A QUANTITATIVE BUCUR-HENROT INEQUALITY

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Abstract. In this paper, we prove a quantitative version of the isoperimetric inequality involving the second non-trivial eigenvalue of the Laplacian with Neumann boundary condition established by Bucur and Henrot [5].

1. Introduction

Given a bounded open Lipschitz set $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, we consider the eigenvalue problem

$$\begin{cases} \Delta u + \mu u = 0, & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases}$$

On such domains, the Laplacian operator with Neumann boundary conditions has discrete spectrum

$$0 = \mu_0 (\Omega) \leq \mu_1 (\Omega) \leq \mu_2 (\Omega) \leq \ldots \to \infty,$$

where the eigenvalues are counted with their multiplicities.

For each $k \geq 1$, the k-th Neumann eigenvalue has the variational characterisation

$$\mu_k (\Omega) = \min_{S \in \mathcal{S}_k} \max_{u \in S} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx}, \tag{1.1}$$

where $\mathcal{S}_k$ is the family of all k-dimensional subspaces in $\{u \in H^1(\Omega) : \int_\Omega u dx = 0\}$. If Ω is connected, then $\mu_1 (\Omega) > 0$.

The classical Szegö-Weinberger inequality for $\mu_1(\Omega)$ asserts that for any bounded open Lipschitz set $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, there holds

$$|\Omega|^{\frac{2}{n}} \mu_1(\Omega) \leq |B|^{\frac{2}{n}} \mu_1(B), \tag{1.2}$$

and if equality occurs, then $\Omega = B$ a.e., where B is (any) ball. In 1954, Szegö [9] proved this inequality for simply connected smooth domains in $\mathbb{R}^2$ by conformal method. Using a topological degree argument to find the test functions for $\mu_1(\Omega)$, Weinberger [10] removed the topological constraint and the dimension restriction in 1956.

Concerning the second non-trivial Neumann eigenvalue, Girouard, Nadirashvili and Polterovich [7] proved that in $\mathbb{R}^2$, the union of two disjoint, equal disks produces a larger $\mu_2(\Omega)$ than any smooth simply connected planar domain of the same

2010 *Mathematics Subject Classification.* 47A75, 49R05.
Key words and phrases. Neumann eigenvalues, spectral inequality, quantitative stability.

measure, and this value is asymptotically attained by two disks with vanishing intersection. Building on Weinberger's strategy, Bucur and Henrot [5] devised a degree argument which enabled them to build test functions for the second non-trivial Neumann eigenvalue $\mu_2(\Omega)$. This is no trivial task because the test functions must be orthogonal to both the constant functions and the unknown first Neumann eigenfunctions on Ω. Consequently, Bucur and Henrot [5] made the breakthrough on the isoperimetric inequality for $\mu_2(\Omega)$ by showing that for an arbitrary domain Ω of prescribed measure in $\mathbb{R}^n$ $(n \geq 2)$, there holds

$$|\Omega|^{\frac{2}{n}} \mu_2(\Omega) \leq (2|B|)^{\frac{2}{n}} \mu_1(B), \tag{1.3}$$

and if equality occurs, then Ω coincides a.e. with the union of two disjoint, equal balls. In this paper, we refer to (1.3) as the Bucur-Henrot inequality.

Concerning the stability of isoperimetric inequalities involving the Neumann eigenvalues, Nadirashvili [8] proved one of the first quantitative improvements of the Szegö-Weinberger inequality for simply-connected sets in the plane. Later, Brasco and Pratelli [4] established the sharp quantitative Szegö-Weinberger inequality for arbitrary open Lipschitz sets in $\mathbb{R}^n$:

$$|B|^{\frac{2}{n}} \mu_1(B) - |\Omega|^{\frac{2}{n}} \mu_1(\Omega) \geq c_n A(\Omega)^2, \tag{1.4}$$

where c_n is a constant depending only on the dimension n. The exponent 2 of $A(\Omega)$ in (1.4) is optimal. Here, $A(\Omega)$ is the Fraenkel asymmetry of a set defined by

$$A(\Omega) := \inf \left\{ \frac{|\Omega \,\Delta\, B|}{|\Omega|} : |B| = |\Omega| \right\},$$

where $\Omega \,\Delta\, B$ denotes the symmetric difference between Ω and B. A related quantity is the Fraenkel 2-asymmetry which measures the distance of Ω from the disjoint union of two equal balls and is defined as

$$A_2(\Omega) := \inf \left\{ \frac{|\Omega \,\Delta\, (B_1 \cup B_2)|}{|\Omega|} : |B_1 \cap B_2| = 0 \text{ and } |B_1| = |B_2| = \frac{|\Omega|}{2} \right\}. \tag{1.5}$$

We note that there is a universal constant $c > 0$ such that $A_2(\Omega) \leq c$.

Inspired by the Bucur-Henrot inequality (1.3) and the sharp quantitative Szegö-Weinberger inequality (1.4) due to Brasco and Pratelli, we prove in this paper the following quantitative Bucur-Henrot inequality.

Theorem 1.1. *For every bounded open Lipschitz set $\Omega \subset \mathbb{R}^n$, we have*

$$(2\,|B|)^{\frac{2}{n}} \mu_1(B) - |\Omega|^{\frac{2}{n}} \mu_2(\Omega) \geq c_n A_2(\Omega)^{n+1}, \tag{1.6}$$

where B is any ball in $\mathbb{R}^n$ and c_n is a positive constant depending only on the dimension n.

Let us relax the definition of the Fraenkel 2-asymmetry to

$$E_2(\Omega) := \inf \left\{ \frac{|\Omega \,\Delta\, (B_1 \cup B_2)|}{|\Omega|} : |B_1| = |B_2| = \frac{|\Omega|}{2} \right\}, \tag{1.7}$$

and call $E_2(\Omega)$ the 2-error of the set Ω in this paper. By definition, $E_2(\Omega) \leq A_2(\Omega)$. As shown by Brasco and Pratelli (cf. [4, Lemma 3.3]), the 2-error controls the Fraenkel 2-asymmetry:

$$A_2(\Omega)^{n+1} \leq c_n E_2(\Omega)^2. \tag{1.8}$$

Theorem 1.1 follows from the following theorem via (1.8).

