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Towards the p-canonical basis in $\tilde{A}_1$

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ABSTRACT. For a universal Coxeter system (W, S) and a realization, we give a topological formula for the local intersection forms $I_{\underline{w},x}$, where $\underline{w}$ is a reduced expression and x is an element of W. This formula is given in terms of the generalized Cartan matrix determined by the realization. For this purpose, we calculate compositions modulo lower terms of degree zero Libedinsky's light leaves. In fields k of characteristic $p > 0$, ranks of local intersection forms over k are used to define the p-canonical basis of the Hecke algebra. Furthermore, these numbers govern the direct sum decomposition of Soergel bimodules over k.

1. INTRODUCTION

1.1. **Representation theory and Coxeter groups.** Representation theory is a branch of mathematics that simplifies the study of algebraic structures, replacing it by the study of linear algebra. This translation is done by "representing" elements of the algebraic structure by linear transformations between vector spaces. Formally, we have the following definition of a representation.

Definition 1.1. Let $\mathbf{F}$ be a field. A *representation* of a group G (respectively, an associative algebra A) on an $\mathbf{F}$-vector space V is a group homomorphism (respectively, an associative algebra homomorphism) $\varphi\colon G \to \mathrm{GL}(V)$ (respectively, $\varphi\colon G \to \mathrm{End}_{\mathbf{F}}(V)$). We denote the image of $g \in G$ respect to φ by $\varphi_g := \varphi(g)$, and for $v \in V$, we denote $\varphi_g(v) = g \cdot v$. If φ is injective, we say that the representation is *faithful*. A representation φ of an algebraic group G is called *rational* if it is a rational map between the algebraic varieties G and $\mathrm{GL}(V)$.

Example 1.2. Let $m, r \in \mathbb{Z}$ with $r \mid m$, let $\mathbb{Z}/m\mathbb{Z}$ be the additive group of integers modulo m, $V = \mathbb{R}^2$ and $\theta = 2\pi/r$. Let φ be the homomorphism given by

$$\varphi\colon \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathrm{GL}(2, \mathbb{R})$$
$$\bar{1} \longmapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The representation is faithful only if $r = m$.

Definition 1.3. Let V be a finite-dimensional vector space and let φ be a group representation of G on V. The *character* of φ is the function $\chi_\varphi\colon G \to \mathbf{F}$ given by $\chi_\varphi(g) = \mathrm{Tr}(\varphi_g)$, where Tr is the matrix trace.

Characters carry essential information about group representations. Furthermore, over the field $\mathbb{C}$ there is a bijective correspondence between characters and finite group representations. Knowledge about characters gives us some valuable data in order to study finite group theory and their classification theorems. Some fundamental objects in the study of representations are the *irreducible* group representations, since one can construct many other representations and characters from these simple ones.

It is of great interest the study of representations of groups arising as symmetries, reflections or root systems. The symmetry group of an object is the group of all isometries under which the object is invariant. For instance, the dihedral group D_{2n} corresponds to the symmetry group of an n-gon. A *reflection group* is a group which is generated by a set of reflections of a finite-dimensional Euclidean space. A *Weyl group* associated with a root system Φ is a group generated by reflections associated with Φ, see [Hum92]. All groups described in this paragraph are particular cases of Coxeter groups. This kind of groups are the central object of this thesis. Its definition is the following.

Definition 1.4. Let $S = \{s, t, \ldots, u\}$ be a finite set, for each pair (s,t) of elements in S we define $m_{s,t} \in \mathbb{N} \cup \{\infty\}$ such that $m_{s,s} = 1$ and $m_{s,t} > 1$ if $s \neq t$. Let W be the group with presentation

$$W = \langle S \mid (st)^{m_{s,t}} = e \quad \text{whenever} \quad m_{s,t} \neq \infty \rangle.$$

We say that W is a *Coxeter group* and the pair (W, S) is a *Coxeter system*.

Notation 1.5. For each element of S, we will associate a different color, e.g. $S = \{s, t, \ldots, u\}$.

Let us introduce a basic object of study in this thesis.

1.2. **The Hecke algebra.** The study of representations of a group W is closely related to the study of representations of the group algebra $\Bbbk[W]$, where $\Bbbk$ is a ring (commonly a field). In representation theory, it is important to study a more general substitute of this algebra.

We focus on a Coxeter system (W, S). The algebra that we will define now is a "quantum" deformation of the group algebra $\mathbb{Z}[W]$.

Definition 1.6. The *Hecke algebra* $\mathbf{H}$ is defined as the unique $\mathbb{Z}[v, v^{-1}]$-algebra generated by $\{H_s\}_{s \in S}$, with relations:

$$H_s^2 = (v - v^{-1})H_s + H_e, \quad \text{for} \quad s \in S,$$

$$\underbrace{H_s H_t H_s H_t \cdots}_{m_{s,t}} = \underbrace{H_t H_s H_t H_s \cdots}_{m_{s,t}}, \quad \text{for} \quad s, t \in S. \tag{1.1}$$

Where H_e is the multiplicative identity of $\mathbf{H}$.

Specializing at $v = 1$ we recover the group algebra $\mathbb{Z}[W]$. For $w \in W$ and a reduced expression $\underline{w} = (s, t, \ldots, u)$ of w, let us define $H_w := H_s H_t \cdots H_u$. As we will see in Section 2.2, this is well-defined (i.e. it does not depend on the reduced expression $\underline{w}$, only on w) thanks to the braid relation (1.1) and the Matsumoto theorem (Theorem 2.22). The set $\{H_w \mid w \in W\}$ is a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathbf{H}$ called the *standard basis*. There is another $\mathbb{Z}[v, v^{-1}]$-basis

$$\{\underline{H}_w \mid w \in W\}$$

of $\mathbf{H}$. The element $\underline{H}_w$ is defined as the unique element in $\mathbf{H}$ which is invariant under the involution map $\iota \colon \mathbf{H} \to \mathbf{H}$, defined by $H_x \mapsto H_{x^{-1}}^{-1}$ and such that

$$\underline{H}_w \in H_w + \sum_{y < w} v\mathbb{Z}[v] H_y,$$

where $<$ is the "Bruhat order" on W, see Chapter 2. This is the *Kazhdan-Lusztig basis* (or just, the *KL-basis*) of the Hecke algebra $\mathbf{H}$. In particular,

$$\underline{H}_w = H_w + \sum_{y<w} vP_{y,w}(v)H_y,$$

where $P_{y,w}$ are the so-called *Kazhdan-Lusztig polynomials*. This is the purely algebraic part of the story. Before continuing, let us recall some definitions.

Definition 1.7. A connected linear algebraic group G over a field $\Bbbk$ is called *reductive* if the radical (maximal connected solvable normal subgroup) of the connected component of the identity G^0 is an algebraic torus. A *split reductive group* over $\Bbbk$ is a reductive group which contains a torus T, where T is maximal among all $\Bbbk$-tori in G. The algebraic subgroup T is called a *split maximal torus* of G. A *Borel subgroup* $B \subset G$ is a maximal smooth connected solvable Zariski closed subgroup of G.

Example 1.8. The general linear group $G \coloneqq \mathrm{GL}(n, \mathbb{C})$ of invertible $n \times n$ matrices is a split reductive group over $\mathbb{C}$. A split maximal torus T is the set of diagonal matrices which have all diagonal entries different from zero. In this case, it is easy to see that $T \cong G_m^n$. One possible Borel subgroup B is given by the set of all upper-triangular invertible $n \times n$ matrices. Clearly $T \subset B \subset G$.

A first description of the Hecke algebra appeared in the study of the irreducible complex characters of a split reductive group G over a finite field $\mathbb{F}_q$, where $q = p^m$, see [Iwa64]. Iwahori studied the algebra

$$\mathrm{Fun}_{B\times B}(G, \mathbb{C}),$$

i.e., the set of B-bi-invariant complex-valued functions on G, with multiplication given by convolution * and $B \subset G$ is a Borel subgroup. Fix a maximal split torus $T \subset B$. Iwahori [Iwa64] observed that this algebra has a description which is in a certain sense independent of the size q of the base field and only relies on the Weyl group. Let (W, S) be the Weyl group associated with (G, B), namely, $W = N_G(T)/T$. The algebra above has a basis given by indicator functions of the subsets $BwB \subset G$. This fact can be proved using the Bruhat decomposition of G,

$$G = \bigsqcup_{w\in W} BwB.$$

Defining $\mathbf{H}_{\mathbb{F}_q} \coloneqq \mathbf{H} \otimes \mathbb{C}$ to be the specialization of v^{-1} at $\sqrt{|\mathbb{F}_q|} \in \mathbb{C}$, we have an isomorphism of algebras

$$\mathbf{H}_{\mathbb{F}_q} \xrightarrow{\sim} \mathrm{Fun}_{B\times B}(G, \mathbb{C}).$$

Sending $T_w \coloneqq v^{-\ell(w)}H_w$ to the indicator function $\mathbf{1}_{BwB}$ of $BwB \subset G$, where ℓ is the length function in the Weyl group W. As we stated above, Iwahori defined the Hecke algebra only in terms of the Coxeter system structure, whether or not it arises as the Weyl group of a reductive group.

The study of linear group representations over different fields is a central subject in representation theory. In 1963, Steinberg proved that all the irreducible representations of the finite groups $\mathrm{GL}(n, \mathbb{F}_q)$ could be obtained from the irreducible representations of $\mathrm{GL}(n, \overline{\mathbb{F}}_p)$ by restriction (see [Ste63]). Sixteen years later, Lusztig conjectured a description of the irreducible characters of highest weight $w \cdot 0$, i.e., those that are in the orbit of 0. From this conjecture and other theorems (Steinberg's theorem of tensor product, the linkage principle of Andersen and the translation principle), one obtain all irreducible rational representations of any reductive group G over $\overline{\mathbb{F}}_p$, when $p > h$, where h is the Coxeter number of the Weyl group associated with G. He gave in [Lus80] the following formulas for those characters,

$$\mathrm{ch}(L_w) = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1)\,\mathrm{ch}(M_y),$$
$$\mathrm{ch}(M_w) = \sum_{y \leq w} P_{w_0 w, w_0 y}(1)\,\mathrm{ch}(L_y).$$

Where w_0 is the longest element in W, M_w is the Verma module of highest weight $w \cdot 0$, L_w its irreducible quotient, and the polynomials P are the Kazhdan-Lusztig polynomials. These are the famous *Kazhdan-Lusztig's conjectures*.

1.3. **The Hecke category.** Due to the seminal work [KL79] of Kazhdan and Lusztig, it was realized that the Hecke algebra admits a categorification. Let G be a complex reductive group with Borel subgroup $B \subset G$ and maximal torus $T \subset B$ as in the previous section. The Hecke category $\mathscr{H}$ is the additive subcategory of semi-simple complexes

$$\mathscr{H} \subset \mathbf{D}^b_{B\times B}(G, \mathbb{C}),$$

in the equivariant bounded derived category[1] of sheaves on G (in the sense of Bernstein and Lunts). In particular, the objects of $\mathscr{H}$ are direct sums of shifts of finitely many $\mathbf{IC}_w \coloneqq \mathbf{IC}(X_w)$, where $\mathbf{IC}$ denotes the "equivariant" intersection cohomology complex[2] of X_w, with $w \in W$. The variety X_w as well as the equivariant cohomology will be defined below. The monoidal structure * on $\mathbf{D}^b_{B\times B}(G, \mathbb{C})$ is given by convolution of complexes, which preserves $\mathscr{H}$. This fact can be proved due to the Decomposition Theorem and the compactness of the flag variety $X \coloneqq G/B$, using ideas from Springer and MacPherson in [Spr82] which are described in a better way by Riche in [Ric10]. The rough idea is the following. Consider the Bruhat decomposition of X:

$$X = \bigsqcup_{w \in W} BwB/B.$$

[1] The *derived category* $\mathbf{D}^b(\mathscr{A})$ of an abelian category $\mathscr{A}$ is the category whose objects are chain complexes of $\mathscr{A}$, but instead of having arrows as the usual morphisms of complexes, this category is a "localization" of all quasi-isomorphisms. A *quasi-isomorphism* between chain complexes is a morphism of complexes which induces isomorphisms between cohomologies.

[2] The *intersection homology* is an analog of the singular homology but it has a better behavior in the study of singular points of a variety. This requires the space X to be "stratified".

which is a Whitney stratification[3] $\mathcal{T}$ of the flag variety. We now define the closure of a strata

$$X_w := \overline{BwB/B} = \bigsqcup_{y \leq w} ByB/B,$$

which is called a *Schubert variety*. The *simple perverse sheaves* are the intersection cohomology complexes $\mathbf{IC}(X_w)$ corresponding to the trivial local system (see the StackExchange discussion [Loc]) on the $B \times B$-orbit BwB/B. The key result is to relate the stalk $\mathbf{IC}(X_w)_y$ with the Kazhdan-Lusztig polynomial $P_{y,w}$ using the anti-involution property of the KL-basis defined in the previous section. The formula given by Kazhdan and Lusztig following ideas of Deligne was the following,

$$P_{y,w}(q) = \sum_i q^i \dim \mathrm{IH}^{2i}_{X_y}(X_w).$$

Here IH is the intersection homology complex. We will not prove this formula in this thesis.

Finally, the split Grothendieck group[4] of $\mathscr{H}$, say $[\mathscr{H}]$, has a $\mathbb{Z}[v, v^{-1}]$ structure given by $v[\mathcal{F}^\bullet] := [\mathcal{F}^\bullet[1]]$, where $[1]$ is the homological shift by 1. A key result is an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras

$$\mathrm{Fun}_{B \times B}(G, \mathbb{C}) \xrightarrow{\sim} [\mathscr{H}],$$

given by sending the Kazhdan-Lusztig basis element corresponding to $\underline{H}_w \in \mathbf{H}$ (defined in a purely algebraic way in the previous section) to the element $[\mathbf{IC}_w]$.

1.4. **The category of Soergel bimodules.**

In 1990, Soergel gave an alternative proof of Kazhdan-Lusztig conjectures using certain modules over the cohomology ring of the flag variety associated with the Weyl group [Soe90]. Later, Soergel introduced an equivariant analog of these modules which were subsequently called *Soergel bimodules* [Soe92]. In that article, Soergel gave a purely algebraic definition of the Hecke category which depends only on a representation of the Coxeter group (see Chapter 2) of the Coxeter system. That is, in the same spirit as Iwahori gave an essentially algebraic definition of the Hecke algebra which relies only on the associated Weyl group. Furthermore, the Hecke category given by Soergel categorifies the Hecke algebra $\mathbf{H}$. Also, it coincides with the Hecke category described in the previous section when the Coxeter group comes from a Weyl group associated with a split reductive group. For the rest of this section, let G be a reductive group as denoted in the previous section.

Let $\mathfrak{h} := \mathrm{Lie}(T)$ be the Lie algebra of the split torus T and let $R := \Bbbk[\mathfrak{h}]$ be the algebra of regular functions on $\mathfrak{h}$. We consider a $\mathbb{Z}$-grading in the polynomial ring

[3] A *stratification* of a singular variety is a decomposition of the space in pieces called *strata*, which are topological manifolds. This decomposition separates a variety in a finite number of parts. First, decomposing the variety in a manifold and a new lower dimensional variety containing all singular points. Then repeating the decomposition of this new lower dimensional variety, and so on. A stratification is needed to define the intersection cohomology. A *Whitney stratification* is a particular type of stratification very common in this context. We will not enter into much details about stratification theory.

[4] The *split Grothendieck group* $[\mathscr{A}]$ of an additive category $\mathscr{A}$ is the abelian group generated by isomorphism classes of objects of $\mathcal{A}$, with relations of the form

$$[A] = [B] + [C],$$

if $A, B, C \in \mathrm{Ob}(\mathscr{A})$ and $A \cong B \oplus C$. If $A \in \mathscr{A}$ we denote its class by $[A] \in [\mathscr{A}]$.

R by decreeing $\deg(\mathfrak{h}^*) = 2$. Due to Borel, we have an isomorphism

$$R \cong H_T^\bullet(\mathrm{pt}, \mathbb{R}).$$

In order to have an insight of this, we will explain what is the meaning of equivariant cohomology of a G-space X and we will do some calculations. A *G-space* is a smooth manifold (respectively, a variety) X with an action of a compact Lie group G (respectively, a reductive algebraic group). The equivariant cohomology can be viewed as a natural generalization of the usual cohomology. The main difficulty is that the action on the space X may not be free and in that situation, we will need to enlarge the space.

Definition 1.9. A space E is said to be a *universal G-space* if E carries a free G action and E is contractible.

It is a well know fact that for a given G, E is unique up to homotopy equivalence. So we will pick one universal G-space and denote it by E_G. Let $X_G \coloneqq (X \times E_G)/G$.

Definition 1.10. The *equivariant cohomology* of a G-space X, denoted by $H_G^\bullet(X)$, is defined as

$$H_G^\bullet(X) \coloneqq H^\bullet(X_G).$$

In other words, it is the ordinary cohomology of the topological space X_G. The quotient space $B_G \coloneqq E_G/G$ is called the *classifying G-space* since it classifies the principal G-bundles.

Remark 1.11. If we have any cohomology functor $H^\bullet(-)$ and a space M, the constant morphism $M \to \mathrm{pt}$ induces a ring homomorphism $H^\bullet(\mathrm{pt}) \to H^\bullet(M)$. This gives $H^\bullet(M)$ an $H^\bullet(pt)$-module structure. Thus the cohomology of a point serves as the coefficient ring in any cohomology theory.

Example 1.12 (Baby example)**.** Let $T = \mathbb{C}^*$ be the 1-dimensional torus. The space

$$\mathbb{C}^\infty \setminus \{0\} \coloneqq \varinjlim \mathbb{C}^n \setminus \{0\},$$

is a direct limit of contractible spaces. By compactness of $\mathbb{S}^1$, the direct limit is also contractible. Furthermore, T acts freely on it by multiplication, then $E_T = \mathbb{C}^\infty \setminus \{0\}$. Let us calculate B_T,

$$B_T = E_T/T = \varinjlim (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^* = \varinjlim \mathbb{CP}^n =: \mathbb{CP}^\infty.$$

In particular, $\mathbb{C}^\infty \setminus \{0\}$ is a line bundle of B_T. Therefore, we can obtain the T-equivariant cohomology of a point pt as follows

$$H_T^\bullet(\mathrm{pt}, \mathbb{R}) = H^\bullet((\mathrm{pt} \times E_T)/T, \mathbb{R}) = H^\bullet(B_T, \mathbb{R}) = H^\bullet(\mathbb{CP}^\infty, \mathbb{R}) \cong \mathbb{R}[u].$$

where $u = -c_1(E_T) \in H^2(B_T, \mathbb{R})$ and c_1 is the first Chern class. In other words, $H_T^\bullet(\mathrm{pt}, \mathbb{R})$ is the polynomial ring with real coefficients with indeterminate u of degree 2.

Example 1.13. Let $T = (\mathbb{C}^*)^\ell$ be the ℓ-dimensional torus. One can compute that the classifying space is $B_T = (\mathbb{P}^\infty)^{\times \ell}$ and $E_T = L_1 \oplus \cdots \oplus L_\ell$, where L_i is the pullback of the canonical bundle on the *i*-th component of B_T. Furthermore, the coefficient ring is given by

$$H_T^\bullet(\mathrm{pt}, \mathbb{R}) = H^\bullet(B_T, \mathbb{R}) = H^\bullet(\mathbb{P}^\infty, \mathbb{R})^{\times n} \cong \mathbb{R}[u_1, \ldots, u_\ell],$$

where $u_i = -c_1(L_i) \in H^2(B_T) \cong \mathrm{Hom}(T, \mathbb{C}^*)$, and the isomorphisms here are canonical.

