q_n}q_1^{n-1},$$

and so $r^{-\lambda_1}a_{\lambda_1,\nu-\lambda_1;\nu}=N_{\lambda_1}^{-1}\sqrt{q_0q_n}q_1^{n-1}$.

Let us now consider $a_{\lambda_1,\nu;\nu}$. We have $a_{\lambda_1,\nu;\nu}=N_{\lambda_1}^{-1}|V_{\lambda_1}(x)\cap V_\nu(y)|$ where $y\in V_\nu(x)$. It follows from (7.7.1) and (7.7.2) (and W_0-invariance) that for ν suitably large

$$a_{\lambda_1,\nu-\mu;\nu}=r^{\mu-\lambda_1}a_{\lambda_1,\nu-\lambda_1;\nu}\quad\text{for all }\mu\in W_0\lambda_1,$$

and so (using (4.4.3) and Corollary 7.5.6)

$$|V_{\lambda_1}(x)\cap V_{\nu-\mu}(y)|=r^{\lambda_1-\mu}|V_{\lambda_1}(x)\cap V_{\nu-\lambda_1}(y)|=r^{\lambda_1-\mu}. \tag{B.3.1}$$

Also note that

$$\sum_{\mu\in\Pi_{\lambda_1}}|V_{\lambda_1}(x)\cap V_{\nu-\mu}(y)|=N_{\lambda_1}. \tag{B.3.2}$$

To see this, by the proof of Theorem 7.7.2 (and in the notation used there) we have $a_{\lambda_1,\nu-\mu;\nu}=r^\mu a_{\lambda_1,\mu}$ for sufficiently large $\nu\in P^+$, and since $a_{\lambda_1,\mu}=0$ if $\mu\notin\Pi_{\lambda_1}$ (see (7.7.1)), it follows from (4.4.3) that $V_{\lambda_1}(x)=\bigcup_{\mu\in\Pi_{\lambda_1}}V_{\nu-\mu}(y)$, and (B.3.2) follows.

Thus by (B.3.2) and (B.3.1) we have

$$\begin{aligned}|V_{\lambda_1}(x)\cap V_\nu(y)|&=N_{\lambda_1}-\sum_{\mu\in W_0\lambda_1}|V_{\lambda_1}(x)\cap V_{\nu-\mu}(y)|\\&=N_{\lambda_1}-r^{\lambda_1}\sum_{\mu\in W_0\lambda_1}r^{-\mu}.\end{aligned} \tag{B.3.3}$$

A short calculation shows that

$$r^\mu=(q_0q_n)^{\frac{1}{2}\langle\mu,\sum_{i=1}^n e_i\rangle}q_1^{\langle\mu,\sum_{j=1}^n(n-j)e_j\rangle}. \tag{B.3.4}$$

Thus by (B.3.3) and (B.3.4) we deduce that

$$\begin{aligned}|V_{\lambda_1}(x)\cap V_\nu(y)|&=N_{\lambda_1}-r^{\lambda_1}\sum_{i=1}^n\left(\sqrt{q_0q_n}\,q_1^{n-i}+\frac{1}{\sqrt{q_0q_n}}\,q_1^{i-n}\right)\\&=N_{\lambda_1}-(q_0q_nq_1^{n-1}+1)(1+q_1+\cdots+q_1^{n-1}).\end{aligned} \tag{B.3.5}$$

The result follows from Lemma B.3.1, (B.3.5), and (4.4.3). □

Recall that $\mathbb{U}'$ consists of those $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ such that $t_1 = -b = -\sqrt{q_n/q_0}$ and $t_2, \ldots, t_n \in \mathbb{T}$. Write $\xi_t = (-b, t_2, \ldots, t_n)$, where $t_2, \ldots, t_n \in \mathbb{T}$. Writing $t_j = e^{i\theta_j}$ for $2 \leq j \leq n$, from Lemma B.3.2 we have

$$P_{\lambda_1}(\xi_t) = c_1 - \frac{q_0 + q_n}{\sqrt{q_0 q_n}} c_2 + 2c_2 \sum_{j=2}^{n} \cos\theta_j, \tag{B.3.6}$$

where $c_1 = N_{\lambda_1}^{-1}(q_0 - 1)(1 + q_1 + \cdots + q_1^{n-1})$ and $c_2 = N_{\lambda_1}^{-1}\sqrt{q_0 q_n} q_1^{n-1}$.

Note that since $\lambda^* = \lambda$ for all $\lambda \in P^+$, $P_\lambda(u) \in \mathbb{R}$ for all $u \in U$.

THEOREM B.3.3. *Let ξ_t be as above. Then*

$$|P_{\lambda_1}(\xi_t)| < P_{\lambda_1}(1).$$

PROOF. Note that $|W_0 \lambda_1| = 2n$, and so $P_{\lambda_1}(1) = c_1 + 2nc_2$, with c_1 and c_2 as above (alternatively, take $t_1 = \cdots = t_n = 1$ in Lemma B.3.2). Since $c_1, c_2 > 0$, the inequality $P_{\lambda_1}(\xi_t) < P_{\lambda_1}(1)$ is clear from (B.3.6). We now show that $-P_{\lambda_1}(\xi_t) < P_{\lambda_1}(1)$. We have

$$\begin{aligned} N_{\lambda_1}\big(P_{\lambda_1}(1) + P_{\lambda_1}(\xi_t)\big) &\geq N_{\lambda_1}\left(2c_1 + 2c_2 - \frac{q_0 + q_n}{\sqrt{q_0 q_n}} c_2\right) \\ &= 2(q_0 - 1)(1 + q_1 + \cdots + q_1^{n-1}) + \sqrt{q_0 q_n}\, q_1^{n-1} + (\sqrt{q_0 q_n} - q_n) q_1^{n-1} - q_0 q_1^{n-1} \\ &> 2(q_0 - 1)(1 + q_1 + \cdots + q_1^{n-1}) + \sqrt{q_0 q_n}\, q_1^{n-1} - q_0 q_1^{n-1} \\ &= 2(q_0 - 1)(1 + q_1 + \cdots + q_1^{n-2}) + (\sqrt{q_0 q_n} - 1) q_1^{n-1} + (q_0 - 1) q_1^{n-1} \\ &\geq 0, \end{aligned}$$

where the strict inequality holds since $q_0 > q_n$, completing the proof. □

LEMMA B.3.4. *If $\lambda \neq 0$ then $a_{\lambda,\lambda;\lambda_1} \neq 0$.*

PROOF. By Lemma 8.2.1 and the fact that $\lambda_1^* = \lambda_1$ we have $a_{\lambda,\lambda_1;\lambda} \neq 0$ for all $\lambda \neq 0$. The result now follows from Proposition 4.4.11, since $\lambda^* = \lambda$ for all $\lambda \in P^+$.

Alternatively the result may be proved in a similar way to Lemma 8.2.1. □

THEOREM B.3.5. *Let ξ_t be as above. For all $\lambda \neq 0$ we have $|P_\lambda(\xi_t)| < P_\lambda(1)$.*

PROOF. For any $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ we have

$$|P_\lambda^2(u)| = |h_u(A_\lambda^2)| \leq \sum_{\mu \in P^+} a_{\lambda,\lambda;\mu} |P_\mu(u)|. \tag{B.3.7}$$

Since the algebra homomorphisms h_{ξ_t} are continuous with respect to the ℓ^2-operator norm (see Corollary 6.3.8), and since $\|A_\mu\| = P_\mu(1)$ (see Theorem 7.7.3) we have

$$|P_\mu(\xi_t)| \leq P_\mu(1) \quad \text{for all } \mu \in P^+.$$

By Lemma B.3.4 we have $a_{\lambda,\lambda;\lambda_1} > 0$, and so Theorem B.3.3 and (B.3.7) imply that $|P_\lambda^2(\xi_t)| < P_\lambda^2(1)$, proving the result. □

Proof of Lemma 8.3.4. Since $\widehat{A}(u)$ is continuous on $\mathbb{U}'$, and since $\mathbb{U}'$ is compact,

$$\rho_2 = \max\{|\widehat{A}(u)|/\widehat{A}(1) : u \in \mathbb{U}'\} = \max\{|\widehat{A}(\xi_t)|/\widehat{A}(1) : t \in \mathbb{T}^{n-1}\} < 1$$

by Theorem B.3.5, and the result easily follows. $\square$

B.4. A Building Proof

In this section we give a 'building theoretic' proof of Lemma 7.1.2. The author would like to thank Donald Cartwright for the results of this section.

Lemma B.4.1. *Let $x \in V_P$ and let $\mathcal{S}$ be a sector in $\mathscr{X}$ based at x. Let*

$$D_0, \ldots, D_\ell \tag{B.4.1}$$

be a reduced gallery in $\mathscr{X}$ such that $x \in D_0$. Then either

- *there is an apartment $\mathcal{A}$ containing $\mathcal{S}$ and $D_0, \ldots, D_\ell$, or*
- *there is an apartment $\mathcal{A}$ containing $\mathcal{S}$ so that, writing $\rho = \rho_{\mathcal{A},\mathcal{S}}$, the sequence $\rho(D_0), \ldots, \rho(D_\ell)$ has $\rho(D_{j-1}) = \rho(D_j)$ for some $1 \le j \le \ell$.*

Proof. By Lemma C.4.3, there is an apartment containing $\mathcal{S}$ and D_0. Suppose that for some $k < \ell$ there is an apartment $\mathcal{A}$ such that $S, D_0, \ldots, D_k \subset \mathcal{A}$, but no apartment containing $\mathcal{S}, D_0, \ldots, D_{k+1}$. Let H be the wall in $\mathcal{A}$ which contains a panel common to D_k and D_{k+1}. Let H^- be the closed half-space of $\mathcal{A}$ containing D_k, and let H^+ be the other closed half-space of $\mathcal{A}$ determined by H.

Let f be the type of the gallery (B.4.1). There is a unique gallery in $\mathcal{A}$ of type f starting at D_0. Let us denote this gallery by

$$D_0' = D_0, D_1', \ldots, D_\ell'. \tag{B.4.2}$$

If g is the word of length k at the beginning of f, then since there is a unique gallery in $\mathcal{A}$ of type g starting at D_0, we must have $D_j = D_j'$ for $j = 0, \ldots, k$. Being reduced, the gallery (B.4.2) crosses H at most once. If $D_k \sim_i D_{k+1}$, then $D_k \sim_i D_{k+1}'$ because galleries (B.4.1) and (B.4.2) are of the same type. So H separates D_k and D_{k+1}'. Therefore (B.4.2) crosses H precisely once, and

$$D_0, \ldots, D_k \subset H^- \quad \text{and} \quad D_{k+1}', \ldots, D_\ell' \subset H^+.$$

By Lemma 7.4.7, either H^- or H^+ contains a subsector of $\mathcal{S}$. If H^- contains a subsector of $\mathcal{S}$, then since $x \in D_0 \subset H^-$, the whole sector $\mathcal{S}$ is contained in H^- [**35**, Lemma 9.7]. By the proof of [**35**, Lemma 9.4], there is an apartment $\mathcal{A}'$ containing H^- and D_{k+1}. But then $\mathcal{S}, D_0, \ldots, D_k, D_{k+1} \subset \mathcal{A}'$, contrary to hypothesis.

So H^+ must contain a subsector $\mathcal{S}_1$ of $\mathcal{S}$. Pick an apartment $\mathcal{A}'$ containing H^+ and D_{k+1}. By the building axioms, there is an isomorphism $\phi : \mathcal{A}' \to \mathcal{A}$ which fixes $\mathcal{A} \cap \mathcal{A}'$, and in particular H^+, and therefore $\mathcal{S}_1$. Since $D_{k+1}' \subset H^+$, we have $\phi(D_{k+1}') = D_{k+1}'$. So $\phi(D_{k+1})$

cannot be D'_{k+1}, and so must be D_k. Let $\rho = \rho_{\mathcal{A},\mathcal{S}}$. Then $\rho(D_{k+1}) = \phi(D_{k+1}) = D_k = \rho(D_k)$, proving the result. $\square$

LEMMA B.4.2. $-\Pi_\lambda = \Pi_{\lambda^*}$ *for all* $\lambda \in P^+$.

PROOF. It is clear from the definition that $-\Pi_\lambda$ is saturated. Furthermore, the highest coweight of $-\Pi_\lambda$ is λ^*. To see this, observe first that $\lambda^* = -w_0\lambda$, and so $\lambda^* \in -\Pi_\lambda$. Let $\nu \in -\Pi_\lambda$. Then $\nu^* = w_0(-\nu) \in \Pi_\lambda$, and so $\lambda - \nu^* \in Q^+$. Since $w_0 B = -B$ we have $\lambda^* - \nu \in Q^+$, and so $\nu \preceq \lambda^*$. $\square$

PROPOSITION B.4.3. *Let* $x, y \in V_P$, *and let* $y \in V_{\lambda+\mu}(x)$, *where* $\lambda, \mu \in P^+$. *Then the set* $V_\lambda(x) \cap V_{\mu^*}(y)$ *contains exactly one element. This element lies in any apartment containing* x *and* y.

PROOF. Let $z \in V_\lambda(x) \cap V_{\mu^*}(y)$. Let $\tau(x) = i$, and let $D_0, \ldots, D_\ell$ be a gallery of type $\sigma_i(f_\lambda)$ such that $x \in D_0$ and $z \in D_\ell$. Choose a sector $\mathcal{S}$ containing x and y, based at x, and such that $y \in \mathcal{S}$. Let $\mathcal{A}$ be any apartment containing $\mathcal{S}$. Let $\rho = \rho_{\mathcal{A},\mathcal{S}}$ and $\psi = \psi_{\mathcal{A},\mathcal{S}}$. Now let $\mathcal{S}^y$ be the subsector of $\mathcal{S}$ based at y. It is clear from their definitions that $\rho_{\mathcal{A},\mathcal{S}^y} = \rho$ and that $\psi_{\mathcal{A},\mathcal{S}^y} = t_{-\lambda-\mu} \circ \psi$.

Let $\nu = (\psi \circ \rho)(z)$. By Theorem 7.4.2(ii), $\nu \in \Pi_\lambda$ and $(\psi_{\mathcal{A},\mathcal{S}^y} \circ \rho_{\mathcal{A},\mathcal{S}^y})(z) \in \Pi_{\mu^*}$, so that $\nu - \lambda - \mu \in \Pi_{\mu^*}$. Hence $\lambda + \mu - \nu \in \Pi_\mu$ by Lemma B.4.2. Since we have both $\nu \preceq \lambda$ and $\lambda + \mu - \nu \preceq \mu$. If either of these "$\preceq$" were "$\prec$", then adding we would have $\lambda + \mu \prec \lambda + \mu$, a contradiction. Hence $\nu = \lambda$.

Writing $\Phi = \psi \circ \rho$, by the proof of Theorem 7.4.1, the sequence $\Phi(D_0), \ldots, \Phi(D_\ell)$ is a pre-gallery in Σ of type f_λ from 0 to λ. So $\rho(D_0), \ldots, \rho(D_\ell)$ is in fact a gallery. Since this is true for any apartment $\mathcal{A}$ containing $\mathcal{S}$, by Lemma B.4.1 there is an apartment containing $\mathcal{S}$ and $D_0, \ldots, D_\ell$, and in particular x, y and z.

So let $\mathcal{A}$ be an apartment containing x, y and z, and let $\mathcal{S}$ be a sector in $\mathcal{A}$ based at x and containing y. Let $\psi = \psi_{\mathcal{A},\mathcal{S}}$. Then $\psi(x) = 0$, $\psi(y) = \lambda + \mu$, and $\psi(z) = w\lambda$ for some $w \in W_0$. Since $y \in V_\mu(z)$, we have $\lambda + \mu = w\lambda + w'\mu$ for some $w' \in W_0$. Since $w\lambda \preceq \lambda$ and $w'\mu \preceq \mu$, we must have $w\lambda = \lambda$ and $w'\mu = \mu$. Hence $\psi(z) = \lambda \in \mathcal{S}_0$, so that $z \in \mathcal{S}$.

So we have shown that any $z \in V_\lambda(x) \cap V_{\mu^*}(y)$ lies in any sector $\mathcal{S}$ containing x and y and based at x. In particular, any such z lies in any apartment containing x and y.

Suppose that $z, z' \in V_\lambda(x) \cap V_{\mu^*}(y)$. Choose any apartment $\mathcal{A}$ containing x and y, and any sector $\mathcal{S}$ in $\mathcal{A}$ based at x and containing y. Let $\psi = \psi_{\mathcal{A},\mathcal{S}}$. Then the above shows that z and z' are both in $\mathcal{S}$, and that $\psi(z) = \lambda = \psi(z')$. Hence $z = z'$. $\square$

APPENDIX C

Some Elementary Calculations in Low Dimension

In this appendix we show how the algebra $\mathscr{A}$ can be studied in an 'elementary' way in low dimensional cases. We also demonstrate how the Macdonald formula for the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ can be computed without the machinery of affine Hecke algebras in these cases.

The method here involves explicitly computing the numbers $a_{\lambda,\lambda_i;\mu}$ for all $\lambda, \mu \in P^+$ and $i \in I_0$, thus providing explicit formulae $A_\lambda A_{\lambda_i} = \sum_{\mu \in P^+} a_{\lambda,\lambda_i;\mu} A_\mu$. This gives an analogue of [**25**, (3.2)] for these low dimensional buildings, and is certainly of some independent interest (for our Hecke algebra methods give no such explicit formulae). Our arguments follow [**11**], where affine buildings of type A_2 are studied.