Theorem 1.2. *For every bounded open Lipschitz set $\Omega \subset \mathbb{R}^n$, we have*

$$(2\,|B|)^{\frac{2}{n}}\,\mu_1\,(B) - |\Omega|^{\frac{2}{n}}\,\mu_2(\Omega) \geq c_n E_2(\Omega)^2, \tag{1.9}$$

where B is any ball in $\mathbb{R}^n$ and c_n is a positive constant depending only on the dimension n.

As we will see in Section 4, the exponent 2 of $E_2(\Omega)$ in the quantitative inequality (1.9) is sharp. In contrast, it is very likely that the exponent $n+1$ of $A_2(\Omega)$ in the quantitative inequality (1.6) is not sharp, but we are not able to prove it here. We expect the sharp exponent of $A_2(\Omega)$ in (1.6) to depend on the dimension n owing to the example constructed by Brasco and Pratelli [4, Example 3.4]. We note that the same phenomenon occurs in the quantitative Hong-Krahn-Szegö inequality for the second non-trivial eigenvalue of the Laplacian with Dirichlet boundary condition, cf. [4, Section 3] and [3, Section 7.6.1].

The study of the optimal value of c_n in a quantitative isoperimetric inequality is not at all trivial. To the best of the authors' knowledge, such a study is the most fruitful in dimension $n = 2$ [1, 2, 6]. In this paper, we do not attempt to estimate the constant c_n in either inequality (1.6) or inequality (1.9).

This paper is organised as follows. In Section 2, we fix the notation and collect some preliminary facts. Section 3 is devoted to the proof of Theorem 1.2. In Section 4, we adapt the construction by Brasco and Pratelli in [4] to establish the sharpness of the exponent 2 of $E_2(\Omega)$ in the quantitative inequality (1.9).

2. Notation and preliminaries

Let B_r denote a ball of radius r centred at the origin $O \in \mathbb{R}^n$ and ω_n the volume of B_1. Then the first non-trivial Neumann eigenvalue rescales according to

$$\mu_1\,(B_1) = r^2 \mu_1\,(B_r)\,. \tag{2.1}$$

We denote by g_1 a non-negative, strictly increasing solution of the following ODE boundary value problem on the interval $(0, 1)$:

$$g_1''(t) + \frac{n-1}{t} g_1'(t) + \left(\mu_1(B_1) - \frac{n-1}{t^2}\right) g_1(t) = 0, \quad g_1(0) = g_1'(1) = 0. \tag{2.2}$$

Then the eigenfunctions of $\mu_1(B_1)$ are given by

$$g_1(|x|)\frac{x_i}{|x|}, \quad i = 1, \ldots, n.$$

Given a bounded open Lipschitz set $\Omega \subset \mathbb{R}^n$, we define

$$r_0 := \left(\frac{|\Omega|}{2\omega_n}\right)^{\frac{1}{n}}. \tag{2.3}$$

Then $|B_{r_0}| = |\Omega|/2$. We now define $g : [0,\infty) \to \mathbb{R}$ by

$$g(t) := \begin{cases} g_1\left(t/r_0\right), & t < r_0, \\ g_1(1), & t \geq r_0. \end{cases} \tag{2.4}$$

Then g is a non-negative, strictly increasing function on $[0, r_0]$, and $g'(t) = 0$ on $[r_0, \infty)$. Since

$$g_1\left(\frac{|x|}{r_0}\right)\frac{x_i}{|x|}, \quad i = 1, \ldots, n$$

are the eigenfunctions of $\mu_1(B_{r_0})$, (1.1) implies

$$\mu_1\left(B_{r_0}\right) = \frac{\int_{B_{r_0}} h\left(r_O(x)\right) dx}{\int_{B_{r_0}} g^2\left(r_O(x)\right) dx}, \tag{2.5}$$

where $h : [0,\infty) \to \mathbb{R}$ is defined by

$$h(t) := \left(g'(t)\right)^2 + \frac{n-1}{t^2} g^2(t), \tag{2.6}$$

and $r_x(y)$ denotes the Euclidean distance between $x, y \in \mathbb{R}^n$. Let us also define

$$h_1(t) := \left(g_1'(t)\right)^2 + \frac{n-1}{t^2} g_1^2(t). \tag{2.7}$$

Then it follows from (2.2) that $h_1'(t) \leq 0$ for $t \in [0,1]$. We note that h rescales by

$$h(t) = \frac{1}{r_0^2} h_1\left(\frac{t}{r_0}\right) \tag{2.8}$$

for $t \in [0, r_0]$, and (2.7) implies that

$$\begin{aligned} h'(t) &= \frac{2(n-1)}{t}\left(g'(t) - \frac{g(t)}{t}\right)^2 - 2\mu_1\left(B_{r_0}\right) g(t) g'(t) \\ &\leq 0 \end{aligned}$$

for $t \in [0, r_0]$. If $t > r_0$, then by definition

$$h(t) = \frac{n-1}{t^2} g_1(1),$$

and hence $h'(t) < 0$ for $t > r_0$.

Let us now recall some results from [5].

Given two different points $A, B \in \mathbb{R}^n$, let H_A and H_B denote the half-spaces determined by the mediator hyperplane Π_{AB} of the segment AB and containing A and B, respectively. We define the map $T_{AB} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T_{AB}(v) := v - 2\left(\overrightarrow{ab} \cdot v\right)\overrightarrow{ab}, \tag{2.9}$$

where $\overrightarrow{ab} = \overrightarrow{AB}/\left|\overrightarrow{AB}\right|$, and the map $g^{AB}:\mathbb{R}^n \to \mathbb{R}^n$ by

$$g^{AB}(x) := \begin{cases} g\left(r_A(x)\right)\nabla r_A(x), & x \in H_A, \\ T_{AB}\left(g\left(r_B(x)\right)\nabla r_B(x)\right), & x \in H_B. \end{cases} \tag{2.10}$$

Let $\{e_i\}_{i=1}^n$ be orthonormal basis vectors of $\mathbb{R}^n$ and u_1 a first eigenfunction of the Neumann Laplacian on Ω. A crucial step in [5], known as the centre-of-mass theorem, states that there exist distinct points $A, B \in \mathbb{R}^n$ such that

$$\int_\Omega g^{AB}\cdot e_i\,dx = \int_\Omega g^{AB}\cdot e_i u_1\,dx = 0 \quad \text{for all } i = 1,\dots,n,$$

and that the second non-trivial Neumann eigenvalue satisfies

$$\mu_2(\Omega) \le \frac{\sum_{i=1}^n \int_\Omega \left|\nabla\left(g^{AB}\cdot e_i\right)\right|^2 dx}{\sum_{i=1}^n \int_\Omega \left|g^{AB}\cdot e_i\right|^2 dx}. \tag{2.11}$$

3. Proof of Theorem 1.2

From now on, c_n and $\widetilde{c}_n$ denote constants which depend only on n but may change from line to line.