We have a canonical identification between T-equivariant and B-equivariant cohomology groups, since T is a deformation retract of B. Then

$$R = \mathrm{Lie}(\mathfrak{h}) = H^\bullet_B(\mathrm{pt}) = H^\bullet_T(\mathrm{pt}).$$

In the bounded equivariant derived category $\mathbf{D}^b_{B\times B}(G,\mathbb{C})$, we have the *hypercohomology* functor $\mathbb{H}^\bullet \colon \mathbf{D}^b_{B\times B}(G,\mathbb{C}) \to R-\mathbf{Bim}$. The hypercohomology is the generalization of the sheaf cohomology, from sheaves to complexes of sheaves, and it coincides with the former one in the complexes of the form

$$\mathcal{F}^\bullet_0 = \cdots 0 \longrightarrow 0 \longrightarrow \mathcal{F} \longrightarrow 0 \longrightarrow 0 \cdots,$$

where $\mathcal{F}$ is a sheaf in degree 0. In other words, we have a natural isomorphism

$$\mathbb{H}^\bullet(\mathcal{F}^\bullet_0) \cong H^\bullet(\mathcal{F}).$$

Let us define a particular case of hypercohomology in the well-known context of Čech cohomology. We define the *Čech hypercohomology* of a complex of sheaves

$$\mathcal{F}^\bullet = \cdots \mathcal{F}^{-2} \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \mathcal{F}^2 \cdots,$$

as the cohomology of a corresponding chain complex. This chain complex comes from a double chain complex. One chain is the usual Čech cohomology chain complex for a fixed sheaf $\mathcal{F}^i$. For a fixed covering $\mathcal{U}$ of the space, the first complex is given by

$$\cdots \check{C}^{-2}(\mathcal{U},\mathcal{F}^i) \longrightarrow \check{C}^{-1}(\mathcal{U},\mathcal{F}^i) \longrightarrow \check{C}^{0}(\mathcal{U},\mathcal{F}^i) \longrightarrow \check{C}^{1}(\mathcal{U},\mathcal{F}^i) \longrightarrow \check{C}^{2}(\mathcal{U},\mathcal{F}^i) \cdots,$$

and the other complex comes from the map induced by $\mathcal{F}^i \longrightarrow \mathcal{F}^{i+1}$. This gives us a double complex $C^{p,q} := \check{C}^p(\mathcal{U},\mathcal{F}^q)$, but a double complex always defines a chain complex by

$$K^n = \bigoplus_{p+q=n} C^{p,q} \qquad \text{and} \qquad d = d_1 + (-1)^p d_2.$$

Applying the cohomology to this complex and then the direct limit over all the coverings $\mathcal{U}$, we obtain the hypercohomology of the original complex.

The hypercohomology ring of $\mathbf{D}^b_{B\times B}(G,\mathbb{C})$ is $H^\bullet_{B\times B}(\mathrm{pt})$, and it is isomorphic to $R\otimes_{\mathbb{C}}R$. As we have seen before, the hypercohomology of any object in $\mathbf{D}^b_{B\times B}(G,\mathbb{C})$ is a $\mathbb{Z}$- graded $R\otimes_{\mathbb{C}} R$-module. Since R is a commutative $\mathbb{Z}$-graded ring, the category of R-graded bimodules $R-\mathbf{Bim}$ is equivalent to the category of R-graded $R\otimes_{\mathbb{C}} R$-modules. $R-\mathbf{Bim}$ is a monoidal category under the tensor product $\otimes_R$ of R-bimodules.

Soergel observed that the decomposition theorem of Beilinson, Berstein, Deligne, and Gabber [BBD82] gives an alternative characterization of $\mathscr{H}$. This characterization comes from the *Bott-Samelson resolution* of the flag variety G/B, which is a resolution of singularities. We will describe this resolution in the same algebraic way as presented in [Dem74]. Let P_s be the group generated by B and BsB in G. Let $\mathrm{Ex}(S)$ be the set of all reduced expressions in a Coxeter system (W,S), see Chapter 2. For a fixed reduced expression $\underline{w} = (s,t,\ldots,z,u) \in \mathrm{Ex}(S)$ of w, we define

$$Z_{\underline{w}} := P_s \times P_t \times \cdots \times P_u / B^{\ell(w)},$$

where ℓ is the length of w in the Weyl group W. This quotient is with respect to the left-action of $B^{\ell(w)}$ defined by

$$(b_s, b_t, \dots, b_z, b_u) \cdot (p_1, p_2 \dots, p_\ell) = (p_1 b_s^{-1}, b_s p_2 b_t^{-1}, \dots, b_z p_\ell b_u^{-1}).$$

One can notice that $Z_{\underline{w}}$ is isomorphic to the Schubert variety X_w defined above. The multiplication of all $\ell(w)$ coordinates of $Z_{\underline{w}}$ is an algebraic map

$$\pi \colon Z_{\underline{w}} \longrightarrow X_w,$$

and it is a resolution of singularities. In particular, $Z_{\underline{w}}$ is a smooth projective scheme. Furthermore, we have $\pi_* \mathcal{O}_{Z_{\underline{w}}} = \mathcal{O}_{X_w}$ and $R^i \pi_* \mathcal{O}_{Z_{\underline{w}}} = 0$, $i \geq 1$. As Schubert varieties are normal, the above means that X_w has just rational singularities, see [Har13].

The characterization of $\mathscr{H}$ given by Soergel is the following. In the first place, he describes the elements $\mathbf{IC}_w$ for $w \in W$. For any reduced expression $\underline{w} = (s, t, \dots, u) \in \mathrm{Ex}(S)$ of w, he noted that $\mathbf{IC}_w$ is an indecomposable direct summand with multiplicity 1 of $\mathbf{IC}_s * \mathbf{IC}_t * \cdots * \mathbf{IC}_u$. Furthermore, if $\underline{v} = (s', t', \dots, u')$ is any reduced expression of an element $v \in W$, such that $v \leq w$ in the Bruhat order, he proved that

$$\mathbf{IC}_w \overset{\oplus}{\not\subset} \mathbf{IC}_{s'} * \mathbf{IC}_{t'} * \cdots * \mathbf{IC}_{u'}.$$

This means that $\mathbf{IC}_w$ does not appear as a direct summand of $\mathbf{IC}_{s'} * \mathbf{IC}_{t'} * \cdots * \mathbf{IC}_{u'}$. We call this characterization "the Bott-Samelson description".

Soergel proved that the hypercohomology functor

$$\mathbb{H}^\bullet_{B \times B} \colon \mathscr{H} \to R - \mathbf{Bim}$$

is fully-faithful and monoidal, the convolution operator of complexes corresponds to the tensor product $\otimes_R$. The essential image of this functor is the *category of Soergel bimodules* and it is equivalent to the category $\mathscr{H}$. It is denoted by $\mathbb{S}$Bim. Since W acts on the torus by conjugation, it acts on $\mathfrak{h} = \mathrm{Lie}(T)$ and by functoriality, it also acts on the graded algebra $R = \mathrm{Sym}(\mathfrak{h}^*)$. For a simple reflection $s \in S$, let R^s denote the subalgebra of all s-invariant elements in R. The inclusion $R^s \subset R$ is a Frobenius extension, see [ESW13]. For $s \in S$, we have

$$\mathbb{H}^\bullet_{B \times B}(\mathbf{IC}_s) = R \otimes_{R^s} R(1)$$

as graded algebra elements, where (1) denotes the grading shift by one. We denote $B_s \coloneqq R \otimes_{R^s} R(1)$. Using the Bott-Samelson description of $\mathscr{H}$, Soergel obtained the following elementary description of $\mathbb{S}$Bim (and therefore, for $\mathscr{H}$ via the hypercohomology equivalence of graded monoidal categories): it is equivalent to the full additive monoidal graded pseudo-abelian[5] subcategory of $R - \mathbf{Bim}$ generated by $\{B_s \mid s \in S\}$.

Since $\mathbb{S}$Bim is a category equivalent to $\mathscr{H}$, it is another incarnation of the Hecke category. Furthermore, this category is defined in a purely algebraic way starting from R. This definition also allows us to define the Hecke category for arbitrary Coxeter systems (W, S), even if there is no geometric context, i.e., when W is not the Weyl group associated to a split reductive group G. In [Soe07], Soergel defined $\mathbb{S}$Bim for any Coxeter system (W, S) starting from a "reflection faithful" representation $\mathfrak{h}$ of W over a ring $\Bbbk$, which replaces the role of the Lie algebra of a torus

[5] A *Karoubian* or *pseudo-abelian* category is a pre-additive category $\mathscr{C}$ such that every idempotent morphism $p \colon A \to A$ has a kernel, and hence also a cokernel.

T. In particular, the role of R is replaced by the symmetric algebra of $\mathfrak{h}^*$. As before, R is also a polynomial ring $R = \Bbbk[\alpha_s, \alpha_t, \dots, \alpha_u]$ in $|S|$ variables, where $s, t, \dots, u \in S$. In this level of generality, Soergel constructed an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras

$$\varepsilon\colon [\mathbb{S}\mathrm{Bim}] \xrightarrow{\sim} \mathbf{H}.$$

Soergel conjectured if $\Bbbk = \mathbb{R}$ that under this isomorphism, the indecomposable objects will descend to the Kazhdan-Luzstig basis, more precisely,

$$\varepsilon([B_w]) = \underline{H}_w,$$

where $w \in W$. This fact is the so-called *Soergel's conjecture*.

Soergel's conjecture was first proved by Soergel, in the geometric context of Weyl groups, see [Soe92]. Recently, in the year 2012, Elias and Williamson proved Soergel's conjecture over $\mathbb{R}$ for any Coxeter group and "nice" realizations [EW12]. This was done by adapting the Hodge-theoretic proof of the decomposition theorem, given by de Cataldo and Migliorini's in the year 2002 [CM02]. Elias and Williamson gave a striking purely algebraic proof of this fact. Furthermore, they gave an algebraic proof of the Kazhdan-Lusztig's conjectures, which also were enunciated in a purely algebraic manner. On the other hand, the analog of Soergel's conjecture over a field of characteristic p is false, and this fact (for weights around the Steinberg weight and p bigger than the Coxeter number, see [Soe00]) is equivalent to the existence of counterexamples of *Lusztig's conjecture*. Geordie Williamson used this fact to give counterexamples of Lusztig conjecture on the characters of simple rational modules for SL_n over fields of positive characteristic [Wil17]. However, the image of the indecomposable objects ${}^kB_w$ over a field k of characteristic p under the isomorphism ε is still a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathbf{H}$. We denote this basis by

$${}^p\underline{H}_w := \varepsilon({}^kB_w),$$

and call it the *p-canonical basis* of the Hecke algebra $\mathbf{H}$. It does not depend on the base field k, only in its characteristic if W is a Weyl or affine Weyl group.

For any Coxeter system, for reflection faithful representations over infinite fields of characteristic $\neq 2$, Soergel proved that the indecomposable bimodules up to grading shifts in $\mathbb{S}\mathrm{Bim}$ are in a bijection correspondence with W, and they are denoted by B_w. He mimicked the Bott-Samelson description. He proved that B_w has multiplicity 1 in $B_{\underline{w}} := B_s \otimes_R B_t \otimes_R \cdots \otimes_R B_u$, for any reduced expression $\underline{w} = (s, t, \dots, u)$ of w. The graded bimodule $B_{\underline{w}}$ is called the *Bott-Samelson bimodule* associated with $\underline{w}$. In particular, for a fixed $i \in \mathbb{Z}$ we have

$$B_w(i) \overset{\oplus}{\subset} B_{\underline{w}}(i) = B_s \otimes_R B_t \otimes_R \cdots \otimes_R B_u(i).$$

Finally, Soergel proved that for a reduced expression $\underline{v} = (s', t', \dots, u')$ of $v \in W$, such that $v \leq w$ in the Bruhat order,

$$B_w \overset{\oplus}{\not\subset} B_{s'} \otimes_R B_{t'} \otimes_R \cdots \otimes_R B_{u'}.$$

This completes the analogy with the Bott-Samelson description of intersection cohomology complexes in $\mathscr{H}$.

These results are known as *Soergel's Categorification Theorem* (or just *SCT*). Soergel's Categorification Theorem comes directly in the case of a Weyl group, by just applying the cohomology functor on $\mathscr{H}$.

Later, in 2008, Libedinsky defined $\mathbb{S}$Bim using the "easier" *geometric representation* $\mathfrak{h}$ of W, instead of the "harder" reflection faithful representation, and proved that the Soergel's conjecture in this setting is equivalent to the Soergel's conjecture in the "harder" setting, see [Lib08a]. He also proved the Soergel's Categorification Theorem for the geometric representation, see [Lib08a].

For Weyl groups, one can prove Soergel's Categorification Theorem easily, by transferring known facts about $\mathscr{H}$ to $\mathbb{S}$Bim using hypercohomology. Soergel's proof for the general case is much trickier but relies only on commutative algebra.

Soergel's theory for the category $\mathbb{S}$Bim not only "lifts" the objects $\underline{H}_w$ to bimodules B_w, but it also lifts relations between $\underline{H}_w$, see Example 1.14. This category cannot lift relations on the standard basis H_w (another categorification of $\mathbf{H}$ called "Rouquier complexes" accomplishes this task). In particular, it is impossible to lift to $\mathbb{S}$Bim the Iwahori presentation of the Hecke algebra $\mathbf{H}$. However, there are presentations of the Hecke algebra in terms of the Kazhdan-Lusztig basis, hence all relations in these presentations can be lifted. Heuristically speaking, $\mathbb{S}$Bim is the algebrization of $\mathscr{H}$. It is the categorical analog of Iwahori's algebrization of $\mathbf{H}$, on the level of objects. Nevertheless, in $\mathbb{S}$Bim there is a whole new layer of structure, with no analog in the Hecke algebra $\mathbf{H}$: morphisms.

Example 1.14. In the category $\mathbb{S}$Bim over the real geometric representation of the symmetric group S_3, the Hecke algebra relations

$$\underline{H}_s\underline{H}_s = v\underline{H}_s + v^{-1}\underline{H}_s,$$
$$\underline{H}_{sts}\underline{H}_{sts} = (v^{-3} + 2v^{-1} + 2v^1 + v^3)\underline{H}_{sts},$$

are lifted to the isomorphisms,

$$B_s \otimes_R B_s \cong B_s(1) \oplus B_s(-1),$$
$$B_{sts} \otimes_R B_{sts} \cong B_{sts}(-3) \oplus B_{sts}(-1)^{\oplus 2} \oplus B_{sts}(1)^{\oplus 2} \oplus B_{sts}(3).$$

1.5. **Local intersection forms.** One of the fundamental tools of de Cataldo and Migliorini's Hodge-theoretic proof of the decomposition theorem [CM02] is the appearance of some bilinear forms, which were called *local intersection forms* and control the behavior of the decomposition theorem. More precisely, intersection forms describe how the direct image of the constant sheaf decomposes into indecomposable objects (indecomposable objects are those with local endomorphism ring in the cohomology of smooth varieties). There is an equivalence between a part of the decomposition theorem and the non-degeneracy of local intersection forms. Juteau, Mautner, and Williamson noted this fact in [JMW14] and they used an analog of those intersection forms with modular coefficients. Furthermore, they proved that the decomposition theorem fails over fields of characteristic p, but it is still possible to determine the multiplicities of parity sheaves in the direct image of the constant sheaf. These multiplicities are just the rank of local intersection forms. In fact, they proved (Lemma 3.1 [JMW14])) Lemma 1.16. Let us make a definition before stating the lemma.

Definition 1.15. Let $\Bbbk$ be a field. Let $\mathscr{C}$ be a Krull-Remak-Schmidt $\Bbbk$-linear category with finite dimensional morphism spaces. Let $a \in \mathscr{C}$ denote an indecomposable object. Given any object $x \in \mathscr{C}$, we can write $x \simeq a^{\oplus m} \oplus y$ such that a is not a direct summand of y. The integer m is called the *multiplicity* of a in x. This multiplicity is well-defined because $\mathscr{C}$ is Krull-Remak-Schmidt.

Lemma 1.16. *Assume that* $\mathrm{End}(a) = \Bbbk$. *Composition gives us a pairing:*

$$B\colon \mathrm{Hom}(a,x) \times \mathrm{Hom}(x,a) \longrightarrow \mathrm{End}(a) = \Bbbk$$
$$(\alpha,\beta) \mapsto \beta \circ \alpha.$$

The multiplicity of a in x is equal to the rank of B.

1.6. **Calculations in the category of Soergel bimodules.** Let us recall the algebraic definition of the category $\mathbb{S}$Bim of Soergel bimodules. We fix a Coxeter system (W,S) and a realization $(\mathfrak{h}, \mathfrak{h}^*, \{\alpha_s\}, \{\alpha_s^\vee\})$ over a commutative integral domain $\Bbbk$ (see Section 2.3)

$$W \to \mathrm{End}\left(\bigoplus_{s\in S} \Bbbk \alpha_s^\vee\right)$$

on the $\Bbbk$-module $\mathfrak{h} := \mathrm{span}(\alpha_s^\vee \mid s \in S)$. Let $R := \Bbbk[\alpha_s, \alpha_t, \ldots, \alpha_u]$ be the graded symmetric algebra of $\mathfrak{h}^*$ with $\deg(\mathfrak{h}^*) = 2$, where α_s is the dual object of $\alpha_s^\vee$. Since W acts on $\mathfrak{h}$ through the realization, by functoriality W acts on R. Given $s \in S$, let R^s be the subalgebra of all s-invariant elements of R. Let B_s denote the graded R-bimodule

$$B_s := R \otimes_{R^s} R(1),$$

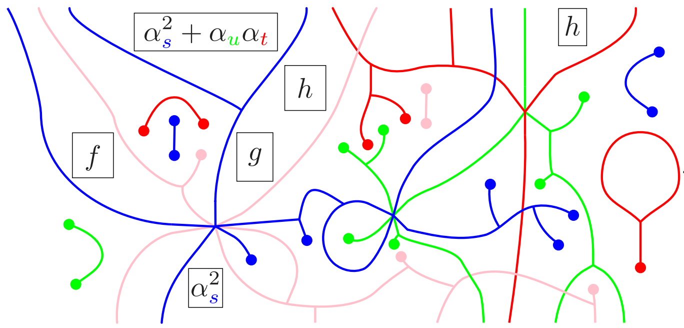
where (1) is the standard grading shift by one to the right. For a reduced expression $\underline{w} = (s,t,\ldots,u)$, we define

$$B_{\underline{w}} := B_s \otimes_R B_t \otimes_R \cdots \otimes_R B_u.$$

The graded R-bimodule $B_{\underline{w}}$ is called the *Bott-Samelson* bimodule associated with $\underline{w}$. Given $w \in W$, let B_w be the unique indecomposable direct summand of $B_{\underline{w}}$ for any reduced expression $\underline{w}$ of w, such that it is not a direct summand of $B_{\underline{v}}$, whenever $\underline{v}$ is a reduced expression of v with $v < w$. The *category of Bott-Samelson bimodules* $\mathbb{BS}$Bim is the full monoidal subcategory of $R-$ **Bim** generated by the set $\{B_s \mid s \in S\}$ of R-bimodules, in particular, whose objects are of the form $B_{\underline{w}}$. *The category of Soergel bimodules* $\mathbb{S}$Bim is the graded additive category whose objects are direct sums of shifts of summands of graded bimodules in $\mathbb{BS}$Bim.

