Thursday, February 28 ∗ **Solutions** ∗ *Curves and integration.*

1. Consider the curve C in $\mathbb{R}^3$ given by

$$\mathbf{r}(t) = \left(e^t \cos t\right)\mathbf{i} + 2\mathbf{j} + \left(e^t \sin t\right)\mathbf{k}$$

   (a) Draw a sketch of C.

   Solution. The sketch of C is the following graph.

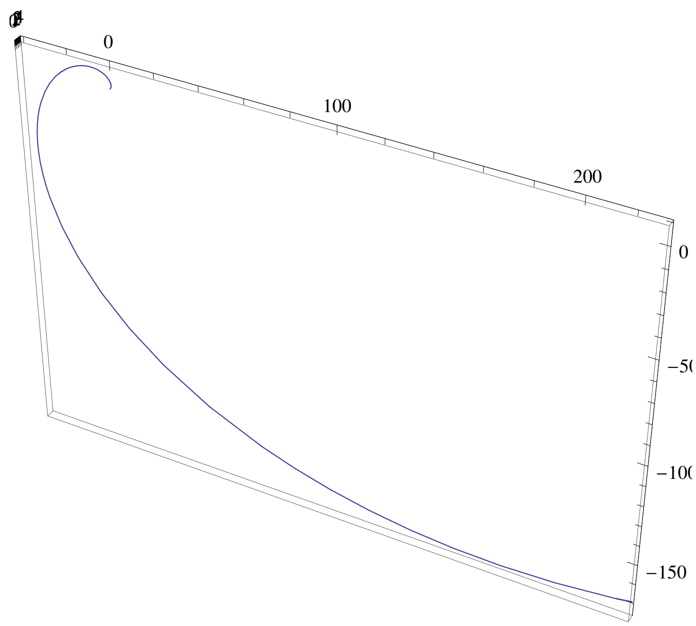

Figure 1: Sketch of C.

   (b) Calculate the arc length function $s(t)$, which gives the length of the segment of C between $\mathbf{r}(0)$ and $\mathbf{r}(t)$ as a function of the time t for all $t \geq 0$. Check your answer with the instructor.

   Solution. Since

$$x'(t) = e^t \cos t - e^t \sin t, \qquad y'(t) = 0, \qquad z'(t) = e^t \sin t + e^t \cos t,$$

   we have

$$|\mathbf{r}'(t)| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} = \sqrt{2}e^t.$$

   Hence the arc length is

$$s(t) = \int_0^t |\mathbf{r}'(u)| du = \int_0^t \sqrt{2}e^u du = \sqrt{2}e^t - \sqrt{2}.$$

(c) Now invert this function to find the inverse function $t(s)$. This gives time as a function of arclength, that is, tells how long you must travel to go a certain distance.

Solution. Solve $s = \sqrt{2}e^t - \sqrt{2}$, which gives $e^t = \frac{s+\sqrt{2}}{\sqrt{2}}$, and so

$$t = t(s) = \ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right).$$

(d) Suppose $h\colon \mathbb{R} \to \mathbb{R}$ is a function. We can get another parameterization of C by considering the composition

$$\mathbf{f}(s) = \mathbf{r}\big(h(s)\big)$$

This is called a *reparametrization*. Find a choice of h so that

i. $\mathbf{f}(0) = \mathbf{r}(0)$

ii. The length of the segment of C between $\mathbf{f}(0)$ and $\mathbf{f}(s)$ is s. (This is called parametrizing by arc length.)

Check your answer with the instructor.

Solution. From (c) we know $t = \ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)$. When $s = 0$, we have $t = \ln 1 = 0$. Then we can choose

$$h(s) = \ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right).$$

(e) Without calculating anything, what is $|\mathbf{f}'(s)|$?

Solution. Since $s = \int_0^s |\mathbf{f}'(u)|du$, then by the fundamental theorem of calculus, we can differentiate both sides with respect to s and get $1 = |\mathbf{f}'(s)|$.

2. Consider the curve C given by the parametrization $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^3$ where $\mathbf{r}(t) = (\sin t, \cos t, \sin^2 t)$.

(a) Show that C is in the intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$.

Solution. Since $x = \sin t, y = \cos t, z = \sin^2 t$, it is very easy to check that $z = x^2$ and $x^2 + y^2 = 1$. So the curve C lies in both these two surfaces, hence is in the intersection of them.

(b) Use (a) to help you sketch the curve C.

Solution. The left graph is the intersection of the two surfaces, while the right one is the curve.