Let $\Omega_A := \Omega\cap H_A$ and $\Omega_B := \Omega \cap H_B$. Since $H_A \sqcup \Pi_{AB} \sqcup H_B = \mathbb{R}^n$ and Ω is a bounded open Lipschitz set of positive measure, Ω_A and Ω_B cannot be both empty[1]. Recall the definition of r_0 in (2.3) and that $|B_{r_0}| = |\Omega|/2$.

Lemma 3.1. *For every bounded open Lipschitz set $\Omega\subset\mathbb{R}^n$, we have*

$$c_n\,|\Omega|\left(\mu_1\left(B_{r_0}\right) - \mu_2(\Omega)\right) \ge 2\mu_1\left(B_{r_0}\right)\int_{B_{r_0}} g^2 dx - \mu_2\left(\Omega\right)\sum_{i=1}^n\int_\Omega \left|g^{AB}\cdot e_i\right|^2 dx. \tag{3.1}$$

Proof. We have

$$\begin{aligned} &2\mu_1\left(B_{r_0}\right)\int_{B_{r_0}} g^2 dx - \mu_2\left(\Omega\right)\sum_{i=1}^n\int_\Omega \left|g^{AB}\cdot e_i\right|^2 dx \\ &= 2\mu_1\left(B_{r_0}\right)\int_{B_{r_0}} g^2 dx - \mu_2\left(\Omega\right)\left(\int_{\Omega_A} g^2\left(r_A(x)\right)dx + \int_{\Omega_B} g^2\left(r_B(x)\right)dx\right) \\ &= \left(\mu_1\left(B_{r_0}\right) - \mu_2\left(\Omega\right)\right)\underbrace{\left(\int_{\Omega_A} g^2\left(r_A(x)\right)dx + \int_{\Omega_B} g^2\left(r_B(x)\right)dx\right)}_{(I)} \\ &\qquad + \mu_1\left(B_{r_0}\right)\underbrace{\left(2\int_{B_{r_0}} g^2 dx - \int_{\Omega_A} g^2\left(r_A(x)\right)dx - \int_{\Omega_B} g^2\left(r_B(x)\right)dx\right)}_{(II)}. \end{aligned}$$

[1]The proof in the rest of this section goes through if either Ω_A or Ω_B is empty.

We now estimate the term I. Since g is non-decreasing for $r>0$, we have

$$\begin{aligned} I &= \int_{\Omega_A} g^2\left(r_A(x)\right)dx + \int_{\Omega_B} g^2\left(r_B(x)\right)dx \\ &\le \int_{\Omega_A} g^2\left(r_0\right)dx + \int_{\Omega_B} g^2\left(r_0\right)dx \\ &= c_n|\Omega|. \end{aligned}$$

The last equality follows from that

$$g(r_0) = (r_0)^{1-\frac{n}{2}}\, J_{\frac{n}{2}}\left(\sqrt{\mu_1\left(B_{r_0}\right)}\right) = c_n,$$

where $J_{\frac{n}{2}}$ is the standard Bessel function.

We now estimate the term II. Let B_{r_1} and B_{r_2} be balls centred at A and B, respectively, such that

$$|\Omega_A| = \left|B_{r_1}\right|, \quad |\Omega_B| = \left|B_{r_2}\right|.$$

Without loss of generality, we assume $r_1 \le r_0 \le r_2$. We note that

$$\left|B_{r_1}\right| + \left|B_{r_2}\right| = |\Omega| = 2\left|B_{r_0}\right|$$

implies

$$r_1^n + r_2^n = 2r_0^n. \tag{3.2}$$

Since $g(t)$ is non-decreasing in t, we have

$$\begin{aligned} \int_{\Omega_A} g^2\left(r_A(x)\right)dx &= \int_{\Omega_A\cap B_{r_1}} g^2\left(r_A(x)\right)dx + \int_{\Omega_A\setminus B_{r_1}} g^2\left(r_A(x)\right)dx \\ &\ge \int_{\Omega_A\cap B_{r_1}} g^2\left(r_A(x)\right)dx + \int_{\Omega_A\setminus B_{r_1}} g^2\left(r_1\right)dx, \end{aligned}$$

and

$$\begin{aligned} \int_{B_{r_1}} g^2(r_A(x))dx &= \int_{B_{r_1}\cap\Omega_A} g^2(r_A(x))dx + \int_{B_{r_1}\setminus\Omega_A} g^2(r_A(x))dx \\ &\le \int_{B_{r_1}\cap\Omega_A} g^2(r_A(x))dx + \int_{B_{r_1}\setminus\Omega_A} g^2(r_1)dx. \end{aligned}$$

Because $|\Omega_A| = |B_{r_1}|$, the above two chains of inequalities yield

$$\begin{aligned} \int_{\Omega_A} g^2\left(r_A(x)\right)dx &\ge \int_{B_{r_1}} g^2(r_A(x))dx \\ &= \sigma_{n-1}\int_0^{r_1} g^2(t)t^{n-1}dt. \end{aligned}$$

Similarly, there holds

$$\begin{aligned} \int_{\Omega_B} g^2\left(r_B(x)\right)dx &\ge \int_{B_{r_2}} g^2(r_B(x))dx \\ &= \sigma_{n-1}\int_0^{r_2} g^2(t)t^{n-1}dt. \end{aligned}$$

As a result, we get the estimate

$$
\begin{aligned}
II &= \left(2\int_{B_{r_0}} g^2 dx - \int_{\Omega_A} g^2\left(r_A(x)\right)dx - \int_{\Omega_B} g^2\left(r_B(x)\right)dx\right)\\
&\leq \sigma_{n-1}\left(2\int_0^{r_0} g^2(t)t^{n-1}dt - \int_0^{r_1} g^2(t)t^{n-1}dt - \int_0^{r_2} g^2(t)t^{n-1}dt\right)\\
&= \sigma_{n-1}\left(\int_{r_1}^{r_0} g^2(t)t^{n-1}dt - \int_{r_0}^{r_2} g^2(t)t^{n-1}dt\right)\\
&\leq \sigma_{n-1}\left(\int_{r_1}^{r_0} g^2\left(r_0\right)t^{n-1}dt - \int_{r_0}^{r_2} g^2\left(r_0\right)t^{n-1}dt\right)\\
&= \omega_n g^2\left(r_0\right)\left(2r_0^n - r_1^n - r_2^n\right)\\
&= 0,
\end{aligned}
$$

where the last equality follows from (3.2).