Despite the elementary definition of the category of Soergel bimodules, calculations here are extremely difficult. Some advances on how to calculate in this category were made by Libedinsky in the year 2008 in [Lib08b]. He presented a combinatorial R-basis for the spaces $\mathrm{Hom}(B_{\underline{w}}, B_{\underline{v}})$ in the category $\mathbb{BS}$Bim, called *Libedinsky's light leaves basis*. This simplifies our work on finding the local intersection forms.

Other advances on how to calculate in $\mathbb{S}$Bim were first made by Libedinsky [Lib10] who gave a presentation by generators and relations of $\mathbb{S}$Bim for right-angled Coxeter systems, later by Elias and Khovanov [EK11] who gave a diagrammatic presentation in type A, by Elias [Eli16] who gave a diagrammatic presentation for the dihedral group, and finally the general result for any Coxeter system by Elias and Williamson [EW16]. This last work was called "Soergel Calculus" introduces the *diagrammatic category*, a powerful tool which presents diagrammatically the category of Soergel bimodules in terms of generators and relations in any Coxeter system for some family of realizations. They describe the space of morphisms $\operatorname{Hom}(B_{\underline{w}}, B_{\underline{v}})$, as the R-module generated by the isotopy classes of S-graphs with some relations. The S-graphs are colored diagrams as in the following picture,



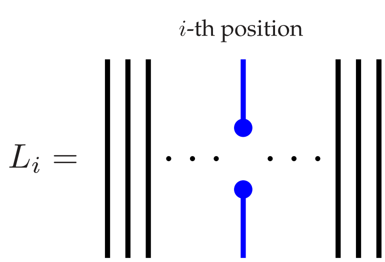
This tool simplifies incredibly some calculations. However, those topological and combinatorial calculations are still extremely complicated to perform in many cases.

1.7. **Recent achievements and related nowadays works.** In spite of the development of "Soergel Calculus", there are few (half a dozen) works performing combinatoric and topological calculations of local intersection forms and decomposition numbers in the category of Soergel bimodules.

In 2013, such calculations in $\tilde{A}_1$ were made by Chen and Unda in [CU13]. Despite that, in their work, they do not compute explicitly the local intersection forms, but they were able to provide a recursive formula for the composition of any two Libedinsky's light leaves modulo lower terms on the category of Bott-Samelson bimodules. They cannot give an explicit expression. The ideas in this thesis allow us to perform better calculations and give a new formula for the local intersection form. This formula holds for the universal Coxeter group U_n. This can be done since we give a combinatorial and topological formula for the composition modulo lower terms of any two degree zero Libedinsky's light leaves. In order to compute intersection forms, we do not need to consider other degrees (see Theorem 3.9 below). Then calculations can be done explicitly in this thesis.

In 2016 He and Williamson reduced the problem of calculating some coefficients of the intersection forms in type A [HW15] to a problem about the nil-Hecke ring. The coefficients they compute are the ones corresponding to compositions modulo lower terms of Libedinsky's light leaves without D1's, see Chapter 3.

In 2017, Ryom-Hansen produced Jucys-Murphy elements for the category of Soergel bimodules for general Coxeter groups [RH16]. The Jucys-Murphy elements are represented diagrammatically by the following picture.



He used them to diagonalize some bilinear forms related to the local intersection forms but considering Libedinsky's light leaves of all degrees. Furthermore, he was able to find a closed expression for the determinant of these forms. However, in a personal conversation, he was skeptical about the possibility of repeating his argument to get the decomposition numbers or just the determinants of the local intersection forms. This may be due to the fact that the Jucys-Murphy elements are of degree two and not of degree zero. Sentinelli worked with the author performing calculations of the decomposition numbers for some realizations, but they have been able to calculate completely only the easier ones. The author will present some of these calculations in this thesis.

The work in this thesis has direct implications for the study of the aforementioned p-canonical basis for the Hecke algebra in type $\tilde{A}_1$. Additional motivation, overview, nowadays works and properties of the p-canonical basis can be found in [JW15].

1.8. **Structure of the thesis.** This thesis contains four parts.

- §2 **Background.** Contains background on Coxeter systems, Hecke algebras, realizations of Coxeter systems, the diagrammatic category of Soergel bimodules and local intersection forms in this category.
- §3 **Libedinsky's light leaves in U_n.** We define the Libedinsky's light leaves basis in U_n (type $\tilde{A}_1$) and recall some of their properties.
- §4 **Properties of $\mathrm{End}^{\bullet}_{\not< x,k}(x)$ in U_n.** We perform calculations in the space of morphisms $\mathrm{End}^{\bullet}_{\not< x,k}(x)$, of the diagrammatic category $k \otimes \mathbb{S}\mathcal{D}^{\not< x}$ in U_n.
- §5 **Intersection forms formula in U_n.** We prove our main result giving a formula for the local intersection forms in U_n.

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2. Background

In this chapter, we will give the basic theorems and definitions to introduce the category of Soergel bimodules. A more detailed background in Coxeter groups and Hecke algebras can be found in [Hum92].

2.1. **Coxeter Groups.** Let $S = \{s, t, \dots, u\}$ be a finite set, for each pair (s,t) of elements in S consider $m_{s,t} \in \mathbb{N} \cup \{\infty\}$ such that $m_{s,s} = 1$ and $m_{s,t} > 1$ if $s \neq t$. Let W be the group with presentation

$$W = \langle S \mid (st)^{m_{s,t}} = e \text{ whenever } m_{s,t} \neq \infty \rangle.$$

We say that W is a *Coxeter group* and the pair (W, S) is a *Coxeter system*. We will use different colors to represent different elements in S, e.g. $S = \{s, t, z, \dots, u\}$.

Remark 2.1. Different Coxeter systems might have isomorphic underlying Coxeter groups, although this cannot happen in finite Coxeter groups.

Definition 2.2. The relation $s^2 = e$ is called the *involution relation*.

Remark 2.3. The involution relation can also be written as

$$s = s^{-1}. \tag{2.1}$$

Definition 2.4. The generators $s \in S$ are called *simple reflections*. The *rank* of a Coxeter system is the cardinality of S.

Definition 2.5. The relation $(st)^{m_{s,t}} = e$ can be written as

$$\underbrace{stst\cdots}_{m_{s,t}} = \underbrace{tsts\cdots}_{m_{s,t}}. \tag{2.2}$$

This relation is called the *braid relation*.

There are many important examples of Coxeter systems, but we recall three of them.

Example 2.6 (Universal Coxeter systems)**.** The *universal Coxeter system of rank* n is the group

$$U_n = \langle s_1, s_2, \dots, s_n \mid s_1^2 = s_2^2 = \cdots = s_n^2 = e \rangle,$$

together with the set $S = s_1, s_2, \dots, s_n$. When $n = 2$, U_2 is called the *infinite dihedral Coxeter system*. We refer to U_2 as the *Weyl group of type* $\tilde{A}_1$.

Example 2.7 (Type A Coxeter systems)**.** The *Weyl group of type* A_{n-1} is the group of all symmetries of an n-simplex. Equivalently, it can be defined as S_n, the symmetric group in n elements. The isomorphism between these two groups is easy to see. It admits a Coxeter presentation given by generators $s_i, 1 \leq i < n$ and relations

$$s_i^2 = e \quad \text{for all} \quad i \in \{i \mid 1 \leq i < n\},$$
$$s_i s_j = s_j s_i \quad \text{for all} \quad (i,j) \in \{(i,j) \mid |i-j| \geq 2\},$$
$$s_i s_j s_i = s_j s_i s_j \quad \text{for all} \quad (i,j) \in \{(i,j) \mid |i-j| = 1\}.$$

The isomorphism from this group to the symmetric group is given by sending s_i to the transposition $(i, i+1) \in S_n$.

Example 2.8 (The rank two or "two color" or "dihedral" Coxeter systems)**.** The *dihedral group of order* n is the finite group D_n of symmetries of a regular n-sided polygon in the plane. This is also a finite Coxeter group denoted by $I_2(n)$ (the subindex 2 refers to the rank of the Coxeter system) a Coxeter presentation is given by

$$I_2(n) = \langle s, t \mid s^2 = t^2 = (st)^n = e \rangle.$$

An isomorphism between D_n and $I_2(n)$ is given by picking any two closest reflections to be s and t respectively. The other rank two Coxeter group is $\tilde{A}_1$, which is infinite.

There are many other examples of Coxeter systems which are much harder to study. In the following example, we will show one, which is not finite nor a Weyl or an affine Weyl Coxeter group, the "triangle group" $\triangle(3,4,5)$.

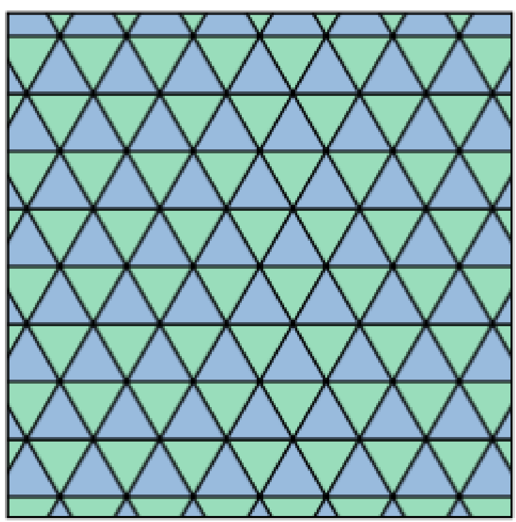
Example 2.9. [Triangle groups $\triangle(p,q,r)$] Let p,q,r be integers greater or equal to 2 or infinite. A *triangle group* $\triangle(p,q,r)$ is a group of reflections of the Euclidean plane, of the two-dimensional sphere, or of the hyperbolic plane generated by the reflections in the sides of a triangle with angles π/p, π/q and π/r. Triangle groups admit the following Coxeter system presentation

$$\triangle(p,q,r) = \langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = e \rangle.$$

(1) For every triplet of integers (p,q,r) such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

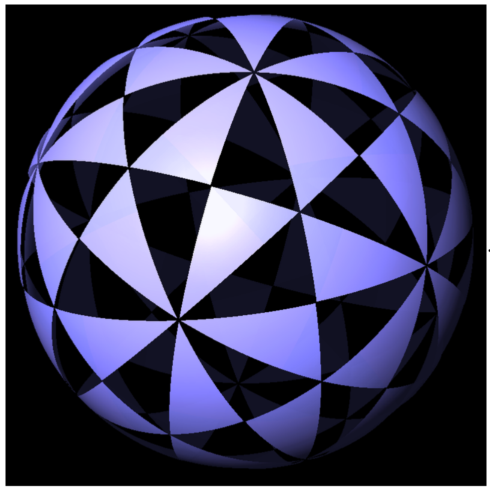
the group $\triangle(p,q,r)$ is an infinite triangle group of reflections of the Euclidean plane. There is a finite number of those triples. In this case $\triangle(p,q,r)$ is also the group of symmetries of a tiling of the Euclidean plane. For example, the group $\triangle(3,3,3)$ is the group of symmetries of the following tiling,


.

(2) For every triplet of integers (p,q,r) such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1,$$

the group $\triangle(p,q,r)$ is a finite triangle group of reflections of the two-dimensional sphere. There are a finite number of those triples. In this case $\triangle(p,q,r)$ is also the group of symmetries of a tiling of the two-dimensional sphere. For example, the group $\triangle(2,3,5)$ is the group of symmetries of the following tiling,



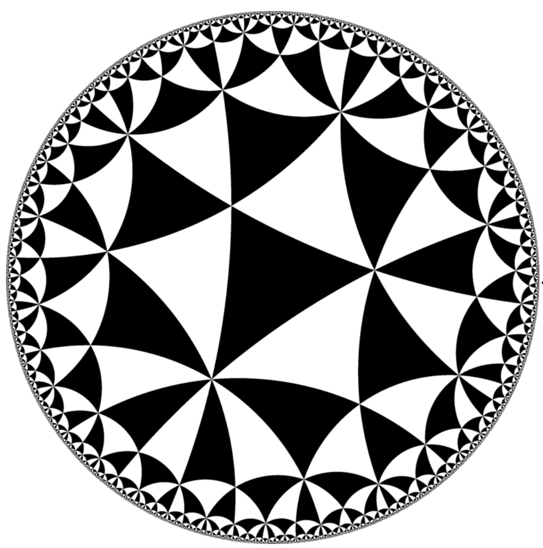
.

(3) For every triplet of integers (p, q, r) such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

the group $\triangle(p, q, r)$ is an infinite triangle group of reflections of the hyperbolic plane. There is an infinite number of those triples. In this case $\triangle(p, q, r)$ is also the group of symmetries of a tiling of the hyperbolic plane. For example, the group $\triangle(3, 4, 5)$ is the group of symmetries of the following tiling,

(2.3)



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Definition 2.10. An *expression* in a Coxeter system (W, S) is a finite sequence of elements of S. We denote by $\mathrm{Ex}(S)$ the set of all expressions of (W, S), i.e,

$$\mathrm{Ex}(S) := \bigsqcup_{i \in \mathbb{N}} S^i.$$

Let $\underline{w} = (s_1, s_2, \cdots, s_r) \in \mathrm{Ex}(S)$. The *length* $\ell(\underline{w})$ of $\underline{w}$ is r. A *subexpression* of $\underline{w}$ is a sequence $(s_1^{e_1}, s_2^{e_2}, \ldots, s_r^{e_r})$, where $e_i \in \{0,1\}$ for all i, $s_i^0 := e$ and $s_i^1 := s_i$ for all i. We call $\underline{e} := (e_1, e_2, \ldots, e_r)$ the associated binary sequence with the subexpression $\underline{w}^{\underline{e}} := (s_1^{e_1}, s_2^{e_2}, \cdots, s_r^{e_r})$. **Caution:** We want to remove all "e" appearing in $\underline{w}^{\underline{e}}$ due to the zero exponents. In particular, $\underline{w}^{\underline{e}}$ will not be considered as an expression of length r. For example, the expression (e, s, t, e, s) should be replaced with (s, t, s) which represents the same element in W.

Definition 2.11. Given an expression $\underline{w} = (s_1, s_2, \cdots, s_r) \in \mathrm{Ex}(S)$, we have an associated element $w = s_1 s_2 \cdots s_r \in W$. We say that $\underline{w}$ is *an expression for* $w \in W$. Every element of W has infinitely many expressions. If the expression $\underline{w} = (s_1, s_2, \cdots, s_r) \in \mathrm{Ex}(S)$ of an element w is such that r is minimal, we say that $\underline{w}$ is a *reduced expression* for w. In this case, we say w has *length* r and we denote it by $\ell(w) := r$. The set of all reduced expressions of an element $w \in W$ is denoted by $\mathrm{Rex}(w)$.

Notation 2.12. The group multiplication gives us a natural map

$$\mathrm{Ex}(S) \to W,$$
$$\underline{w} = (s_1, s_2, \ldots, s_r) \mapsto \underline{w}_\bullet := s_1 s_2 \cdots s_r.$$

Example 2.13. In the triangle group $\triangle(3,3,3)$ generated by $S = (a,b,c)$, we have $m_{a,b} = m_{a,c} = m_{b,c} = 3$. Let $\underline{w} = (a,b,a,b,c,c,c) \in S^7 \subset \mathrm{Ex}(S)$. We define $x := \underline{w}_\bullet \in W$. Note that $\underline{w}$ is not a reduced expression. We have

$$\begin{aligned}(a,b,a,b,c,c,c)_\bullet &= ababccc = (aba)b(cc)c \\ &= (bab)b(cc)c = ba(bb)(cc)c \\ &= ba(e)(e)c = bac.\end{aligned}$$

Where we have used the braid relation 2.2 and two times the involution relation 2.1. Therefore, $\underline{v} := (b,a,c) \in S^3$ is another expression for x. Furthermore, it is not hard to show that it is a reduced expression. In particular, $\ell(x) = 3$ even though $\ell(\underline{w}) = 7$.

Definition 2.14. Notice that $\ell(\underline{w}_\bullet) \leq \ell(\underline{w})$. If the equality holds, then we call $\underline{w}$ a *reduced expression*. In particular, note that if $\underline{w} \in \mathrm{Ex}(S)$ is a reduced expression, then it is a reduced expression for $\underline{w}_\bullet \in W$.

Every Coxeter system has a partial order $\leq$ called the *Bruhat order* of (W,S). The definition is as follows. Given x an element of W, pick some reduced expression of x, say $\underline{w}$, then $y < x$ for $y \in W$ whenever $y = (\underline{w}^{\underline{e}})_\bullet$ for some subexpression $\underline{w}^{\underline{e}}$ of $\underline{w}$. In other words, all elements below x are those elements that can be obtained by removing a number of simple reflections from some reduced expression of x, and then multiplying the remaining simple reflections.

Definition 2.15. Given an expression $\underline{w} = (s_1, s_2, \ldots, s_r) \in \mathrm{Ex}(S)$ and a binary sequence $\underline{e} = (e_1, e_2, \ldots, e_r) \in \{0,1\}^r$, for $1 \leq k \leq r$ we set $\underline{w}_{\leq k} := (s_1, \ldots, s_k)$, $\underline{e}_{\leq k} := (e_1, \ldots, e_k)$ and $w_k := (\underline{w}_{\leq k}^{\underline{e}_{\leq k}})_\bullet$. We set $w_0 := e$. We will define the *decoration* (or *coloration*) of the subexpression $\underline{w}^{\underline{e}}$ as the sequence $\underline{d} \in \{\mathrm{U0}, \mathrm{U1}, \mathrm{D0}, \mathrm{D1}\}^r$ defined as follows. For $1 \leq i \leq r$, we set

$$c_i = \begin{cases} \mathrm{U} & \text{if } w_{i-1}s_i > w_{i-1}, \\ \mathrm{D} & \text{if } w_{i-1}s_i < w_{i-1}. \end{cases}$$

Where $<$ is the Bruhat order, U refers to the word “Up” and D refers to the word “Down”. Then, the decoration $\underline{d}$ is given by,

$$(c_1e_1, c_2e_2, \dots, c_re_r) \in \{\mathrm{U0}, \mathrm{U1}, \mathrm{D0}, \mathrm{D1}\}^r.$$

If $\underline{w}$ is clear from the context, we will just refer to $\underline{d}$ as the decoration of the binary sequence $\underline{e}$.

Definition 2.16. The *Bruhat stroll* of the subexpression $\underline{w}^{\underline{e}}$ is the sequence

$$\mathrm{bs}(\underline{w}^{\underline{e}}) := (w_0, w_1, w_2, \dots, w_r) \in W^{r+1}.$$

We call w_r the *end-point* of the Bruhat stroll.

Another important definition is the following.

Definition 2.17. The *defect* of the subexpression $\underline{w}^{\underline{e}}$ is defined to be

$$\mathrm{df}(\underline{w}^{\underline{e}}) := \sharp\{i; d_i = \mathrm{U0}\} - \sharp\{i; d_i = \mathrm{D0}\},$$

where $\underline{d} = (d_1, d_2, \dots, d_r)$ is the decoration associated with the subexpression. If $\underline{w}$ is clear from the context, we will just refer to $\mathrm{df}(\underline{e})$ as the defect of the binary sequence $\underline{e}$ and we will not mention $\underline{w}$.

