MATH2965

Fourier Transforms

Fourier transforms

Recall

The **complex Fourier series** of $f(x)$ is given by:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{in\pi x}{L}}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(k) e^{\frac{in\pi k}{L}} dk$$

Fourier transforms

Substitute the integral for the coefficient into the sum,

$$f(x) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2L} \int_{-L}^{L} f(k) e^{\frac{in\pi k}{L}} \, dk \right] e^{-\frac{in\pi x}{L}}$$

Introduce the variable $\omega_n = \frac{n\pi}{L}$

Then $\boxed{\frac{\Delta\omega}{2\pi} = \frac{\omega_{n+1}}{2\pi} - \frac{\omega_n}{2\pi} = \frac{1}{2L}}$

$$f(x) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-L}^{L} f(k) e^{i\omega_n k} \, dk \right] e^{-i\omega_n x} \, \Delta\omega$$

Fourier transforms

Consider now the limit as $L \to \infty$, then $\boxed{\Delta\omega \equiv \frac{\pi}{L} \to 0}$

We can therefore interpret the sum as a
Riemann sum and in the limit it is replaced by an integral
with respect to the continuous variable ω

Ie, $$f(x) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(k) e^{i\omega k} \, dk \right] e^{-i\omega x} \, d\omega$$

This is the **Fourier Integral Identity**

Fourier transforms

We now define the **Fourier Transform** of $f(x)$ as

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$$

Note the change of dummy variable from *k* to *x*.

Then from the integral identity we define the **Inverse Transform**

$$f(x) = \int_{-\infty}^{\infty} F(\omega)\,e^{-i\omega k}\,d\omega$$

Existence: $f(x)$ must be piecewise smooth and absolutely integrable: $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$

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Fourier transforms

Alternative notation

$$F(\omega) = \mathcal{F}\{f(x)\}$$

$$f(x) = \mathcal{F}^{-1}\{F(\omega)\}$$

Important:

There are variations in the definition of the Fourier transform and its inverse, especially in the placement of the $\frac{1}{2\pi}$ factor and the sign of the complex exponential.

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Inverse Fourier transform of a Gaussian

To find the solution of the heat equation and other problems, we will need to find the inverse Fourier transform of the function

$$\boxed{G(\omega) = e^{-\alpha\omega^2}}$$

This is the well-known bell-shaped curve known as a **Gaussian**.

By definition, the inverse transform is given by

$$g(x) = \int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega x}\, d\omega$$
$$= \int_{-\infty}^{\infty} e^{-\alpha\omega^2}\, e^{-i\omega x}\, d\omega$$

Inverse Fourier transform of a Gaussian

To evaluate this integral we use the following "trick":
First, differentiate with respect to x

$$g'(x) = \int_{-\infty}^{\infty} (-i\omega) e^{-\alpha\omega^2}\, e^{-i\omega x}\, d\omega$$

Next, integrate by parts:

$$g'(x) = \frac{i}{2\alpha}\int_{-\infty}^{\infty} \frac{d}{d\omega}\left(e^{-\alpha\omega^2}\right) e^{-i\omega x}\, d\omega$$
$$= \frac{i}{2\alpha}\left[e^{-i\omega x} e^{-\alpha\omega^2}\right]_{-\infty}^{\infty}$$
$$- \frac{i}{2\alpha}\int_{-\infty}^{\infty} e^{-\alpha\omega^2}(-ix)\, e^{-i\omega x}\, d\omega$$

Inverse Fourier transform of a Gaussian

The first term vanishes as $\omega \to \pm\infty$

Therefore $$g'(x) = -\frac{x}{2\alpha}\int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x}\, d\omega$$

Ie, $$\boxed{g'(x) = -\frac{x}{2\alpha} g(x)}$$

This is a simple separable first order ODE for $g(x)$.

$$\frac{g'(x)}{g(x)} = -\frac{x}{2\alpha} \quad \Longrightarrow \quad g(x) = g(0)e^{-x^2/4\alpha}$$

Inverse Fourier transform of a Gaussian

But $$g(0) = \int_{-\infty}^{\infty} e^{-\alpha\omega^2}\, d\omega$$

and it can be shown that $\int_{-\infty}^{\infty} e^{-\alpha\omega^2}\, d\omega = \sqrt{\frac{\pi}{\alpha}}$

Therefore, the final expression for the inverse transform of the Gaussian is:

$$\boxed{g(x) = \sqrt{\frac{\pi}{\alpha}} e^{-x^2/4\alpha}}$$

which itself is another Gaussian.

The Dirac delta function

Let us consider the sequence of functions

$$\delta_n(x) = \sqrt{\frac{n}{\pi}}e^{-nx^2}, \quad n = 1, 2, 3, \ldots$$

We can show that $\int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}} e^{-nx^2}\, dx = 1, \quad n = 1, 2, 3, \ldots$

We define the Dirac delta function $\delta(x)$ to be the limit of that sequence as $n \to \infty$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}} e^{-nx^2}\, dx = \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

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The Dirac delta function

We can think of the delta function as an infinitely concentrated pulse which is zero everywhere, except at $x = x_0$ where it is ∞

Ie,

$$\delta(x - x_0) = \begin{cases} 0, & x \neq x_0 \\ \infty, & x = x_0 \end{cases}$$

The Dirac delta function has a physical analogy of an "impulsive" force acting for a short time only.