Therefore, we have

$$
\begin{aligned}
&2\mu_1\left(B_{r_0}\right)\int_{B_{r_0}} g^2(r)dx - \mu_2\left(\Omega\right)\sum_{i=1}^n\int_\Omega\left|g^{AB}\cdot e_i\right|^2dx\\
=&\ \left(\mu_1\left(B_{r_0}\right) - \mu_2(\Omega)\right)\cdot(I) + \mu_1\left(B_{r_0}\right)\cdot(II)\\
\leq&\ c_n\left|\Omega\right|\left(\mu_1\left(B_{r_0}\right) - \mu_2(\Omega)\right),
\end{aligned}
$$

which proves the lemma. □

We now prove Theorem 1.2, whence follows Theorem 1.1 by (1.8).

Proof of Theorem 1.2. By (2.1), inequality (1.6) is equivalent to

$$
(2\omega_n)^{\frac{2}{n}}\mu_1\left(B_1\right) - |\Omega|^{\frac{2}{n}}\mu_2(\Omega) \geq c_n E_2(\Omega)^2. \tag{3.3}
$$

By Lemma 3.1, we have

$$
c_n\left|\Omega\right|\left(\mu_1\left(B_{r_0}\right) - \mu_2(\Omega)\right) \geq 2\mu_1\left(B_{r_0}\right)\int_{B_{r_0}} g^2dx - \mu_2\left(\Omega\right)\sum_{i=1}^n\int_\Omega\left|g^{AB}\cdot e_i\right|^2dx. \tag{3.4}
$$

We prove (3.3) by estimating the right hand side of (3.4).

From (2.5) and (2.11) we deduce that

$$
\begin{aligned}
&2\mu_1\left(B_{r_0}\right)\int_{B_{r_0}} g^2(r(x))\,dx - \mu_2\left(\Omega\right)\sum_{i=1}^n\int_\Omega\left|g^{AB}\cdot e_i\right|^2dx\\
\geq&\ \ 2\int_{B_{r_0}} h(r(x))\,dx - \sum_{i=1}^n\int_\Omega\left|\nabla\left(g^{AB}\cdot e_i\right)\right|^2dx.
\end{aligned}
$$

Using the expression of g^{AB} in (2.10), we have that

$$
\sum_{i=1}^n\int_\Omega\left|\nabla\left(g^{AB}\cdot e_i\right)\right|^2dx = \int_{\Omega_A}\sum_{i=1}^n\left|\nabla\left(g(r_A)\nabla r_A\cdot e_i\right)\right|^2dx
$$

$$+\int_{\Omega_B}\sum_{i=1}^n\left|\nabla\left(T_{AB}(g(r_B)\nabla r_B)\cdot e_i\right)\right|^2dx.$$

Since

$$\begin{aligned}\sum_{i=1}^n\left|\nabla\left(g(r_A)\nabla r_A\cdot e_i\right)\right|^2&=\sum_{i=1}^n\left|g'(r_A)(\nabla r_A\cdot e_i)\nabla r_A+g(r_A)\nabla^2 r_A(e_i)\right|^2\\&=(g'(r_A))^2+\frac{n-1}{r_A^2}g^2(r_A)\\&=h(r_A),\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^n\left|\nabla\left(T_{AB}((g(r_B)\nabla r_B))\cdot e_i\right)\right|^2\\=&\sum_{i=1}^n\left|\nabla\left(g(r_B)(\nabla r_B\cdot e_i)-2g(r_B)(\overrightarrow{ab}\cdot\nabla r_B)(\overrightarrow{ab}\cdot e_i)\right)\right|^2\\=&\sum_{i=1}^n\left|\nabla\left((g(r_B)\nabla r_B)\cdot e_i\right)\right|^2\\=&h(r_B),\end{aligned}$$

we then have that

$$\begin{aligned}&2\mu_1\left(B_{r_0}\right)\int_{B_{r_0}}g^2dx-\mu_2\left(\Omega\right)\sum_{i=1}^n\int_\Omega\left|g^{AB}\cdot e_i\right|^2dx\\\geq&\,2\int_{B_{r_0}}h(r(x))dx-\int_{\Omega_A}h\left(r_A(x)\right)dx-\int_{\Omega_B}h\left(r_B(x)\right)dx\\=&\int_{B_{r_0}(A)}h(r_A(x))dx-\int_{\Omega_A}h\left(r_A(x)\right)dx\\&+\int_{B_{r_0}(B)}h\left(r_B(x)\right)dx-\int_{\Omega_B}h\left(r_B(x)\right)dx\\=&\int_{B_{r_0}(A)\setminus\Omega_A}h\left(r_A(x)\right)dx-\int_{\Omega_A\setminus B_{r_0}(A)}h\left(r_A(x)\right)dx\\&+\int_{B_{r_0}(B)\setminus\Omega_B}h\left(r_B(x)\right)dx-\int_{\Omega_B\setminus B_{r_0}(B)}h\left(r_B(x)\right)dx\\=:&\,III,\end{aligned}$$

where $B_{r_0}(A)$ and $B_{r_0}(B)$ denote balls or radius r_0 centred at A and B, respectively.