Remark 2.18. If we consider $\underline{w}$ to be fixed then, we can define three sequences of the same length: a binary expression $\underline{e}$, the decoration expression $\underline{d}$, and the subexpression $\underline{w}^{\underline{e}}$. Each of these three sequences determines the other two. The same could be said of the Bruhat stroll and the other three sequences.

Here we give some examples of subexpressions, colorations, Bruhat strolls, and defects. These examples help to get familiarity with our notations.

Example 2.19. Let (W, S) be any Coxeter system and let s be an element of S. Let $\underline{w} = (s, s, s)$ to be the expression of length three consisting entirely of s. There are four binary strings (or subexpressions) with end-point e. We have:

- $\underline{e} = (1, 1, 0)$ with coloration $\underline{d} = (\mathrm{U1}, \mathrm{D1}, \mathrm{U0})$ with defect 1, and Bruhat stroll (e, s, e, e).
- $\underline{e} = (0, 1, 1)$ with coloration $\underline{d} = (\mathrm{U0}, \mathrm{U1}, \mathrm{D1})$ with defect 1, and Bruhat stroll (e, e, s, e).
- $\underline{e} = (1, 0, 1)$ with coloration $\underline{d} = (\mathrm{U1}, \mathrm{D0}, \mathrm{D1})$ with defect -1, and Bruhat stroll (e, s, s, e).
- $\underline{e} = (0, 0, 0)$ with coloration $\underline{d} = (\mathrm{U0}, \mathrm{U0}, \mathrm{U0})$ with defect 3, and Bruhat stroll (e, e, e, e).

There are four binary strings (or subexpressions) with end-point s. We have:

- $\underline{e} = (1, 1, 1)$ with coloration $\underline{d} = (\mathrm{U1}, \mathrm{D1}, \mathrm{U0})$ with defect 0, and Bruhat stroll (e, s, e, s).
- $\underline{e} = (0, 1, 0)$ with coloration $\underline{d} = (\mathrm{U0}, \mathrm{U1}, \mathrm{D1})$ with defect 0, and Bruhat stroll (e, e, s, s).
- $\underline{e} = (1, 0, 0)$ with coloration $\underline{d} = (\mathrm{U1}, \mathrm{D0}, \mathrm{D1})$ with defect -2, and Bruhat stroll (e, s, s, s).
- $\underline{e} = (0, 0, 1)$ with coloration $\underline{d} = (\mathrm{U0}, \mathrm{U0}, \mathrm{U0})$ with defect 2, and Bruhat stroll (e, e, e, s).

Example 2.20. Let (W, S) be the dihedral Coxeter system $I_2(3)$ or be the Coxeter system $\tilde{A}_1$, with generators s, t and $m_{s,t} = 3$. Let $\underline{w} = (s, t, s)$ to be the reduced expression (for the longest element $sts = tst$ in $I_2(3)$). There are unique subexpressions with end-points sts, ts, st and t. We have:

- $\underline{e} = (1,1,1)$ with coloration $\underline{d} = (\mathrm{U1}, \mathrm{U1}, \mathrm{U1})$ with defect 0, and Bruhat stroll (e, s, st, sts).
- $\underline{e} = (0,1,1)$ with coloration $\underline{d} = (\mathrm{U0}, \mathrm{U1}, \mathrm{U1})$ with defect 1, and Bruhat stroll (e, e, t, ts).
- $\underline{e} = (1,1,0)$ with coloration $\underline{d} = (\mathrm{U1}, \mathrm{U1}, \mathrm{U0})$ with defect 1, and Bruhat stroll (e, s, st, st).
- $\underline{e} = (0,1,0)$ with coloration $\underline{d} = (\mathrm{U0}, \mathrm{U1}, \mathrm{U0})$ with defect 2, and Bruhat stroll (e, e, t, t).

There are two binary strings (or subexpressions) with end-point s. We have:

- $\underline{e} = (1,0,0)$ with coloration $\underline{d} = (\mathrm{U1}, \mathrm{U0}, \mathrm{D0})$ with defect 0, and Bruhat stroll (e, s, s, s).
- $\underline{e} = (0,0,1)$ with coloration $\underline{d} = (\mathrm{U0}, \mathrm{U0}, \mathrm{U1})$ with defect 2, and Bruhat stroll (e, e, e, s).

There are two binary strings (or subexpressions) with end-point s. We have:

- $\underline{e} = (1,0,1)$ with coloration $\underline{d} = (\mathrm{U1}, \mathrm{U0}, \mathrm{D1})$ with defect 0, and Bruhat stroll (e, s, s, e).
- $\underline{e} = (0,0,0)$ with coloration $\underline{d} = (\mathrm{U0}, \mathrm{U0}, \mathrm{U0})$ with defect 3, and Bruhat stroll (e, e, e, e).

Remark 2.21. The last example illustrates a more general fact about universal Coxeter systems. The fact is that every subexpression of a reduced expression in U_n has positive or zero defect. This fact is equivalent to Lemma 3.5 but in the language of subexpressions instead of the language of Libedinsky's light leaves.

2.2. **The Hecke Algebra.** Let (W, S) be a Coxeter system, with length function ℓ and Bruhat order $\leq$. The *Hecke Algebra* $\mathbf{H}$ is the $\mathbb{Z}[v, v^{-1}]$-algebra generated by the elements $\{H_s \mid s \in S\}$, satisfying the following relations:

$$\begin{aligned} H_s^2 &= (v - v^{-1})H_s + H_e, \quad \text{for} \quad s \in S, \\ \underbrace{H_s H_t H_s H_t \cdots}_{m_{s,t}} &= \underbrace{H_t H_s H_t H_s \cdots}_{m_{s,t}}, \quad \text{for} \quad s, t \in S. \end{aligned} \tag{2.4}$$

Where H_e is the identity element of the algebra. For $w \in W$ and a reduced expression $\underline{w} = (s, t, \ldots, u)$ of w, i.e. $\underline{w}_\bullet = w$, let $H_w \coloneqq H_s H_t \cdots H_u$. This is well-defined as a consequence of the following theorem.

Theorem 2.22 (Matsumoto, 1964)**.** *Let $x \in W$ be an element of the Coxeter system (W, S). Every reduced expression in $\mathrm{Rex}(x)$ is obtained from any other by applying a finite number of times braid relations.*

Notation 2.23. For an expression $\underline{w} = (s, t, \ldots, u)$ we set

$$H_{\underline{w}} \coloneqq H_s H_t \cdots H_u.$$

Since the expression $\underline{w}$ may be non-reduced, one could have $H_{\underline{w}} \neq H_{\underline{v}}$ even when $\underline{w}_\bullet = \underline{v}_\bullet \in W$. In other words, $H_{\underline{w}}$ depends on the expression.

Remark 2.24. If we specialize at $v = 1$ in $\mathbf{H}$, we have an isomorphism

$$\begin{aligned} \mathbf{H}|_{v=1} &\longrightarrow \mathbb{Z}W \\ H_w &\longmapsto w, \end{aligned}$$

between the Hecke algebra $\mathbf{H}|_{v=1}$ and the group algebra $\mathbb{Z}W$.

Remark 2.25. The set $\{H_w \mid w \in W\}$ is a $\mathbb{Z}[v^{\pm 1}]$-basis of $\mathbf{H}$. We call this basis the *standard basis* of $\mathbf{H}$. The set $\{H_s \mid s \in S\}$ is a set of generators of $\mathbf{H}$ as a $\mathbb{Z}[v^{\pm 1}]$-algebra.

Let us introduce a more interesting combinatorial basis, the *Kazhdan-Lusztig basis* of $\mathbf{H}$. First of all, we note by a simple computation that

$$H_s^{-1} = H_s + (v - v^{-1}H_e).$$

In particular, all H_w are invertible. We define the *involution map* $h \mapsto \overline{h}$ by sending $H_x \mapsto H_{x^{-1}}^{-1}$ and $\overline{p}(v) = p(v^{-1})$ for $p \in \mathbb{Z}[v, v^{-1}]$. A simple calculation shows that $H_s + v = H_s^{-1} + v^{-1}$, so that the element $\underline{H}_s := H_s + v$ is $(-)$-invariant. We have the following theorem.

Theorem 2.26 (Kazhdan-Lusztig [KL79])**.** *There exists a unique basis* $\{\underline{H}_w \mid w \in W\}$ *of* $\mathbf{H}$ *as a* $\mathbb{Z}[v^{\pm 1}]$*-module, called the* Kazhdan-Lusztig basis *(or just, the* KL-basis*), which satisfies:*

- $\overline{\underline{H}_w} = \underline{H}_w$*;*
- $\underline{H}_w = H_w + \sum_{x<w} P_{x,w} H_x$ *where* $P_{x,w} \in v\mathbb{Z}[v]$*.*

The elements $P_{y,w}$ *are the so-called* Kazhdan-Lusztig polynomials *(or just, the* KL-polynomials*).*

Notation 2.27. For an expression $\underline{w} = (s, t, \ldots, u) \in \mathrm{Ex}(S)$ we set

$$\underline{H}_{\underline{w}} := \underline{H}_s \ldots \underline{H}_t \underline{H}_u.$$

Note that $\underline{H}_{\underline{w}} \neq \underline{H}_w$ in general, even if the expression $\underline{w}$ is reduced.

Example 2.28 (KL-basis in $I_2(3)$)**.** Let $I_2(3)$ to be the group of symmetries of an equilateral triangle, with Coxeter presentation determined by, s, t, $m_{s,t} = 3$. It is easy to see that $\underline{H}_e = 1$, $\underline{H}_s = H_s$, and $\underline{H}_{st} = \underline{H}_s \underline{H}_t$ (these identities always hold). Let us calculate the $(-)$-invariant element $\underline{H}_s \underline{H}_t \underline{H}_s$,

$$\begin{aligned}
\underline{H}_s \underline{H}_t \underline{H}_s &= (H_{st} + vH_s + vH_t + v^2)(H_s + v) \\
&= (H_{sts} + vH_s^2 + vH_{ts} + v^2 H_s) + (vH_{st} + v^2 H_s + v^2 H_t + v^3) \\
&= H_{sts} + (H_s - v^2 H_s + v) + vH_{ts} + v^2 H_s) + (vH_{st} + v^2 H_s + v^2 H_t + v^3. \\
&= H_{sts} + (H_s + v) + vH_{ts} + vH_{st} + v^2 H_s + v^2 H_t + v^3.
\end{aligned}$$

We can see that $\underline{H}_s \underline{H}_t \underline{H}_s \notin H_{sts} + \sum v\mathbb{Z}[v] H_y$, since there is a term H_s that cannot be factorized by v, but if we subtract $\underline{H}_s$ we obtain

$$\underline{H}_s \underline{H}_t \underline{H}_s - \underline{H}_s = H_{sts} + vH_{ts} + vH_{st} + v^2 H_s + v^2 H_t + v^3.$$

This element is obviously $(-)$-invariant and it is of the desired form. By uniqueness given in Theorem 2.26 we have that $\underline{H}_{sts} = \underline{H}_s \underline{H}_t \underline{H}_s - \underline{H}_s$. It is an easy exercise to see that $\underline{H}_s \underline{H}_t \underline{H}_s - \underline{H}_s = \underline{H}_t \underline{H}_s \underline{H}_t - \underline{H}_t$, this is due to the braid relation $sts = tst$. However, if $m_{s,t} \neq 3$ then $\underline{H}_s \underline{H}_t \underline{H}_s - \underline{H}_s \neq \underline{H}_t \underline{H}_s \underline{H}_t - \underline{H}_t$.

Example 2.29 (KL-basis in Type $I_2(n)$)**.** Generalizing the previous example, we can take $I_2(n)$ for each $n \in \mathbb{N}$, i.e. $m_{s,t} = n$. The first few examples are:

$$\begin{aligned}
m_{s,t} = 2 : \underline{H}_{st} &= \underline{H}_t \underline{H}_s \\
&= \underline{H}_s \underline{H}_t \\
m_{s,t} = 3 : \underline{H}_{sts} &= \underline{H}_s \underline{H}_t \underline{H}_s - \underline{H}_s \\
&= \underline{H}_t \underline{H}_s \underline{H}_t - \underline{H}_t \\
m_{s,t} = 4 : \underline{H}_{stst} &= \underline{H}_s \underline{H}_t \underline{H}_s \underline{H}_t - 2\underline{H}_s \underline{H}_t \\
&= \underline{H}_t \underline{H}_s \underline{H}_t \underline{H}_s - 2\underline{H}_t \underline{H}_s \\
m_{s,t} = 5 : \underline{H}_{ststs} &= \underline{H}_s \underline{H}_t \underline{H}_s \underline{H}_t \underline{H}_s - 3\underline{H}_s \underline{H}_t \underline{H}_s + \underline{H}_s \\
&= \underline{H}_t \underline{H}_s \underline{H}_t \underline{H}_s \underline{H}_t - 3\underline{H}_t \underline{H}_s \underline{H}_t + \underline{H}_t .
\end{aligned}$$

The following example is the most important calculation for this thesis.

Example 2.30 (KL-basis in Universal type U_n)**.** In 1988, Matthew Dyer gave a formula [Dye88] to calculate inductively the Kazhdan-Lusztig basis for a Universal Coxeter system. By "induction" on a Coxeter group, we will always mean induction over the length ℓ. In this case, every element $w \in U_n$ has only one reduced expression, since we do not have any braid relation. The Dyer's formula is the following:

Theorem 2.31 (Dyer's Formula)**.** *Let $w \in U_n$ and $w = st \cdots u$ with $s, t, \ldots u \in S$. Then we have the following recursive formula*

$$\begin{aligned}
&\underline{H}_s \underline{H}_w = (v + v^{-1}) \underline{H}_w, \\
&\underline{H}_z \underline{H}_w = \underline{H}_{zw} \qquad \text{if} \quad z \neq s, t, \\
&\underline{H}_t \underline{H}_w = \underline{H}_{tw} + \underline{H}_{sw} \quad \text{if} \quad t \neq s.
\end{aligned} \tag{2.5}$$

2.3. **Realizations.** In this section, we will define what a "realization" is, in order to define the "nice" diagrammatic category $\mathbb{S}\mathcal{D}$. We will work in $\mathbb{S}\mathcal{D}$ and it will be the "good" substitute of the category $\mathbb{S}$Bim of Soergel bimodules. The original definition of a realization given by Elias and Williamson [EW16, §3.1.] is very technical and requires some unnecessary background for this thesis. However, we will try to keep it as general as possible for our practical calculations and purposes. Some technicalities concerning the good choice of positive roots, "Frobenius realizations" and "root realizations" are developed in [Eli16, §A.4] only in the dihedral situation, we will not mention them here.

First of all, we need a preliminary definition.

Definition 2.32. A *generalized Cartan matrix* is a square matrix $A = (a_{ij})$, such that:

- $a_{ij} \in \mathbb{Z}_{\leq 0} \cup \{2\}$.
- $a_{ij} = 2$ if and only if $i = j$.
- $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Let us define a realization of a Coxeter system (W, S).

Definition 2.33. Let $\Bbbk$ be a commutative integral domain and let (W, S) be a Coxeter system. A *realization* of W over $\Bbbk$ consist of the following data:

- A finite-rank $\Bbbk$-module $\mathfrak{h}$.
- A *root datum* $(\Phi, \Phi^\vee)$, i.e., a set of *roots* $\Phi = \{\alpha_s\} \subset \mathfrak{h}^*$ indexed by S, and a set of *co-roots* $\Phi^\vee = \{\alpha_s^\vee\} \subset \mathfrak{h}$ also indexed by S, such that:

– $\langle \alpha_s, \alpha_s^\vee \rangle = 2$ for all $s \in S$.
– The map

$$S \longmapsto \mathrm{End}_{\Bbbk}(\mathfrak{h}^*)$$
$$s \longmapsto s\colon \mathfrak{h}^* \to \mathfrak{h}^*,$$

determined by

$$s(v) := v - \langle v, \alpha_s^\vee \rangle \alpha_s,$$

induces a representation of W.

Assumption 2.34. For practical purposes, we will always assume that a realization $\mathfrak{h}$ is obtained by the simple extension of scalars (that is, by applying $\otimes_{\Bbbk}\mathbf{F}$, where $\mathbf{F}$ is a field), from a realization whose matrix

$$(a_{st}) := (\langle \alpha_t, \alpha_s^\vee \rangle \mid s, t \in S)$$

is a generalized Cartan matrix.

Assumption 2.35. Moreover, we always assume *Demazure surjectivity* which states the maps $\alpha_s\colon \mathfrak{h} \to \Bbbk$ and $\alpha_s^\vee\colon \mathfrak{h}^* \to \Bbbk$ are surjective, [EW16, Assumption 3.9]. For example, it is satisfied when 2 is invertible in $\Bbbk$.

Definition 2.36. We call a realization *faithful* if the action of W on $\mathfrak{h}$ (and hence the contragredient action on $\mathfrak{h}^*$) is faithful. We call a faithful realization *reflection faithful* if there is a bijection between the set of reflections (i.e., the conjugates in W of S) and the subset of elements of W that act as reflections, i.e., that fix a co-dimension 1 space.

Remark 2.37. If a realization is faithful. By applying a change of basis or an extension of scalars, the resulting realization could be not faithful.

Example 2.38 (Geometric representation)**.** We define the *geometric representation* of any Coxeter system (W, S) of finite rank. It is given by $\Bbbk = \mathbb{R}$ and the vector space $\mathfrak{h}$ is defined by

$$\mathfrak{h} = \bigoplus_{s\in S} \mathbb{R}\alpha_s^\vee.$$

The dual space $\mathfrak{h}^*$ is defined by

$$\langle \alpha_t, \alpha_s^\vee \rangle = -2\cos(\pi/m_{st}), \tag{2.6}$$

where $\cos(\pi/\infty) := 1$. Then $\mathfrak{h}$ is called the *geometric representation* of (W, S), see [Hum92, §5.3]. Note that the subset $\{\alpha_s\} \subset \mathfrak{h}^*$ is linearly independent if and only if W is finite. This is a symmetric realization and faithful when W is finite.

Definition 2.39. We call a realization *symmetric* if $\langle \alpha_s^\vee, \alpha_t \rangle = \langle \alpha_t^\vee, \alpha_s \rangle$ for all $s, t \in S$. For example, the geometric representation is symmetric.

Let $R := \mathrm{Sym}(\mathfrak{h}^*)$ be the graded symmetric algebra of $\mathfrak{h}^*$ over $\Bbbk$, graded such that $\deg \mathfrak{h}^* = 2$. Then,

$$R \cong \Bbbk[\alpha_s \mid s \in S]$$

is the polynomial $\Bbbk$-algebra where each α_s is on degree 2. Since W acts on $\mathfrak{h}^*$, it also acts on R by functoriality. Let R^s be the $\Bbbk$-subalgebra of s-invariant elements of R.