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The Dirac delta function - Properties

1) Filtering property: $\boxed{\int_{-\infty}^{\infty} \delta(x-x_0)f(x)\, dx = f(x_0)}$

2) Operates like an even function:

$$\boxed{\delta(x-x_0) = \delta(x_0-x)}$$

3) Derivative of Heaviside step function $H(x-x_0) = \begin{cases} 0, & x < x_0 \\ 1, & x > x_0 \end{cases}$ is the delta function:

$$\boxed{H'(x-x_0) = \delta(x-x_0)}$$

This can be seen by realizing that $\int_{-\infty}^{x} \delta(\lambda - x_0)\, d\lambda = \begin{cases} 0, & x < x_0 \\ 1, & x > x_0 \end{cases}$

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The Dirac delta function

4) Fourier transform of the delta function

$$\mathcal{F}\{\delta(x-x_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x-x_0)e^{i\omega x}\, dx \;= \frac{1}{2\pi}e^{i\omega x_0}$$

Alternatively: $\mathcal{F}^{-1}\left\{\frac{1}{2\pi}e^{i\omega x_0}\right\} = \delta(x-x_0)$

Ie, $\boxed{\int_{-\infty}^{\infty} e^{-i\omega(x-x_0)}\, d\omega = 2\pi\delta(x-x_0)}$

In the special case where $x_0 = 0$ we get

$$\mathcal{F}\{\delta(x)\} = \frac{1}{2\pi} \quad \text{or} \quad \boxed{\int_{-\infty}^{\infty} e^{-i\omega x}\, d\omega = 2\pi\delta(x)}$$

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Derivatives and Convolution

5) Fourier transforms of derivatives

$$\boxed{\mathcal{F}\{f^{(n)}(x)\} = (-i\omega)^n \mathcal{F}\{f(x)\}}$$

(Proof by integration by parts)

6) Convolution: if $F(\omega) = \mathcal{F}\{f(x)\}$ *and* $G(\omega) = \mathcal{F}\{g(x)\}$

then

$$\boxed{\mathcal{F}^{-1}\{F(\omega)G(\omega)\} = f * g}$$

where

$$f * g = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\lambda) f(x-\lambda)\, d\lambda$$

Is the convolution of f and g.

Heat equation using Fourier transforms

Use Fourier transforms to solve the 1-D Heat equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t > 0$$

with initial condition $u(x,0) = f(x)$

There are implicit physical conditions such as

$$u(x,t) \to 0 \quad as \quad x \to \pm\infty$$

Take the FT with respect to x.

Define

$$U(\omega,t) = \mathcal{F}\{u(x,t)\} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\, dx$$

Heat equation using Fourier transforms

Then $\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{\partial U(\omega,t)}{\partial t}$ $\quad \mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = (-i\omega)^2 U(\omega,t)$

Substitute into the PDE ⟹ Simple ODE: $\frac{\partial U}{\partial t} + k\omega^2 U = 0$

Solving the ODE ⟹ $U(\omega,t) = C(\omega)e^{-k\omega^2 t}$ (*)

Inverting ⟹ $u(x,t) = \int_{-\infty}^{\infty} C(\omega)e^{-k\omega^2 t}\, e^{-i\omega x}\, d\omega$

Heat equation using Fourier transforms

Letting $t = 0$ ⟹ $u(x,0) = \int_{-\infty}^{\infty} C(\omega)\, e^{-i\omega x}\, d\omega$

Then, the initial condition $u(x,0) = f(x)$ gives

$$f(x) = \int_{-\infty}^{\infty} C(\omega)\, e^{-i\omega x}\, d\omega$$

from which we can calculate $C(\omega)$ by inversion.

Ie,

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\, dx \equiv \mathcal{F}\{f(x)\}$$

Heat equation using Fourier transforms

Summarising, the solution of the heat equation in the infinite domain $-\infty < x < \infty$ with IC $u(x,0) = f(x)$ is given by

$$u(x,t) = \int_{-\infty}^{\infty} C(\omega) e^{-k\omega^2 t} e^{-i\omega x}\, d\omega$$

where

$$C(\omega) = \mathcal{F}\{f(x)\}$$

We can now rewrite this as a convolution, but alternatively we can use (*) from the outset:.

Recall that $U(\omega, t) = C(\omega) e^{-k\omega^2 t}$

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Heat equation using Fourier transforms

Hence,
$$u(x,t) = \mathcal{F}^{-1}\left\{C(\omega) e^{-k\omega^2 t}\right\}$$
$$= \mathcal{F}^{-1}\{C(\omega)\} * \mathcal{F}^{-1}\left\{e^{-k\omega^2 t}\right\}$$

But $\mathcal{F}^{-1}\{C(\omega)\} = f(x)$

and (from the inverse of a Gaussian:

$$\mathcal{F}^{-1}\left\{e^{-k\omega^2 t}\right\} = \sqrt{\frac{\pi}{kt}}\, e^{-x^2/4kt}$$

Therefore

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\overline{x}) \sqrt{\frac{\pi}{kt}}\, e^{-(x-\overline{x})^2/4kt}\, d\overline{x}$$

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Heat equation using Fourier transforms

In the special case where $f(x) = \delta(x)$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\bar{x}) \sqrt{\frac{\pi}{kt}} e^{-(x-\bar{x})^2/4kt} \, d\bar{x}$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{kt}} e^{-x^2/4kt}$$

This is called the

fundamental solution of the heat equation

It is the response at time t and position x to an initial input concentrated at $x = 0 \quad and \quad t = 0$