To estimate the term III, we define r_1 and r_2 such that

$$\left|B_{r_0}(A)\cup\Omega_A\right|=\omega_n r_1^n,\tag{3.5}$$

$$\left|B_{r_0}(A)\setminus\Omega_A\right|=\omega_n\left(r_0^n-r_1^n\right),\tag{3.6}$$

$$\left|\Omega_A\setminus B_{r_0}(A)\right|=\omega_n\left(r_2^n-r_0^n\right).\tag{3.7}$$

Similarly, we define r_3 and r_4 such that

$$|B_{r_0}(B) \cup \Omega_B| = \omega_n r_3^n, \tag{3.8}$$

$$|B_{r_0}(B) \setminus \Omega_B| = \omega_n \left(r_0^n - r_3^n\right), \tag{3.9}$$

$$|\Omega_B \setminus B_{r_0}(B)| = \omega_n \left(r_4^n - r_0^n\right). \tag{3.10}$$

Then

$$|\Omega_A| + |\Omega_B| = |\Omega| = 2\,|B_{r_0}|$$

implies

$$r_1^n + r_2^n + r_3^n + r_4^n = 4r_0^n. \tag{3.11}$$

Since $h(t)$ is non-increasing in t, we have

$$\int_{B_{r_0}(A)\setminus\Omega_A} h\left(r_A(x)\right) dx \geq \sigma_{n-1} \int_{r_1}^{r_0} h(t) t^{n-1} dt,$$

$$\text{and } \int_{\Omega_A\setminus B_{r_0}(A)} h\left(r_A(x)\right) dx \leq \sigma_{n-1} \int_{r_0}^{r_2} h(t) t^{n-1} dt,$$

and likewise,

$$\int_{B_{r_0}(B)\setminus\Omega_B} h\left(r_B(x)\right) dx \geq \sigma_{n-1} \int_{r_3}^{r_0} h(t) t^{n-1} dt,$$

$$\text{and } \int_{\Omega_B\setminus B_{r_0}(B)} h\left(r_B(x)\right) dx \leq \sigma_{n-1} \int_{r_0}^{r_4} h(t) t^{n-1} dt.$$

As a result, we arrive at the estimate

$$\begin{aligned}
III &\geq \sigma_{n-1} \left(\int_{r_1}^{r_0} h(t)t^{n-1}dt + \int_{r_3}^{r_0} h(t)t^{n-1}dt - \int_{r_0}^{r_2} h(t)t^{n-1}dt - \int_{r_0}^{r_4} h(t)t^{n-1}dt \right) \\
&= \sigma_{n-1} \left[\int_{r_1}^{r_0} \left(h(t) - h(r_0)\right) t^{n-1}dt + \int_{r_1}^{r_0} h(r_0)t^{n-1}dt \right] \\
&\quad + \sigma_{n-1} \left[\int_{r_3}^{r_0} \left(h(t) - h(r_0)\right) t^{n-1}dt + \int_{r_3}^{r_0} h(r_0)t^{n-1}dt \right] \\
&\quad - \sigma_{n-1} \left[\int_{r_0}^{r_2} \left(h(t) - h(r_0)\right) t^{n-1}dt + \int_{r_0}^{r_2} h(r_0)t^{n-1}dt \right] \\
&\quad - \sigma_{n-1} \left[\int_{r_0}^{r_4} \left(h(t) - h(r_0)\right) t^{n-1}dt + \int_{r_0}^{r_4} h(r_0)t^{n-1}dt \right] \\
&= \sigma_{n-1} \left[\int_{r_1}^{r_0} \left(h(t) - h(r_0)\right) t^{n-1}dt + \int_{r_3}^{r_0} \left(h(t) - h(r_0)\right) t^{n-1}dt \right] \\
&\quad - \sigma_{n-1} \left[\int_{r_0}^{r_2} \left(h(t) - h(r_0)\right) t^{n-1}dt + \int_{r_0}^{r_4} \left(h(t) - h(r_0)\right) t^{n-1}dt \right] \\
&=: IV,
\end{aligned}$$

where in the first equality we have used

$$-\int_{r_1}^{r_0} t^{n-1}dt - \int_{r_3}^{r_0} t^{n-1}dt + \int_{r_0}^{r_2} t^{n-1}dt + \int_{r_0}^{r_4} t^{n-1}dt = \frac{r_1^n + r_2^n + r_3^n + r_4^n - 4r_0^n}{n} = 0$$

because of (3.11).

We continue the proof by estimating the term

$$\begin{aligned} IV = \sigma_{n-1} &\left[\int_{r_1}^{r_0} (h(t) - h(r_0))\, t^{n-1}dt + \int_{r_3}^{r_0} (h(t) - h(r_0))\, t^{n-1}dt\right] \\ &- \sigma_{n-1}\left[\int_{r_0}^{r_2} (h(t) - h(r_0))\, t^{n-1}dt + \int_{r_0}^{r_4} (h(t) - h(r_0))\, t^{n-1}dt\right]. \end{aligned}$$

Recall that

$$\begin{aligned} h(t) &= \left(g'(t)\right)^2 + \frac{n-1}{t^2} g^2(t), \\ h(r_0) &= \frac{n-1}{r_0^2} g^2(r_0), \\ g(t) &= g(r_0) \quad \text{and} \quad g'(t) = 0 \quad \text{for } \ t \geq r_0. \end{aligned}$$

Then we have

$$\begin{aligned} -\int_{r_0}^{r_2} (h(t) - h(r_0))\, t^{n-1}dt &= -\int_{r_0}^{r_2} \left(\left(g'(t)\right)^2 + \frac{n-1}{t^2} g^2(t) - \frac{n-1}{r_0^2} g^2(r_0)\right) t^{n-1}dt \\ &= g^2(r_0) \int_{r_0}^{r_2} \left(\frac{n-1}{r_0^2} - \frac{n-1}{t^2}\right) t^{n-1}dt \end{aligned}$$

since $g'(t) = 0$ for $t \geq r_0$; similarly, there holds

$$-\int_{r_0}^{r_4} (h(t) - h(r_0))\, t^{n-1}dt = g^2(r_0) \int_{r_0}^{r_4} \left(\frac{n-1}{r_0^2} - \frac{n-1}{t^2}\right) t^{n-1}dt.$$

So then

(3.12)

$$\begin{aligned} &III \geq IV \\ &\quad = \sigma_{n-1}\left[\int_{r_1}^{r_0} (h(t) - h(r_0))\, t^{n-1}dt + \int_{r_3}^{r_0} (h(t) - h(r_0))\, t^{n-1}dt\right] \\ &\quad\quad + \sigma_{n-1} g^2(r_0)\left[\int_{r_0}^{r_2} \left(\frac{n-1}{r_0^2} - \frac{n-1}{t^2}\right) t^{n-1}dt + \int_{r_0}^{r_4} \left(\frac{n-1}{r_0^2} - \frac{n-1}{t^2}\right) t^{n-1}dt\right]. \end{aligned}$$

We note that all the integrands in IV are non-negative.