Definition 2.40. For any $s \in S$, let the *Demazure operator* be the graded linear map defined by

$$\partial_s \colon R \longrightarrow R^s(-2) \tag{2.7}$$

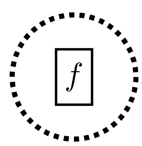
$$f \longmapsto \frac{(f - s(f))}{\alpha_s}. \tag{2.8}$$

Remark 2.41. This definition might not be very convincing. However, it is well defined since $p := (f - s(f))$ is an s-anti-invariant element, i.e., $s(x) = -x$. As was proved in [EW16, §3.3], the set of s-anti-invariant elements of R (under all the above assumptions) lies in the subspace $\alpha_s R$. Therefore, the expression $(f - s(f))/\alpha_s$ lies in R. It also lies in R^s, since α_s is s-anti-invariant. It is well defined as a graded map because dividing by α_s decreases the degree by 2.

2.4. S**-graphs.** Let $S = \{s, t, z, \ldots, u\}$ be the decorated finite set associated with a Coxeter system (W, S), where elements of S have different colors. Sometimes we will refer to an element of S as a *color*.

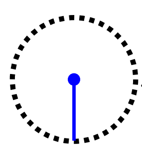
Definition 2.42. An *S-graph with boundary* (or just an *S-graph*) is an isotopy class of a decorated, planar, finite graph $\mathbb{G}$ embedded in the planar strip $M := \mathbb{R} \times [0, 1]$. The edges of such a graph $\mathbb{G}$ are colored by the elements of S. An edge may terminate with a vertex in the boundary $\partial M := \mathbb{R} \times \{0, 1\}$. We require that each vertex in the boundary has only one edge. We also require that the edges can touch the boundary only at its endpoints. Henceforth, vertices in the boundary are not considered as vertices. The (interior) vertices of $\mathbb{G}$ can be of the following four types, with their respective degree:

(1) The *zero-valent vertex* (a "box"), without edges, associated with a homogeneous element $f \in R$,



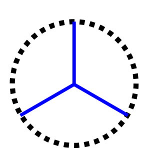
Its associated degree is $\deg f$.

(2) The *univalent vertex* (a "dot"), colored by its unique edge,



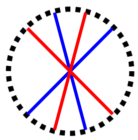
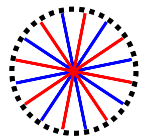
Its associated degree is 1.

(3) The *trivalent vertex*, with all its edges decorated with the same element $s \in S$,



Its associated degree is -1.

(4) The $2m_{s,t}$*-valent vertex* with $2m_{s,t}$ edges, with coloration counterclockwise alternating between s and t. For example,

 , if $m_{s,t} = 4$,  , if $m_{s,t} = 8$.

Its associated degree is 0.

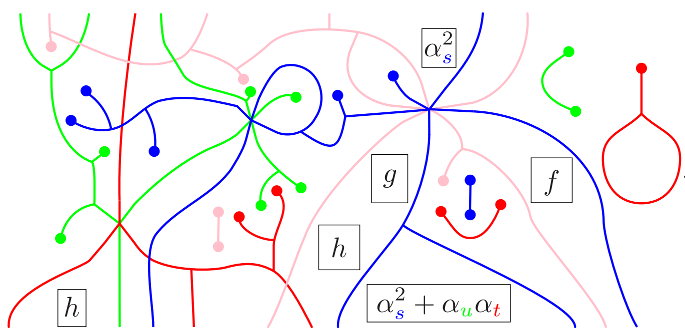
Definition 2.43. Let $\mathcal{D}$ be an S-graph. The *bottom boundary* $\mathrm{bot}(\mathcal{D})$ (respectively, *top boundary* $\mathrm{top}(\mathcal{D})$) of an S-graph is the ordered sequence of points in $\mathbb{R} \times \{0\}$ (respectively, $\mathbb{R} \times \{1\}$), read from the left to the right. Each boundary point has associated a color given by its unique edge, then we can identify $\mathrm{bottom}(\mathcal{D}), \mathrm{top}(\mathcal{D})$ inside of $\mathrm{Ex}(S)$.

Definition 2.44. The *degree* of an S-graph with boundary is the sum of the degrees of its vertices.

Example 2.45 (S-graph)**.** Let (W, S) be the rank four Coxeter system given by $S = \{s, t, z, u\}$ and relations

$$\begin{aligned} m_{s,t} &= 2, \quad & m_{z,u} &= 2, \\ m_{s,u} &= 4, \quad & m_{t,u} &= 3, \\ m_{s,z} &= 5, \quad & m_{t,z} &= 2. \end{aligned}$$

The following diagram represents an S-graph $\mathcal{D}$ for (W, S),



Where $f, g, h \in R$ are homogeneous polynomials. In order to calculate the degree of $\mathcal{D}$, we need to count all univalent, trivalent and zero-valent vertices, since the other are degree zero vertices. There are eight green dot vertices, five red dot vertices, seven blue dot vertices, five pink dot vertices, each one of degree one. Then we have a total of twenty-five univalent vertices. There are five green trivalent vertices, four red trivalent vertices, five blue trivalent vertices, six pink trivalent vertices, each one of degree -1. Then we have a total of twenty trivalent vertices. The two determined box vertices are of degree 4 each one. The indeterminate zero-valent vertices sum a total degree of $\deg f + \deg g + 2\deg h$. Then the degree of $\mathcal{D}$ is

$$\begin{aligned} \deg(\mathcal{D}) &= 25 - 20 + 4 + 4 + \deg f + \deg g + 2\deg h \\ &= 13 + \deg f + \deg g + 2\deg h. \end{aligned}$$

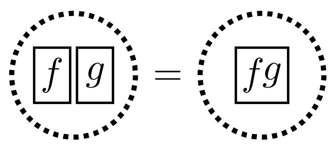
It is easy to determine the bottom and top boundaries from the picture (the usual order in $\mathbb{R} \times \{0\}$ is taken from the left to the right),

$$\mathrm{top}(\mathcal{D}) = (u, z, u, t, u, z, z, s, z) \in S^9,$$
$$\mathrm{bot}(\mathcal{D}) = (t, u, s, t, z, t, s, s, z, s) \in S^{10}.$$

Notation 2.46. Given an S-graph $\mathcal{D}$ we denote by $\overline{\mathcal{D}}$ the S-graph obtained by flipping the diagram vertically. This operation induces a contravariant equivalence on the graded monoidal category $\mathbb{S}\mathcal{D}$ defined in the next section.

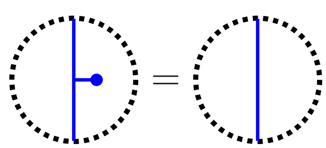
2.5. **Diagrammatic category of Soergel bimodules.** The *diagrammatic category of Soergel bimodules* denoted by $\mathbb{S}\mathcal{D}$ is the monoidal category defined as follows. Objects are expressions $\underline{w} \in \mathrm{Ex}(S)$ with the monoidal operation given by the horizontal concatenation of expressions in $\mathrm{Ex}(S)$. This category is enriched over the category $\Bbbk$-mod of $\Bbbk$-modules. For each $\underline{x}, \underline{y} \in \mathrm{Ex}(S)$ the set of arrows $\mathrm{Hom}_{\mathbb{S}\mathcal{D}}(\underline{x}, \underline{y})$ is defined to be the $\Bbbk$-module generated by the set of isotopy classes of S-graphs with top boundary $\underline{x}$ and bottom boundary $\underline{y}$ modulo the relations below.

(1) Zero-color relation.
   (a) *Multiplication law.*

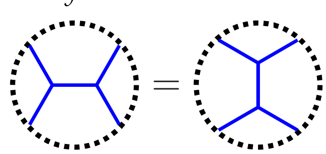


Where f, g are homogeneous elements of R.

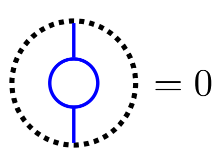
(2) One-color relations.
   (a) *Frobenius unit.*



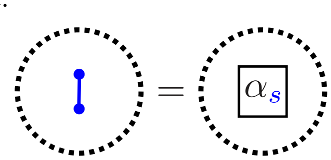
   (b) *Frobenius associativity.*



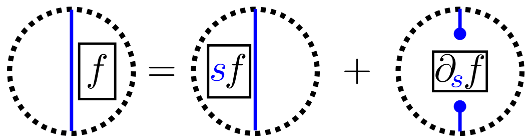
   (c) *Needle relation.*



   (d) *Barbell relation.*



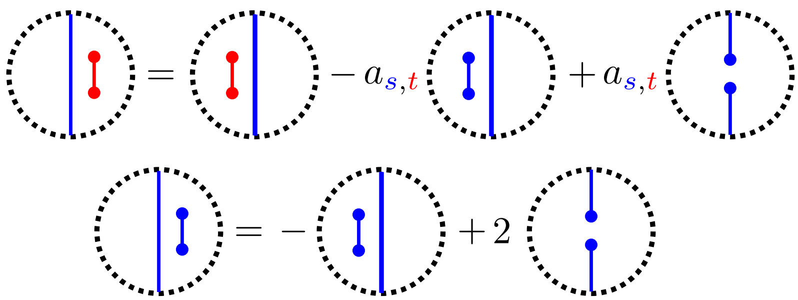
(e) *Nil Hecke relation*.



(3) Two-color and three-color relations. We are not going to show or use those relations in this thesis, so we will not write them down.

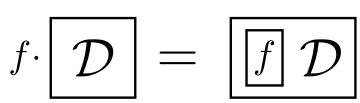
The *barbell forcing relations* are particular cases of the Nil Hecke relation.

(2.9)

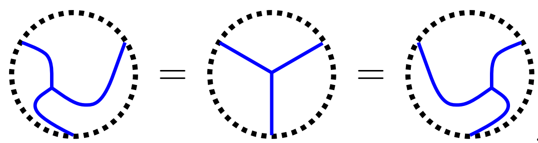


The composition of arrows is defined to be the isotopy class of the vertical juxtaposition of their underlying S-graphs. In order to get a well defined S-graph as a result, vertical juxtaposition should be made matching vertices of the boundary of one S-graph with the other in a bijective pairing. This composition is denoted by $\circ$, for example, $f \circ g$ can be depicted by the S-graph of f just above of the S-graph of g. The monoidal structure of arrows in $\mathbb{S}\mathcal{D}$ is given by the isotopy class of the horizontal juxtaposition of their underlying S-graphs. It is denoted by $\otimes_R$ or just by $\otimes$, for example, $f \otimes g$ can be depicted by the S-graph of g placed "very far to the right" of the S-graph of f. By "very far to the right" of f, we mean the place at the right where f is empty, this place exists since S-graphs are finite graphs, then they are compact as subsets of $\mathbb{R}^2$.

All relations presented here are homogeneous for the grading on S-graphs. Thus, the category $\mathbb{S}\mathcal{D}$ is enriched over the category of graded R-modules by the action of R-modules induced by the following action on homogeneous elements:



Remark 2.47. In a diagrammatic category like this, there are some extra implicit relations. As stated in [Eli16] we need *cyclicity* which states that taking any morphism and using adjunction maps to rotate it by 360 degrees will not change the morphism. This is required in order to be able to draw morphisms on a plane because any labeling we use to depict the morphism is evidently invariant under a 360-degree rotation. Furthermore, for each generator morphism for which we used labeling with some non-trivial rotation invariances (this excludes the box label for the zero-valent vertices, which is considered to have the same group of invariance transformations as a single point, i.e., every rotation is an invariance) those labels carry hidden and non-stated relations called the *symmetry labeling relations* which formulates that rotating this morphism in a certain way does nothing. For example, the trivalent vertex above has an invariance under a non-trivial 120-degree rotation, then we have the following relation



Those relations extend to any diagrams having non-trivial symmetries, i.e., not only the generating ones.

2.6. **Local intersection forms.** Let x be an element of W. Let $\mathbb{S}\mathcal{D}^{\not< x}$ be the quotient category of $\mathbb{S}\mathcal{D}$ by the ideal $\mathbb{S}\mathcal{D}_{<x}$ generated by lower terms than x, i.e., by all morphisms which can be factorized through a morphism with domain or codomain $\underline{y} \in \mathrm{Rex}(y)$ for some $y < x$ in the Bruhat order of W. Let $\mathrm{Hom}_{\not< x}(-,-)$ denote the set of morphisms in this category. In the category $\mathbb{S}\mathcal{D}^{\not< x}$, the images of any two reduced expressions for x are canonically isomorphic, see [EW16, §6.5]. Therefore, we denote the image of any reduced expression for x in $\mathbb{S}\mathcal{D}^{\not< x}$ by x just as well. Furthermore as it is mentioned in [EW16, §6.5] we have $\mathrm{End}_{\not< x}(x) \cong R$. For any $\underline{w} \in \mathrm{Ex}(S)$ we have the following R-bilinear pairing of graded R-modules

$$I_{x,\underline{w}} \colon \mathrm{Hom}_{\not< x}(x, \underline{w}) \times \mathrm{Hom}_{\not< x}(\underline{w}, x) \to \mathrm{End}_{\not< x}(x) \cong R.$$
$$(f,g) \longmapsto g \circ f.$$

In order to define the intersection forms, we need to make an extension of scalars. Let $\mathrm{Hom}_{\not< x,k}(-,-)$ be the set of arrows in $k \otimes_R \mathbb{S}\mathcal{D}^{\not< x}$, where we are killing the action of each polynomial of positive degree in R. The category $k \otimes_R \mathbb{S}\mathcal{D}^{\not< x}$ is a Krull-Remak-Schmidt k-linear graded monoidal category. In particular, we have $\mathrm{End}_{\not< x,k}(x) \cong k \otimes_R R \cong k$.

Notation 2.48. In a graded category, we denote by

$$\mathrm{Hom}^{\bullet}(a,b) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(a, b(n)),$$

to the set of arrows in the induced category which forgets the grades.

Definition 2.49. For any $\underline{w}$ reduced expression in S, the *local intersection form of $\underline{w}$ at x* is the k-bilinear pairing

$$I_{\underline{w},x} \colon \mathrm{Hom}^{\bullet}_{\not< x,k}(x, \underline{w}) \times \mathrm{Hom}^{\bullet}_{\not< x,k}(\underline{w}, x) \longrightarrow \mathrm{End}^{\bullet}_{\not< x,k}(x) \cong k$$
$$(f,g) \longmapsto g \circ f.$$

Remark 2.50. Observe that $\mathrm{End}^{\bullet}_{\not< x,k}(x)$ has only degree zero elements. A morphism in $\mathrm{Hom}_{\not< x,k}(x, \underline{w}(d))$ for some $d \in \mathbb{Z}$ can only be paired non-trivially with elements of $\mathrm{Hom}_{\not< x,k}(\underline{w}(-d), x)$.

Definition 2.51. The *d-th grading piece of the local intersection form* is defined as the k-bilinear pairing

$$I^d_{\underline{w},x} \colon \mathrm{Hom}^{\bullet}_{\not< x,k}(x(d), \underline{w}) \times \mathrm{Hom}^{\bullet}_{\not< x,k}(\underline{w}, x(-d)) \longrightarrow \mathrm{End}^{\bullet}_{\not< x,k}(x) \cong k$$
$$(f,g) \longmapsto g \circ f.$$

Finally, the *graded rank* of $I_{\underline{w},x}$ is denoted by $n_{x,w} \in \mathbb{Z}[v^{\pm 1}]$ and defined as

$$n_{x,w} := \sum_{d \in \mathbb{Z}} \operatorname{rk}\left(I^d_{\underline{w},x}\right) v^d.$$

Then we can restate Lemma 1.16 as:

Lemma 2.52. *The graded multiplicity of x in $\underline{w}$ in $\mathbb{S}\mathcal{D}$ is given by the graded rank of $I_{\underline{w},x}$.*

In order to understand what a graded multiplicity means in terms of powers of v, we need to note that v acts by the grading shift functor (1) on the graded category $\mathbb{S}\mathcal{D}$. This is the analog of what happens in the category of Soergel bimodules $\mathbb{S}$Bim defined in the introduction.

Example 2.53. In the category $\mathbb{S}$Bim defined over the real geometric representation of the symmetric group S_3, we saw in the introduction that the Hecke algebra relation

$$\underline{H}_{sts}\underline{H}_{sts} = (v^{-3} + 2v^{-1} + 2v^1 + v^3)\underline{H}_{sts}, \tag{2.10}$$

is lifted by the isomorphism

$$B_{sts} \otimes_R B_{sts} \cong B_{sts}(-3) \oplus B_{sts}(-1)^{\oplus 2} \oplus B_{sts}(1)^{\oplus 2} \oplus B_{sts}(3). \tag{2.11}$$

In this case, we say that the graded multiplicity of B_{sts} in $B_{sts} \otimes_R B_{sts}$ is $v^{-3} + 2v^{-1} + 2v^1 + v^3$ to mean the following four statements:

(a) $B_{sts}(-3)$ has multiplicity one in $B_{sts} \otimes_R B_{sts}$.
(b) $B_{sts}(-1)$ has multiplicity two in $B_{sts} \otimes_R B_{sts}$.
(c) $B_{sts}(1)$ has multiplicity two in $B_{sts} \otimes_R B_{sts}$.
(d) $B_{sts}(3)$ has multiplicity one in $B_{sts} \otimes_R B_{sts}$.

2.7. **The p-canonical basis and the relation with Soergel bimodules.** Before introducing the p-canonical basis, we have to fix some notations which will be used in this thesis.

Notation 2.54. Let $\mathbb{O}$ denote a complete discrete valuation ring of characteristic zero (e.g., a finite extension of $\mathbb{Z}_p$), $\mathbb{K}$ its field of fractions (e.g., a finite extension of $\mathbb{Q}_p$), and k its residue field (e.g., a finite field $\mathbb{F}_q$ $(q = p^m)$). Unless stated otherwise, $\Bbbk$ denotes a complete local principal ideal domain, which may be for example $\mathbb{K}$, $\mathbb{O}$ or k.

Notation 2.55. Let

$$M = \sum_{i \in \mathbb{Z}} M_i$$

be a $\mathbb{Z}$-graded module. The *graded rank* of M is given by

$$\operatorname{grk}(M) = \sum_{i \in \mathbb{Z}} \dim (M_i) v^i.$$

Let $\mathbb{S}\mathcal{D}_\oplus$ denote the graded idempotent completion of $\mathbb{S}\mathcal{D}$. That is, objects of $\mathbb{S}\mathcal{D}_\oplus$ are shifts of finitely many direct sums of elements of the form $(\underline{w}, p)$, where $\underline{w}$ is an object of $\mathbb{S}\mathcal{D}$ and $p \in \operatorname{End}(\underline{w})$ is such that $p \circ p = \mathbb{1}_{\underline{w}}$. On the other hand, arrows from $(\underline{w}, p)$ to $(\underline{v}, q)$ are triples (e, f, h), where

$$f \in \operatorname{Hom}_{\mathbb{S}\mathcal{D}}(\underline{w}, \underline{v}), \quad e \in \operatorname{End}_{\mathbb{S}\mathcal{D}}(\underline{w}), \quad h \in \operatorname{End}_{\mathbb{S}\mathcal{D}}(\underline{v}), \tag{2.12}$$

such that, $f \circ e = h \circ f = f$, $e \circ e = \mathbb{1}_{\underline{w}}$ and $h \circ h = \mathbb{1}_{\underline{v}}$. In this category, we have the following theorem known as *Soergel Categorification Theorem* (or just *SCT*) which is the main theorem of [EW16, Theorem 6.25].