In Section C.1 we consider affine buildings of type BC_1 and A_1, where the calculations are relatively straightforward. In the context of semi-homogeneous trees (the BC_1 case) and homogeneous trees (the A_1 case), these calculations have a rather long history: see [**16**] for semi-homogeneous trees, and see [**8**], [**36**], and [**14**] for homogeneous trees. See also [**44**] for calculations involving infinite distance regular graphs (cf. Remark 3.8.3).

In Sections C.4 and C.7 we discuss affine buildings of types BC_2 and G_2, where the technique becomes rather complicated. We will deduce the results for affine buildings of type C_2 from the BC_2 case. For the sake of completeness we also list some results from [**11**] in the A_2 case.

C.1. The BC_1 Case

Let R be a root system of type BC_1. Thus we may take $E = \mathbb{R}$, $B = \{e_1\}$, and $R^+ = \{e_1, 2e_1\}$. We have $\lambda_1 = e_1$, and so $P = \{ke_1 \mid k \in \mathbb{Z}\}$ and $P^+ = \{ke_1 \mid k \in \mathbb{N}\}$. The fundamental chamber of Σ is $C_0 = (0, 1/2)$, and the vertices of Σ are the elements $k/2$, where $k \in \mathbb{Z}$. The set of good vertices of Σ is $\mathbb{Z}$; these vertices are shown as solid circles in Figure C.1.1 (note that hyperplanes are zero dimensional).

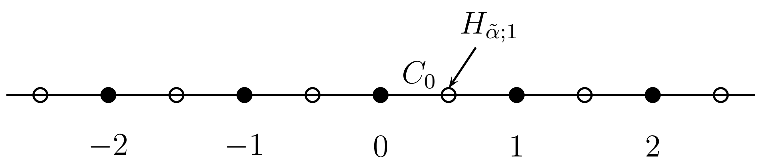


FIGURE C.1.1

Let $\mathscr{X}$ be an affine building of type BC_1 with parameters $q_0 = p$ and $q_1 = q$. Thus $\mathscr{X}$ is a semi-homogeneous tree, as shown in Figure C.1.2 for the case $p = 3$ and $q = 2$.

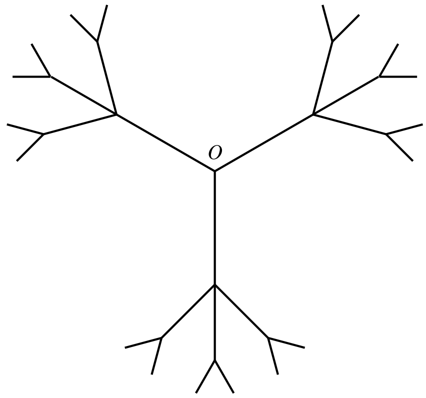


FIGURE C.1.2

The vertex o in Figure C.1.2 is a good vertex (since $q_0 = 3$), and writing V for the vertex set of $\mathscr{X}$, $V_P = \{x \in V \mid d(o,x) \in 2\mathbb{Z}\}$.

For $x \in V_P$ and $k \geq 0$, write $V_k(x)$ in place of $V_{k\lambda_1}(x)$. Thus

$$V_k(x) = \{y \in V \mid d(x,y) = 2k\}. \tag{C.1.1}$$

Let us study the algebra $\mathscr{A}$ in elementary terms.

LEMMA C.1.1. *The numbers $N_k = |V_k(x)|$ are independent of $x \in V_P$, and are given by $N_0 = 1$, and for $k \geq 1$*

$$N_k = (q+1)p(pq)^{k-1}.$$

PROOF. Using (C.1.1) we have $|V_0(x)| = 1$ and $|V_{k+1}(x)| = |V_k(x)|pq$ if $k \geq 1$. Since $|V_1(x)| = p(q+1)$ the result follows. $\square$

LEMMA C.1.2. *Let $x \in V_P$ and $k \geq 1$. Then*

$$|V_k(x) \cap V_1(z)| = \begin{cases} pq & \text{if } z \in V_{k-1}(x) \text{ and } k \geq 2 \\ p(q+1) & \text{if } z \in V_{k-1}(x) \text{ and } k = 1 \\ p-1 & \text{if } z \in V_k(x) \\ 1 & \text{if } z \in V_{k+1}(x). \end{cases}$$

PROOF. These counts are obvious from (C.1.1). $\square$

COROLLARY C.1.3. *For $k \geq 1$ and $l = k-1, k$ or $k+1$, let*

$$a_{k,1;l} = \frac{N_l}{N_k N_1}|V_k(x) \cap V_1(z)| \quad \text{where } z \in V_l(x).$$

The numbers $a_{k,1;l}$ depend only on k, l, p and q, and are given by

$$a_{k,1;l} = \begin{cases} \frac{1}{p(q+1)} & \text{if } l = k-1 \\ \frac{p-1}{p(q+1)} & \text{if } l = k \\ \frac{q}{q+1} & \text{if } l = k+1. \end{cases}$$

PROOF. This is immediate from Lemmas C.1.1 and C.1.2. $\square$

For $k \in \mathbb{N}$, write A_k in place of $A_{k\lambda_1}$ (see Definition 4.4.1). Thus

$$(A_k f)(x) = \frac{1}{N_k} \sum_{y \in V_k(x)} f(y) \qquad \text{for all } f : V_P \to \mathbb{C} \text{ and } x \in V_P.$$

For all $m \in \mathbb{N}$, $u \in V_m(v)$ if and only if $v \in V_m(u)$, and so for $k, l \in \mathbb{N}$ we compute

$$\begin{aligned} (A_k A_l f)(x) &= \frac{1}{N_k} \sum_{y \in V_k(x)} (A_l f)(y) \\ &= \frac{1}{N_k N_l} \sum_{y \in V_k(x)} \sum_{z \in V_l(y)} f(z) \\ &= \frac{1}{N_k N_l} \sum_{z \in V_P} |V_k(x) \cap V_l(z)| f(z). \end{aligned} \tag{C.1.2}$$

The key to this section is the following theorem.

THEOREM C.1.4. *Let $k \geq 1$. Then*

$$A_k A_1 = \frac{1}{p(q+1)} A_{k-1} + \frac{p-1}{p(q+1)} A_k + \frac{q}{q+1} A_{k+1}$$

PROOF. Let $x, z \in V_P$. If $y \in V_k(x) \cap V_1(z)$, then since $|d(x,z) - d(x,y)| \leq d(y,z)$ we have $d(x,z) = 2k-2, 2k-1, 2k, 2k+1$ or $2k+2$. Thus, since $z \in V_P$, we have $z \in V_{k-1}(x) \cup V_k(x) \cup V_{k+1}(x)$. Using Corollary C.1.3 and (C.1.2) we have

$$(A_k A_1 f)(x) = \sum_{l=k-1}^{k+1} a_{k,1;l} \left(\frac{1}{N_l} \sum_{z \in V_l(x)} f(z) \right),$$

and the result follows. $\square$

Let $\mathscr{A}$ be the linear span over $\mathbb{C}$ of $\{A_k\}_{k \in \mathbb{N}}$.

COROLLARY C.1.5. *$\mathscr{A}$ is a commutative algebra, generated by A_1.*

PROOF. This is a simple induction using Theorem C.1.4. $\square$

LEMMA C.1.6. *For each $z \in \mathbb{C}$ there is a unique algebra homomorphism $h^{(z)} : \mathscr{A} \to \mathbb{C}$ such that $h^{(z)}(A_1) = z$.*

PROOF. Let $\mathbb{C}[X]$ be the algebra of polynomials in the indeterminate X. We claim that $\mathscr{A} \cong \mathbb{C}[X]$. Since $\mathscr{A}$ is generated by A_1, there is a unique surjective algebra homomorphism $\varphi : \mathbb{C}[X] \to \mathscr{A}$ such that $\varphi(X) = A_1$. To see that φ is injective, by Theorem C.1.4 we easily see that for each $k \in \mathbb{N}$ there exists a number $c_k > 0$ such that

$$A_1^k = c_k A_k + \text{a linear combination of the } A_l \text{ with } l < k.$$

Thus φ maps nonzero $f = \sum_{l\in\mathbb{N}} a_l X^l \in \mathbb{C}[X]$ to

$$a_k c_k A_k + \text{a linear combination of the } A_l \text{ with } l < k,$$

where $k \in \mathbb{N}$ is maximal amongst the $l \in \mathbb{N}$ such that $a_l \neq 0$. Thus, since $\{A_k\}_{k\in\mathbb{N}}$ is linearly independent, $\varphi(f) \neq 0$. The result follows. $\square$

In order to give a formula for the homomorphism $h^{(z)} : \mathscr{A} \to \mathbb{C}$ from Lemma C.1.6, we introduce a parameter $u \in \mathbb{C}^\times$ related to $z \in \mathbb{C}$ by

$$z = \frac{\sqrt{q}}{\sqrt{p}(q+1)}(u + u^{-1}) + \frac{p-1}{p(q+1)}, \tag{C.1.3}$$

and we write h_u in place of $h^{(z)}$. It is clear that $h_u = h_v$ if and only if $v = u$ or $v = u^{-1}$. Thus the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ are indexed by the set $\mathbb{C}^\times/\sim$ of equivalence classes in $\mathbb{C}^\times$ of the relation $u \sim u^{-1}$.

We may now prove the *Macdonald formula* for the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$.

THEOREM C.1.7. *If $u \neq \pm 1$ then*

$$h_u(A_k) = \frac{(pq)^{-k/2}}{1+q^{-1}}\big(c(u)u^k + c(u^{-1})u^{-k}\big), \tag{C.1.4}$$

where, writing $a = \sqrt{pq}$ and $b = \sqrt{q/p}$,

$$c(u) = \frac{(1 - a^{-1}u^{-1})(1 + b^{-1}u^{-1})}{1 - u^{-2}}. \tag{C.1.5}$$

If $u = \pm 1$ the value of h_u may be found by taking an appropriate limit

PROOF. Let $z \in \mathbb{C}$, and for each $k \in \mathbb{N}$ write $x_k^{(z)} = (pq)^{k/2} h^{(z)}(A_k)$. Applying $h^{(z)}$ to the formula in Theorem C.1.4 we have

$$z x_k^{(z)} = \frac{\sqrt{q}}{\sqrt{p}(q+1)} x_{k-1}^{(z)} + \frac{p-1}{p(q+1)} x_k^{(z)} + \frac{\sqrt{q}}{\sqrt{p}(q+1)} x_{k+1}^{(z)} \quad \text{for all } k \geq 1,$$

and so

$$x_{k+2}^{(z)} - \left(\frac{\sqrt{p}(q+1)}{\sqrt{q}} z - \frac{p-1}{\sqrt{pq}}\right) x_{k+1}^{(z)} + x_k^{(z)} = 0 \quad \text{for all } k \geq 0. \tag{C.1.6}$$

Let u be a root of the auxiliary equation of the recurrence relation (C.1.6). Then u^{-1} is also a root, and

$$u + u^{-1} = \frac{\sqrt{p}(q+1)}{\sqrt{q}} z - \frac{p-1}{\sqrt{pq}}. \tag{C.1.7}$$

Thus if $u \neq \pm 1$, the solution of (C.1.6) is

$$x_k^{(z)} = a(u)u^k + b(u)u^{-k} \quad \text{for all } k \geq 0$$

for some $a(u), b(u)$ depending on z (and hence on u). Observe that $x_0^{(z)} = h^{(z)}(A_0) = 1$ and by (C.1.7)

$$x_1^{(z)} = \sqrt{pq}z = \frac{q}{q+1}(u + u^{-1}) + \frac{\sqrt{q}(p-1)}{\sqrt{p}(q+1)}.$$

An elementary calculation shows that $a(u) = qc(u)/(q+1)$ and $b(u) = qc(u^{-1})/(q+1)$, where $c(u)$ is as in (C.1.5). Since $h_u(A_1) = h^{(z)}(A_1)$ (see (C.1.7) and (C.1.3)) the result follows. $\square$

C.2. The A_1 Case

Let $\mathscr{X}$ be an affine building of type A_1 with parameter $q_0 = q_1 = q$. Thus $\mathscr{X}$ is a homogeneous tree with degree $q+1$, which can be considered as an affine building of type BC_1 with parameters $q_0 = 1$ and $q_1 = q$ by adding a vertex in the middle of each edge. For example, in Figure C.2.1 an affine building of type A_1 with parameter $q_0 = q_1 = 3$ is considered as a BC_1 building with parameters $q_0 = 1$ and $q_1 = 3$.

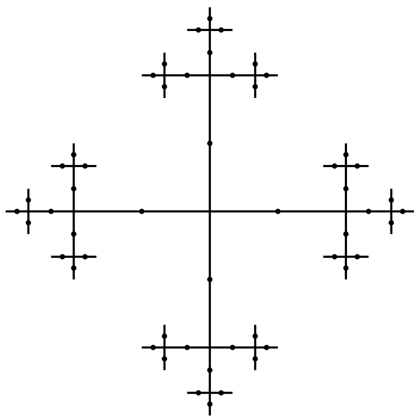


FIGURE C.2.1

Thus the results of the previous section are applicable. We have

$$A_k A_1 = \frac{q}{q+1} A_{k+1} + \frac{1}{q+1} A_{k-1} \qquad \text{for } k \geq 1.$$

C.3. The A_2 Case

In the interests of completeness, in this section we will list some results from [**11**]. Let R be a root system of type A_2, and let $\mathscr{X}$ be an affine building of type A_2 with parameter $q_0 = q_1 = q_2 = q$. If $\lambda = m\lambda_1 + n\lambda_2$, $m, n \in \mathbb{N}$, write $V_{m,n}(x)$ and $A_{m,n}$ in place of $V_\lambda(x)$ and A_λ. In [**11**, Corollary 2.2] it is shown that $N_{m,n} = |V_{m,n}(x)|$ is independent of $x \in V_P$, and is given by $N_{0,0} = 1$, and for $m, n \geq 1$,

$$\begin{aligned}
N_{m,n} &= (q^2+q+1)(q^2+q)q^{2(m+n-2)}\\
N_{m,0} &= (q^2+q+1)q^{2(m-1)}\\
N_{0,n} &= (q^2+q+1)q^{2(n-1)}.
\end{aligned}$$

In [**11**, Proposition 2.3] it is shown that for $m, n \geq 1$, the operators $A_{m,n}$ satisfy

$$\begin{aligned}
A_{m,n}A_{1,0} &= \frac{q^2}{q^2+q+1}A_{m+1,n} + \frac{q}{q^2+q+1}A_{m-1,n+1} + \frac{1}{q^2+q+1}A_{m,n-1}\\
A_{m,n}A_{0,1} &= \frac{q^2}{q^2+q+1}A_{m,n+1} + \frac{q}{q^2+q+1}A_{m+1,n-1} + \frac{1}{q^2+q+1}A_{m-1,n}\\
A_{m,0}A_{1,0} &= \frac{q^2}{q^2+q+1}A_{m+1,0} + \frac{q+1}{q^2+q+1}A_{m-1,1}\\
A_{0,n}A_{1,0} &= \frac{q^2+q}{q^2+q+1}A_{1,n} + \frac{1}{q^2+q+1}A_{0,n-1}\\
A_{m,0}A_{0,1} &= \frac{q^2+q}{q^2+q+1}A_{m,1} + \frac{1}{q^2+q+1}A_{m-1,0}\\
A_{0,n}A_{0,1} &= \frac{q^2}{q^2+q+1}A_{0,n+1} + \frac{q+1}{q^2+q+1}A_{1,n-1}.
\end{aligned} \tag{C.3.1}$$

Let $\mathscr{A}$ be the linear span of $\{A_{m,n}\}_{m,n\in\mathbb{N}}$ over $\mathbb{C}$. A simple induction using the equations (C.3.1) shows that $\mathscr{A}$ is a commutative algebra, generated by $A_{1,0}$ and $A_{0,1}$ (see [**11**, Proposition 2.3] for details). The Macdonald formula for the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ can be deduced from (C.3.1) in a similar (although more complicated) way as in Theorem C.1.7. See [**11**, Proposition 3.1] for details. The final result is as follows.

THEOREM C.3.1. *Let $u_1, u_2, u_3 \in \mathbb{C}$ be pairwise distinct numbers with $u_1u_2u_3 = 1$. Then for any $n_1, n_2, n_3 \in \mathbb{Z}$ with $m = n_1 - n_2$ and $n = n_2 - n_3$,*

$$h_{u_1,u_2,u_3}(A_{m,n}) = \frac{q^{-m-n}}{(1+q^{-1})(1+q^{-1}+q^{-2})}\sum_{\sigma\in\mathfrak{S}_3} c(u_{\sigma(1)}, u_{\sigma(2)}, u_{\sigma(3)})u_{\sigma(1)}^{n_1}u_{\sigma(2)}^{n_2}u_{\sigma(3)}^{n_3},$$

where

$$c(u_1,u_2,u_3) = \prod_{1\leq i<j\leq 3}\frac{u_i - q^{-1}u_j}{u_i - u_j}.$$

In the singular cases the formula for h_{u_1,u_2,u_3} may be obtained from the above by taking an appropriate limit.