The proof now proceeds in two cases.

Case 1. Let us suppose that $|r_0 - r_i| > r_0/2$ for some $i \in \{1, 2, 3, 4\}$.

Suppose that $|r_0 - r_1| > r_0/2$, i.e., $r_1 < r_0/2$. Then using (2.6) and that $h_1'(t) \leq 0$ on $[0, 1]$, we see that (3.12) implies that

$$III \geq IV \geq \int_{r_1}^{r_0} (h(t) - h(r_0))\, t^{n-1}dt$$

$$
\begin{aligned}
&= r_0^n \int_{r_1/r_0}^{1} \left(h(r_0 t) - h(r_0)\right) t^{n-1} dt \\
&= r_0^n \int_{r_1/r_0}^{1} \frac{1}{r_0^2} \left(h_1(t) - h_1(1)\right) t^{n-1} dt \\
&\geq r_0^{n-2} \int_{1/2}^{1} \left(h_1(t) - h_1(1)\right) t^{n-1} dt \\
&= \frac{\widetilde{c}_n}{r_0^2} |\Omega|.
\end{aligned}
$$

Similarly, suppose that $|r_0 - r_3| > r_0/2$, i.e., $r_3 < r_0/2$, then we have

$$
III \geq \frac{\widetilde{c}_n}{r_0^2} |\Omega|.
$$

Suppose that $|r_0 - r_2| > r_0/2$, i.e., $r_2 > 3r_0/2$. Then from (3.12) we get

$$
\begin{aligned}
III \geq IV &(n-1)\sigma_{n-1} g^2(r_0) \int_{r_0}^{r_2} \left(\frac{1}{r_0^2} - \frac{1}{t^2}\right) t^{n-1} dt \\
&\geq (n-1)\sigma_{n-1} (g_1(1))^2 \int_{r_0}^{\frac{3}{2} r_0} \left(\frac{1}{r_0^2} - \frac{1}{t^2}\right) t^{n-1} dt \\
&= c_n r_0^{n-2} \int_{1}^{3/2} \left(1 - \frac{1}{u^2}\right) u^{n-1} du \\
&= \frac{\widetilde{c}_n}{r_0^2} |\Omega|.
\end{aligned}
$$

Similarly, suppose that $|r_0 - r_4| > r_0/2$, i.e., $r_4 > 3r_0/2$, then we have

$$
III \geq \frac{\widetilde{c}_n}{r_0^2} |\Omega|.
$$

Combining the previous estimates with (3.4) yields

$$
c_n \left|\Omega\right| \left(\mu_1\left(B_{r_0}\right) - \mu_2\left(\Omega\right)\right) \geq III \geq \frac{\widetilde{c}_n}{r_0^2} \left|\Omega\right|;
$$

that is,

$$
\mu_1\left(B_{r_0}\right) - \mu_2\left(\Omega\right) \geq \frac{c_n}{r_0^2},
$$

where $r_0 = (|\Omega|/(2\omega_n))^{1/n}$. Thus, by (2.1) we have

$$
(2\omega_n)^{\frac{2}{n}} \mu_1(B_1) - |\Omega|^{\frac{2}{n}} \geq c_n \geq c_n E_2(\Omega)^2,
$$

proving the stability inequality (3.3) in Case 1.

Case 2. Let us suppose that $|r_0 - r_i| \leq r_0/2$ for $i = 1, 2, 3, 4$.

The goal is to estimate

(3.13)
$$IV = \sigma_{n-1}\left[\int_{r_1}^{r_0}\left(h(t)-h(r_0)\right)t^{n-1}dt+\int_{r_3}^{r_0}\left(h(t)-h(r_0)\right)t^{n-1}dt\right]$$
$$+\sigma_{n-1}g^2(r_0)\left[\int_{r_0}^{r_2}\left(\frac{n-1}{r_0^2}-\frac{n-1}{t^2}\right)t^{n-1}dt+\int_{r_0}^{r_4}\left(\frac{n-1}{r_0^2}-\frac{n-1}{t^2}\right)t^{n-1}dt\right].$$

By the Mean Value Theorem, we have

$$\int_{r_1}^{r_0}\left(h(t)-h(r_0)\right)t^{n-1}dt = r_0^n\int_{r_1/r_0}^{1}\left(h(tr_0)-h(r_0)\right)t^{n-1}dt$$
$$= r_0^n\int_{r_1/r_0}^{1}\left(h'(\xi)(t-1)r_0\right)t^{n-1}dt$$

for some $\xi\in(r_1,r_0)$. Recall that $h(t)$ defined by (2.6) rescales according to (2.8). So $h(t)$ and $h'(t)$ rescale according to

$$h(t)=\frac{1}{r^2}h_1\left(\frac{t}{r}\right),\quad h'(t)=\frac{1}{r^3}h_1'\left(\frac{t}{r}\right),$$

respectively, and hence there exists $\theta=\xi/r_0\in(1/2,1)$ such that

$$h'(\xi)=\frac{1}{r_0^3}h_1'(\theta)$$
$$=-\frac{1}{r_0^3}\left[-\frac{2(n-1)}{\theta}\left(g_1'(\theta)-\frac{g_1(\theta)}{\theta}\right)^2+2\mu_1\left(B_1\right)g_1(\theta)g_1'(\theta)\right]$$
$$\le -\frac{c_n}{r_0^3}$$

for some constant c_n. It then follows that

$$\int_{r_1}^{r_0}\left(h(t)-h(r_0)\right)t^{n-1}dt \ge r_0^n\int_{r_1/r_0}^{1}\frac{c_n}{r_0^3}(1-t)r_0t^{n-1}dt.$$

Since $r_1/r_0\ge 1/2$ and $c_n(1-t)\ge 0$, it then follows that

$$\int_{r_1}^{r_0}\left(h(t)-h(r_0)\right)t^{n-1}dt \ge \frac{c_n}{2^{n-1}}r_0^{n-2}\int_{r_1/r_0}^{1}(1-t)dt$$
$$=\frac{c_n}{r_0^4}r_0^n\left(r_0-r_1\right)^2;$$

that is,

$$\int_{r_1}^{r_0}\left(h(t)-h(r_0)\right)t^{n-1}dt \ge \frac{c_n}{r_0^4}r_0^n\left(r_0-r_1\right)^2. \tag{3.14}$$