Theorem 2.56 (Soergel Categorification Theorem)**.** *Let $\Bbbk$ be a complete local principal ideal domain. Let $\mathbb{S}\mathcal{D}_{\oplus}$ be the additive diagrammatic category defined over $\Bbbk$, then:*

(1) $\mathbb{S}\mathcal{D}_{\oplus}$ is a Krull-Remak-Schmidt $\Bbbk$-linear category with a grading shift functor (1).
(2) For all $w \in W$ there exists a unique indecomposable direct summand

$$b_w \overset{\oplus}{\subset} \underline{w}$$

in $\mathbb{S}\mathcal{D}_{\oplus}$ for any reduced expression $\underline{w}$ of w and which is not isomorphic to a grading shift of any direct summand of any expression $\underline{v}$ for an element $v < w$. The object b_w does not depend on the reduced expression $\underline{w}$ of w.
(3) The set $\{b_w \mid w \in W\}$ gives a complete set of representatives of the isomorphism classes of indecomposable objects in $\mathbb{S}\mathcal{D}_{\oplus}$ up to grading shift.
(4) There exists a unique isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras

$$\begin{aligned} \varepsilon\colon \langle \mathbb{S}\mathcal{D}_{\oplus} \rangle &\longrightarrow \mathbf{H} \\ [b_s] &\longmapsto H_s, \end{aligned}$$

where $s \in S$, and $\langle \mathbb{S}\mathcal{D}_{\oplus} \rangle$ denotes the split Grothendieck group of $\mathcal{H}$. Above, the group $\langle \mathbb{S}\mathcal{D}_{\oplus} \rangle$ has a $\mathbb{Z}[v^{\pm 1}]]$-algebra structure as follows: the monoidal structure on $\mathbb{S}\mathcal{D}_{\oplus}$ induces a unital, associative multiplication and v acts via $v[b] := [b(1)]$ for an object b of $\mathcal{H}$. Furthermore, the explicit isomorphism is given by

$$\begin{aligned} \varepsilon\colon \langle \mathbb{S}\mathcal{D}_{\oplus} \rangle &\longrightarrow \mathbf{H} \\ [b] &\longmapsto \sum_{w \in W} \operatorname{grk}\left(\operatorname{Hom}^{\bullet}_{\not< x,k}(b, w)\right) H_w. \end{aligned}$$

The second part of this theorem is analog to the *Bott-Samelson* description presented in the introduction in the category $\mathbb{S}$Bim of Soergel bimodules.

Remark 2.57*.* Despite the fact we have a diagrammatic presentation of $\mathbb{S}\mathcal{D}$, we do not have a diagrammatic presentation of $\mathbb{S}\mathcal{D}_{\oplus}$ as determining the idempotents in $\mathbb{S}\mathcal{D}$ is usually extremely difficult. In [EL14], it is presented a recursion formula for the idempotents in type $\tilde{A}_1$ over fields $\mathbb{K}$ of characteristic zero, but over characteristic p this is still an interesting open question which strongly motivates the results in this thesis.

For any $\underline{w} \in \operatorname{Ex}(S)$, by the Lemma 2.52 one has in $[\mathbb{S}\mathcal{D}_{\oplus}]$ the identity

$$[\underline{w}] = \sum_{x \in W} m_x [b_x],$$

where m_x denotes the graded rank of the intersection form $I_{x,\underline{w}}$. This explains the central importance of the intersection forms.

On the one hand, in [EW12] it is proved (using Soergel bimodules) that if $\mathbb{K} \cong \mathbb{R}$ then ε sends the Kazhdan-Lusztig basis element $\underline{H}_w$ to the class of b_w. On the other hand, over fields k of characteristic p, the set $\{\varepsilon(\underline{H}_w) \mid w \in W\}$ is still a $\mathbb{Z}[v^{\pm 1}]$-basis of $\mathbf{H}$ and it is called the *p-canonical basis of* $\mathbf{H}$. We denote

$${}^p\underline{H}_w := \varepsilon(\underline{H}_w).$$

Let $\mathbb{S}$Bim the category of Soergel bimodules described in the introduction. In [EW16], a functor $\mathscr{F}: \mathbb{S}\mathcal{D} \to \mathbb{S}\mathrm{Bim}$ is constructed and it is proved that it induces an equivalence of graded monoidal categories $\mathbb{S}\mathcal{D}_{\oplus} \cong \mathbb{S}\mathrm{Bim}$. In particular, $\mathscr{F}$ maps b_w to B_w, for any $w \in W$.

Example 2.58. This is the analog of the example 2.53 but in the diagrammatic category. Let $\mathbb{S}\mathcal{D}$ be the diagrammatic category defined over the real geometric representation of the symmetric group S_3. By the equation 2.11 and the equivalence described in the previous paragraph, we have that

$$\underline{H}_{sts}\underline{H}_{sts} = (v^{-3} + 2v^{-1} + 2v^{1} + v^{3})\underline{H}_{sts},$$

is lifted by the isomorphism

$$b_{sts} \otimes_R b_{sts} \cong b_{sts}(-3) \oplus b_{sts}(-1)^{\oplus 2} \oplus b_{sts}(1)^{\oplus 2} \oplus b_{sts}(3). \tag{2.13}$$

We say that the graded multiplicity of b_{sts} in $b_{sts} \otimes_R b_{sts}$ is $v^{-3} + 2v^{-1} + 2v^{1} + v^{3}$ to mean the following four statements:

(a) $b_{sts}(-3)$ has multiplicity one in $b_{sts} \otimes_R b_{sts}$.
(b) $b_{sts}(-1)$ has multiplicity two in $b_{sts} \otimes_R b_{sts}$.
(c) $b_{sts}(1)$ has multiplicity two in $b_{sts} \otimes_R b_{sts}$.
(d) $b_{sts}(3)$ has multiplicity one in $b_{sts} \otimes_R b_{sts}$.

3. Libedinsky's light leaves in U_n

As a requirement of Lemma 1.16 (from which we have deduced Lemma 2.52), we have the fact that morphisms spaces are finitely generated as k-modules. Here is where we need to introduce the Libedinsky's light leaves in the category $\mathbb{S}\mathcal{D}$. Henceforth, for simplicity, we will work in the universal Coxeter system U_n (Coxeter systems of type $\tilde{A}_1$). More general light leaves are not required in this thesis and can be found in a diagrammatic version in [EW16, §6]. They are not canonically constructed. Their construction depends on a finite number of arbitrary choices. They are the diagrammatic translation of the Libedinsky's light leaves defined over the category $\mathbb{S}$Bim of Soergel bimodules, see [Lib15, §3].

Notation 3.1. In the category $\mathbb{S}\mathcal{D}$, the identity morphism in $\underline{s} = (s)$ will be denoted by $\mathbb{1}_s$. Its class as an S-graph consist of the class of just a vertical s-colored line, as in the right-hand side of the Frobenius unit relation, see Section 2.5. Secondly, the s-colored trivalent vertex corresponds to a morphism from (s, s) to (s) and it will be denoted by j_s. Finally, the s-colored univalent vertex corresponds to a morphism from (s) to (e) and it will be denoted by m_s.

Notation 3.2. The identity of an expression $\underline{w} = (s_1, \dots, s_n)$ will be denoted by $\mathbb{1}_{\underline{w}}$ or just $\mathbb{1}$ if the domain is clear from the context. It is equal to $\mathbb{1}_{s_1} \otimes \mathbb{1}_{s_2} \otimes \cdots \otimes \mathbb{1}_{s_n}$, in particular, its graph will sometimes be depicted with n colored vertical and parallel segments.

3.1. **Diagrammatic construction.** Let $\underline{w} = (s_1, \dots, s_n)$ be any expression (reduced or not) of length n. Recall from Chapter 2 that given a binary string $\underline{e}$ of length n, we have a coloration $\underline{d} = d_1, \dots, d_n$ for the subexpression $\underline{w}^{\underline{d}}$. For $1 \leq k \leq n$ we set $\underline{w}_{\leq k} := (s_1, \dots, s_k)$, $\underline{e}_{\leq k} := (e_1, \dots, e_k)$ and $w_k := (\underline{w}_{\leq k}^{\underline{e}_{\leq k}})_{\bullet}$ as before. We set $\underline{w_k}$ as the unique[6] reduced expression of w_k. We will construct a canonical

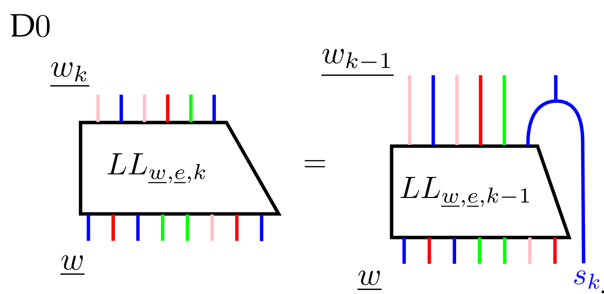
[6] In the universal Coxeter group U_n, all reduced expressions are unique.

morphism $LL_{\underline{w},\underline{e}} \in \mathrm{Hom}(\underline{w}, \underline{w}^{\underline{e}})$ associated with this subexpression. This will be done constructing its S-graph. In order to do this, we will construct for each $0 \leq k \leq n$ an intermediate diagram $LL_{\underline{w},\underline{e},k}$. Let $LL_{\underline{w},\underline{e},0}$ be the empty diagram. Suppose that $LL_{\underline{w},\underline{e},k-1} \in \mathrm{Hom}(\underline{w}, \underline{w_{k-1}})$ has already been constructed, it is well defined and $\underline{w_{k-1}}$ is its codomain. Then we need to construct a well-defined map $LL_{\underline{w},\underline{e},k} \in \mathrm{Hom}(\underline{w}, \underline{w_k})$ and prove that $\underline{w_k}$ is a reduced expression. There are only four cases:

(1) Case $d_k =$ D0. This case occurs when $e_k = 0$ and $w_{k-1}s_k < w_{k-1}$. The last condition can only happen if $\underline{w_{k-1}}$ ends with s_k. Let

$$LL_{\underline{w},\underline{e},k} := (\mathbb{1} \otimes j_{s_k}) \circ (LL_{\underline{w},\underline{e},k-1} \otimes \mathbb{1}_{s_k}).$$

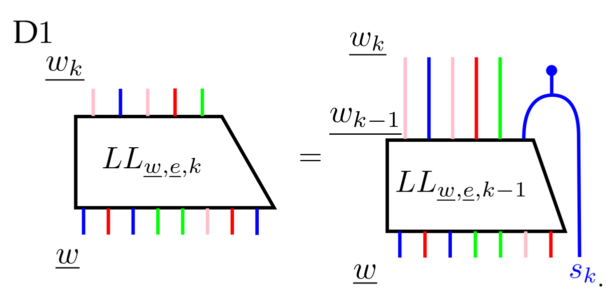
It is easy to show that the codomain of $LL_{\underline{w},\underline{e},k}$ is $\underline{w_k}$ which in this case equals to the reduced expression $\underline{w_{k-1}}$. This can be depicted by the following picture



(2) Case $d_k =$ D1. This case occurs when $e_k = 1$ and $w_{k-1}s_k < w_{k-1}$. The last condition can only happen if $\underline{w_{k-1}}$ ends with s_k. Let

$$LL_{\underline{w},\underline{e},k} := \big(\mathbb{1} \otimes (m_{s_k} \circ j_{s_k})\big) \circ (LL_{\underline{w},\underline{e},k-1} \otimes \mathbb{1}_{s_k}).$$

We have that the codomain of $LL_{\underline{w},\underline{e},k}$ is $\underline{w_k}$. If follows by definition of $\underline{w_k}$, since $\underline{w_{k-1}}$ ends with s_k, we have that $\underline{w_k}$ is just $\underline{w_{k-1}}$ without the last s_k. This can be depicted by the following picture

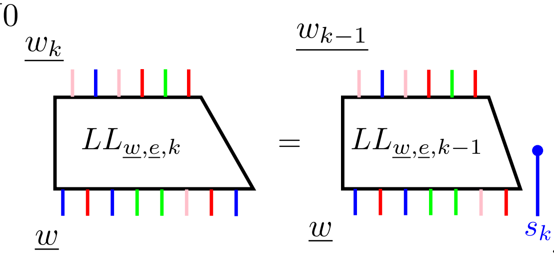


(3) Case $d_k =$ U0. This case occurs when $e_k = 0$ and $w_{k-1}s_k > w_{k-1}$. The last condition can only happen if $\underline{w_{k-1}}$ ends with s_i for some $i \neq k$. Let

$$LL_{\underline{w},\underline{e},k} := (\mathbb{1} \otimes m_{s_k}) \circ (LL_{\underline{w},\underline{e},k-1} \otimes \mathbb{1}_{s_k}).$$

It is easy to show that the codomain of $LL_{\underline{w},\underline{e},k}$ is $\underline{w_k}$ which in this case equals to the reduced expression $\underline{w_{k-1}}$. This can be depicted by the following picture

U0

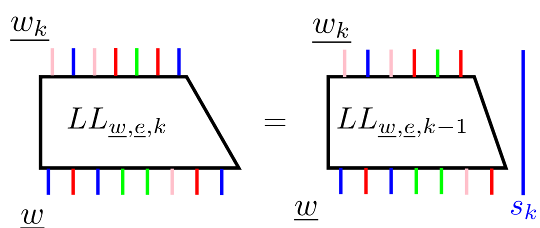


(4) Case $d_k = \text{U1}$. This case occurs when $e_k = 1$ and $w_{k-1}s_k > w_{k-1}$. The last condition can only happen if $\underline{w_{k-1}}$ ends with s_i for some $i \neq k$. Let

$$LL_{\underline{w},\underline{e},k} := LL_{\underline{w},\underline{e},k-1} \otimes \mathbb{1}_{s_k}.$$

We have that the codomain of $LL_{\underline{w},\underline{e},k}$ is $\underline{w_k}$. If follows by definition of $\underline{w_k}$, since w_{k-1} ends with s_i, we have that $\underline{w_k}$ is just w_{k-1} with a s_k at the end. This can be depicted by the following picture

U1



Define $LL_{\underline{w},\underline{e}} := LL_{\underline{w},\underline{e},n} \in \text{Hom}(\underline{w}, \underline{w}^{\underline{e}})$ to be the desired morphism. It turns by construction that the top of $LL_{\underline{w},\underline{e}} := LL_{\underline{w},\underline{e},n}$ is the reduced expression $\underline{w}^{\underline{e}}$.

Definition 3.3. The morphism $LL_{\underline{w},\underline{e}}$ in the diagrammatic category $\mathbb{S}\mathcal{D}$ is called the *Libedinsky light leaf* from $\underline{w}$ to $\underline{w}^{\underline{e}}$. Let $x \in W$, the set of light leaves from $\underline{w}$ to $\underline{w}^{\underline{e}}$ such that $(\underline{w}^{\underline{e}})_\bullet = x$ is denoted by $\mathbb{LL}_{\underline{w}}(x)$. The set of light leaves from $\underline{w}$ to anywhere is denoted by $\mathbb{LL}_{\underline{w}}$. In particular,

$$\mathbb{LL}_{\underline{w}} = \bigsqcup_{x \leq (\underline{w})_\bullet} \mathbb{LL}_{\underline{w}}(x).$$

3.2. **Basic properties.**

Lemma 3.4. *The degree of the map $LL_{\underline{w},\underline{e}}$ equals to the defect of the subexpression $\underline{w}^{\underline{e}}$.*

Proof. This follows by induction in the k-th step of construction $LL_{\underline{w},\underline{e},k}$ of $LL_{\underline{w},\underline{e}}$. Notice that $d_k = \text{U0}$ adds just a degree 0 line and $d_k = \text{D1}$ adds a trivalent vertex of degree -1 and a dot of degree 1. Therefore, the only contributions to the degree are the cases $d_k = \text{D0}$ and $d_k = \text{U0}$, which adds a trivalent vertex of degree -1 or a dot of degree 1 respectively. □

The following lemma is standard and follows easily from [EL14, Proposition 3.7].

Lemma 3.5. *If $\underline{w}$ is a reduced expression, then every light leaf in $\mathbb{LL}_{\underline{w}}$ has non-negative degree. If $l \in \mathbb{LL}_{\underline{w}}(e)$ then $\deg l \geq 1$.*

The main theorem of [EW16] is the following.

Theorem 3.6 (Double-leaves basis)**.** *In the category $\mathbb{SD}$, the set*

$$\{\overline{l_2} \circ l_1 \mid l_1 \in \mathbb{LL}_{\underline{w}}, l_2 \in \mathbb{LL}_{\underline{v}}, \mathrm{top}(l_1) = \mathrm{top}(l_2)\}$$

is an R-basis for $\mathrm{Hom}(\underline{w}, \underline{v})$.

We conclude the following corollaries.

Corollary 3.7. *In the category $\mathbb{SD}$, if $\underline{w}$ and $\underline{v}$ are reduced expressions. Then there are no negative degree morphisms in $\mathrm{Hom}(\underline{w}, \underline{v})$.*

Corollary 3.8. *In the category $\mathbb{SD}^{\not< x}$, the set $\mathbb{LL}_{\underline{w}}(x)$ is an R-basis for $\mathrm{Hom}_{\not< x}(\underline{w}, x)$. In particular, in the category $k \otimes \mathbb{SD}^{\not< x}$, the set $\mathbb{LL}_{\underline{w}}(x)$ is a k-basis for the k-vector space $\mathrm{Hom}_{\not< x,k}(\underline{w}, x)$.*

Finally, we have the main theorem of this section.

Theorem 3.9. *The local intersection form $I_{\underline{w},x}$ in the category $k \otimes \mathbb{SD}^{\not< x}$ is completely determined by the matrix*

$$\{LL_{\underline{w},\underline{f}} \circ \overline{LL_{\underline{w},\underline{e}}} \mid (\underline{w}^{\underline{f}})_\bullet = (\underline{w}^{\underline{e}})_\bullet = x, \text{ and } \mathrm{df}(\underline{w}^{\underline{f}}) = \mathrm{df}(\underline{w}^{\underline{e}}) = 0\}.$$

Proof. By Corollary 3.8 and $\Bbbk$-bi-linearity of $I_{\underline{w},x}$, we know that the set

$$\{LL_{\underline{w},\underline{f}} \circ \overline{LL_{\underline{w},\underline{e}}} \mid (\underline{w}^{\underline{f}})_\bullet = (\underline{w}^{\underline{e}})_\bullet = x\}$$

corresponds to the image of a basis of the domain of $(I_{\underline{w},x})$. Then this set determines completely $I_{\underline{w},x}$. However, this set is still too big. By Remark 2.50, we know that each degree d light leaf can be paired non-trivially only with a degree $-d$ light leaf. But there does not exist light leaves of negative degree by Lemma 3.5. Then it suffices with degree zero (defect zero) light leaves. ☐

4. Properties of $\mathrm{End}^\bullet_{\not< x,k}(x)$ in U_n.

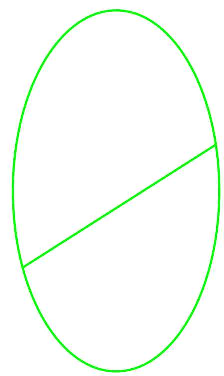
Fix $x = (t_1, t_2, \ldots, t_r)$ a reduced expression of $x \in W$. Consider the k-vector space $\mathrm{End}^\bullet_{\not< x,k}(x)$ defined above. Note that morphisms in this space have $2r$ vertices on its boundary. There are r vertices on the bottom boundary, they are colored from left to right by $(t_i \mid 1 \leq i \leq r)$ and indexed by $(i \mid 1 \leq i \leq r)$, we denote them by a_i. Analogously, there are r vertices on the top, we denote them by $b_i := a_{r+i}$. In particular, for each i, both a_i and b_i are colored by the same element t_i of S. We will refer to an arbitrary boundary vertex by a_i or just a, where $i \in \{1, 2, \ldots, 2r\}$.

