REVISION OF FLUID FLOW

Consider a fluid with velocity $\underset{\sim}{u}$ and density ρ.

CONSERVATION OF MASS is a fundamental property, expressed by the equation

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{u}) = 0} \qquad (1)$$

Also called CONTINUITY EQN

For steady flow $\partial/\partial t \equiv 0$

For INCOMPRESSIBLE fluid $\rho(x,y,z) =$ constant

$\Rightarrow$ (1) becomes

$$\boxed{\nabla \cdot \underset{\sim}{u} = 0}$$

Then, we introduce a STREAM FUNCTION $\psi = \psi(x,y)$ defined by

$$\boxed{\frac{\partial \psi}{\partial y} = u} \quad \text{and} \quad \boxed{\frac{\partial \psi}{\partial x} = -v}$$

Then

$$\nabla \cdot \underset{\sim}{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0$$

i.e: $\nabla \cdot \underset{\sim}{u} = 0$ is automatically satisfied.

STREAMLINES are lines of constant ψ. They are curves that have the same direction as the velocity field $\underset{\sim}{u}$.

In STEADY FLOW, the streamline pattern is the same at all times and PARTICLE PATHS are along streamlines.

Also, assume there is no rotation of fluid particles in this flow, i.e: <u>IRROTATIONAL FLOW</u>

$$\Rightarrow \boxed{\nabla \times \underset{\sim}{u} = 0}$$

$\nabla \times \underset{\sim}{u}$ = VORTICITY of the fluid.

Now $\nabla \times \underset{\sim}{u} = \underset{\sim}{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \underset{\sim}{0}$

$$\Rightarrow \boxed{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0} \quad (2)$$

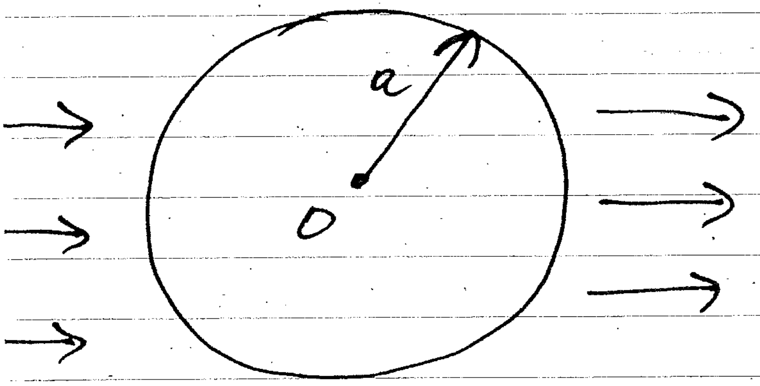
But from the definition of STREAM FN:

(2) becomes $\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$

$$\Rightarrow \boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0}$$

i.e: ψ satisfies LAPLACE'S EQN.

FLOW PAST A CYLINDER



The flow is independent of z

i.e: is 2D.

We want to solve $\boxed{\nabla^2 \psi = 0}$

OUTSIDE a DISK of radius a, with centre at O.

Consider the simplest possible flow, going from left to right

at constant speed U

$$\boxed{\underset{\sim}{u} = (u, v) = (U, 0)}$$

far from the cylinder

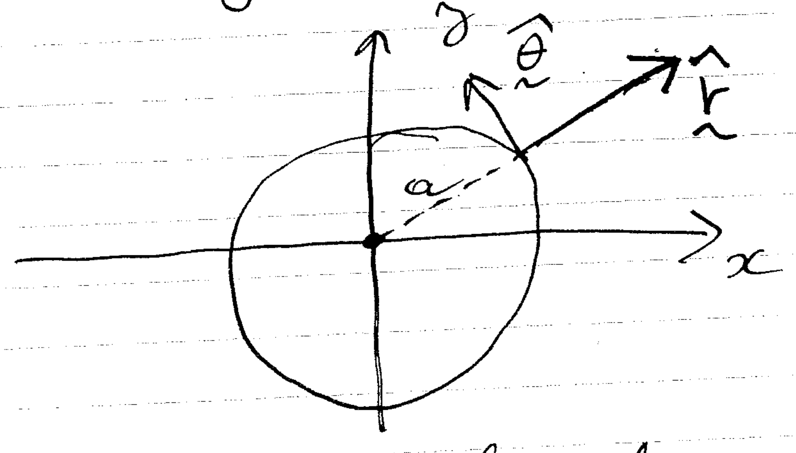
$\therefore$ Far from the cylinder

$u = \psi_y = U \quad , \quad v = \psi_x = 0$

as $r \to \infty$ $\quad \therefore \quad \boxed{\psi = Uy = Ur \sin\theta \text{ as } r \to \infty}$