Similarly, we have

$$\int_{r_3}^{r_0}\left(h(t)-h(r_0)\right)t^{n-1}dt \ge \frac{c_n}{r_0^4}r_0^n\left(r_0-r_3\right)^2 \tag{3.15}$$

To estimate the remaining integrals in (3.13), we let $u(t) := (n-1)/t^2$. Then again the Mean Value Theorem implies that for some $\xi \in (1, r_2/r_0)$, there holds

$$\begin{aligned}
g^2(r_0)\int_{r_0}^{r_2}\left(\frac{n-1}{r_0^2}-\frac{n-1}{t^2}\right)t^{n-1}dt &= g^2(r_0)\int_{r_0}^{r_2}\left(u(r_0)-u(t)\right)t^{n-1}dt\\
&= g^2(r_0)r_0^n\int_1^{r_2/r_0}\left(u(r_0)-u(r_0t)\right)t^{n-1}dt\\
&= g^2(r_0)r_0^n\int_1^{r_2/r_0}u'(\xi)(1-t)r_0t^{n-1}dt\\
&\geq -\frac{c_n}{r_0^2}r_0^n\int_1^{r_2/r_0}(1-t)dt\\
&= \frac{c_n}{r_0^4}r_0^n\left(r_0-r_2\right)^2
\end{aligned}$$

for some constant c_n; that is,

$$g^2(r_0)\int_{r_0}^{r_2}\left(\frac{n-1}{r_0^2}-\frac{n-1}{t^2}\right)t^{n-1}dt \geq \frac{c_n}{r_0^4}r_0^n\left(r_0-r_2\right)^2. \tag{3.16}$$

Likewise, we have

$$g^2(r_0)\int_{r_0}^{r_4}\left(\frac{n-1}{r_0^2}-\frac{n-1}{t^2}\right)t^{n-1}dt \geq \frac{c_n}{r_0^4}r_0^n\left(r_0-r_4\right)^2. \tag{3.17}$$

Estimates (3.14)–(3.17) imply that

$$\begin{aligned}
\widetilde{c}_n\,|\Omega|\left(\mu_1\left(B_{r_0}\right)-\mu_2(\Omega)\right) &\geq 2\mu_1\left(B_{r_0}\right)\int_{B_{r_0}}g^2(r)dx-\mu_2\left(\Omega\right)\sum_{i=1}^n\int_\Omega\left|g^{AB}\cdot e_i\right|^2dx\\
&\geq III \geq IV\\
&\geq \frac{c_n}{r_0^4}\left[\sum_{i=1}^4\left(r_0-r_i\right)^2\right]r_0^n.
\end{aligned}$$

By (2.3), $|\Omega| = 2\omega_n r_0^n$, so then

$$\mu_1\left(B_{r_0}\right)-\mu_2(\Omega) \geq \frac{c_n}{r_0^4}\sum_{i=1}^4\left(r_0-r_i\right)^2. \tag{3.18}$$

To estimate the right hand side of (3.18), we use (3.6) to get

$$\begin{aligned}
\left|B_{r_0}(A)\setminus\Omega_A\right| &= \omega_n\left(r_0-r_1\right)^n\\
&\leq c_n r_0^{n-1}\left(r_0-r_1\right),
\end{aligned}$$

where the inequality follows from the assumption $|r_0-r_1| \leq r_0/2$ in Case 2. So we have proved the following inequality

$$\frac{\left|B_{r_0}(A)\setminus\Omega_A\right|}{|\Omega|} \leq c_n\frac{r_0-r_1}{r_0}. \tag{3.19}$$

Similar estimates hold for the remaining terms on the right hand side of (3.18):

$$\frac{|\Omega_A \setminus B_{r_0}(A)|}{|\Omega|} \le c_n \frac{r_2 - r_0}{r_0}, \tag{3.20}$$

$$\frac{|B_{r_0}(B) \setminus \Omega_B|}{|\Omega|} \le c_n \frac{r_0 - r_3}{r_0}, \tag{3.21}$$

$$\frac{|\Omega_B \setminus B_{r_0}(B)|}{|\Omega|} \le c_n \frac{r_4 - r_0}{r_0}. \tag{3.22}$$

Thus, from (3.18)–(3.22) we deduce

$$\begin{aligned} \mu_1(B_{r_0}) - \mu_2(\Omega) &\ge \frac{c_n}{r_0^2}\left(\frac{|B_{r_0}(A)\,\Delta\,\Omega_A| + |B_{r_0}(B)\,\Delta\,\Omega_B|}{|\Omega|}\right)^2 \\ &\ge \frac{c_n}{|\Omega|^{\frac{2}{n}}} E_2(\Omega)^2, \end{aligned}$$

where $\mu_1(B_{r_0}) = \mu_1(B_1)/r_0^2$ and $r_0 = (|\Omega|/(2\omega_n))^{1/n}$. Therefore, we have

$$(2\omega_n)^{\frac{2}{n}} \mu_1(B_1) - |\Omega|^{\frac{2}{n}} \mu_2(\Omega) \ge c_n E_2(\Omega)^2,$$

proving the stability inequality (3.3) in Case 2.

The proof of Theorem 1.2 is now complete. □

4. Sharpness of the exponent of $E_2(\Omega)$ in (1.9)

In [4], Brasco and Pratelli proved the sharp quantitative Szegö-Weinberger inequality

$$|B|^{\frac{2}{n}} \mu_1(B) - |\Omega|^{\frac{2}{n}} \mu_2(\Omega) \ge c_n A(\Omega)^2.$$

The authors established, through non-trivial work, the sharpness of the exponent 2 of $A(\Omega)$ by exhibiting sets[2] $B_\varepsilon \subset \mathbb{R}^n$ for $\varepsilon > 0$ small such that

$$|B_\varepsilon| = |B|, \tag{4.1}$$

$$A(B_\varepsilon) \approx \frac{|B_\varepsilon\,\Delta\,B|}{|B|} = O(\varepsilon), \tag{4.2}$$

$$\mu_1(B) - \mu_1(B_\varepsilon) = O(\varepsilon^2). \tag{4.3}$$

We now adapt the Brasco-Pratelli construction in [4] to show that the exponent 2 of $E_2(\Omega)$ in the quantitative inequality (1.9) is sharp. Let B^1, B^2 be two disjoint balls of unit radius in $\mathbb{R}^n$ such that the distance between B^1 and B^2 is large (e.g., ≥ 20). We take B_ε in the Brasco-Pratelli construction and define

$$\Omega_\varepsilon = B_\varepsilon^1 \cup B_\varepsilon^2,$$

where $B_\varepsilon^1 = B_\varepsilon^2 = B_\varepsilon$. Since B^1 and B^2 are far away from each other, we have $B_\varepsilon^1 \cap B_\varepsilon^2 = \emptyset$.