4.1. Topological behavior of $\mathbb{SD}$.

Definition 4.1. A *curve* of an S-graph $\mathcal{D}$ is a non-empty connected subset of $\mathcal{D} \subset \mathbb{R} \times [0, 1]$. A *loop of an S-graph $\mathcal{D}$* is a curve of $\mathcal{D}$ homeomorphic to the circle $\mathbb{S}^1$. A *loop of a curve γ of an S-graph $\mathcal{D}$* is a loop of $\mathcal{D}$ contained in γ.

Definition 4.2. Let γ be a curve of an S-graph $\mathcal{D}$. A *cycle of γ in $\mathcal{D}$* is a minimal loop $\mathfrak{n} \subset \gamma$ in γ, i.e., there are no other loops of γ such that its interior region (as a subset of $\mathbb{R}^2$) is contained in the interior region of $\mathfrak{n}$. A *cycle* in $\mathcal{D}$ is a cycle for some curve in $\mathcal{D}$.

Example 4.3. In the following picture, we can appreciate three different loops.

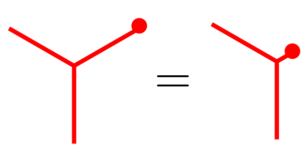


But the big one is the only which is not minimal, therefore there are only two cycles.

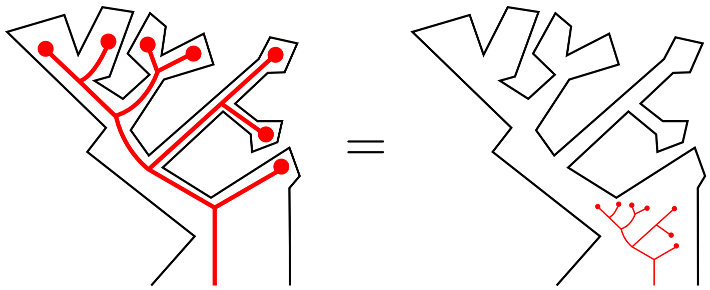
Definition 4.4. Given an S-graph $\mathcal{D}$. The *connected portion* or just a *portion* of a point $p \in \mathcal{D}$ (respectively, of a cycle $\mathfrak{n}$ or a curve γ) is the connected component Γ which contains it, united with the interior region (as a subset of $\mathcal{D}$) of any cycle $\mathfrak{n}$ belonging to Γ. We will denote it by $\mathrm{por}_{\mathcal{D}}(p)$ (respectively, by $\mathrm{por}_{\mathcal{D}}(\mathfrak{n})$ or by $\mathrm{por}_{\mathcal{D}}(\gamma)$). We can omit the label $\mathcal{D}$ if the graph is clear from the context. A *portion of* $\mathcal{D}$ is the portion of some point, cycle or curve of $\mathcal{D}$.

Remark 4.5. Since S-graphs are finite graphs, S-graphs and portions can be expressed as a finite union of curves. Furthermore, curves of an S-graph $\mathcal{D}$ can be expressed as a finite union of 1-manifolds.

Remark 4.6. Recall that an open edge of an S-graph $\mathcal{D}$ is a curve homeomorphic to the closed interval $[0, 1]$, whose two endpoints correspond to two different vertices. An edge of an S-graph with at least one non-boundary endpoint can be contracted in a way as small as one wants in $\mathbb{S}\mathcal{D}$ by the isotopy relations (see Section 2.5), as in the following picture.

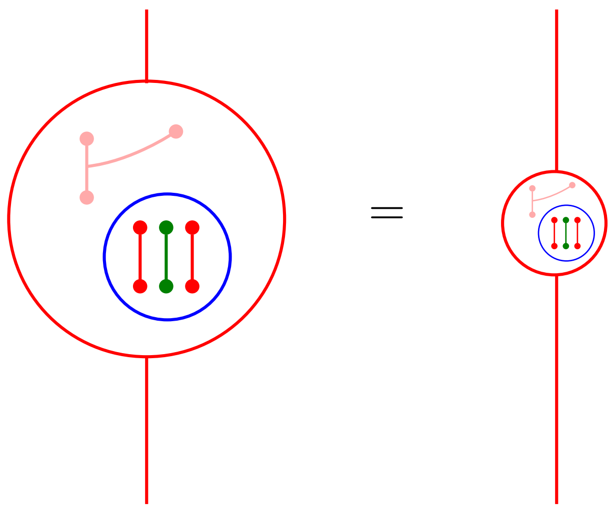
(4.1) 

Caution: This relation should be not confused with the Frobenius unit relation (see Section 2.5) which is stronger and eliminates the dot. Furthermore, any portion with no cycles (called a *tree*), which does not contain any boundary vertex can be contracted as in the following picture.



(4.2)

This relation is obtained using a finite number of times the relation 4.1 for a small distance. This can be done without touching the dark line by compactness of portions in $\mathcal{D}$. Furthermore a cycle and its respective interior region can be contracted.



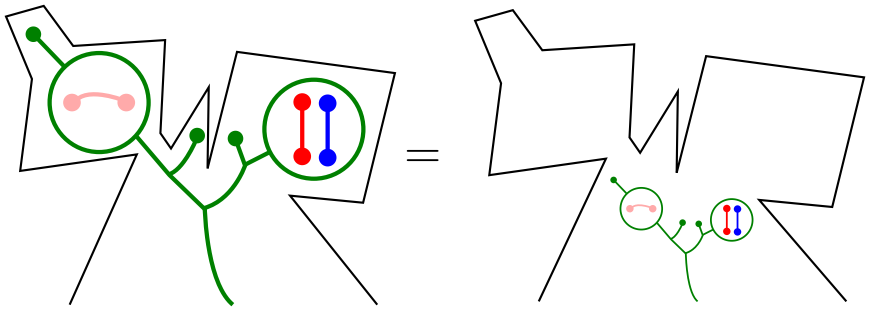
(4.3)

All relations in the remark above can be translated in a single topological claim.

Claim 4.7 (Contraction to a neighborhood of a point)**.** *Let $\varepsilon > 0$ and a a point of $\mathcal{D}$. Let $\mathcal{D}$ be an S-graph. If $\mathrm{por}_{\mathcal{D}}(a)$ does not contain any boundary vertex (except maybe for a itself), then there is an S-graph $\mathcal{E}$ such that:*

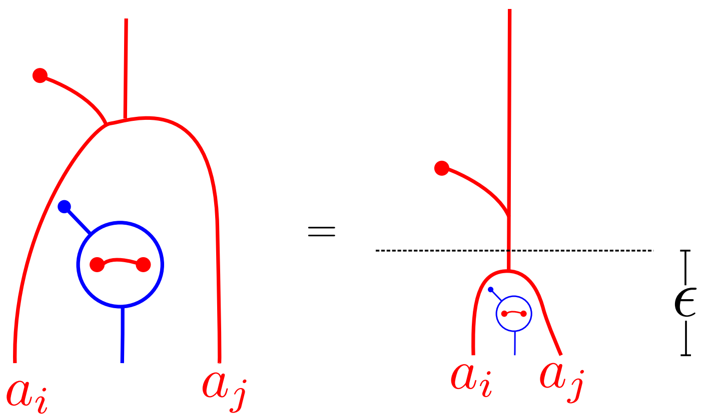
(1) The images of $\mathcal{D}$ and $\mathcal{E}$ are equal in $\mathbb{S}\mathcal{D}$.
(2) The connected portion $\mathrm{por}_{\mathcal{E}}(a)$ is completely contained in the open ball with radius ε and center a.
(3) The complement of $\mathrm{por}_{\mathcal{E}}(a)$ relative to $\mathcal{E}$ is equal (as S-graphs) to the complement of $\mathrm{por}_{\mathcal{D}}(a)$ relative to $\mathcal{D}$.

The proof of claim 4.7 follows by the application of the isotopy relations in $\mathbb{S}\mathcal{D}$ and compactness of $\mathbb{S}\mathcal{D}$. This is a standard topological fact about tubular neighborhoods and we will not prove it here. An example of an application of this claim is the following identity.



(4.4)

Analogously, we have another useful relation of contraction, we will not state it. It enables us to move some parts of our S-graph near to the bottom (or top) boundary, i.e., in a region of the form $\mathbb{R} \times [0, \epsilon]$ for a small ϵ. We will just show a picture of this.



(4.5)

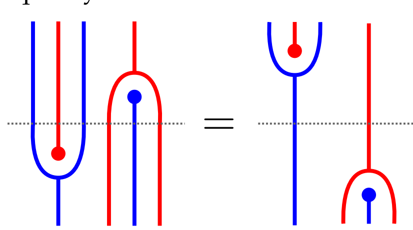
Here we also used the Frobenius associativity relation (see Section 2.5), in order to move the red dot to the top of the diagram.

4.2. **Criterion for being zero.** We will start with the following illustrative example:

Example 4.8. Let U_2 the universal dihedral group with generators s and t. Let $x = stst$ and $\underline{w} = (s, t, s, t, s, t)$. Consider the binary strings $\underline{d} = (1, 1, 1, 1, 0, 0)$ and $\underline{e} = (1, 0, 0, 1, 1, 1)$. We will look at the composition of the two light leaves

$$LL_{\underline{w},\underline{e}} \circ \overline{LL_{\underline{w},\underline{d}}} \in \mathrm{End}^{\bullet}_{\not< x}(x).$$

We have the following equality.



On the left-hand side, we have used relation 4.5 two times. We also used Claim 4.7 two times, the first time in a_3 and the second time in a_6. This was done in order to obtain the right-hand side. The middle gray line represents $\underline{w}$ on the left-hand side and (s,t) on the right-hand side. Since $st < x$ we conclude that

$$LL_{\underline{w},\underline{e}} \circ \overline{LL_{\underline{w},\underline{d}}} = 0$$

in $\mathbb{SD}^{\not< x}$. In particular

$$I_{\underline{w},x}(LL_{\underline{w},\underline{f}} \circ \overline{LL_{\underline{w},\underline{e}}}) = 0.$$

Let us characterize those phenomena by the following lemma.

Lemma 4.9 (Zero criterion). *Let $\mathcal{D}$ be the S-graph representing a morphism f in $\mathrm{End}^{\bullet}_{\not< x,k}(x)$ and $\ell(x) = r$. If there is some $j \in \{1, 2, \ldots, r\}$ such that*

$$\mathrm{por}(a_j) \neq \mathrm{por}(a_{r+j})$$

then $f = 0$. (Recall that $b_j := a_{r+j}$)

Proof. We will analyze the three different cases in which the hypothesis holds.

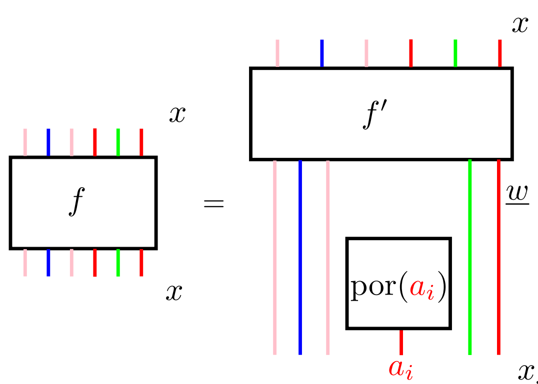
(1) Suppose that there is an $i \in \{1, 2, \ldots, r\}$ such that for all $k \in \{1, 2, \ldots, 2r\}$ with $k \neq i$

$$\mathrm{por}(a_i) \neq \mathrm{por}(a_k).$$

or for all $k \in \{1, 2, \ldots, 2r\}$ with $k \neq i$

$$\mathrm{por}(a_{r+i}) \neq \mathrm{por}(a_k).$$

By Claim 4.7 applied to a_i we have the following equality



Where $\underline{w}$ is the subexpression of x by suppressing a_i and f' is a morphism from $\underline{w}$ to x. Then clearly $\underline{w}_{\bullet} < x$. Therefore f factors through a lower term, then $f = 0$.

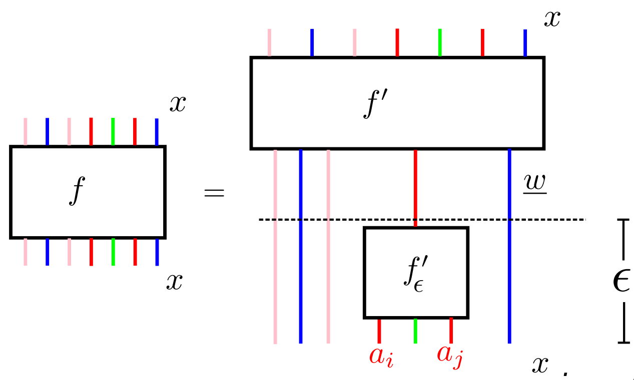
(2) Suppose there are $i, j \in \{1, 2, \ldots, r\}$ such that $i \neq j$ and

$$\mathrm{por}(a_i) = \mathrm{por}(a_j)$$

or

$$\mathrm{por}(a_{r+i}) = \mathrm{por}(a_{r+j}).$$

Then we have two bottom vertices or two top vertices connected. By relation 4.5 applied to some trivalent vertex connecting a_i and a_j we have the following equality



Where $\underline{w}$ is the subexpression of x by suppressing $a_{i+1}, \dots, a_j$ and f' is a morphism from $\underline{w}$ to x. Then clearly $\underline{w}_\bullet < x$. Therefore f factors trough a lower term, then $f = 0$. We cannot assure that we can suppress a_i itself, since in this case a_i may be connected with other boundary vertices.

(3) Suppose that the first two cases do not occur. Then f induces a well-defined function f^* from the ordered r-tuple (a_i) to the ordered r-tuple (b_i) (remember that $b_i := a_{r+i}$), where $f(a_i)$ is b_j if a_i is connected with a_j. This is a well-defined function since, by assumptions of this case, each a_j is forced to be connected to some b_j. This function is non-decreasing by the intermediate value theorem because curves cannot cross between them in a universal S-graph, the last sentence is true since there are no $m_{s,t}$-valent vertices. The unique function f^* satisfying these conditions is the function such that for all $j \in \{1, 2, \dots, r\}$

$$f^*(a_i) = b_i.$$

In particular, for all $j \in \{1, 2, \dots, r\}$ we have

$$\mathrm{por}(a_j) = \mathrm{por}(a_{r+j}).$$

Which is a contradiction with the hypothesis.

There are no more cases if the hypothesis holds. □

Remark 4.10. When the hypothesis of the previous lemma fail (i.e., f^* defined above is determined by the rule $f^*(a_i) = b_i$), the graph is divided into $r+1$ regions R_j, which are the connected components of the complement of $\mathcal{D}$ in $\mathbb{R} \times [0,1]$. Where R_j is the connected component containing the point $a_i - \epsilon \times [0,1]$ if $j < r+1$ and R_{r+1} is the connected component containing the point $a_r + \epsilon \times [0,1]$ for ϵ small. Then, we have for all $i \in \{1, 2, \dots, r\}$

$$\mathrm{por}(a_i) = \mathrm{por}(a_{r+i}).$$

Furthermore, since x is a reduced expression, we have for all $i, j \in \{1, 2, \dots, r\}$ such that $i \neq j$ that

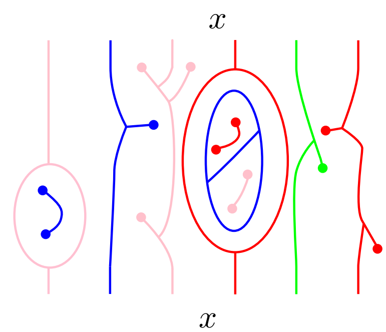
$$\mathrm{por}(a_i) \neq \mathrm{por}(a_j)$$

and

$$\mathrm{por}(a_i) \neq \mathrm{por}(a_{r+j}).$$

This tells us that the connections between the bottom and top vertices are one to one, and there are no more connections.

Example 4.11. Let U_4 be the universal group generated by the set $\{s, t, u, z\}$. Let $x = zsztut$. Here we give an example of a non-zero morphism in $\mathbb{SD}^{\not< x}$.



Note that the strip $\mathbb{R} \times [0, 1]$ is divided into seven different disjoint regions. There are also four interior regions, but they are trapped in cycles. Three of them are part of the big red portion in the middle $\mathrm{por}(a_4)$. The other one lies in the pink portion of a_1.

4.3. **Evil cycles in $\mathrm{End}^{\bullet}_{\not< x,k}(x)$.** Let us make some definitions in order to characterize the cycles appearing in the graphs representing morphisms in $\mathrm{End}^{\bullet}_{\not< x,k}(x)$.

Definition 4.12. Let $\mathcal{D}$ be an S-graph. A *maximal free portion* is a maximally portion which is not connected to any vertex of the boundary. A *free cycle* is a cycle contained in a maximally free portion.

Definition 4.13. Let $\mathfrak{n}$ be a cycle. We define a *portion inside* $\mathfrak{n}$ as a portion contained in the interior region of $\mathfrak{n}$. A *main portion of* $\mathfrak{n}$ is a portion inside $\mathfrak{n}$ which is maximal with respect to this property. The set of main portions of a cycle $\mathfrak{n}$ is denoted by $M(\mathfrak{n})$.

Definition 4.14. An *evil cycle* $\mathfrak{e}$ is either a free cycle (in which case we call it a *free evil cycle*) or a cycle $\mathfrak{e}$ that satisfies

$$\sharp M(\mathfrak{e}) > 1.$$

The *evilness* $E(\mathfrak{e})$ of a free evil cycle $\mathfrak{e}$ is defined by

$$E(\mathfrak{e}) := \sharp M(\mathfrak{e}).$$

The *evilness* $E(\mathfrak{e})$ of a non-free evil cycle $\mathfrak{e}$ is defined by

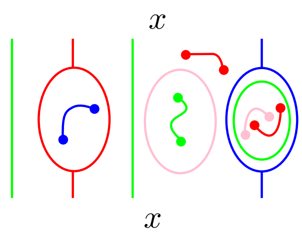
$$E(\mathfrak{e}) := \sharp M(\mathfrak{e}) - 1.$$

Definition 4.15. A cycle $\mathfrak{n}$ is *contained* in a cycle $\mathfrak{m}$ if the interior region of $\mathfrak{n}$ is contained in the interior region of $\mathfrak{m}$.

Definition 4.16. The *depth* of a cycle is defined recursively. We say that a cycle has depth 1 if it is not contained in any other cycle. A cycle has depth $k + 1$ if it is contained in a cycle $\mathfrak{n}_k$ of depth k and it is not contained in any cycle contained in $\mathfrak{n}_k$.

Definition 4.17. Let $\mathcal{D}$ be an S-graph. A *barbell* is an edge with a dot vertex in each of its two endpoints, the edge and the dots have all the same coloration. A barbell is called an *evil barbell* if it is not contained in any interior region of a cycle.

Example 4.18. Let U_4 be the universal group generated by the set $\{s, t, u, z\}$. Let $x = utus$. Here we give an example of a non-zero morphism in $\mathbb{SD}^{\not< x}$.