C.4. The BC_2 Case

Let R be a root system of type BC_2, and let $\mathscr{X}$ be a building of type $\widetilde{BC}_2$, with parameters $p = q_0, q = q_1$ and $r = q_2$. Let V_P be the set of good vertices (that is, type 0 vertices). An apartment of $\mathscr{X}$ can be viewed as in Figure C.4.1, where the dashed lines

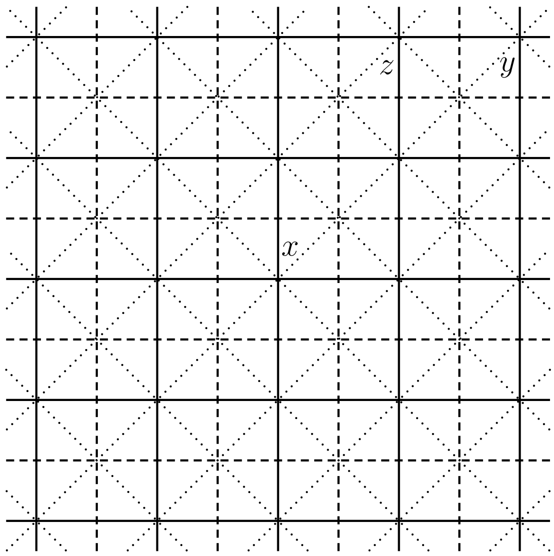


FIGURE C.4.1

represent walls of valency $p+1$, the dotted lines represent walls of valency $q+1$, and the solid lines represent walls of valency $r+1$. Let us fix this convention throughout this section.

For $x \in V_P$, define $V_\lambda(x)$ as in Definition 4.2. If $\lambda = k\lambda_1 + l\lambda_2$, write $V_{k,l}(x)$ in place of $V_\lambda(x)$. Thus, in Figure C.4.1, $y \in V_{0,2}(x)$, $z \in V_{1,1}(x)$, and $z \in V_{1,0}(y)$.

Let us give a general lemma which will be useful in the following.

LEMMA C.4.1. *Let $\mathscr{X}$ be a building of type $\bullet\overset{2k}{\text{——}}\bullet$, with parameters q_1 and q_2. Then*

$$|\mathcal{C}| = (1+q_1)(1+q_2)(1+q_1q_2+\cdots+(q_1q_2)^{k-1}).$$

PROOF. Let $W_{(k)} = \langle \{s_1, s_2\} \mid s_1^2 = s_2^2 = (s_1s_2)^{2k} = 1\rangle$. In the notation of Section 1.7, for fixed $c \in \mathcal{C}$ we have $\mathcal{C} = \bigcup_{w\in W_{(k)}} \mathcal{C}_w(c)$ where the union is disjoint. Thus

$$|\mathcal{C}| = \sum_{w\in W_{(k)}} |\mathcal{C}_w(c)| = \sum_{w\in W_{(k)}} q_w = W_{(k)}(q).$$

For $k \geq 1$ we have $W_{(k+1)}(q) = W_{(k)}(q) + q_1^kq_2^k + q_1^{k+1}q_2^k + q_2^{k+1}q_1^k + q_1^{k+1}q_2^{k+1}$, and the result follows by induction. $\square$

LEMMA C.4.2. *The numbers $N_{k,l} = |V_{k,l}(x)|$ are independent of $x \in V_P$ and are given by $N_{0,0} = 1$, and for $k, l \geq 1$*

$$\begin{aligned}
N_{k,l} &= (q+1)(r+1)(qr+1)(pq^2r)^{k-1}(p^2q^2r^2)^{l-1}p^3q^2r \\
N_{k,0} &= (q+1)(qr+1)(pq^2r)^{k-1}p \\
N_{0,l} &= (r+1)(qr+1)(p^2q^2r^2)^{l-1}p^2q\,.
\end{aligned}$$

PROOF. Let $x \in V_P$. We have

$$|V_{1,0}(x)| = (q+1)(qr+1)p \tag{C.4.1}$$
$$|V_{0,1}(x)| = (r+1)(qr+1)p^2q \tag{C.4.2}$$
$$|V_{i+1,j}(x)| = pq^2r|V_{i,j}(x)| \qquad i \geq 1,\ j \geq 0 \tag{C.4.3}$$
$$|V_{1,j}(x)| = pqr(q+1)|V_{0,j}(x)| \qquad j \geq 1 \tag{C.4.4}$$
$$|V_{i,j+1}(x)| = p^2q^2r^2|V_{i,j}(x)| \qquad i \geq 0,\ j \geq 1 \tag{C.4.5}$$
$$|V_{i,1}(x)| = p^2q^2r(r+1)|V_{i,0}(x)| \qquad i \geq 1\,. \tag{C.4.6}$$

We will prove (C.4.1) and (C.4.3), the other proofs are similar.

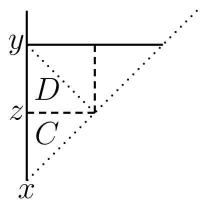


FIGURE C.4.2

By Lemma C.4.1 there are $(q+1)(r+1)(qr+1)$ chambers in $\mathscr{X}$ containing x, and so we count $(q+1)(r+1)(qr+1)p$ chambers in the position of D as shown in Figure C.4.2. The 'good' vertex of each such D chamber is in $V_{1,0}(x)$, and for each $y \in V_{1,0}(x)$ there are exactly $r+1$ chambers in the position of D containing y (since the edge zy is common to $r+1$ chambers). Thus $|V_{1,0}(x)| = (q+1)(qr+1)p$.

Let us prove (C.4.3) in the case $i, j \geq 1$. Let $y \in V_{i,j}(x)$; there are $|V_{i,j}(x)|$ such y's. There are q choices for the chamber C as shown in Figure C.4.3. Having chosen C, there are then r choices for the chamber D, then q choices for the chamber E, and finally p choices for the chamber F. Thus for each $y \in V_{i,j}(x)$ we have counted pq^2r (pairwise distinct) vertices w in the configuration shown. Since each $z \in V_{i+1,j}(x)$ may be reached from some $y \in V_{i,j}(x)$ by a gallery C, D, E, F as considered, we have $|V_{i+1,j}(x)| = pq^2r|V_{i,j}(x)|$.

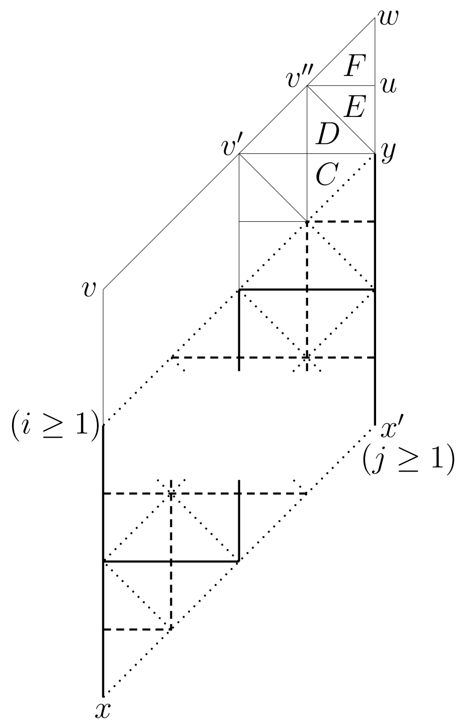


FIGURE C.4.3

Formulae (C.4.1)-(C.4.6) and induction show that

$$|V_{m,n}(x)| = \begin{cases} (pq^2r)^{m-1}(p^2q^2r^2)^{n-1}|V_{1,1}(x)| & \text{if } m, n \geq 1 \\ (pq^2r)^{m-1}|V_{1,0}(x)| & \text{if } m \geq 1, n = 0 \\ (p^2q^2r^2)^{n-1}|V_{0,1}(x)| & \text{if } m = 0,\ n \geq 1 \end{cases}$$

By (C.4.4) and (C.4.2) we have $|V_{1,1}(x)| = (q+1)(r+1)(qr+1)p^3q^2r$, and the result follows. $\square$

For each $k, l \in \mathbb{N}$, define an operator $A_{k,l}$ by

$$(A_{k,l}f)(x) = \frac{1}{N_{k,l}} \sum_{y \in V_{k,l}(x)} f(y) \quad \text{for all } x \in V_P.$$

Since $y \in V_{k,l}(x)$ if and only if $x \in V_{k,l}(y)$, we have

$$(A_{k,l}A_{m,n}f)(x) = \frac{1}{N_{k,l}N_{m,n}} \sum_{z \in V_P} |V_{k,l}(x) \cap V_{m,n}(z)| f(z). \tag{C.4.7}$$

The following lemma is stated for all affine buildings.

LEMMA C.4.3. *Let $\mathcal{S}$ be a sector based at $x \in V_P$ in an affine building, and let c be a chamber such that $x \in c$. Then there exists an apartment containing $\mathcal{S} \cup c$.*

PROOF. By [**35**, Lemma 9.4] there exists a subsector $\mathcal{S}' \subset \mathcal{S}$ such that $\mathcal{S}' \cup c$ lies in an apartment $\mathcal{A}$, say. In particular, $x \in \mathcal{A}$, and so there is a sector $\mathcal{S}'' \subset \mathcal{A}$ based at x in the class of $\mathcal{S}'$. Now $\mathcal{S}'' = \mathcal{S}$ by [**35**, Lemma 9.7], completing the proof. $\square$

We have the following analogue of Lemma C.1.2.

LEMMA C.4.4. *Let $x \in V_P$ and $z \in V_{k,\ell}(x)$.*

(i) *If $k, l \geq 1$, then*

$$|V_{i,j}(x) \cap V_{1,0}(z)| = \begin{cases} 1 & \textit{if } (i,j) = (k-1,l) \\ (q+1)(p-1) & \textit{if } (i,j) = (k,l) \\ q & \textit{if } (i,j) = (k+1,l-1) \\ pq^2r & \textit{if } (i,j) = (k+1,l) \\ pqr & \textit{if } (i,j) = (k-1,l+1) \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) *If $k = 0$ and $l \geq 1$, then*

$$|V_{i,j}(x) \cap V_{1,0}(z)| = \begin{cases} q+1 & \textit{if } (i,j) = (1,l-1) \\ (q+1)(p-1) & \textit{if } (i,j) = (0,l) \\ pqr(q+1) & \textit{if } (i,j) = (1,l) \\ 0 & \textit{otherwise.} \end{cases}$$

(iii) *If $k \geq 1$ and $l = 0$, then*

$$|V_{i,j}(x) \cap V_{1,0}(z)| = \begin{cases} 1 & \textit{if } (i,j) = (k-1,0) \\ p-1 & \textit{if } (i,j) = (k,0) \\ pq(r+1) & \textit{if } (i,j) = (k-1,1) \\ pq^2r & \textit{if } (i,j) = (k+1,0) \\ 0 & \textit{otherwise.} \end{cases}$$

(iv) *If $k \geq 2$ and $l \geq 1$, then*

$$|V_{i,j}(x) \cap V_{0,1}(z)| = \begin{cases} 1 & \text{if } (i,j) = (k, l-1) \\ p-1 & \text{if } (i,j) = (k-1, l) \\ p(q-1)(r+1) + q(p-1)^2 & \text{if } (i,j) = (k,l) \\ (p-1)q & \text{if } (i,j) = (k+1, l-1) \\ pq^2r & \text{if } (i,j) = (k+2, l-1) \\ pq^2r(p-1) & \text{if } (i,j) = (k+1, l) \\ p^2q^2r^2 & \text{if } (i,j) = (k, l+1) \\ pr & \text{if } (i,j) = (k-2, l+1) \\ pqr(p-1) & \text{if } (i,j) = (k-1, l+1) \\ 0 & \text{otherwise.} \end{cases}$$

(v) *If $k = 1$ and $l \geq 1$ the counts are as in (iv), with the definition $V_{-1,l+1}(x) = V_{1,l}(x)$.*

(vi) *If $k = 0$ and $l \geq 1$, then*

$$|V_{i,j}(x) \cap V_{0,1}(z)| = \begin{cases} 1 & \text{if } (i,j) = (0, l-1) \\ (q+1)(p-1) & \text{if } (i,j) = (1, l-1) \\ (q-1)p + q(p-1)^2 & \text{if } (i,j) = (0,l) \\ pqr(q+1) & \text{if } (i,j) = (2, l-1) \\ (q+1)(p-1)pqr & \text{if } (i,j) = (1, l) \\ p^2q^2r^2 & \text{if } (i,j) = (0, l+1) \\ 0 & \text{otherwise.} \end{cases}$$

(vii) *If $k \geq 2$ and $l = 0$, then*

$$|V_{i,j}(x) \cap V_{0,1}(z)| = \begin{cases} (r+1)p & \text{if } (i,j) = (k-2, 1) \\ (r+1)(q-1)p & \text{if } (i,j) = (k, 0) \\ (r+1)(p-1)pq & \text{if } (i,j) = (k-1, 1) \\ (r+1)p^2q^2r & \text{if } (i,j) = (k, 1) \\ 0 & \text{otherwise.} \end{cases}$$

(viii) *If $k = 1$ and $l = 0$ the counts are as in (vii), with the definition $V_{-1,1}(x) = V_{1,0}(x)$.*

Proof. We will prove (i). Numbers (ii) and (iii) are of a similar difficulty. Numbers (iv)-(viii) are a little more complicated.

Let $y \in V_{1,0}(z)$. This is shown in Figure C.4.4 (possibly in two different apartments).

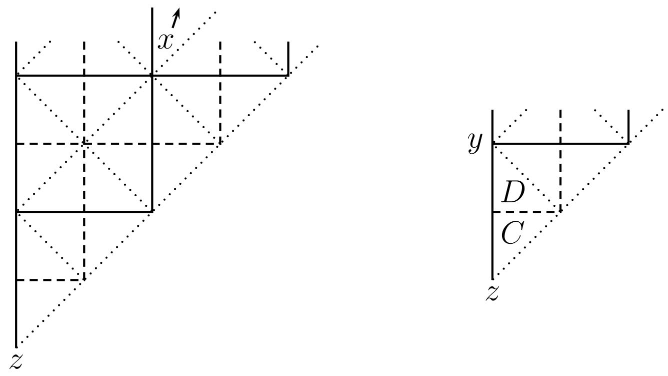


FIGURE C.4.4

Let $\mathcal{S}$ be a sector based at z which contains x (as shown in Figure C.4.4). By Lemma C.4.3 there exists an apartment $\mathcal{A}$ containing $\mathcal{S}$ and C, and so we may suppose that the chamber C is in the position of one of C_i, $i = 1, 2, \dots, 8$ as shown in Figure C.4.5.

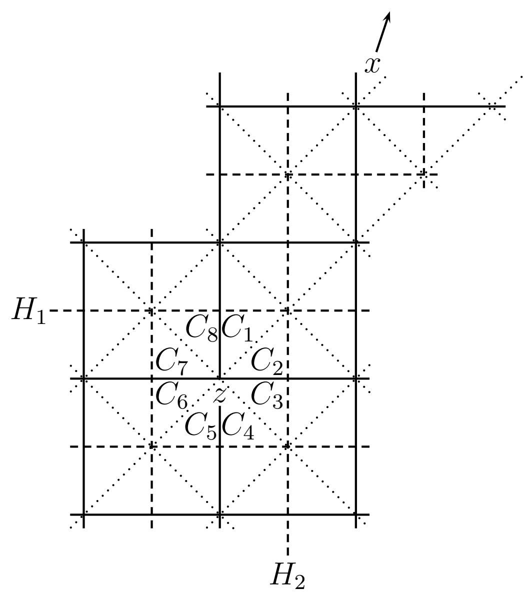


FIGURE C.4.5

It is clear that every $y \in V_P$ reachable from a gallery $C \sim_0 D$ (see Figure C.4.4) with $C = C_1$, is also reachable from a gallery $C' \sim_0 D'$ with $C' = C_8$. Similarly for galleries with $C = C_2, C_3$, $C = C_4, C_5$ and $C = C_6, C_7$. Thus we need only consider those galleries with $C = C_1, C_2, C_4$ or C_7.

Suppose $C = C_1$. There is exactly one chamber in the position of C_1, and so there are p choices for the D chamber. One of these chambers has its type 0 vertex in $V_{k-1,l}(x)$, and the remaining $p - 1$ of the chambers can be folded back across the wall H_1, and so their type 0 vertices are in $V_{k,l}(x)$.

Suppose now that $C = C_2$. There are q chambers in the position of C_2, each with p possible D chambers. Thus there are pq (distinct) vertices $y \in V_{1,0}(z)$ reachable by a gallery of $C \sim_0 D$ with C in the position of C_2. Of these vertices, exactly q are in $V_{k+1,l-1}(x)$, and the remaining $q(p-1)$ vertices may be folded 'leftwards' across H_2, placing them in $V_{k,l}(x)$.

Now, considering galleries with C in the position of C_4 we find pq^2r vertices $y \in V_{1,0}(z)$ which are in $V_{k+1,l}(x)$, and for galleries with C in the position of C_7 we find pqr vertices $y \in V_{1,0}(z)$ which are in $V_{k-1,l+1}(x)$.