BC at $r = a$

Physically, it is clear there can be no flow through the surface of the cylinder.
Mathematically it means that the RADIAL COMPONENT of the fluid velocity must be zero at $r = a$



Therefore we have to express the velocity $\underset{\sim}{u} = u\underset{\sim}{i} + v\underset{\sim}{j}$ as $\underset{\sim}{u} = u_r \hat{\underset{\sim}{r}} + u_\theta \hat{\underset{\sim}{\theta}}$

u_r = RADIAL COMPONENT

u_θ = TANGENTIAL COMPONENT

It may be shown (see P - 5.17 NOTES,

It may be shown (P 5.17 NOTES) that the relevant BC at $r=a$ is that: $\boxed{\psi(a,\theta)=0}$

IN SUMMARY

Have to solve $\nabla^2\psi$ in polar coord. with BCs

a) $\psi = Ur\sin\theta, \quad r\to\infty$

b) $\psi(a,\theta)=0, \quad r=a$

The general solution we derived earlier is:

$$\psi(r,\theta) = c_1 \ln r + c_2 + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n\cos n\theta + D_n\sin n\theta\right)$$

Notes: $r=0$ does NOT belong to the solution domain.

$\boxed{C_n = 0}$ (NO $\cos n\theta$ TERM because of a)

$\therefore$

$$\psi(r,\theta) = c_1 \ln r + c_2 + \sum_{n=1}^{\infty} \left(A_n r^n + B_n r^{-n}\right) \sin n\theta$$

Now $\psi(a,\theta) = 0$

$$\Rightarrow \quad c_1 \ln a + c_2 = 0$$

$$\Rightarrow \quad c_2 = -c_1 \ln a$$

Also $A_n a^n + B_n a^{-n} = 0 \qquad n = 1, 2, 3 \ldots$

$$\Rightarrow \quad B_n = -A_n a^{2n}$$

$\therefore$

$$\psi(r,\theta) = c_1 \ln\left(\frac{r}{a}\right) + \sum_{n=1}^{\infty} A_n \left(r^n - \frac{a^{2n}}{r^n}\right) \sin n\theta$$

Finally, since $\lim_{r \to \infty} \psi = U r \sin\theta$

$$\Rightarrow \quad A_n = \begin{cases} U, & n = 1 \\ 0, & n = 2, 3, \cdots \end{cases}$$

$\Rightarrow$ FINAL ANSWER IS

$$\boxed{\psi(r,\theta) = C_1 \ln \frac{r}{a} + U\left(r - \frac{a^2}{r}\right)\sin\theta}$$

$$\psi(r,\theta) = c_1 \ln \frac{r}{a} + \sum_{n=1}^{\infty} A_n (r^n - \frac{a^{2n}}{r^n}) \sin n\theta. \tag{2.5.54}$$

In order for the fluid velocity to be approximately a constant at infinity with $\psi \approx Uy = Ur \sin\theta$ for large r, $A_n = 0$ for $n \geq 2$ and $A_1 = U$. Thus,

$$\boxed{\psi(r,\theta) = c_1 \ln \frac{r}{a} + U \left(r - \frac{a^2}{r} \right) \sin\theta.} \tag{2.5.55}$$

It can be shown in general that the fluid velocity in polar coordinates can be obtained from the stream function: $u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, u_\theta = -\frac{\partial\psi}{\partial r}$. Thus, the θ-component of the fluid velocity is $u_\theta = -\frac{c_1}{r} - U(1+\frac{a^2}{r^2})\sin\theta$. The **circulation** is defined to be $\int_0^{2\pi} u_\theta r \, d\theta = -2\pi c_1$. For a given velocity at infinity, different flows depending on the circulation around a cylinder are illustrated in Figure 2.5.3.