This morphism does not belong to $\mathrm{End}^{\bullet}_{\not< x}(x)$ since it is a degree six morphism, then it lives in $\mathrm{Hom}_{\not< x}(x, x(6))$. It divides the strip into five free regions. The red cycle more to the left is non-evil and non-free cycle since it contains only one blue main portion and the red portion which contains it is connected to two red boundary vertices. This red cycle has a depth equal to one since there are no other cycles containing it. The other depth-one cycles are the free pink cycle and the non-free blue cycle at the right. The blue cycle is not evil since it contains only one green main portion. The green cycle is contained in the blue cycle, then the green cycle has depth two. It is an evil cycle since contains two main portions: a pink and a red barbell. The lonely red barbell at the top is free, then it is an evil barbell. Finally, the pink cycle is evil, even though it has only one main green portion, it is an evil cycle since it is free, i.e., its associated portion does not touch any boundary vertex.

We have the following lemma.

Lemma 4.19. *In $\mathbb{SD}$, let α_s be the map from e to e given by the left multiplication by α_s, where $s \in S$, then any map of the form $\mathbb{1}_{\underline{w}} \otimes \alpha_s \otimes \mathbb{1}_{\underline{v}}$ where $\underline{w}$ and $\underline{v}$ are two expressions (not necessarily reduced) has degree two.*

Proof. Since α_s is a barbell and $\mathbb{1}_{\underline{w}}$ consist of a finite number $\ell\underline{w}$ of vertical lines, then $\mathbb{1}$ has degree zero and α_s has degree two. Since the degree of a tensor of morphisms is the sum of degrees. The resulting map has degree two. □

Corollary 4.20. *If $\mathcal{D}$ is an S-graph of a non-zero morphism $f \in \mathrm{End}^{\bullet}_{\not< x}(x)$, then $\mathcal{D}$ has no evil barbells.*

Proof. Suppose that $\mathcal{D}$ has at least one evil barbell. Let $\varepsilon > 0$. We can contract the barbell by Claim 4.7, making the diameter of it less than ε and move it to the top of the diagram without touching any portion of $\mathcal{D}$. Then we can write

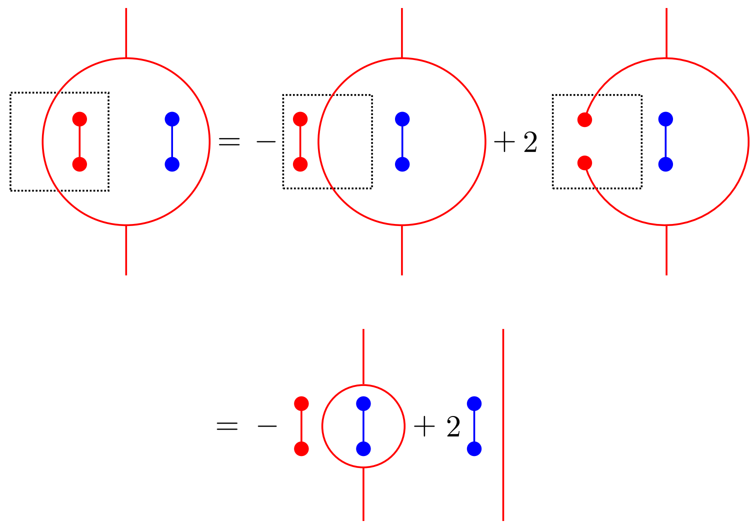
$$f = (\mathbb{1}_{\underline{w}} \otimes \alpha_s \otimes \mathbb{1}_{\underline{v}}) \circ g,$$

where g has an S-graph given by the complement of the evil barbell. However, by Lemma 4.19, g is forced to have degree equal to -2, which contradicts Corollary 3.7. □

Notation 4.21. Let $\mathcal{D}$ be an S-graph. We denote by $E(d, e)$ to the number of evil cycles of $\mathcal{D}$ with evilness e and depth d. Define $n(d, e)$ to be the cardinal of $E(d, e)$.

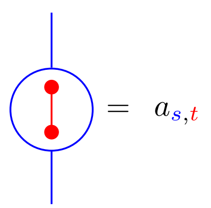
Using the relations 2.9 we can reduce evilness and depth of cycles. As in the following example.

Example 4.22. Let U_2 the universal dihedral group generated by $\{s, t\}$. We have the following identity:



(4.6)

Where in the dashed box we replaced the left-hand side of the first equation of 2.9 by the right-hand side of it. Also, we used the Frobenius unit relation (see Section 2.5) to contract those two red dots. Analogously, we have the following relation.



(4.7)

But we used here the second equation of 2.9 and the neddle relation to kill the second term, see Section 2.5.

Let $\mathcal{D}$ an S-graph. For r, p in S let us define the set $I^{\mathcal{D}}_{r,p}$ as follows:

$$I^{\mathcal{D}}_{r,p} := \{\text{non evil cycles of } \mathcal{D} \text{ colored by } r \text{ and main portion colored by } p\}.$$

As a consequence, we have the main lemma of this section.

Lemma 4.23. *If $\mathcal{D}$ is an S-graph of a non-zero morphism $f \in \mathrm{End}^{\bullet}_{\not< x}(x)$ with non-evil cycles. Then we have the following formula:*

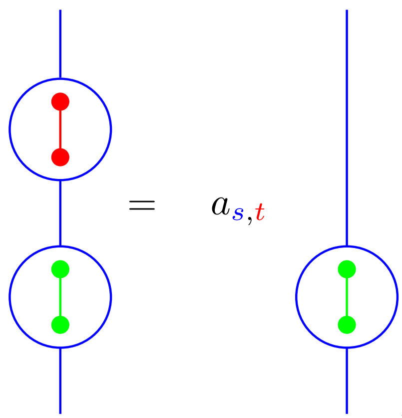
$$f = \prod_{(r,p)\in S^2} (a_{r,p})^{\sharp I^{\mathcal{D}}_{r,p}} \cdot \mathbb{1}_x$$

Proof. We will proceed by induction in the number of cycles. The set of cycles is equal to

$$\bigcup_{(r,p)\in S^2} I_{r,p}.$$

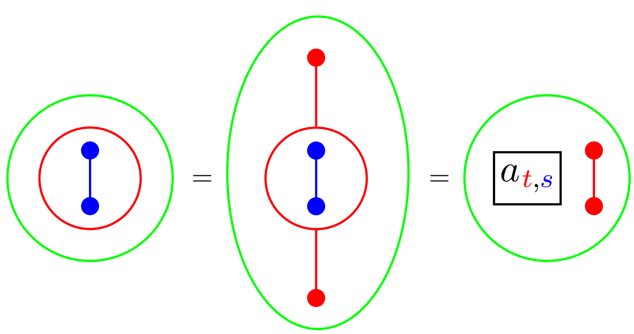
The base case is when we have no cycles, then by Remark 4.10 and Corollary 4.20 we have no free components and all boundary vertices are connected in a bijective correspondence $a_j \leftrightarrow a_{\ell(x)+j}$. Then by isotopy relations, we have that f has only $\ell(x)$ vertical parallel lines without cycles, then $f = \mathbb{1}$. Let $\mathcal{D}$ be an S-graph as in the hypothesis. Let d be the maximal depth of a cycle in $\mathcal{D}$. We pick a cycle $\mathfrak{g}$ of depth d colored by r, since it is non-evil it contains only one main portion colored by r. That main portion cannot have a cycle by maximality of depth, therefore it is a tree. By the Frobenius unit relation (see Section 2.5), we can reduce it to a barbell. We have two possible situations:

(1) If $d = 1$, then we apply we have the following situation

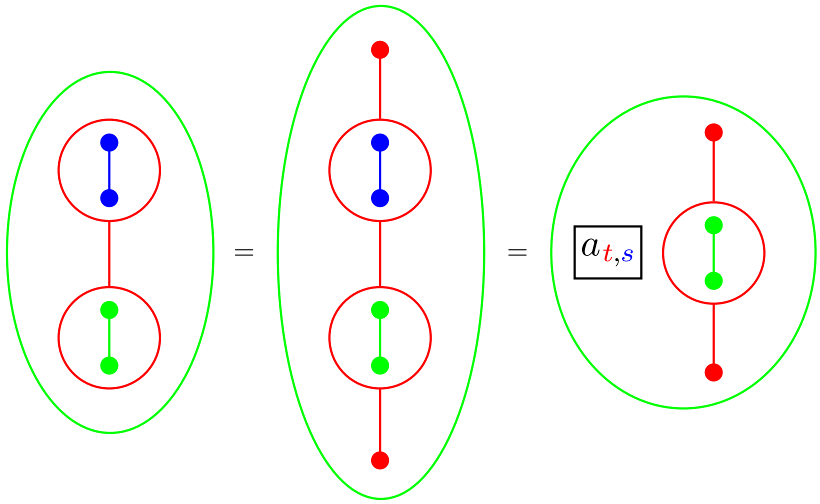


Where we have applied 4.7 to the cycle near the top. In the picture, $r = s$ and $p = t$.

(2) If $d > 1$, then the cycle $\mathfrak{g}$ is contained as a maximal cycle inside another cycle $\mathfrak{n}$ colored by q, then we have two possible sub-scenarios
   (a) The cycle $\mathfrak{g}$ is the unique cycle contained in $\mathfrak{n}$.



(b) The cycle $\mathfrak{g}$ is not the unique cycle contained in $\mathfrak{n}$.



.

Where in both sub-scenarios we have applied 4.7 to the red cycle (of course, with colors changed). In the pictures, $q = u$, $r = t$ and $p = s$. In both cases the cycle $\mathfrak{n}$ remains being colored by the same way, it still is colored by q and contains a unique main portion colored by r.

In all cases, we obtained a graph $\mathcal{E}$ with:

(a) The same cycles as $\mathcal{D}$, with same colorations except a cycle belonging to $I_{r,p}$.
(b) An additional boxed $a_{r,p}$ in some part of the diagram. Denote by $\mathcal{B}$ to the diagram $\mathcal{E}$ without the boxed $a_{r,p}$.

By the Nil Hecke relation (see Section 2.5), the number $a_{r,p}$ can pass through all lines and go to the left free region of the diagram. Then $f = a_{r,p} \cdot g$ where g can be represented by the diagram $\mathcal{B}$. By induction hypothesis,

$$g = (a_{r,p})^{\sharp I^{\mathcal{D}}_{r,p} - 1} \cdot \prod_{(q,t)\in S^2} (a_{q,t})^{\sharp I^{\mathcal{D}}_{q,t}} \cdot \mathbb{1}_x.$$

But $f = a_{r,p} \cdot g$. By replacing g in this equation we finally obtain the formula.

□

We will also prove a kind of converse of the last lemma.

Lemma 4.24. *If $\mathcal{D}$ is an S-graph of a non-zero morphism $f \in \mathrm{End}^{\bullet}_{\not<x}(x)$, then $\mathcal{D}$ has no evil cycles.*

Proof. The conclusion is equivalent to say that for all $d \in \mathbb{N}$ and for all $e \in \mathbb{N}$ the set $E(d,e)$ is empty. We will proceed by contradiction, in other words, we will assume that

$$\bigcup_{d\in\mathbb{N}} \bigcup_{e\in\mathbb{N}} E(d,e) \neq \emptyset.$$

Since graphs are finite, there is a maximum d_0 such that

$$\bigcup_{e\in\mathbb{N}} E(d_0,e) \neq \emptyset.$$

By the same reason we can take e_0 such that

$$e_0 = \max\{e \in \mathbb{N} : E(d_0, e) \neq \emptyset\}.$$

We will prove that there is a function g with a lower degree than g with just evil cycles in the set $E(1,1)$. But this follows straightforward since $e_0 > 1$ we pick a cycle in $E(d_0, e_0)$, and delete all $e_0 - 1$ maximal portions contained in this cycle except one. This gives us a valid new graph with

$$e_1 = \max\{e \in \mathbb{N} : E(d_0, e) \neq \emptyset\} < e_0.$$

As a consequence of 3.5, the deleted portions cannot have a negative degree (they have degree two), then the degree of the new diagram must be less than or equal to the original degree which is zero, then this new diagram has a negative degree We can repeat this argument until

$$\bigcup_{e \in \mathbb{N}} E(d_0, e) = \emptyset.$$

Then we can define.

$$d_1 = \max\left\{d \in \mathbb{N} : \bigcup_{e \in \mathbb{N}} E(d, e) \neq \emptyset\right\} < d_0.$$

And repeat this argument until

$$\bigcup_{d \in \mathbb{N}} \bigcup_{e \in \mathbb{N}} E(d, e) \neq \emptyset.$$

Then the resulting graph has no evilness and has a strictly negative degree. But by the Lemma 4.23 this resulting graph has a formula, which corresponds to a degree zero morphism, a contradiction. □

5. Intersection forms formula in U_n

In this section, we will prove the main theorem of this thesis.

Theorem 5.1 (Main theorem). *Let $\underline{w}$ be a reduced expression of length r. Let $x \in U_n$. Let $\underline{f}$ and $\underline{e}$ be two binary expressions of length r. Let $\mathcal{D}$ an S-graph associated to $LL_{\underline{w},\underline{f}}$. Then*

$$I_{\underline{w},x}\left(LL_{\underline{w},\underline{f}} \circ \overline{LL_{\underline{w},\underline{e}}}\right) = \begin{cases} 0, & \text{if } \exists j \mid \operatorname{por}_{\mathcal{D}}(a_j) \neq \operatorname{por}_{\mathcal{D}}(a_{r+j}), \\ \displaystyle\prod_{(r,p) \in S^2} a_{r,p}^{\sharp I^{\mathcal{D}}_{r,p}}, & \text{otherwise}. \end{cases}$$

The intersection form $I_{\underline{w},x}$ is completely determined by those numbers.

Proof. The first case comes from Corollary 4.9. The second case follows by Lemma 4.23 and Lemma 4.24. The last phrase comes from Theorem 3.9. □

References

[BBD82] Alexander A Beilinson, Joseph Bernstein, and Pierre Deligne. "Faisceaux pervers. Analysis and topology on singular spaces, I (Luminy, 1981), 5–171". In: *Astérisque* 100 (1982).

[CM02] Mark Andrea A de Cataldo and Luca Migliorini. "The hard Lefschetz theorem and the topology of semismall maps". In: *Annales Scientifiques de lécole Normale Supérieure*. Vol. 35. 5. Elsevier. 2002, pp. 759–772.

[CU13] Evan Chen and Francisco Unda. "Diagrammatic Computation of Morphisms Between Bott-Samelson Bimodules via Libedinsky's Light Leaves". In: (2013).

[Dem74] Michel Demazure. "Désingularisation des variétés de Schubert généralisées". In: *Ann. Sci. École Norm. Sup.(4)* 7.1 (1974), pp. 53–88.

[Dye88] Matthew Dyer. "On some generalisations of the Kazhdan-Lusztig polynomials for universal Coxeter systems". In: *Journal of Algebra* 116.2 (1988), pp. 353–371.

[EK11] Ben Elias and Mikhail Khovanov. "Diagrammatics for Soergel categories". In: *International Journal of Mathematics and Mathematical Sciences* 2010 (2011).

[EL14] Ben Elias and Nicolas Libedinsky. "Soergel bimodules for universal Coxeter groups". In: *arXiv preprint arXiv:1401.2467* (2014).

[Eli16] Ben Elias. "The two-color Soergel calculus". In: *Compositio Mathematica* 152.2 (2016), pp. 327–398.

[ESW13] Ben Elias, Noah Snyder, and Geordie Williamson. "On cubes of Frobenius extensions". In: *arXiv preprint arXiv:1308.5994* (2013).

[EW12] Ben Elias and Geordie Williamson. "The Hodge theory of Soergel bimodules". In: *arXiv preprint arXiv:1212.0791* (2012).

[EW16] Ben Elias and Geordie Williamson. "Soergel calculus". In: *Representation Theory of the American Mathematical Society* 20.12 (2016), pp. 295–374.

[Har13] Robin Hartshorne. *Algebraic geometry*. Vol. 52. Springer Science & Business Media, 2013.

[Hum92] James E Humphreys. *Reflection groups and Coxeter groups*. Vol. 29. Cambridge university press, 1992.

[HW15] Xuhua He and Geordie Williamson. "Soergel calculus and Schubert calculus". In: *arXiv preprint arXiv:1502.04914* (2015).

[Iwa64] Nagayoshi Iwahori. "On the structure of a Hecke ring of a Chevalley group over a finite field". In: (1964).

[JMW14] Daniel Juteau, Carl Mautner, and Geordie Williamson. "Parity sheaves". In: *Journal of the American Mathematical Society* 27.4 (2014), pp. 1169–1212.

[JW15] Lars Thorge Jensen and Geordie Williamson. "The p-Canonical Basis for Hecke Algebras". In: *arXiv preprint arXiv:1510.01556* (2015).

[KL79] David Kazhdan and George Lusztig. "Representations of Coxeter groups and Hecke algebras". In: *Inventiones mathematicae* 53.2 (1979), pp. 165–184.

[Lib08a] Nicolas Libedinsky. "Équivalences entre conjectures de Soergel". In: *Journal of Algebra* 320.7 (2008), pp. 2695–2705.

[Lib08b] Nicolas Libedinsky. "Sur la catégorie des bimodules de Soergel". In: *Journal of Algebra* 320.7 (2008), pp. 2675–2694.

[Lib10] Nicolas Libedinsky. "Presentation of right-angled Soergel categories by generators and relations". In: *Journal of Pure and Applied Algebra* 214.12 (2010), pp. 2265–2278.

[Lib15] Nicolas Libedinsky. "Light leaves and Lusztig's conjecture". In: *Advances in Mathematics* 280 (2015), pp. 772–807.

[Loc] *What local system really is*. URL: https://math.stackexchange.com/q/13332 (version: 2012-04-11).

[Lus80] George Lusztig. "Some problems in the representation theory of finite Chevalley groups". In: *Proc. Symp. in Pure Math.* Vol. 37. 1980, pp. 313–317.

[RH16] Steen Ryom-Hansen. "Jucys-Murphy operators for Soergel bimodules". In: *arXiv preprint arXiv:1607.03470* (2016).

[Ric10] Simon Riche. "Perverse sheaves on flag manifolds and Kazhdan-Lusztig polynomials (after Kazhdan-Lusztig, MacPherson, Springer,...)" In: *Notes disponibles en ligne à ladresse http://math. univ-bpclermont. fr/ riche* (2010).

[Soe00] Wolfgang Soergel. "On the relation between intersection cohomology and representation theory in positive characteristic". In: *Journal of Pure and Applied Algebra* 152.1 (2000), pp. 311–335.

[Soe07] Wolfgang Soergel. "Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln" uber Polynomringen". In: *arXiv preprint math/0403496* (2007).

[Soe90] Wolfgang Soergel. "Kategorie , perverse Garben und Moduln über den Koinvarianten zur Weylgruppe". In: *Journal of the American Mathematical Society* 3.2 (1990), pp. 421–445.

[Soe92] Wolfgang Soergel. "The combinatorics of Harish-Chandra bimodules". In: *J. reine angew. Math* 429.49-74 (1992), p. 4.

[Spr82] TA Springer. "Quelque applications de la cohomologie d'intersection, Seminare Bourbaki, expose no 589". In: *Astérisque* 92 (1982), pp. 249–273.

[Ste63] Robert Steinberg. "Representations of algebraic groups". In: *Nagoya Mathematical Journal* 22 (1963), pp. 33–56.

[Wil17] Geordie Williamson. "Schubert calculus and torsion explosion". In: *Journal of the American Mathematical Society* 30.4 (2017), pp. 1023–1046.