Combining these results proves (i). To check that no vertices have been overlooked, note that the sum of the contributions equals $N_{1,0}$. $\square$

THEOREM C.4.5. *Write $A'_{1,0} = N_{1,0}A_{1,0}$ and $A'_{0,1} = N_{0,1}A_{0,1}$. Then*

$$A_{1,0}A_{0,1} = A_{0,1}A_{1,0} \tag{C.4.8}$$

$$\begin{aligned} A_{m,n}A'_{1,0} = {} & A_{m-1,n} + pqrA_{m-1,n+1} + (q+1)(p-1)A_{m,n} \\ & + qA_{m+1,n-1} + pq^2rA_{m+1,n} \end{aligned} \tag{C.4.9}$$

$$A_{m,0}A'_{1,0} = A_{m-1,0} + pq(r+1)A_{m-1,1} + (p-1)A_{m,0} + pq^2rA_{m+1,0} \tag{C.4.10}$$

$$A_{0,n}A'_{1,0} = (q+1)\big[A_{1,n-1} + (p-1)A_{0,n} + pqrA_{1,n}\big] \tag{C.4.11}$$

$$\begin{aligned} A_{m,n}A'_{0,1} = {} & pq^2r(p-1)A_{m+1,n} + (p-1)A_{m-1,n} + pqr(p-1)A_{m-1,n+1} \\ & + q(p-1)A_{m+1,n-1} + pq^2rA_{m+2,n-1} + p^2q^2r^2A_{m,n+1} \\ & \quad + prA_{m-2,n+1} + A_{m,n-1} + \big[(q-1)(r+1)p + q(p-1)^2\big]A_{m,n} \end{aligned} \tag{C.4.12}$$

$$A_{m,0}A'_{0,1} = p(r+1)\big[(p-1)qA_{m-1,1} + (q-1)A_{m,0} + A_{m-2,1} + pq^2rA_{m,1}\big] \tag{C.4.13}$$

$$\begin{aligned} A_{0,n}A'_{0,1} = {} & (q+1)\big[pqrA_{2,n-1} + (p-1)A_{1,n-1} + (p-1)pqrA_{1,n}\big] \\ & + A_{0,n-1} + p^2q^2r^2A_{0,n+1} + \big[(p-1)^2q + p(q-1)\big]A_{0,n} \end{aligned} \tag{C.4.14}$$

$$\begin{aligned} A_{1,n}A'_{0,1} = {} & A_{1,n-1} + q(p-1)A_{2,n-1} + pq^2rA_{3,n-1} + p^2q^2r^2A_{1,n+1} \\ & + (p-1)A_{0,n} + pqr(p-1)A_{0,n+1} + pq^2r(p-1)A_{2,n} \\ & \quad + \big[(q-1)p + (p-1)^2q + pqr\big]A_{1,n} \end{aligned} \tag{C.4.15}$$

where in each case the indices m, n are required to be large enough so that the indices appearing on the right are all at least 0. For example in (C.4.12) we require $m \geq 2$ and $n \geq 1$.

PROOF. These formulae follow from (C.4.7) and Lemma C.4.4. For example, we will derive (C.4.9), so suppose that $m, n \geq 1$. From (C.4.7) we have

$$(A_{m,n}A_{1,0}f)(x) = \frac{1}{N_{m,n}N_{1,0}} \sum_{k,l\in\mathbb{N}} \sum_{z\in V_{k,l}(x)} |V_{m,n}(x) \cap V_{1,0}(z)|\, f(z)\,.$$

By Lemma C.4.4 we see that $|V_{m,n}(x) \cap V_{1,0}(z)| = 0$ unless:

- $z \in V_{m-1,n}(x)$; if $m \geq 2$, Lemma C.4.4(i) implies that $|V_{m,n}(x) \cap V_{1,0}(z)| = pq^2r$, whereas if $m = 1$ we use Lemma C.4.4(ii) to see that $|V_{m,n}(x) \cap V_{1,0}(z)| = pqr(q+1)$.
- $z \in V_{m-1,n+1}(x)$; if $m \geq 2$ then $|V_{m,n}(x) \cap V_{1,0}(z)| = q$, whereas if $m = 1$ then $|V_{m,n}(x) \cap V_{1,0}(z)| = q + 1$.
- $z \in V_{m,n}(x)$; in this case $|V_{m,n}(x) \cap V_{1,0}(z)| = (q + 1)(p - 1)$ for all $m, n \geq 1$.
- $z \in V_{m+1,n-1}(x)$; if $n \geq 2$ then $|V_{m,n}(x) \cap V_{1,0}(z)| = pqr$, whereas if $n = 1$, $|V_{m,n}(x) \cap V_{1,0}(z)| = pq(r + 1)$.
- $z \in V_{m+1,n}(x)$; in this case $|V_{m,n}(x) \cap V_{1,0}(z)| = 1$ for all $m, n \geq 1$.

Equation (C.4.9) follows by recalling the formulae for $N_{k,l}$ in Lemma C.4.2. □

Let $\mathscr{A}$ denote the linear span of $\{A_{k,l}\}_{k,l\in\mathbb{N}}$ over $\mathbb{C}$.

COROLLARY C.4.6. *$\mathscr{A}$ is a commutative algebra, generated by $A_{1,0}$ and $A_{0,1}$.*

PROOF. Declare $(k, l) \prec (m, n)$ if either $k+l < m+n$, or $k+l = m+n$ and $l < n$, and write $(k, l) \preceq (m, n)$ if $(k, l) \prec (m, n)$ or $(k, l) = (m, n)$. This defines a total order on $\mathbb{N}^2$.

Assume that $m + n \geq 2$ and that for each $(k, l) \prec (m, n)$ we can write $A_{k,l}$ as a polynomial in $A_{1,0}$ and $A_{0,1}$. We claim that $A_{m,n}$ itself is a polynomial in $A_{1,0}$ and $A_{0,1}$. If $m \geq 2$ and $n \geq 1$ one sees this from (C.4.9), with m there replaced by $m - 1$. If $m \geq 2$ and $n = 0$ we use (C.4.10), and if $m \geq 1$ and $n \geq 1$ we use (C.4.11) in the same way. Finally, if $m = 0$ and $n \geq 2$, we use (C.4.14), with n replaced by $n - 1$, noting that $(2, n-2) \prec (1, n-1) \prec (0, n)$. Thus each $A_{m,n}$ is a polynomial in $A_{1,0}$ and $A_{0,1}$, and so $\mathscr{A}$ is commutative by (C.4.8). □

Let $\prec$ be the total order defined in the proof of Corollary C.4.6.

LEMMA C.4.7. *For each $(m, n) \in \mathbb{N}^2$ there is a number $c_{m,n} > 0$ so that*

$$A_{1,0}^m A_{0,1}^n = c_{m,n}A_{m,n} + \text{a linear combination of those } A_{k,l} \text{ with } (k, l) \prec (m, n). \tag{C.4.16}$$

PROOF. First note that for each $(m, n) \in \mathbb{N}^2$ there exist $c'_{m,n}, c''_{m,n} > 0$ so that

$$\begin{aligned} A_{m,n}A_{1,0} &= c'_{m,n}A_{m+1,n} \\ &\qquad + \text{a linear combination of the } A_{k,l} \text{ with } (k, l) \prec (m + 1, n), \text{ and} \end{aligned} \tag{C.4.17}$$

$$\begin{aligned} A_{m,n}A_{0,1} &= c''_{m,n}A_{m,n+1} \\ &\qquad + \text{a linear combination of the } A_{k,l} \text{ with } (k, l) \prec (m, n + 1). \end{aligned} \tag{C.4.18}$$

We see (C.4.17) using (C.4.9) through (C.4.11), and (C.4.18) using (C.4.12) through (C.4.15), noting that $(m + 2, n - 1) \prec (m + 1, n) \prec (m, n + 1)$.

Now (C.4.16) is obvious if $(m, n) = (0, 0)$, $(1, 0)$ or $(0, 1)$, and so assume that $m+n \geq 2$ and that formula (C.4.16) holds when (m, n) is replaced by any $(m', n') \prec (m, n)$. If $m \geq 2$

and $n \geq 1$, then by (C.4.9) we can write

$$\begin{aligned} A_{1,0}^m A_{0,1}^n &= A_{1,0}(A_{1,0}^{m-1} A_{0,1}^n) \\ &= A_{1,0}\big(c_{m-1,n} A_{m-1,n} + \text{terms in } A_{k,l} \text{ with } (k,l) \prec (m-1,n)\big) \\ &= c_{m,n} A_{m,n} + \text{terms in } A_{k,l} \text{ with } (k,l) \prec (m,n), \quad \text{by (C.4.17)}, \end{aligned}$$

where $c_{m,n} = c_{m-1,n} pq^2 r / N_{1,0}$. If $m \geq 2$ and $n = 0$, we use (C.4.10) instead. If $m = 1$ and $n \geq 1$ we use (C.4.11). Finally, if $m = 0$ and $n \geq 2$, we use (C.4.14). $\square$

COROLLARY C.4.8. *Let $\mathbb{C}[X,Y]$ denote the algebra of polynomials in two commuting indeterminates X and Y. Then there is an isomorphism $\varphi : \mathbb{C}[X,Y] \to \mathscr{A}$ such that $\varphi(X) = A_{1,0}$ and $\varphi(Y) = A_{0,1}$.*

PROOF. Since $\mathscr{A}$ is commutative (see Corollary C.4.6), there is a unique algebra homomorphism $\varphi : \mathbb{C}[X,Y] \to \mathscr{A}$ such that $\varphi(X) = A_{1,0}$ and $\varphi(Y) = A_{0,1}$. Since $A_{1,0}$ and $A_{0,1}$ generate $\mathscr{A}$, φ is surjective. To see that φ is injective, suppose $f(X,Y) = \sum_{k,\ell} a_{k,\ell} X^k Y^\ell \in \ker \varphi$ is nonzero, and suppose that (m,n) is maximal with respect to $\prec$ amongst the (k,l) for which $a_{k,l} \neq 0$. By Lemma C.4.7, φ maps $f(X,Y)$ to

$$a_{m,n} c_{m,n} A_{m,n} + \text{a linear combination of the } A_{k,l} \text{ with } (k,l) \prec (m,n)$$

and this cannot be zero because the $A_{k,l}$'s are linearly independent. $\square$

Since the algebra homomorphisms $\mathbb{C}[X,Y] \to \mathbb{C}$ are just the evaluation maps $f(X,Y) \mapsto f(z_1,z_2)$, where $z_1, z_2 \in \mathbb{C}$, we see that the algebra homomorphisms $\mathscr{A} \to \mathbb{C}$ are indexed by $\mathbb{C}^2$. That is,

COROLLARY C.4.9. *For each pair $(z_1,z_2) \in \mathbb{C}^2$, there is a unique algebra homomorphism $h = h^{(z_1,z_2)} : \mathscr{A} \to \mathbb{C}$ such that $h(A_{1,0}) = z_1$ and $h(A_{0,1}) = z_2$.*

In order to give a formula for $h^{(z_1,z_2)}(A_{m,n})$, we introduce parameters $u_1, u_2 \in \mathbb{C}^\times$ related to z_1, z_2 by the equations

$$z_1 = \frac{1}{N_{1,0}}\Big[(q+1)(p-1) + q\sqrt{pr}(u_1 + u_1^{-1} + u_2 + u_2^{-1})\Big], \tag{C.4.19}$$

$$\begin{aligned} z_2 = \frac{1}{N_{0,1}}\Big[&pqr(u_1 + u_1^{-1})(u_2 + u_2^{-1}) + (p-1)q\sqrt{pr}(u_1 + u_1^{-1} + u_2 + u_2^{-1}) \\ &+ q(p-1)^2 + (q-1)(r+1)p\Big]. \end{aligned} \tag{C.4.20}$$

The reason for this parametrisation will become apparent in the following theorem.

The group C_2 of signed permutations of $\{1,2\}$ acts on $(\mathbb{C}^\times)^2$ by $\sigma\cdot(u_1,u_2) = (u_{\sigma(1)}, u_{\sigma(2)})$, where we use the convention that $u_{-j} = u_j^{-1}$ for $j = 1,2$. Given any $z_1, z_2 \in \mathbb{C}$, it is elementary that we can find $(u_1,u_2) \in (\mathbb{C}^\times)^2$ so that (C.4.19) and (C.4.20) hold, and any other pair (u_1', u_2') satisfying these equations is equal to $(u_{\sigma(1)}, u_{\sigma(2)})$ for some $\sigma \in C_2$.

NOTATION. If (u_1, u_2) are related to (z_1, z_2) as in (C.4.19) and (C.4.20), we shall write h_{u_1,u_2} in place of $h^{(z_1,z_2)}$. Clearly $h_{u_1',u_2'} = h_{u_1,u_2}$ if and only if $(u_1', u_2') = (u_{\sigma(1)}, u_{\sigma(2)})$ for some $\sigma \in C_2$. Hence the algebra homomorphisms $\mathscr{A} \to \mathbb{C}$ are indexed by the set $(\mathbb{C}^\times)^2/C_2$ of orbits in $(\mathbb{C}^\times)^2$ under the action of C_2. In symbols;

$$\mathrm{Hom}(\mathscr{A}, \mathbb{C}) \cong (\mathbb{C}^\times)^2/C_2 \,.$$

Let $\rho_1 = q\sqrt{pr}$, $\rho_2 = pqr$, $a = \sqrt{pr}$ and $b = \sqrt{r/p}$.

THEOREM C.4.10. *If u_1, u_1^{-1}, u_2 and u_2^{-1} are pairwise distinct we have*

$$h_{u_1,u_2}(A_{m,n}) = \frac{\rho_1^{-m}\rho_2^{-n}}{(1+q^{-1})(1+r^{-1})(1+q^{-1}r^{-1})} \sum_{\sigma \in C_2} c(u_{\sigma(1)}, u_{\sigma(2)}) u_{\sigma(1)}^{m+n} u_{\sigma(2)}^{n}$$

where $c(u_1, u_2)$ equals

$$\frac{(1-q^{-1}u_1^{-1}u_2^{-1})(1-q^{-1}u_1^{-1}u_2)(1-a^{-1}u_1^{-1})(1+b^{-1}u_1^{-1})(1-a^{-1}u_2^{-1})(1+b^{-1}u_2^{-1})}{(1-u_1^{-1}u_2^{-1})(1-u_1^{-1}u_2)(1-u_1^{-2})(1-u_2^{-2})} \,.$$

PROOF. Let $h : \mathscr{A} \to \mathbb{C}$ be an algebra homomorphism, and apply h to both sides of the equations in Theorem C.4.5. For each pair m, n, form $x_{m,n} = \rho_1^m \rho_2^n \, h(A_{m,n})$. Write

$$\begin{aligned} &\alpha = N_{1,0}\rho_1^{-2}x_{1,0} - (q+1)(p-1)\rho_1^{-1}, \quad \text{and} \\ &\beta = 2 + N_{0,1}\rho_2^{-2}x_{0,1} - q(p-1)N_{1,0}\rho_1^{-3}x_{1,0} + \big((p-1)^2 - (q-1)(r+1)p\big)\,\rho_2^{-1} \,. \end{aligned}$$

Then

$$x_{m+4,n} - \alpha x_{m+3,n} + \beta x_{m+2,n} - \alpha x_{m+1,n} + x_{m,n} = 0 \qquad \text{for } m, n \geq 0. \tag{C.4.21}$$

Let us derive (C.4.21) in the case $n \geq 1$. From (C.4.9) we have

$$(a_1 x_{1,0} - a_2)x_{m,n} = x_{m-1,n} + x_{m-1,n+1} + x_{m+1,n-1} + x_{m+1,n}, \tag{C.4.22}$$

where $a_1 = \rho_1^{-2}N_{1,0}$ and $a_2 = \rho_1^{-1}(q+1)(p-1)$. Thus if $m, n \geq 1$ we have

$$x_{m+1,n-1} + x_{m-1,n+1} = (a_1 x_{1,0} - a_2)x_{m,n} - x_{m-1,n} - x_{m+1,n} \,. \tag{C.4.23}$$

The important thing to notice is that the terms on the right hand side of (C.4.23) all have second index n. Supposing $m \geq 2$, from (C.4.12) we may write

$$\begin{aligned} (a_3 x_{0,1} - a_4)x_{m,n} = {} & x_{m+2,n-1} + x_{m,n-1} + x_{m,n+1} + x_{m-2,n+1} \\ & + a_5(x_{m-1,n} + x_{m-1,n+1} + x_{m+1,n-1} + x_{m+1,n}) \,, \end{aligned} \tag{C.4.24}$$

where $a_3 = \rho_2^{-2}N_{0,1}$, $a_4 = \rho_2^{-1}\left((q-1)(r+1)p + q(p-1)^2\right)$ and $a_5 = \rho_1^{-1}q(p-1)$. From (C.4.22) and (C.4.24) we have

$$\begin{aligned} (a_3 x_{0,1} - a_4)x_{m,n} = {} & (x_{m+2,n-1} + x_{m,n+1}) + (x_{m,n-1} + x_{m-2,n+1}) \\ & + a_5(a_1 x_{1,0} - a_2)x_{m,n} \,, \end{aligned} \tag{C.4.25}$$

and by using (C.4.23), firstly with m replaced with $m+1$, and secondly with m replaced by $m-1$, we find from (C.4.25) that

$$x_{m+2,n} - (a_1x_{1,0} - a_2)x_{m+1,n} + (2 + a_3x_{0,1} - a_1a_5x_{1,0} - a_4 + a_2a_5)x_{m,n} \\ - (a_1x_{1,0} - a_2)x_{m-1,n} + x_{m-2,n} = 0$$

valid for $n \geq 1$ and $m \geq 2$. Equation (C.4.21) follows by replacing m by $m+2$ and recalling the formulae for the constants $a_1, a_2, \ldots, a_5$. In the case where $n = 0$, a similar calculation works using (C.4.10) and (C.4.13).