[2] In [4, Section 6], $|B| - |B_\varepsilon| = O(\varepsilon^2)$. Rescaling B_ε so that (4.1) holds introduces error $O(\varepsilon^2)$ to (4.2) and (4.3).

Lemma 4.1. *There holds the following equality*

$$\mu_1\left(B_\varepsilon^1\right)=\mu_2\left(\Omega_\varepsilon\right). \tag{4.4}$$

Proof. We first note that

$$\begin{aligned}\mu_0\left(\Omega_\varepsilon\right)=0 \quad &\text{with eigenfunction } u_0^\varepsilon=\chi_{B_\varepsilon^1},\\ \mu_1\left(\Omega_\varepsilon\right)=0 \quad &\text{with eigenfunction } u_0^\varepsilon=\chi_{B_\varepsilon^2},\end{aligned}$$

where χ_Ω is the characteristic function on Ω.

On the one hand, let u_2^ε be an eigenfunction for $\mu_2\left(\Omega_\varepsilon\right)$, then

$$0=\int_{\Omega_\varepsilon}u_2^\varepsilon(x)u_0^\varepsilon(x)dx=\int_{B_\varepsilon^1}u_2^\varepsilon(x).$$

So u_2^ε is a test function for $\mu_1\left(D_\varepsilon^1\right)$, and hence

$$\mu_1\left(B_\varepsilon^1\right)\le\frac{\int_{B_\varepsilon^1}\left|\nabla u_2^\varepsilon(x)\right|^2}{\int_{B_\varepsilon^1}\left|u_2^\varepsilon(x)\right|^2}=\mu_2\left(\Omega_\varepsilon\right).$$

On the other hand, let v_1^ε be an eigenfunction of $\mu_1\left(B_\varepsilon^1\right)$ and define

$$v_2^\varepsilon(x)=v_1^\varepsilon(x)\chi_{B_\varepsilon^1}.$$

Then we have

$$\begin{aligned}\int_{\Omega_\varepsilon}v_2^\varepsilon(x)u_0^\varepsilon(x)dx&=\int_{B_\varepsilon^1}v_1^\varepsilon(x)dx=0,\\ \int_{\Omega_\varepsilon}v_2^\varepsilon(x)u_1^\varepsilon(x)dx&=\int_{B_\varepsilon^1}v_1^\varepsilon(x)\chi_{B_\varepsilon^2}(x)dx=0.\end{aligned}$$

So v_2^ε is a testing function for $\mu_2\left(\Omega_\varepsilon\right)$, and thus

$$\mu_2\left(\Omega_\varepsilon\right)\le\frac{\int_{\Omega_\varepsilon}\left|\nabla v_2^\varepsilon(x)\right|^2}{\int_{\Omega_\varepsilon}\left(v_2^\varepsilon(x)\right)^2}=\frac{\int_{B_\varepsilon^1}\left|\nabla v_1^\varepsilon(x)\right|^2}{\int_{B_\varepsilon^1}\left(v_1^\varepsilon(x)\right)^2}=\mu_1\left(B_\varepsilon^1\right).$$

Therefore, the lemma is proved. ∎

By construction, we have

$$E_2(\Omega_\varepsilon)\approx\frac{|\Omega_\varepsilon\,\Delta\,\Omega|}{|\Omega|}=O(\varepsilon).$$

By (4.3) and Lemma 4.1, we have

$$\mu_1(B)-\mu_2\left(\Omega_\varepsilon\right)=\mu_1(B)-\mu_1(B_\varepsilon)=O(\varepsilon^2).$$

Therefore, the exponent 2 of $E_2(\Omega)$ in inequality (1.9) is sharp.

Acknowledgements. The authors thank Professor Lorenzo Brasco and the anonymous referees for valuable comments on the manuscript. K. Wang is partially supported by NSF of China under Grant No. 11601359; H. Wu is supported by ARC Grant DE180101348. Both authors thank the Sydney Mathematical Research Institute and the mathematical research institute MATRIX in Australia for providing excellent working conditions.

References

[1] Angelo Alvino, Vincenzo Ferone, and Carlo Nitsch, *A sharp isoperimetric inequality in the plane*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 1, 185–206. MR2735080

[2] T. Bonnesen, *Über das isoperimetrische Defizit ebener Figuren*, Math. Ann. **91** (1924), no. 3-4, 252–268. MR1512192

[3] Lorenzo Brasco and Guido De Philippis, *Spectral inequalities in quantitative form*, Shape optimization and spectral theory, 2017, pp. 201–281.

[4] Lorenzo Brasco and Aldo Pratelli, *Sharp stability of some spectral inequalities*, Geom. Funct. Anal. **22** (2012), no. 1, 107–135.

[5] Dorin Bucur and Antoine Henrot, *Maximization of the second non-trivial Neumann eigenvalue*, Acta Math. **222** (2019), no. 2, 337–361.

[6] Marco Cicalese and Gian Paolo Leonardi, *Best constants for the isoperimetric inequality in quantitative form*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 3, 1101–1129. MR3085102

[7] Alexandre Girouard, Nikolai Nadirashvili, and Iosif Polterovich, *Maximization of the second positive Neumann eigenvalue for planar domains*, J. Differential Geom. **83** (2009), no. 3, 637–661.

[8] Nikolai Nadirashvili, *Conformal maps and isoperimetric inequalities for eigenvalues of the Neumann problem*, Proceedings of the Ashkelon Workshop on Complex Function Theory (1996), 1997, pp. 197–201.

[9] G. Szegö, *Inequalities for certain eigenvalues of a membrane of given area*, J. Rational Mech. Anal. **3** (1954), 343–356.

[10] H. F. Weinberger, *An isoperimetric inequality for the N-dimensional free membrane problem*, J. Rational Mech. Anal. **5** (1956), 633–636. MR79286

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