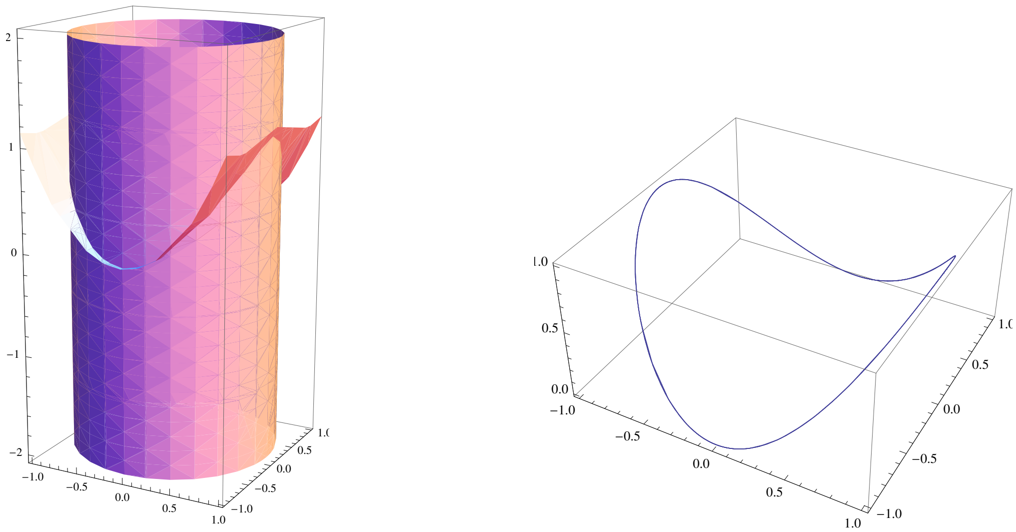

Figure 2: Two surfaces and the curve C.

3. (a) Sketch the top half of the sphere $x^2 + y^2 + z^2 = 5$. Check that $P = \left(1, 1, \sqrt{3}\right)$ is on this sphere and add this point to your picture.

   Solution. The top half of the sphere is shown in Figure 3 (the black dot is P). Since $1^2 + 1^2 + (\sqrt{3})^2 = 5$, we know P is on this sphere.

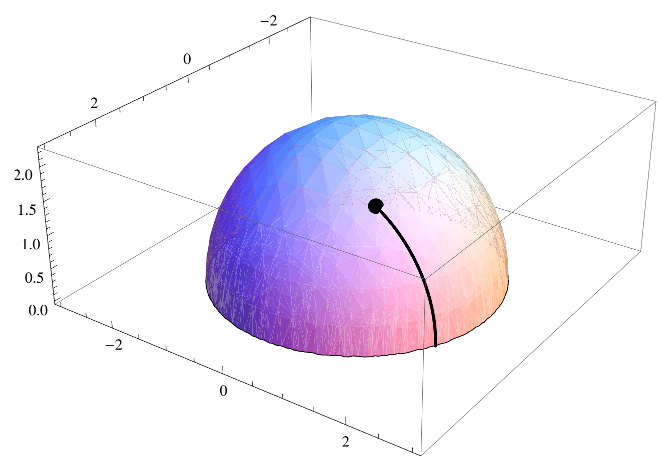

Figure 3: Half sphere and the path.

   (b) Find a function $f(x, y)$ whose graph is the top-half of the sphere. Hint: solve for z.

   Solution. Since $x^2 + y^2 + z^2 = 5$, we have $z^2 = 5 - x^2 - y^2$, and so $z = \pm\sqrt{5 - x^2 - y^2}$. As we only want the top half of the sphere, we can let $f(x, y) = \sqrt{5 - x^2 - y^2}$.

   (c) Imagine an ant walking along the surface of the sphere. It walks *down* the sphere along the path C that passes through the point P in the direction parallel to the yz-plane. Draw this path in your picture.

Solution. The black curve in Figure 3 is the path.

(d) Find a parametrization $\mathbf{r}(t)$ of the ant's path along the portion of the sphere shown in your picture. Specify the domain for $\mathbf{r}$, i.e. the initial time when the ant is at P and the final time when it hits the xy-plane.

Solution. $x = 1$ along the path and $f(1, y) = \sqrt{4 - y^2}$, so setting $y = t$ we get the parametrization

$$\mathbf{r}(t) = (1, t, \sqrt{4 - t^2}).$$

4. As in 1(d), consider a reparametrization

$$\mathbf{f}(s) = \mathbf{r}\big(h(s)\big)$$

of an arbitrary vector-valued function $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^3$. Use the chain rule to calculate $|\mathbf{f}'(s)|$ in terms of $\mathbf{r}'$ and h'.

Solution. By the chain rule, $\mathbf{f}'(s) = \mathbf{r}'(h(s))\ h'(s)$. Taking magnitudes of both sides we have $|\mathbf{f}'(s)| = |\mathbf{r}'(h(s))| \cdot |h'(s)|$.