Fixing n we are led to consider the auxiliary equation of (C.4.21), namely

$$\lambda^4 - \alpha\lambda^3 + \beta\lambda^2 - \alpha\lambda + 1 = 0\,. \tag{C.4.26}$$

Let $u_1, u_2, u_3, u_4 \in \mathbb{C}^\times$ be the roots of (C.4.26). Notice that

$$\lambda^4 - \alpha\lambda^3 + \beta\lambda^2 - \alpha\lambda + 1 = \left(\lambda^2 + \xi_1\lambda + 1\right)\left(\lambda^2 + \xi_2\lambda + 1\right)\,,$$

where $\xi_1, \xi_2 = -\frac{\alpha}{2} \pm \sqrt{2 + \frac{\alpha^2}{4} - \beta}$, and so if u_1 and u_3 are the roots of the first quadratic, and u_2 and u_4 are the roots of the second, we have $u_3 = u_1^{-1}$ and $u_4 = u_2^{-1}$. Furthermore, since $\alpha = u_1 + u_2 + u_1^{-1} + u_2^{-1}$, we find $x_{1,0} = \rho_1 z_1$, where z_1 is as in (C.4.19). Similarly, since $\beta = 2 + (u_1 + u_1^{-1})(u_2 + u_2^{-1})$ we find $x_{0,1} = \rho_2 z_2$, where w is as in (C.4.20). Thus $h(A_{m,n}) = h_{u_1,u_2}(A_{m,n})$ for $(m,n) = (1,0)$ or $(0,1)$ and hence for all (m,n) by Corollary C.4.6, where h_{u_1,u_2} was defined in the notation section before the theorem.

Now, if u_1, u_2 and their inverses are pairwise distinct we have

$$x_{m,n} = C_{1,n}u_1^m + C_{2,n}u_2^m + C_{3,n}u_1^{-m} + C_{4,n}u_2^{-m} \qquad \text{for all } m,n \geq 0 \tag{C.4.27}$$

for suitable functions $C_{i,n} = C_{i,n}(u_1, u_2)$ which are independent of m. For each $\sigma \in C_2$, $x_{m,n} = \rho_1^m\rho_2^n h_{u_1,u_2}(A_{m,n})$ is unchanged if (u_1, u_2) is replaced by $(u_{\sigma(1)}, u_{\sigma(2)})$, and so

$$C_{2,n} = C_{1,n}(u_2, u_1), \quad C_{3,n} = C_{1,n}(u_1^{-1}, u_2^{-1}), \text{ and } C_{4,n} = C_{1,n}(u_2^{-1}, u_1^{-1})\,.$$

For the same reason, $C_{1,n}(u_1, u_2^{-1}) = C_{1,n}(u_1, u_2)$. Write $C_n = C_{1,n}$.

Using (C.4.22) with n replaced by $n+1$ we find

$$u_1^{-1}C_{n+2} - (u_2 + u_2^{-1})C_{n+1} + u_1C_n = 0 \quad \text{for } n \geq 0. \tag{C.4.28}$$

The roots of the auxiliary equation of (C.4.28) are u_1u_2 and $u_1u_2^{-1}$, which are distinct by hypothesis. Thus

$$C_n = D(u_1, u_2)u_1^n u_2^n + D'(u_1, u_2)u_1^n u_2^{-n}$$

for suitable functions $D(u_1, u_2)$ and $D'(u_1, u_2)$ independent of m and n. Since $C_n(u_1, u_2) = C_n(u_1, u_2^{-1})$, we have $D'(u_1, u_2) = D(u_1, u_2^{-1})$. Thus

$$x_{m,n} = \sum_{\sigma \in C_2} D(u_{\sigma(1)}, u_{\sigma(2)})u_{\sigma(1)}^{m+n}u_{\sigma(2)}^n\,.$$

All that remains is to evaluate $D(u_1, u_2)$. Unfortunately this step, although straightforward, is computationally messy. One proceeds as follows. First find C_0 by writing $x_{0,0}$, $x_{1,0}$, $x_{2,0}$ and $x_{3,0}$ in terms of u_1 and u_2. We have $x_{0,0} = 1$ since h is required to map the identity $A_{0,0}$ of $\mathscr{A}$ to 1, and we have already noted that $x_{1,0} = \rho_1 z_1$ and $x_{0,1} = \rho_2 z_2$. The required formulae for $x_{2,0}$ and $x_{3,0}$ follow from the recursive formulae in Theorem C.4.5. Once we have these we know the four initial conditions of the recurrence (C.4.21) when $n = 0$, and thus we calculate

$$C_0(u_1, u_2) = \frac{(qu_1u_2 - 1)(qu_1 - u_2)(au_1 - 1)(bu_1 + 1)}{(q+1)(qr+1)(u_1u_2 - 1)(u_1 - u_2)(u_1^2 - 1)}\,.$$

Furthermore, from (C.4.10) and (C.4.27) we evaluate

$$C_1(u_1, u_2) = \frac{\rho_2 u_1(u_2 + q(p-1)\rho_1^{-1} + u_2^{-1})}{(r+1)pq} C_0(u_1, u_2)\,,$$

and so we know the initial conditions of the recurrence (C.4.28). Thus we calculate

$$D(u_1, u_2) = \frac{c(u_1, u_2)}{(q^{-1}+1)(r^{-1}+1)(q^{-1}r^{-1}+1)}$$

where $c(u_1, u_2)$ is as in the statement of the theorem. □

C.5. The C_2 Case

Let $\mathscr{X}$ be an affine building of type C_2 with parameter system (q_0, q_1, q_0). As Figure C.5.1 suggests, by adding new walls $\mathscr{X}$ may be considered as an affine building of type BC_2 with parameter system $(1, q_0, q_1)$ (see Section C.2 for a similar discussion). Thus the results of the previous section are applicable.

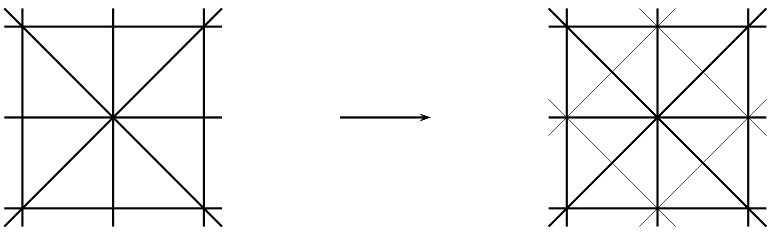


Figure C.5.1

(Note that we have strayed from our conventions regarding dotted, dashed and solid lines).

C.6. Another Algebra in the C_2 Case

Let $\mathscr{X}$ be an affine building of type C_2 with parameters (q_0, q_1, q_0). For $i = 0, 1, 2$, let V_i denote the set of type i vertices of $\mathscr{X}$. Thus V_1 consists of all *non-special* vertices of $\mathscr{X}$. We will explain below how it is possible to study vertex set averaging operators acting on the space of all functions $f : V_1 \to \mathbb{C}$.

As in Section C.5 we add new walls into $\mathscr{X}$, making a new building $\mathscr{X}'$ of affine type BC_2. The vertex set of $\mathscr{X}'$ is $V' = V_0' \cup V_1' \cup V_2'$, where $V_0' = V_1$, $V_2' = V_0 \cup V_2$, and V_1' consists of *new vertices* resulting from the new walls. For example, in Figure C.6.1, $x \in V_1$, $y \in V_0$ and $z \in V_2$ (considered as vertices in $\mathscr{X}$), and $x \in V_0'$, $y, z \in V_2'$ and $v \in V_1'$ (considered as vertices in $\mathscr{X}'$).

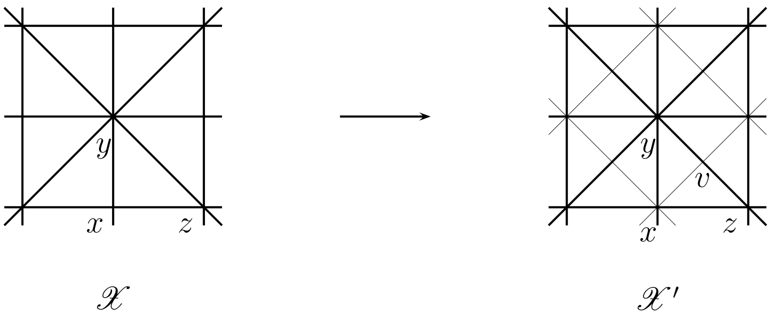


FIGURE C.6.1

For each $k, l \in \mathbb{N}$ we define an averaging operator $A_{k,l}$, acting on the space of all functions $f : V_1 = V_0' \to \mathbb{C}$, as in Section C.4. The results of that section are now applicable (the parameter system of $\mathscr{X}'$ is $(q_0', q_1', q_2') = (q_1, q_2, 1)$).

C.7. The G_2 Case

We conclude with the most difficult 'low dimension' case, the affine buildings of type G_2. We give a large diagram of the $\tilde{G}_2$ Coxeter complex in Figure C.7.1, which the reader may find helpful if they wish to verify any of the counts we make.

Let $\mathscr{X}$ be an affine building of type G_2 with parameters $q_0 = q_2 = q$ and $q_1 = r$.

LEMMA C.7.1. *The numbers $N_{m,n} = |V_{m,n}(x)|$ are independent of $x \in V_P$, and are given by $N_{0,0} = 1$ and for $m, n \geq 1$*

$$\begin{aligned}
N_{m,n} &= (q+1)(r+1)(q^2r^2 + qr + 1)(q^6r^4)^{m-1}(q^4r^2)^{n-1}q^7r^3 \\
N_{m,0} &= (r+1)(q^2r^2 + qr + 1)(q^6r^4)^{m-1}q^4r \\
N_{0,n} &= (q+1)(q^2r^2 + qr + 1)(q^4r^2)^{n-1}q.
\end{aligned}$$

PROOF. This is very similar to Lemma C.4.2. Figure C.7.1 is useful here. $\square$

The following rather complicated counts are proved as in Lemma C.4.4.

LEMMA C.7.2. *Lex $x \in V_P$ and $z \in V_{k,l}(x)$. Write $\alpha^{k,l}_{i,j} = |V_{i,j}(x) \cap V_{0,1}(z)|$ and $\beta^{k,l}_{i,j} = |V_{i,j}(x) \cap V_{1,0}(z)|$. Then the numbers $\alpha^{k,l}_{i,j}$ and $\beta^{k,l}_{i,j}$ are independent of the particular pair x, z with $z \in V_{k,l}(x)$, and are given by the following.*

(i) *If $k \geq 1$ and $l \geq 2$, then*

$$\alpha^{k,l}_{i,j} = \begin{cases} q & \text{if } (i,j) = (k-1, l+1) \\ q^3 r & \text{if } (i,j) = (k-1, l+2) \\ 1 & \text{if } (i,j) = (k, l-1) \\ (q-1)(qr+q+1) & \text{if } (i,j) = (k, l) \\ q^4 r^2 & \text{if } (i,j) = (k, l+1) \\ qr & \text{if } (i,j) = (k+1, l-2) \\ q^3 r^2 & \text{if } (i,j) = (k+1, l-1) \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If $k \geq 1$ and $l = 1$ the counts are as in (i), with the definition $V_{k+1,-1}(x) = V_{k,1}(x)$.*

(iii) *If $k \geq 1$ and $l = 0$, then*

$$\alpha^{k,l}_{i,j} = \begin{cases} q+1 & \text{if } (i,j) = (k-1, 1) \\ q^2 r(q+1) & \text{if } (i,j) = (k-1, 2) \\ (q-1)(q+1) & \text{if } (i,j) = (k, 0) \\ q^3 r^2 (q+1) & \text{if } (i,j) = (k, 1) \\ 0 & \text{otherwise.} \end{cases}$$

(iv) *If $k = 0$ and $l \geq 2$, then*

$$\alpha^{k,l}_{i,j} = \begin{cases} 1 & \text{if } (i,j) = (0, l-1) \\ (q-1)(qr+q+1) & \text{if } (i,j) = (0, l) \\ q^4 r^2 & \text{if } (i,j) = (0, l+1) \\ (r+1)q & \text{if } (i,j) = (1, l-2) \\ q^3 r(r+1) & \text{if } (i,j) = (1, l-1) \\ 0 & \text{otherwise.} \end{cases}$$

(v) *If $k = 0$ and $l = 1$ the counts are as in (iv) with the definition $V_{1,-1}(x) = V_{0,1}(x)$.*

(vi) *If $k \geq 2$ and $l \geq 3$, then*

$$\beta_{i,j}^{k,l} = \begin{cases} (q-1)(r+1)q^3r & \textit{if } (i,j) = (k-1, l+2) \\ (q-1)^2(r+1)qr + (r-1)q^2(qr+r+1) & \textit{if } (i,j) = (k,l) \\ (q-1)(r+1)qr & \textit{if } (i,j) = (k+1, l-2) \\ (q-1)(r+1)q & \textit{if } (i,j) = (k-1, l+1) \\ (q-1)(r+1) & \textit{if } (i,j) = (k, l-1) \\ 1 & \textit{if } (i,j) = (k-1, l) \\ (q-1)(r+1)q^4r^2 & \textit{if } (i,j) = (k, l+1) \\ q^3r & \textit{if } (i,j) = (k-2, l+3) \\ q^6r^3 & \textit{if } (i,j) = (k-1, l+3) \\ q^6r^4 & \textit{if } (i,j) = (k+1, l) \\ (q-1)(r+1)q^3r^2 & \textit{if } (i,j) = (k+1, l-1) \\ q^3r^3 & \textit{if } (i,j) = (k+2, l-3) \\ r & \textit{if } (i,j) = (k+1, l-3) \\ 0 & \textit{otherwise.} \end{cases}$$

(vii) *If $k \geq 2$ and $l = 2$ the counts are as in (vi) with the definition $V_{k+1,-1}(x) = V_{k,1}(x)$.*

(viii) *If $k \geq 2$ and $l = 1$, then*

$$\beta_{i,j}^{k,l} = \begin{cases} (q-1)(r+1)q^3r & \textit{if } (i,j) = (k-1, 3) \\ 2q^2r(qr-1) + q^2(r-1) & \textit{if } (i,j) = (k,1) \\ q(qr+q-1) & \textit{if } (i,j) = (k-1, 2) \\ q-1 & \textit{if } (i,j) = (k, 0) \\ 1 & \textit{if } (i,j) = (k-1, 1) \\ q^4r^2(qr+q-1) & \textit{if } (i,j) = (k, 2) \\ q^3r & \textit{if } (i,j) = (k-2, 4) \\ q^6r^3 & \textit{if } (i,j) = (k-1, 4) \\ q^6r^4 & \textit{if } (i,j) = (k+1, 1) \\ (q-1)q^3r^2 & \textit{if } (i,j) = (k+1, 0) \\ 0 & \textit{otherwise.} \end{cases}$$

(ix) *If $k \geq 2$ and $l = 0$, then*

$$\beta_{i,j}^{k,l} = \begin{cases} (q-1)(q+1)(r+1)q^2r & \text{if } (i,j) = (k-1,2) \\ (r-1)(qr+r+1)q^2 & \text{if } (i,j) = (k,0) \\ (q-1)(q+1) & \text{if } (i,j) = (k-1,1) \\ 1 & \text{if } (i,j) = (k-1,0) \\ (q-1)(q+1)q^3r^2 & \text{if } (i,j) = (k,1) \\ (q+1)q^2r & \text{if } (i,j) = (k-2,3) \\ (q+1)q^5r^3 & \text{if } (i,j) = (k-1,3) \\ q^6r^4 & \text{if } (i,j) = (k+1,0) \\ 0 & \text{otherwise.} \end{cases}$$

(x) *If $k = 1$ and $l \geq 3$, then*

$$\beta_{i,j}^{k,l} = \begin{cases} (q-1)(r+1)q^3r & \text{if } (i,j) = (0,l+2) \\ (q-1)^2(r+1)qr + (r-1)(r+1)q^2 + q^3r^2 & \text{if } (i,j) = (1,l) \\ (q-1)(r+1)qr & \text{if } (i,j) = (2,l-2) \\ (q-1)(r+1)q & \text{if } (i,j) = (0,l+1) \\ (q-1)(r+1) & \text{if } (i,j) = (1,l-1) \\ 1 & \text{if } (i,j) = (0,l) \\ (q-1)(r+1)q^4r^2 & \text{if } (i,j) = (1,l+1) \\ q^6r^3 & \text{if } (i,j) = (0,l+3) \\ q^6r^4 & \text{if } (i,j) = (2,l) \\ (q-1)(r+1)q^3r^2 & \text{if } (i,j) = (2,l-1) \\ q^3r^3 & \text{if } (i,j) = (3,l-3) \\ r & \text{if } (i,j) = (2,l-3) \\ 0 & \text{otherwise.} \end{cases}$$

(xi) *If $k=1$ and $l=2$, then*

$$\beta_{i,j}^{k,l} = \begin{cases} (q-1)(r+1)q^3r & \text{if } (i,j)=(0,4) \\ (q-1)^2(r+1)qr+(r-1)(r+1)q^2+q^3r^2 & \text{if } (i,j)=(1,2) \\ (q-1)(r+1)qr & \text{if } (i,j)=(2,0) \\ (q-1)(r+1)q & \text{if } (i,j)=(0,3) \\ qr+q-1 & \text{if } (i,j)=(1,1) \\ 1 & \text{if } (i,j)=(0,2) \\ (q-1)(r+1)q^4r^2 & \text{if } (i,j)=(1,3) \\ q^6r^3 & \text{if } (i,j)=(0,5) \\ q^6r^4 & \text{if } (i,j)=(2,2) \\ q^3r^2(qr+r-1) & \text{if } (i,j)=(2,1) \\ 0 & \text{otherwise.} \end{cases}$$