The **pressure** p of the fluid exerts a force in the direction opposite to the outward normal to the cylinder $(\frac{x}{a}, \frac{y}{a}) = (\cos\theta, \sin\theta)$. The **drag** ($x$-direction) and **lift** (y-direction) forces (per unit length in the z direction) exerted by the fluid on the cylinder are

$$\boldsymbol{F} = -\int_0^{2\pi} p\,(\cos\theta, \sin\theta)\, a\, d\theta. \tag{2.5.56}$$

For steady flows such as this one, the pressure is determined from **Bernoulli's condition**

$$p + \frac{1}{2}\rho\, |\boldsymbol{u}|^2 = \text{ constant}. \tag{2.5.57}$$

10

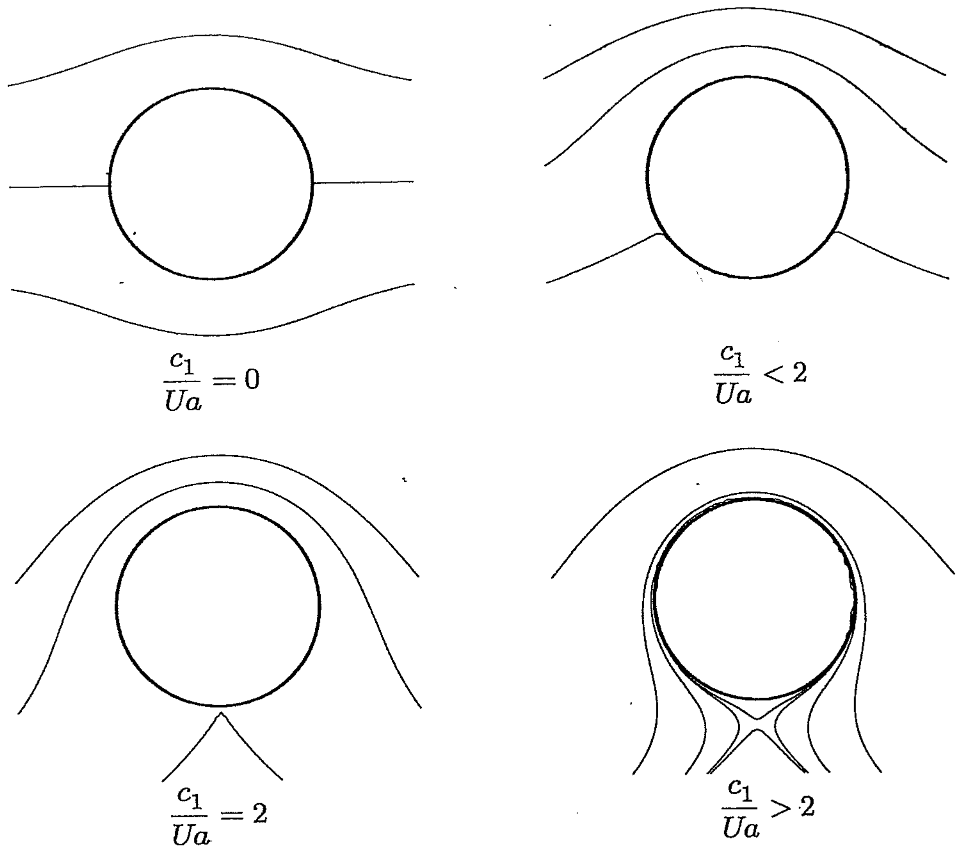


Figure 2.5.3 Flow past cylinder and lift = $2\pi\rho c_1 U$.

Thus, the pressure is lower where the velocity is higher. If the circulation is clockwise around the cylinder (a negative circulation), then intuitively (which can be verified) the velocity will be higher above the cylinder than below and the pressure will be lower on the top of the cylinder and hence lift (a positive force in the y-direction) will be generated. At the cylinder $u_r = 0$, so that there $|\boldsymbol{u}|^2 = u_\theta^2$. It can be shown that the x-component of the force, the drag, is zero, but the y-component the lift is given by (since the integral involving the constant vanishes)

$$F_y = \frac{1}{2}\rho \int_0^{2\pi} \left[-\frac{c_1}{r} - U\left(1 + \frac{a^2}{r^2}\right)\sin\theta \right]^2 \sin\theta \, a \, d\theta. \tag{2.5.58}$$

$$F_y = \rho \frac{c_1}{a} U 2 \int_0^{2\pi} \sin^2\theta \, a \, d\theta = \rho 2\pi c_1 U, \tag{2.5.59}$$

which has been simplified since $\int_0^{2\pi} \sin\theta \, d\theta = \int_0^{2\pi} \sin^3\theta \, d\theta = 0$ due to the oddness of the sin function. The lift vanishes if the circulation is zero. A negative circulation (positive c_1) results in a lift force on the cylinder by the fluid.

In the real world the drag is more complicated. Boundary layers exist due to the viscous nature of the fluid. The pressure is continuous across the boundary layer so that the preceding analysis is still often valid. However, things get much more complicated when the boundary layer separates from the cylinder, in which case a more substantial drag force occurs (which has been ignored in this elementary