(xii) *If $k=1$ and $l=1$*

$$\beta_{i,j}^{k,l} = \begin{cases} (q-1)(r+1)q^3r & \text{if } (i,j)=(0,3) \\ 2(q-1)q^2r+(r-1)(qr+1)q^2+q^3r^2 & \text{if } (i,j)=(1,1) \\ q(qr+q-1) & \text{if } (i,j)=(0,2) \\ q-1 & \text{if } (i,j)=(1,0) \\ 1 & \text{if } (i,j)=(0,1) \\ q^4r^2(qr+q-1) & \text{if } (i,j)=(1,2) \\ q^6r^3 & \text{if } (i,j)=(0,4) \\ q^6r^4 & \text{if } (i,j)=(2,1) \\ (q-1)q^3r^2 & \text{if } (i,j)=(2,0) \\ 0 & \text{otherwise.} \end{cases}$$

(xiii) *If $k=1$ and $l=0$, then*

$$\beta_{i,j}^{k,l} = \begin{cases} (q-1)(q+1)(r+1)q^2r & \text{if } (i,j)=(0,2) \\ (r-1)q^2+(q+1)q^2r^2 & \text{if } (i,j)=(1,0) \\ (q-1)(q+1) & \text{if } (i,j)=(0,1) \\ 1 & \text{if } (i,j)=(0,0) \\ (q-1)(q+1)q^3r^2 & \text{if } (i,j)=(1,1) \\ (q+1)q^5r^3 & \text{if } (i,j)=(0,3) \\ q^6r^4 & \text{if } (i,j)=(2,0) \\ 0 & \text{otherwise.} \end{cases}$$

(xiv) *If $k=0$ and $l\geq 3$, then*

$$\beta_{i,j}^{k,l} = \begin{cases} (r+1)^2(q-1)q^3r & \text{if } (i,j)=(1,l-1) \\ (r+1)(q-1)^2qr+(r+1)(r-1)q^2 & \text{if } (i,j)=(0,l) \\ (r+1)^2(q-1)q & \text{if } (i,j)=(1,l-2) \\ (r+1)(q-1) & \text{if } (i,j)=(0,l-1) \\ r+1 & \text{if } (i,j)=(1,l-3) \\ (r+1)(q-1)q^4r^2 & \text{if } (i,j)=(0,l+1) \\ (r+1)q^3r^2 & \text{if } (i,j)=(2,l-3) \\ (r+1)q^6r^3 & \text{if } (i,j)=(1,l) \\ 0 & \text{otherwise.} \end{cases}$$

(xv) *If $k=0$ and $l=2$, then*

$$\beta_{i,j}^{k,l} = \begin{cases} (r+1)(qr+q-1)q^3r & \text{if } (i,j)=(1,1) \\ (q-1)^2(r+1)qr+(r-1)(r+1)q^2 & \text{if } (i,j)=(0,2) \\ (q-1)(r+1)^2q & \text{if } (i,j)=(1,0) \\ (r+1)q & \text{if } (i,j)=(0,1) \\ (r+1)(q-1)q^4r^2 & \text{if } (i,j)=(0,3) \\ (r+1)q^6r^3 & \text{if } (i,j)=(1,2) \\ 0 & \text{otherwise.} \end{cases}$$

(xvi) *If $k=0$ and $l=1$, then*

$$\beta_{i,j}^{k,l} = \begin{cases} (r+1)(q-1)q^3r & \text{if } (i,j)=(1,0) \\ (r+1)q^3r & \text{if } (i,j)=(0,1) \\ (r+1)q^5r^2 & \text{if } (i,j)=(0,2) \\ (r+1)q^6r^3 & \text{if } (i,j)=(1,1) \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$\begin{aligned} c_1 &= (q-1)(qr+q+1) \\ c_2 &= (q-1)^2(r+1)qr + (r-1)(qr+r+1)q^2 \quad \text{and} \\ c_3 &= (q-1)^2qr + (r-1)q^2\,. \end{aligned}$$

Using the Lemma C.7.2 we obtain the following theorem.

THEOREM C.7.3. *The operators $A_{m,n}$ satisfy*

$$\begin{aligned} N_{0,1}A_{m,n}A_{0,1} &= qA_{m-1,n+1} + q^3rA_{m-1,n+2} + A_{m,n-1} + q^4r^2A_{m,n+1} \\ &\quad + qrA_{m+1,n-2} + q^3r^2A_{m+1,n-1} + c_1A_{m,n} \end{aligned} \tag{C.7.1}$$

$$\begin{aligned} N_{1,0}A_{m,n}A_{1,0} &= q^3rA_{m-2,n+3} + A_{m-1,n} + q^6r^3A_{m-1,n+3} + q^3r^3A_{m+2,n-3} \\ &\quad + q^6r^4A_{m+1,n} + rA_{m+1,n-3} + (q-1)(r+1)\big[qA_{m-1,n+1} + q^4r^2A_{m,n+1} \\ &\qquad + q^3rA_{m-1,n+2} + A_{m,n-1} + qrA_{m+1,n-2} + q^3r^2A_{m+1,n-1}\big] + c_2A_{m,n}\,, \end{aligned} \tag{C.7.2}$$

where $m \geq 1$ and $n \geq 2$ in (C.7.1) and $m \geq 2$ and $n \geq 3$ in (C.7.2).

The special cases of $A_{m,n}A_{0,1}$ are given by

$$\begin{aligned} N_{0,1}A_{m,1}A_{0,1} &= qA_{m-1,2} + q^3rA_{m-1,3} + A_{m,0} + q^4r^2A_{m,2} + q^3r^2A_{m+1,0} \\ &\quad + (c_1+qr)A_{m,1} \end{aligned} \tag{C.7.3}$$

$$N_{0,1}A_{m,0}A_{0,1} = (q+1)\big[A_{m-1,1} + q^2rA_{m-1,2} + (q-1)A_{m,0} + q^3r^2A_{m,1}\big] \tag{C.7.4}$$

$$\begin{aligned} N_{0,1}A_{0,n}A_{0,1} &= A_{0,n-1} + c_1A_{0,n} + q^4r^2A_{0,n+1} + q(r+1)A_{1,n-2} \\ &\quad + q^3r(r+1)A_{1,n-1} \end{aligned} \tag{C.7.5}$$

$$N_{0,1}A_{0,1}A_{0,1} = A_{0,0} + q^4r^2A_{0,2} + (r+1)q^3rA_{1,0} + \big[c_1 + q(r+1)\big]A_{0,1}\,, \tag{C.7.6}$$

where in each case m and n are required to be large enough so that the indices appearing on the right are all at least 0.

The special cases of $A_{m,n}A_{1,0}$ are given by

$$\begin{aligned}
N_{1,0}A_{m,2}A_{1,0} &= q^3 r A_{m-2,5} + A_{m-1,2} + q^6 r^3 A_{m-1,5} + q^6 r^4 A_{m+1,2} \\
&\quad + (q-1)(r+1)\big[qA_{m-1,3} + q^3 r A_{m-1,4} + q^4 r^2 A_{m,3} + qr A_{m+1,0}\big] \\
&\quad + (qr+q-1)(A_{m,1} + q^3 r^2 A_{m+1,1}) + c_2 A_{m,2}
\end{aligned} \tag{C.7.7}$$

$$\begin{aligned}
N_{1,0}A_{m,1}A_{1,0} &= q^3 r A_{m-2,4} + A_{m-1,1} + q^6 r^3 A_{m-1,4} + q^6 r^4 A_{m+1,1} \\
&\quad + (qr+q-1)(qA_{m-1,2} + q^4 r^2 A_{m,2}) + (q-1)(A_{m,0} + q^3 r^2 A_{m+1,0}) \\
&\quad + (q-1)(r+1)q^3 r A_{m-1,3} + \big[c_2 + (q-1)(r+1)qr\big]A_{m,1}
\end{aligned} \tag{C.7.8}$$

$$\begin{aligned}
N_{1,0}A_{m,0}A_{1,0} &= A_{m-1,0} + q^6 r^4 A_{m+1,0} + (q+1)(q^2 r A_{m-2,3} + q^5 r^3 A_{m-1,3}) \\
&\quad + (q^2-1)(A_{m-1,1} + q^3 r^2 A_{m,1}) + (q^2-1)(r+1)q^2 r A_{m-1,2} \\
&\quad + (r-1)(qr+r+1)q^2 A_{m,0}
\end{aligned} \tag{C.7.9}$$

$$\begin{aligned}
N_{1,0}A_{1,n}A_{1,0} &= A_{0,n} + q^6 r^3 A_{0,n+3} + rA_{2,n-3} + q^6 r^4 A_{2,n} + q^3 r^3 A_{3,n-3} \\
&\quad + (q-1)(r+1)\big[A_{1,n-1} + q^4 r^2 A_{1,n+1} + qr A_{2,n-2} + q^3 r^2 A_{2,n-1} \\
&\quad + qA_{0,n+1} + q^3 r A_{0,n+2}\big] + (c_2 + q^3 r)A_{1,n}
\end{aligned} \tag{C.7.10}$$

$$\begin{aligned}
N_{1,0}A_{0,n}A_{1,0} &= (r+1)\big[(q-1)(A_{0,n-1} + q^4 r^2 A_{0,n+1}) + c_3 A_{0,n} + A_{1,n-3} \\
&\quad + (q-1)(r+1)(qA_{1,n-2} + q^3 r A_{1,n-1}) + q^6 r^3 A_{1,n} + q^3 r^2 A_{2,n-3}\big]
\end{aligned} \tag{C.7.11}$$

$$\begin{aligned}
N_{1,0}A_{1,2}A_{1,0} &= A_{0,2} + q^6 r^3 A_{0,5} + q^6 r^4 A_{2,2} + (qr+q-1)(A_{1,1} + q^3 r^2 A_{2,1}) \\
&\quad + (c_2 + q^3 r)A_{1,2} + (q-1)(r+1)(qA_{0,3} + q^3 r A_{0,4} + q^4 r^2 A_{1,3} + qr A_{2,0})
\end{aligned} \tag{C.7.12}$$

$$\begin{aligned}
N_{1,0}A_{0,2}A_{1,0} &= (r+1)\big[qA_{0,1} + q^6 r^3 A_{1,2} + (q-1)q^4 r^2 A_{0,3} + c_3 A_{0,2} \\
&\quad + (q-1)(r+1)qA_{1,0} + (qr+q-1)q^3 r A_{1,1}\big]
\end{aligned} \tag{C.7.13}$$

$$\begin{aligned}
N_{1,0}A_{1,1}A_{1,0} &= A_{0,1} + q^6 r^3 A_{0,4} + q^6 r^4 A_{2,1} + (q-1)(A_{1,0} + q^3 r^2 A_{2,0}) \\
&\quad + (qr+q-1)(qA_{0,2} + q^4 r^2 A_{1,2}) + (q-1)(r+1)q^3 r A_{0,3} \\
&\quad + \big[c_2 + q^3 r + (q-1)(r+1)qr\big]A_{1,1}
\end{aligned} \tag{C.7.14}$$

$$N_{1,0}A_{0,1}A_{1,0} = (r+1)q^3 r\big[A_{0,1} + q^2 r A_{0,2} + (q-1)A_{1,0} + q^3 r^2 A_{1,1}\big] \tag{C.7.15}$$

$$\begin{aligned}
N_{1,0}A_{1,0}A_{1,0} &= A_{0,0} + (q^2-1)\big[A_{0,1} + (r+1)q^2 r A_{0,2} + q^3 r^2 A_{1,1}\big] \\
&\quad + (q+1)q^5 r^3 A_{0,3} + \big[(r-1)q^2 + (q+1)q^2 r^2\big]A_{1,0} + q^6 r^4 A_{2,0}\,,
\end{aligned} \tag{C.7.16}$$

where again in each case m and n are required to be large enough so that the indices appearing on the right are all at least 0.

Let $\mathscr{A}$ be the linear span over $\mathbb{C}$ of $\{A_{k,l}\}_{k,l\in\mathbb{N}}$.

Corollary C.7.4. *$\mathscr{A}$ is a commutative algebra, generated by $A_{1,0}$ and $A_{0,1}$.*

PROOF. It is an induction on (k,l) to see that $\mathscr{A}$ is the algebra generated by $A_{1,0}$ and $A_{0,1}$. By comparing (C.7.4) (with $m=1$) and (C.7.15) we see that $A_{1,0}A_{0,1}=A_{0,1}A_{1,0}$, and so $\mathscr{A}$ is commutative. $\square$

As in the BC_2 case, it is easy to see that the algebra homomorphisms $h:\mathscr{A}\to\mathbb{C}$ are indexed by X/G_2, where $X=\{(z_1,z_2,z_3)\in(\mathbb{C}^\times)^3 : z_1z_2z_3=1\}$, and the action of G_2 on X is given by permutations along with taking inverses of all entries.

Let $\rho_1=q^3r^2$ and $\rho_2=q^2r$. Suppose $h:\mathscr{A}\to\mathbb{C}$ is an algebra homomorphism, and write $x_{m,n}=\rho_1^m\rho_2^n h(A_{m,n})$. Let $\xi_1=N_{1,0}\rho_1^{-2}x_{1,0}$ and $\xi_2=N_{0,1}\rho_2^{-2}x_{0,1}$.

The following definitions will be useful.

$$\begin{aligned}
a_1&=(q-1)(qr+q+1)\rho_2^{-1}=c_1\rho_2^{-1}\,,\\
a_2&=\big[(q-1)^2(r+1)qr+(r-1)(qr+r+1)q^2\big]\rho_1^{-1}=c_2\rho_1^{-1}\,,\\
a_3&=(q-1)(r+1)q^{-1}r^{-1}\,.
\end{aligned}$$

LEMMA C.7.5. *The complex numbers $x_{m,n}$ satisfy*

$$x_{m,n+6}-\alpha x_{m,n+5}+\beta x_{m,n+4}+\gamma x_{m,n+3}+\beta x_{m,n+2}-\alpha x_{m,n+1}+x_{m,n}=0\,,$$

where

$$\begin{aligned}
\alpha&=\xi_2-a_1\\
\beta&=\xi_1-(a_3-1)(\xi_2-a_1)+3-a_2\\
\gamma&=2\xi_1-(\xi_2-a_1+2a_3)(\xi_2-a_1)+4-2a_2\,.
\end{aligned}$$

PROOF. From (C.7.1) we have

$$\begin{aligned}
(\xi_2-a_1)x_{k,l}=x_{k-1,l+1}&+x_{k-1,l+2}+x_{k,l-1}\\
&+x_{k,l+1}+x_{k+1,l-2}+x_{k+1,l-1}
\end{aligned}\tag{C.7.17}$$

for $k\geq 1$ and $l\geq 2$, and using (C.7.2) and the above we have

$$Kx_{k,l}=x_{k-2,l+3}+x_{k-1,l}+x_{k-1,l+3}+x_{k+1,l-3}+x_{k+1,l}+x_{k+2,l-3}\tag{C.7.18}$$

for $k\geq 2$ and $l\geq 3$, where $K=\xi_1-a_3\xi_2+a_1a_3-a_2$. We will use variations of the fundamental formulae (C.7.17) and (C.7.18) to prove the lemma. Our aim is to give a formula with all the first indices being m.

Adding two copies of equation (C.7.18), one with $(k,l)=(m,n)$ and one with $(k,l)=(m,n+1)$ gives

$$\begin{aligned}
K(x_{m,n}+x_{m,n+1})=x_{m-2,n+3}&+x_{m-2,n+4}+x_{m-1,n}+x_{m-1,n+1}\\
&+x_{m-1,n+3}+x_{m-1,n+4}+x_{m+1,n-3}+x_{m+1,n-2}\\
&+x_{m+1,n}+x_{m+1,n+1}+x_{m+2,n-3}+x_{m+2,n-2}
\end{aligned}\tag{C.7.19}$$

valid for $m \geq 2$ and $n \geq 3$. Using (C.7.17), with $(k,l) = (m-1, n+2)$, on the first two terms on the right hand side of (C.7.19), and with $(k,l) = (m+1, n-1)$ on the last two terms gives

$$(K+2)(x_{m,n} + x_{m,n+1}) = (\xi_2 - a_1)(x_{m-1,n+2} + x_{m+1,n-1}) \\ + x_{m-1,n} + x_{m+1,n-3} + x_{m-1,n+4} + x_{m+1,n+1} \tag{C.7.20}$$

valid for $m \geq 2$ and $n \geq 3$. Adding two copies of (C.7.20), one with (m,n) replaced by $(m, n-1)$, gives

$$\begin{aligned}
&(K+2)(2x_{m,n} + x_{m,n+1} + x_{m,n-1}) \\
&\quad = (\xi_2 - a_1)(x_{m-1,n+2} + x_{m+1,n-1} + x_{m-1,n+1} + x_{m+1,n-2}) \\
&\qquad + x_{m-1,n} + x_{m+1,n-3} + x_{m-1,n+4} + x_{m+1,n+1} \\
&\qquad\quad + x_{m-1,n-1} + x_{m+1,n-4} + x_{m-1,n+3} + x_{m+1,n}\,,
\end{aligned}$$

valid for $m \geq 2$ and $n \geq 4$, and so by using (C.7.17) with $(k,l) = (m,n)$ we have

$$\begin{aligned}
&(K+2)(2x_{m,n} + x_{m,n+1} + x_{m,n-1}) \\
&\quad = (\xi_2 - a_1)^2 x_{m,n} - (\xi_2 - a_1)(x_{m,n-1} + x_{m,n+1}) \\
&\qquad + x_{m-1,n} + x_{m+1,n-3} + x_{m-1,n+4} + x_{m+1,n+1} \\
&\qquad\quad + x_{m-1,n-1} + x_{m+1,n-4} + x_{m-1,n+3} + x_{m+1,n}
\end{aligned} \tag{C.7.21}$$

for $m \geq 2$ and $n \geq 4$. Now, the last 8 terms on the right hand side of (C.7.21) may be handled using (C.7.17), once with $(k,l) = (m, n+2)$ and once with $(k,l) = (m, n-2)$. The result is

$$\begin{aligned}
&(K+2)(2x_{m,n} + x_{m,n+1} + x_{m,n-1}) \\
&\quad = (\xi_2 - a_1)^2 x_{m,n} - (\xi_2 - a_1)(x_{m,n-1} + x_{m,n+1}) \\
&\qquad + (\xi_2 - a_1)x_{m,n+2} - x_{m,n+1} - x_{m,n+3} \\
&\qquad\quad + (\xi_2 - a_1)x_{m,n-2} - x_{m,n-3} - x_{m,n-1}
\end{aligned}$$

valid for $m \geq 2$ and $n \geq 4$, which, after replacing n by $n+4$, proves the lemma in the case $m \geq 2$. Similar arguments work using the lower order formulae in the cases $m = 0, 1$. $\square$

THEOREM C.7.6. *The algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ are indexed by X/G_2. If z_1, z_2, z_3 and their inverses are pairwise distinct complex numbers with $z_1 z_2 z_3 = 1$, then*

$$h_{z_1,z_2,z_3}(A_{m,n}) = \frac{\rho_1^{-m}\rho_2^{-n}}{(1+q^{-1})(1+r^{-1})(1+q^{-1}r^{-1}+q^{-2}r^{-2})} \sum_{\sigma \in G_2} c(\sigma \cdot z) z_{\sigma 1}^{2m+n} z_{\sigma 2}^{m}\,,$$

where

$$c(z) = \frac{(1-q^{-1}z_1^{-1})(1-q^{-1}z_2)(1-q^{-1}z_3)(1-r^{-1}z_1^{-1}z_2)(1-r^{-1}z_1^{-1}z_3)(1-r^{-1}z_2^{-1}z_3)}{(1-z_1^{-1})(1-z_2)(1-z_3)(1-z_1^{-1}z_2)(1-z_1^{-1}z_3)(1-z_2^{-1}z_3)}\,.$$

PROOF. With α, β and γ as in the previous lemma, observe that $\gamma = 2\beta - 2 - \alpha^2 - 2\alpha$. Thus if we set ω to be a root of the quadratic $x^2 - \alpha x + \beta - \alpha = 0$, we have

$$\begin{aligned}&\lambda^6 - \alpha\lambda^5 + \beta\lambda^4 + \gamma\lambda^3 + \beta\lambda^2 - \alpha\lambda + 1 \\ &\qquad = \big(\lambda^3 - \omega\lambda^2 - (\omega-\alpha)\lambda - 1\big)\big(\lambda^3 + (\omega-\alpha)\lambda^2 + \omega\lambda - 1\big)\end{aligned} \tag{C.7.22}$$

and so if z_1, z_2 and z_3 are the roots of the first cubic, we have $z_1z_2z_3 = 1$. Since

$$z^{-3} + (\omega - \alpha)z^{-2} + \omega z^{-1} - 1 = -z^{-3}\left(z^3 - \omega z_i^2 - (\omega-\alpha)z - 1\right) = 0$$

for $z = z_1, z_2, z_3$, we see that z_1^{-1}, z_2^{-1} and z_3^{-1} are the roots of the second cubic on the right hand side of (C.7.22). Thus the six roots of the sextic polynomial on the left hand side of (C.7.22) are $z_1, z_2, z_3, z_1^{-1}, z_2^{-1}, z_3^{-1}$. Furthermore $z_1z_2z_3 = 1$.

Thus if $z_1, z_2, z_3, z_1^{-1}, z_2^{-1}, z_3^{-1}$ are pairwise distinct, then

$$x_{m,n}(z_1,z_2,z_3) = C_m^{(1)}z_1^n + C_m^{(2)}z_2^n + C_m^{(3)}z_3^n + C_m^{(4)}z_1^{-n} + C_m^{(5)}z_2^{-n} + C_m^{(6)}z_3^{-n}\,, \tag{C.7.23}$$

for all $m, n \geq 0$. By root-coefficient relations of (C.7.22) we calculate

$$\begin{aligned}\xi_1 &= \frac{z_1}{z_2} + \frac{z_2}{z_1} + \frac{z_2}{z_3} + \frac{z_3}{z_2} + \frac{z_3}{z_1} + \frac{z_1}{z_3} + a_3\left(z_1 + z_2 + z_3 + \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) + a_2 \\ \xi_2 &= z_1 + z_2 + z_3 + \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + a_1,\end{aligned}$$

thus giving formulae for $x_{1,0}$ and $x_{0,1}$. The symmetries in these formulae imply that

$$\begin{aligned}C_m^{(2)}(z_1,z_2,z_3) &= C_m^{(1)}(z_2,z_1,z_3) & C_m^{(3)}(z_1,z_2,z_3) &= C_m^{(1)}(z_3,z_1,z_2) \\ C_m^{(4)}(z_1,z_2,z_3) &= C_m^{(1)}(z_1^{-1},z_2^{-1},z_3^{-1}) & C_m^{(5)}(z_1,z_2,z_3) &= C_m^{(1)}(z_2^{-1},z_1^{-1},z_3^{-1}) \\ C_m^{(6)}(z_1,z_2,z_3) &= C_m^{(1)}(z_3^{-1},z_1^{-1},z_2^{-1}) & C_m^{(1)}(z_1,z_2,z_3) &= C_m^{(1)}(z_1,z_3,z_2)\,,\end{aligned}$$

reducing the calculation to finding $C_m^{(1)}$.

By (C.7.17) we see that for $m \geq 1$

$$(\xi_2 - a_1)C_m^{(1)} = z_1C_{m-1}^{(1)} + z_1^2C_{m-1}^{(1)} + z_1^{-1}C_m^{(1)} + z_1C_m^{(1)} + z_1^{-2}C_{m+1}^{(1)} + z_1^{-1}C_{m+1}^{(1)}$$

and so with some simplification we have

$$z_1^{-2}C_{m+2}^{(1)} - (z_2 + z_3)C_{m+1}^{(1)} + z_1C_m^{(1)} = 0\,, \quad m \geq 0\,.$$

Solving the associated auxiliary equation we see that

$$C_m^{(1)} = d_1(z_1,z_2,z_3)z_1^{2m}z_2^m + d_2(z_1,z_2,z_3)z_1^{2m}z_3^m \tag{C.7.24}$$

for suitable functions $d_1(z_1,z_2,z_3)$ and $d_2(z_1,z_2,z_3)$. Since $C_m^{(1)}(z_1,z_2,z_3) = C_m^{(1)}(z_1,z_3,z_2)$ we see that $d_2(z_1,z_2,z_3) = d_1(z_1,z_3,z_2)$.

From (C.7.5) and (C.7.23) we have

$$(r+1)r^{-1}z_1^{-2}(z_1+1)C_1^{(1)} = \left[\xi_2 - a_1 - z_1 - z_1^{-1}\right]C_0^{(1)}$$

and so

$$C_1^{(1)} = \frac{rz_1(z_1z_2^2+1)}{z_2(r+1)}C_0^{(1)}\,. \tag{C.7.25}$$

The next thing to do is explicitly calculate $C_0^{(1)}$. To do this, we need the initial conditions of the recurrence in Lemma C.7.5, with $m = 0$. Thus we need to find $x_{0,k}$ for $k = 0, 1, 2, 3, 4$ and 5. Firstly we have $x_{0,0} = 1$, $x_{0,1} = \rho_2^2 N_{0,1}^{-1}\xi_2$ and $x_{1,0} = \rho_1^2 N_{1,0}^{-1}\xi_1$. The other formulae may be read off the following list;

$$\begin{aligned}
(\text{C.7.6}) \Longrightarrow x_{0,2} &= \xi_2 x_{0,1} - \big[c_1 + q(r+1)\big]\rho_2^{-1}x_{0,1} - (r+1)r^{-1}x_{1,0} - x_{0,0}\\
(\text{C.7.4}) \Longrightarrow x_{1,1} &= q(q+1)^{-1}\xi_2 x_{1,0} - x_{0,1} - x_{0,2} - (q-1)q^{-1}r^{-1}x_{1,0}\\
(\text{C.7.5}) \Longrightarrow x_{0,3} &= \xi_2 x_{0,2} - x_{0,1} - c_1\rho_2^{-1}x_{0,2} - (r+1)\big[q\rho_2\rho_1^{-1}x_{1,0} - r^{-1}x_{1,1}\big]\\
(\text{C.7.13}) \Longrightarrow x_{1,2} &= r(r+1)^{-1}\xi_1 x_{0,2} - x_{0,1} - (q-1)q^{-1}x_{0,3} - r\rho_1^{-1}c_5 x_{0,2}\\
&\quad - (q-1)(r+1)q^{-1}r^{-1}x_{1,0} - (qr+q-1)q^{-1}r^{-1}x_{1,1}\\
(\text{C.7.5}) \Longrightarrow x_{0,4} &= \xi_2 x_{0,3} - x_{0,2} - c_1\rho_2^{-1}x_{0,3} - (r+1)r^{-1}x_{1,1} - (r+1)r^{-1}x_{1,2}\\
(\text{C.7.16}) \Longrightarrow x_{2,0} &= \xi_1 x_{1,0} - x_{0,0} - (q^2-1)\rho_2^{-1}\big[x_{0,1} + (r+1)x_{0,2} + x_{1,1}\big]\\
&\quad - (q+1)q^{-1}x_{0,3} - \big[(r-1)q^2 + (q+1)q^2r^2\big]\rho_1^{-1}x_{1,0}\\
(\text{C.7.11}) \Longrightarrow x_{1,3} &= r(r+1)^{-1}\xi_1 x_{0,3} - (q-1)q^{-1}x_{0,2} - (q-1)q^{-1}x_{0,4}\\
&\quad - c_3\rho_2^{-3}\rho_1 x_{0,3} - x_{1,0} - (q-1)(r+1)q^{-1}r^{-1}x_{1,1}\\
&\quad\quad - (q-1)(r+1)q^{-1}r^{-1}x_{1,2} - x_{2,0}\\
(\text{C.7.5}) \Longrightarrow x_{0,5} &= \xi_2 x_{0,4} - x_{0,3} - c_1\rho_2^{-1}x_{0,4} - (r+1)r^{-1}x_{1,2} - (r+1)r^{-1}x_{1,3}\,.
\end{aligned}$$

Plugging all of this information into Mathematica we see that

$$C_0^{(1)} = \frac{(qz_1-1)(z_2-q)(rz_1-z_2)(qz_1z_2-1)(rz_1^2z_2-1)}{(q+1)(q^2r^2+qr+1)(z_1-1)(z_1-z_2)(z_2-1)(z_1z_2-1)(z_1^2z_2-1)}$$

Thus by (C.7.24) and (C.7.25) we are able to find

$$d_1(z_1,z_2,z_3) = M^{-1}\frac{(qz_1-1)(z_2-q)(z_3-q)(rz_1-z_2)(rz_1-z_3)(rz_2-z_3)}{(z_1-1)(z_2-1)(z_3-1)(z_1-z_2)(z_1-z_3)(z_2-z_3)}$$

where $M = (q+1)(r+1)(q^2r^2+qr+1)$. The result follows. □

REMARK C.7.7. Let us recover the above formula from the general formula for $h_u(A_\lambda)$. As in Appendix D we take $E = \{\xi \in \mathbb{R}^3 \mid \xi_1+\xi_2+\xi_3 = 0\}$, $\alpha_1 = e_1 - e_2$, and $\alpha_2 = -2e_1 + e_2 + e_3$, and

$$R^+ = \{e_1-e_2, -e_1+e_3, -e_2+e_3, -2e_1+e_2+e_3, -2e_2+e_1+e_3, 2e_3-e_1-e_2\}.$$

Thus $\lambda_1 = e_3 - e_2$ (note the typo in [**5**, Plate IX]) and $\lambda_2 = \frac{1}{3}(2e_3 - e_1 - e_2)$.

Let $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ and write $u_1 = u^{\lambda_1}$ and $u_2 = u^{\lambda_2}$. Let $t_3 = u_2$, $t_2 = u_2u_1^{-1}$, and $t_1 = t_2^{-1}t_3^{-1}$. Thus $t_1, t_2, t_3 \in \mathbb{C}^\times$, $t_1t_2t_3 = 1$, and if $ae_1 + be_2 + ce_3 \in P$ then $u^{ae_1+be_2+ce_3} = t_1^a t_2^b t_3^c$.

Recall from (6.1.1) that

$$c(u) = \prod_{\alpha \in R^+} \frac{1 - q_\alpha^{-1}u^{-\alpha^\vee}}{1 - u^{-\alpha^\vee}},$$

and so we need to compute $u^{-\alpha^\vee}$ in terms of t_1, t_2, t_3, for $\alpha \in R^+$. For example, $u^{-(e_1-e_2)^\vee} = u^{-e_1+e_2} = t_1^{-1}t_2$, and $u^{-(-2e_1+e_2+e_3)^\vee} = u^{\frac{1}{3}(2e_1-e_2-e_3)} = u^{\lambda_1-2\lambda_2} = u_1u_2^{-2} = t_2^{-1}t_3^{-1} = t_1$. Thus we see that

$$c(u) = \frac{(1-q^{-1}t_1)(1-q^{-1}t_2)(1-q^{-1}t_3^{-1})(1-r^{-1}t_1t_2^{-1})(1-r^{-1}t_1t_3^{-1})(1-r^{-1}t_2t_3^{-1})}{(1-t_1)(1-t_2)(1-t_3^{-1})(1-t_1t_2^{-1})(1-t_1t_3^{-1})(1-t_2t_3^{-1})}.$$

Now $u^{m\lambda_1+n\lambda_2} = u_1^m u_2^n = t_1^m t_3^{2m+n}$. By Proposition B.1.5 we see that $q_{t_{m\lambda_1+n\lambda_2}}^{-1/2} = \rho_1^{-m}\rho_2^{-n}$, and by Lemma C.4.1 we have $W_0(q^{-1}) = (1+q^{-1})(1+r^{-1})(1+q^{-1}r^{-1}+q^{-2}r^{-2})$.

Finally, after permuting $(t_1, t_2, t_3) \to (t_3, t_1, t_2)$ (which is fine, since h_u is G_2-symmetric) we see that the formula (6.1.1) agrees with the formula we obtained in Theorem C.7.6.

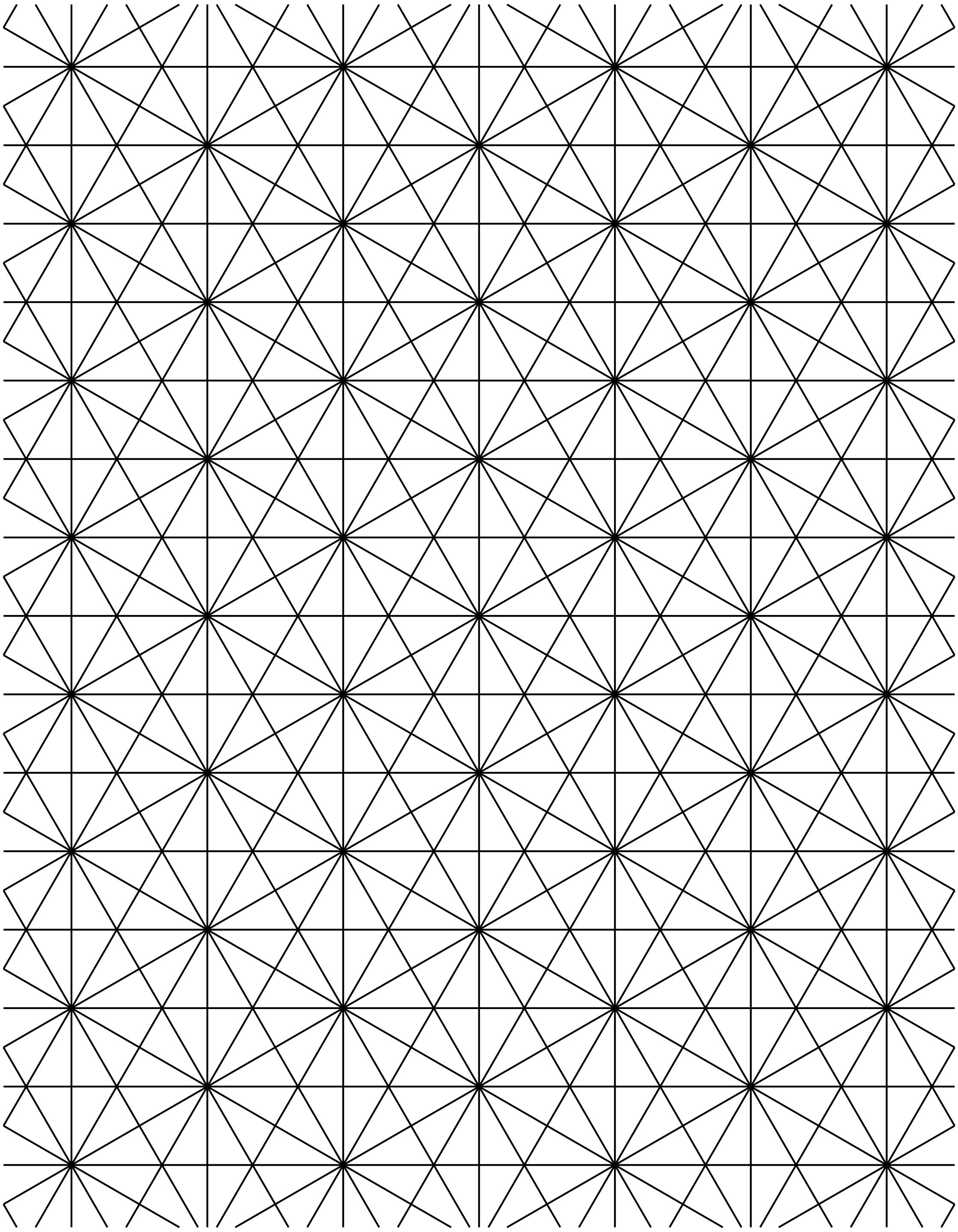


FIGURE C.7.1

APPENDIX D

The Irreducible Root Systems

The material in this appendix is taken from [**5**, Plates I-IX]. We give the following data.

(i) The underlying vector space E, and the root system R.
(ii) A base $B = \{\alpha_i\}_{i=1}^n$ of R.
(iii) The set R^+ of positive roots (relative to B).
(iv) The highest root $\tilde{\alpha}$ (relative to B), and $\tilde{\alpha}^\vee$.
(v) The set $\{\lambda_i\}_{i=1}^n$ of fundamental coweights (relative to B).
(vi) A description of P/Q.

We write $\{e_i\}_{i=1}^n$ for the canonical basis of $\mathbb{R}^n$. In expressions like $\pm e_i \pm e_j$, the $\pm$ signs are to be taken independently.

D.1. Systems of Type A_n $(n \geq 1)$

(i) $E = \{\xi \in \mathbb{R}^{n+1} \mid \xi_1 + \cdots + \xi_{n+1} = 0\}$, and $R = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n+1\}$.
(ii) $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n$.
(iii) $R^+ = \{e_i - e_j \mid 1 \leq i < j \leq n+1\}$.
(iv) $\tilde{\alpha} = \tilde{\alpha}^\vee = e_1 - e_{n+1} = \alpha_1 + \cdots + \alpha_n = \lambda_1 + \lambda_n (= 2\lambda_1$ if $n = 1)$.
(v) $\lambda_i = \frac{n+1-i}{n+1}(e_1 + \cdots + e_i) - \frac{i}{n+1}(e_{i+1} + \cdots + e_{n+1})$ for $1 \leq i \leq n$.
(vi) $P/Q \cong \mathbb{Z}/(n+1)\mathbb{Z}$, and is generated by $\lambda_1 + Q$.

D.2. Systems of Type B_n $(n \geq 2)$

(i) $E = \mathbb{R}^n$, and $R = \{\pm e_i, \pm e_j \pm e_k \mid 1 \leq i \leq n,\ 1 \leq j < k \leq n\}$.
(ii) $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$, and $\alpha_n = e_n$.
(iii) $R^+ = \{e_i, e_j \pm e_k \mid 1 \leq i \leq n,\ 1 \leq j < k \leq n\}$.
(iv) $\tilde{\alpha} = \tilde{\alpha}^\vee = e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n = \lambda_2$.
(v) $\lambda_i = e_1 + \cdots + e_i$ for $1 \leq i \leq n$.
(vi) $P/Q \cong \mathbb{Z}/2\mathbb{Z}$, and is generated by $\lambda_1 + Q$.

D.3. Systems of Type C_n $(n \geq 2)$

(i) $E = \mathbb{R}^n$, and $R = \{\pm 2e_i, \pm e_j \pm e_k \mid 1 \leq i \leq n,\ 1 \leq j < k \leq n\}$.
(ii) $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$, and $\alpha_n = 2e_n$.
(iii) $R^+ = \{2e_i, e_j \pm e_k \mid 1 \leq i \leq n,\ 1 \leq j < k \leq n\}$.
(iv) $\tilde{\alpha} = 2e_1 = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$, and $\tilde{\alpha}^\vee = e_1 = \lambda_1$.

(v) $\lambda_i = e_1 + \cdots + e_i$ for $1 \le i \le n-1$, and $\lambda_n = \frac{1}{2}(e_1 + \cdots + e_n)$.

(vi) $P/Q \cong \mathbb{Z}/2\mathbb{Z}$, and is generated by $\lambda_n + Q$.

D.4. Systems of Type D_n $(n \ge 4)$

(i) $E = \mathbb{R}^n$, and $R = \{\pm e_i \pm e_j \mid 1 \le i < j \le n\}$.

(ii) $\alpha_i = e_i - e_{i+1}$ for $1 \le i \le n-1$, and $\alpha_n = e_{n-1} + e_n$.

(iii) $R^+ = \{e_i \pm e_j \mid 1 \le i < j \le n\}$.

(iv) $\tilde{\alpha} = \tilde{\alpha}^\vee = e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n = \lambda_2$.

(v) $\lambda_i = e_1 + \cdots + e_i$ for $1 \le i \le n-2$, $\lambda_{n-1} = \frac{1}{2}(e_1 + \cdots + e_{n-1} - e_n)$, and $\lambda_n = \frac{1}{2}(e_1 + \cdots + e_n)$.

(vi) If n is *odd*, then $P/Q \cong \mathbb{Z}/4\mathbb{Z}$, and is generated by $\lambda_n + Q$. If n is *even*, then $P/Q \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, and is generated by $\lambda_{n-1} + Q$ and $\lambda_n + Q$.

D.5. Systems of Type E_6

(i) $E = \{\mathbb{R}^8 \mid \xi_6 = \xi_7 = -\xi_8\}$, and R consists of $\pm e_i \pm e_j$ $(1 \le i < j \le 5)$, and

$$\pm\frac{1}{2}\left(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{\nu(i)}e_i\right) \quad \text{with } \sum_{i=1}^{5}\nu(i) \text{ is even.}$$

(ii) $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$, $\alpha_2 = e_1 + e_2$, $\alpha_3 = e_2 - e_1$, $\alpha_4 = e_3 - e_2$, $\alpha_5 = e_4 - e_3$, and $\alpha_6 = e_5 - e_4$.

(iii) R^+ consists of the vectors $\pm e_i + e_j$ $(1 \le i < j \le 5)$, and

$$\frac{1}{2}\left(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{\nu(i)}e_i\right) \quad \text{with } \sum_{i=1}^{5}\nu(i) \text{ is even.}$$

(iv) $\tilde{\alpha} = \tilde{\alpha}^\vee = \frac{1}{2}(e_1+e_2+e_3+e_4+e_5-e_6-e_7+e_8) = \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6 = \lambda_2$.

(v) The fundamental coweights are

$$\begin{aligned}
\lambda_1 &= \frac{2}{3}(e_8 - e_6 - e_7) \\
\lambda_2 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8) \\
\lambda_3 &= \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5) + \frac{5}{6}(-e_6 - e_7 + e_8) \\
\lambda_4 &= e_3 + e_4 + e_5 - e_6 - e_7 + e_8 \\
\lambda_5 &= e_4 + e_5 + \frac{2}{3}(-e_6 - e_7 + e_8) \\
\lambda_6 &= e_5 + \frac{1}{3}(-e_6 - e_7 + e_8).
\end{aligned}$$

(vi) $P/Q \cong \mathbb{Z}/3\mathbb{Z}$, and is generated by $\lambda_1 + Q$.

D.6. Systems of Type E_7

(i) $E = \{\xi \in \mathbb{R}^8 \mid \xi_7 = -\xi_8\}$, and R consists of $\pm e_i \pm e_j$ $(1 \leq i < j \leq 6)$, $\pm(e_7 - e_8)$, and

$$\pm\frac{1}{2}\left(e_7 - e_8 + \sum_{i=1}^{6}(-1)^{\nu(i)}e_i\right) \quad \text{with } \sum_{i=1}^{6}\nu(i) \text{ odd.}$$

(ii) $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$, $\alpha_2 = e_1 + e_2$, $\alpha_3 = e_2 - e_1$, $\alpha_4 = e_3 - e_2$, $\alpha_5 = e_4 - e_3$, $\alpha_6 = e_5 - e_4$, and $\alpha_7 = e_6 - e_5$.

(iii) R^+ consists of $\pm e_i + e_j$ $(1 \leq i < j \leq 6)$, $-e_7 + e_8$, and

$$\frac{1}{2}\left(e_7 - e_8 + \sum_{i=1}^{6}(-1)^{\nu(i)}e_i\right) \quad \text{with } \sum_{i=1}^{6}\nu(i) \text{ odd.}$$

(iv) $\tilde{\alpha} = \tilde{\alpha}^\vee = e_8 - e_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \lambda_1$.

(v) The fundamental coweights are

$$\begin{aligned}
\lambda_1 &= e_8 - e_7 \\
\lambda_2 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 - 2e_7 + 2e_8) \\
\lambda_3 &= \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 - 3e_7 + 3e_8) \\
\lambda_4 &= e_3 + e_4 + e_5 + e_6 - 2e_7 + 2e_8 \\
\lambda_5 &= e_4 + e_5 + e_6 + \frac{3}{2}(e_8 - e_7) \\
\lambda_6 &= e_5 + e_6 - e_7 + e_8 \\
\lambda_7 &= e_6 + \frac{1}{2}(e_8 - e_7).
\end{aligned}$$

(vi) $P/Q \cong \mathbb{Z}/2\mathbb{Z}$, and is generated by $\lambda_7 + Q$.

D.7. Systems of Type E_8

(i) $E = \mathbb{R}^8$, and R consists of $\pm e_i \pm e_j$ $(1 \leq i < j \leq 8)$, and $\frac{1}{2}\sum_{i=1}^{8}(-1)^{\nu(i)}e_i$ with $\sum_{i=1}^{8}\nu(i)$ even.

(ii) $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$, $\alpha_2 = e_1 + e_2$, $\alpha_3 = e_2 - e_1$, $\alpha_4 = e_3 - e_2$, $\alpha_5 = e_4 - e_3$, $\alpha_6 = e_5 - e_4$, $\alpha_7 = e_6 - e_5$, and $\alpha_8 = e_7 - e_6$.

(iii) R^+ consists of $\pm e_i + e_j$ $(1 \leq i < j \leq 8)$ and

$$\frac{1}{2}\left(e_8 + \sum_{i=1}^{7}(-1)^{\nu(i)}e_i\right) \quad \text{with } \sum_{i=1}^{7}\nu(i) \text{ even.}$$

(iv) $\tilde{\alpha} = \tilde{\alpha}^\vee = e_7 + e_8 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 = \lambda_8$.

(v) The fundamental coweights are

$$\begin{aligned}
\lambda_1 &= 2e_8 \\
\lambda_2 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + 5e_8) \\
\lambda_3 &= \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + 7e_8) \\
\lambda_4 &= e_3 + e_4 + e_5 + e_6 + e_7 + 5e_8 \\
\lambda_5 &= e_4 + e_5 + e_6 + e_7 + 4e_8 \\
\lambda_6 &= e_5 + e_6 + e_7 + 3e_8 \\
\lambda_7 &= e_6 + e_7 + 2e_8 \\
\lambda_8 &= e_7 + e_8.
\end{aligned}$$

(vi) $P = Q$, and so P/Q is trivial.

D.8. Systems of Type F_4

(i) $E = \mathbb{R}^4$, and $R = \{\pm e_i, \pm e_j \pm e_k, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \mid 1 \leq i \leq 4, 1 \leq j < k \leq 4\}$.
(ii) $\alpha_1 = e_2 - e_3$, $\alpha_2 = e_3 - e_4$, $\alpha_3 = e_4$, and $\alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$.
(iii) $R^+ = \{e_i, e_j \pm e_k, \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \mid 1 \leq i \leq 4,\ 1 \leq j < k \leq 4\}$.
(iv) $\tilde{\alpha} = \tilde{\alpha}^\vee = e_1 + e_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \lambda_1$.
(v) $\lambda_1 = e_1 + e_2$, $\lambda_2 = 2e_1 + e_2 + e_3$, $\lambda_3 = 3e_1 + e_2 + e_3 + e_4$, and $\lambda_4 = 2e_1$.
(vi) $P = Q$, and so P/Q is trivial.

D.9. Systems of Type G_2

(i) $E = \{\xi \in \mathbb{R}^3 \mid \xi_1 + \xi_2 + \xi_3 = 0\}$, and R consists of $\pm(e_i - e_j)$ $(1 \leq i < j \leq 3)$, $\pm(2e_1 - e_2 - e_3)$, $\pm(2e_2 - e_1 - e_3)$, and $\pm(2e_3 - e_1 - e_2)$.
(ii) $\alpha_1 = e_1 - e_2$, and $\alpha_2 = -2e_1 + e_2 + e_3$.
(iii) $R^+ = \{e_1 - e_2, -e_1 + e_3, -e_2 + e_3, -2e_1 + e_2 + e_3, -2e_2 + e_1 + e_3, 2e_3 - e_1 - e_2\}$.
(iv) $\tilde{\alpha} = 2e_3 - e_1 - e_2$, and $\tilde{\alpha}^\vee = \frac{1}{3}(2e_3 - e_1 - e_2) = \lambda_2$.
(v) $\lambda_1 = -e_2 + e_3$, and $\lambda_2 = \frac{1}{3}(2e_3 - e_1 - e_2)$.
(vi) $P = Q$, and so P/Q is trivial.

D.10. Systems of Type BC_n $(n \geq 1)$

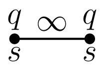
(i) $E = \mathbb{R}^n$, and $R = \{\pm e_i, \pm 2e_i, \pm e_j \pm e_k \mid 1 \leq i \leq n,\ 1 \leq j < k \leq n\}$.
(ii) $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$, and $\alpha_n = e_n$.
(iii) $R^+ = \{e_i, 2e_i, e_j \pm e_k \mid 1 \leq i \leq n,\ 1 \leq j < k \leq n\}$.
(iv) $\tilde{\alpha} = 2e_1 = 2(\alpha_1 + \cdots + \alpha_n)$, and $\tilde{\alpha}^\vee = e_1 = \lambda_1$.
(v) $\lambda_i = e_1 + \cdots + e_i$ for $1 \leq i \leq n$.
(vi) $P = Q$, and so P/Q is trivial.

APPENDIX E

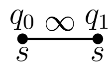
Parameter Systems of Irreducible Affine Buildings

For an $\widetilde{X}_n$ building there $n+1$ vertices in the Coxeter graph. The special vertices are marked with an s. If all the parameters are equal we write $q_i = q$.

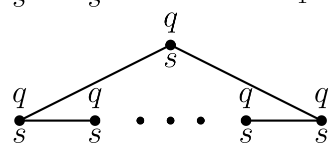
$\widetilde{A}_1$:



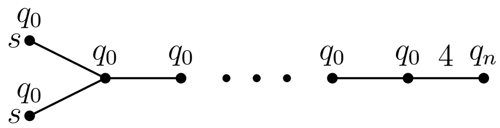
$\widetilde{BC}_1$:



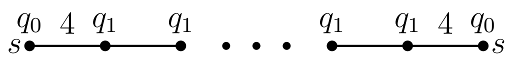
$\widetilde{A}_n (n \geq 2)$:



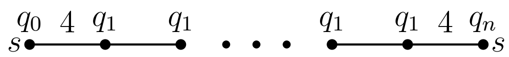
$\widetilde{B}_n (n \geq 3)$:



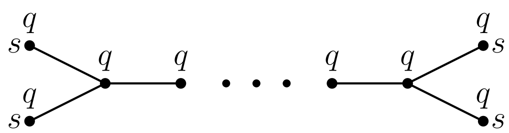
$\widetilde{C}_n (n \geq 2)$:



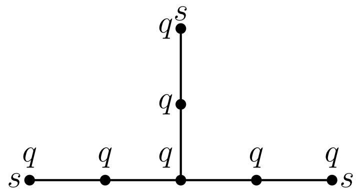
$\widetilde{BC}_n (n \geq 2)$:



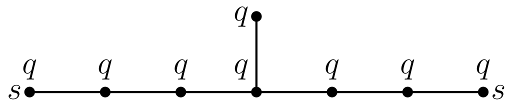
$\widetilde{D}_n (n \geq 4)$:



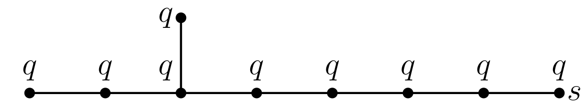
$\widetilde{E}_6$:



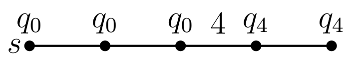
$\widetilde{E}_7$:



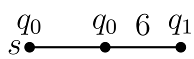
$\widetilde{E}_8$:



$\widetilde{F}_4$:



$\widetilde{G}_2$:



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